Letter

Going beyond soft plus virtual

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We present a formalism that sums up the soft-virtual (SV) and next-to-SV (NSV) diagonal contributions to inclusive colorless productions in hadron colliders to all orders in perturbative QCD. Using the factorization theorem and renormalization group invariance as well as employing the transcendental structure of perturbative results, we show the exponential behavior of soft-collinear function. This allows us to predict certain SV and NSV terms to all orders from lower order information. We also present an integral representation for the coefficient functions that is suitable for Mellin N-space resummation.

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I. INTRODUCTION

The tests [1] of the Standard Model (SM) of high energy physics has been going on at the Large Hadron Collider (LHC) with an unparalleled accuracy. Together, precise theoretical predictions of several of the observables with strong and electroweak radiative corrections are also available. Perturbative quantum chromodynamics (QCD) results for both inclusive [2–5] and differential observables to third order in the strong coupling constant play an important role in precision studies. These perturbative results improve our understanding of ultraviolet and infrared structures of the underlying quantum field theory [6–11]. In particular, the factorization properties of amplitudes and cross sections and the corresponding renormalization group (RG) equations shed light on certain universal structures of the underlying dynamics which help us to sum up certain dominant contributions to all orders in perturbation theory [12–20]. The factorization of ultraviolet (UV) and infrared (IR) sensitive terms in Green's function or in the observables bring in unphysical scales, and their RGs are controlled by the universal anomalous dimensions. In the seminal works by Sterman [21] and by Catani and Trentadue [22] the contributions from large logarithms from soft gluons were shown to exponentiate in a systematic fashion. Remarkable success [23–28] in the resummation of soft gluons lead to questions related to the summing up of subleading threshold logarithms, for example logarithms resulting from next to soft-virtual (NSV) contributions. In QCD and in soft-collinear effective theory there have been significant developments to resum NSV terms to all orders [29–41]. In this article, restricting to the diagonal channels of the inclusive production of a colorless particle, we provide a framework to resum next to soft-virtual terms to all orders in perturbation theory using the mass factorization, RG invariance, and transcendentality structure of fixed order predictions. We provide an elaborate discussion on the structure of NSV logarithms and on the resummation formalism in the longer version [42].

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II. FACTORIZATION

We consider the inclusive cross sections for the production of color-singlet final states, such as the production of a single scalar Higgs boson in gluon fusion or in bottom quark annihilation and lepton pair production in the Drell-Yan (DY) process. In the QCD improved parton model, thanks to the well established factorization theorem for the inclusive cross sections, the hadronic cross section $\sigma(q^2,\tau)$ can be expressed in terms of mass factorized partonic coefficient functions (CFs), $\Delta_{ab}(q^2,\mu_R^2,\mu_F^2,z)$, and parton distribution functions (PDFs), $f_c(x_i,\mu_F^2)$, of incoming partons:

$$\sigma(q^2,\tau) = \sigma_0(\mu_R^2)\sum_{ab}\int dx_1\int dx_2 f_a(x_1,\mu_F^2) f_b(x_2,\mu_F^2) \times \Delta_{ab}(q^2,\mu_R^2,\mu_F^2,z), \quad (1)$$

with σ_0 being the born level cross section. The hadronic scaling variable is defined by $\tau = q^2/S$, where S is the square of the hadronic center of mass energy. A similar scaling variable of CF at the partonic level is denoted by $z = q^2/\hat{s}$, with $\hat{s}$ being the partonic center of mass energy. Here q^2 refers to the invariant mass of final state leptons, $M^2_{l^+l^-}$, for DY and for the Higgs boson productions $q^2 = m_H^2$, with m_H being the Higgs boson mass. The subscripts a, b in Δ_{ab} and c in f_c collectively denote the type of parton (quark, antiquark, and gluon), their flavor, etc. The hadronic and partonic scaling variables are related through $\hat{s} = x_1 x_2 S$, which in turn implies $z = \tau/(x_1 x_2)$ with x_i refers to the momentum fraction of the incoming partons.

The inclusive cross sections beyond leading order in perturbation theory contain collinear singularities resulting from massless initial states. The mass factorization theorem allows one to decompose such cross sections in terms of collinear singular but universal/process independent Altarelli-Parisi (AP) kernels [43], Γ_{ab}, and process dependent finite CFs, Δ_{ab}, at an arbitrary factorization scale μ_F:

$$\frac{1}{z}\hat{\sigma}_{ab}(q^2,z,\epsilon) = \sigma_0(\mu_R^2)\sum_{a'b'}\Gamma^T_{aa'}(z,\mu_F^2,) \otimes \Delta_{a'b'}(q^2,\mu_R^2,\mu_F^2,z,\epsilon) \otimes \Gamma_{b'b}(z,\mu_F^2,\epsilon). \quad (2)$$

The PDFs given in (1) are related to bare PDFs $\hat{f}_b$ by AP kernels, i.e., $f_a(\mu_F^2) = \Gamma_{ab}(\mu_F^2) \otimes \hat{f}_b$. The CFs are expanded in powers of coupling constant $a_s(\mu_R^2) = g_s^2(\mu_R^2)/16\pi^2$ as $\Delta_{ab} = \sum_i a_s^i(\mu_R^2)\Delta^{(i)}_{ab}(\mu_R^2)$. The g_s is a renormalized strong coupling constant of QCD and μ_R is the renormalization scale.

The CFs, Δ_{ab}, can be classified into two categories, viz., diagonal (CF$_d$) when $b = \bar{a}$ and off-diagonal (CF$_{nd}$). These CFs depend on two unphysical scales μ_F, μ_R, a physical scale q^2, and the scaling variable z. In the following, we will investigate the all order perturbative structure of CFs in terms of q^2 and the scaling variable z by setting up a Sudakov type of differential equation for CFs in the kinematic region where z is closer to threshold limit $z = 1$. Let us begin with the mass factorization for CF$_d$'s, for example, of the DY process:

$$\frac{\hat{\sigma}_{q\bar{q}}}{z\sigma_0} = \Gamma^T_{qq} \otimes \Delta_{q\bar{q}} \otimes \Gamma_{\bar{q}\bar{q}} + \Gamma^T_{qq} \otimes \Delta_{qg} \otimes \Gamma_{g\bar{q}} + \cdots. \quad (3)$$

If we restrict only to distributions such as $\mathcal{D}_k(z) = (\ln^k(1-z)/(1-z))_+, k \geq 0$, and $\delta(1-z)$, the SV terms and $L_z^k = \ln^k(1-z)$, with $k \geq 0$, called NSV terms, then only the first term in the above expansion survives. The rest of the terms in (3) contains at least one pair of nondiagonal pieces which upon convolutions will give terms of the form $(1-z)^l \ln^k(1-z), l > 0, k \geq 0$. They are called beyond NSV contributions and are not considered in our study.

Further, the diagonal Γ_{qq}'s in the first term in (3) also contain beyond NSV terms, dropping the latter will give rise to a simple form for the Γ_{qq}'s containing only diagonal AP splitting functions, P_{cc}. This is true for $\hat{\sigma}_{b\bar{b}}$ and $\hat{\sigma}_{gg}$ in the threshold limit. In summary, for diagonal channels, the mass factorized result given in (2) contains only diagonal terms $\hat{\sigma}_{c\bar{c}}$, $\Delta_{c\bar{c}}$, and AP kernels Γ_{cc}, and the sum over ab is dropped:

$$\frac{\hat{\sigma}^{\rm sv+nsv}_{c\bar{c}}}{z\sigma_0} = \Gamma^T_{cc} \otimes \Delta^{\rm sv+nsv}_{c\bar{c}} \otimes \Gamma_{\bar{c}\,\bar{c}}. \quad (4)$$

We will show below that this remarkable simplification happens only for the diagonal CFs, allowing us to explore their perturbative structure with the help of the Sudakov $K+G$ type of first order differential equation with respect to q^2.

For an off-diagonal channel, say $\hat{\sigma}_{qg}$, we find

$$\frac{\hat{\sigma}_{qg}}{z\sigma_0} = \Gamma^T_{qq} \otimes \Delta_{qq} \otimes \Gamma_{qg} + \Gamma^T_{qq} \otimes \Delta_{qg} \otimes \Gamma_{gg} + \cdots. \quad (5)$$

In the above expansion, no single term produces distributions after the convolution, since each term contains at least one off-diagonal term. If we then restrict to NSV contributions, those involving at least two off-diagonal pieces do not contribute and hence results in

$$\frac{\hat{\sigma}^{\rm sv+nsv}_{qg}}{z\sigma_0} = \Gamma^T_{qq} \otimes \Delta^{\rm sv+nsv}_{q\bar{q}} \otimes \Gamma_{\bar{q}g} + \Gamma^T_{qq} \otimes \Delta^{\rm sv+nsv}_{qg} \otimes \Gamma_{gg}. \quad (6)$$

Note that the off-diagonal Δ_{qg} receives contributions from $\hat{\sigma}_{qg}$ as well as from $\Delta_{q\bar{q}}$ unlike the diagonal $\Delta_{q\bar{q}}$ which receives only from $\hat{\sigma}_{q\bar{q}}$. This feature makes the diagonal ones simpler than the rest. The rest of the article will only deal with CF$_d$'s, unless otherwise stated.

III. COEFFICIENT FUNCTION

The CF$_d$'s of inclusive cross sections get contributions from form factor (FF) type processes, where the final state contains only colorless particle(s), and from those processes that involve at least one real parton emission. The former from the FF, such as $\hat{F}_c, c = q, g, b$, is proportional to $\delta(1-z)$ and hence can be factored out from $\hat{\sigma}^{\rm sv+nsv}_{c\bar{c}}$ along with the square of UV renormalization constant $Z_{UV,c}$, if any. We call the resulting one by soft-collinear function, that is,

$$\mathcal{S}_c(\hat{a}_s,\mu^2,q^2,z,\epsilon) = (\sigma_0(\mu_R^2))^{-1}(Z_{UV,c}(\hat{a}_s,\mu_R^2,\mu^2,\epsilon))^{-2} \times |\hat{F}_c(\hat{a}_s,\mu^2,-q^2,\epsilon)|^{-2}\delta(1-z) \otimes \hat{\sigma}^{\rm sv+nsv}_{c\bar{c}}(q^2,z,\epsilon). \quad (7)$$

Note that the function $\mathcal{S}_c$ is computable in perturbation theory in powers of $\hat{a}_s$ and is RG invariant with respect to μ_R. Substituting for $\hat{\sigma}_{c\bar{c}}$ from (7) in terms of $\mathcal{S}^c$, in (2) and keeping only the diagonal terms in AP kernels, we obtain $\Delta^{\rm sv+nsv}_{c\bar{c}} \equiv \Delta_c$:

$$\begin{aligned}\Delta_c(q^2,\mu_R^2,\mu_F^2,z) = &(Z_{UV,c}(\hat{a}_s,\mu_R^2,\mu^2,\epsilon))^2\\ &\times|\hat{F}_c(\hat{a}_s,\mu^2,-q^2,\epsilon)|^2\delta(1-z)\\ &\otimes(\Gamma^T)^{-1}_{cc}(z,\mu_F^2,\epsilon)\otimes\mathcal{S}_c(\hat{a}_s,\mu^2,q^2,z,\epsilon)\\ &\otimes\Gamma^{-1}_{\bar{c}\bar{c}}(z,\mu_F^2,\epsilon). \end{aligned} \tag{8}$$

So far, we have shown that if we restrict ourselves to SV + NSV terms in the partonic CFs, the diagonal CFs take a simpler form compared to nondiagonal ones. For the diagonal ones, CF_d's decompose into building blocks such as squares of FF and of UV renormalization constant, soft-collinear function, and diagonal AP kernels.

There is a great deal of understanding of the infrared and UV structure of the FFs through the Sudakov $K+G$ equation [13,44–50] and of the AP kernels through the AP evolution equation in terms of universal anomalous dimensions. For the FF, the factorization of IR singularity implies that $\hat{F}_c(q^2) = Z_{\hat{F}_c}(q^2,\mu_s^2)F_{c,fin}(q^2,\mu_s^2)$, where $Z_{\hat{F}_c}$ is IR singular, $F_{c,fin}$ is IR finite, and the scale μ_s is the IR factorization scale. Differentiation with respect to q^2 leads to the Sudakov $K+G$ differential equation, namely $d\ln\hat{F}_c/d\ln(q^2)=(K_c+G_c)/2$, where the IR singular kernel $K_c(\mu_s^2) = d\ln Z_{\hat{F}_c}/d\ln(q^2)$ and IR finite $G_c(q^2,\mu_s^2) = d\ln F_{c,fin}/d\ln(q^2)$. The solution to the $K+G$ equation

$$\hat{F}_c(-q^2,\epsilon) = \exp\left(\int_0^{-q^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\hat{F},c}(\lambda^2,\epsilon)\right) \tag{9}$$

with $\hat{F}_c(-q^2=0,\epsilon)=1$ and $\Gamma_{\hat{F},c}=(K_c+G_c)/2$ is the kernel. The UV renormalization constant $Z_{UV,c}$ admits a similar exponential solution governed by anomalous dimension $\gamma_{UV,c}$. The latter are known to third order in QCD for $c=b$ (see [51]) and for $c=g$ (see [52]). The AP kernel Γ_{cc} satisfies the AP evolution equation, and in the approximation we work with, they are controlled only by diagonal AP slitting functions P_{cc}. Hence, the all order solution takes the simple form

$$\Gamma_{cc}(\mu_F^2,z,\epsilon) = \mathcal{C}\exp\left(\frac{1}{2}\int_0^{\mu_F^2}\frac{d\lambda^2}{\lambda^2}P_{cc}(\lambda^2,z,\epsilon)\right). \tag{10}$$

The symbol $\mathcal{C}$ is defined in [13]. The AP splitting function is known to third order in perturbation theory and the SV distributions and NSV logarithms present in them are controlled by universal cusp and collinear anomalous dimensions.

IV. SOFT-COLLINEAR FUNCTION

Our next task is to unravel the factorization properties of the soft-collinear function by setting up a differential equation in dimensional regularization. Differentiating both sides of (8) with respect to q^2 and using the $K+G$ equation for the FF, we obtain

$$q^2\frac{d\mathcal{S}_c(q^2,z)}{dq^2} = \Gamma_{\mathcal{S},c}(q^2,z)\otimes\mathcal{S}_c(q^2,z), \tag{11}$$

where

$$\begin{aligned}\Gamma_{\mathcal{S},c} = &q^2\frac{d}{dq^2}(\mathcal{C}\ln\Delta_c(q^2,\mu_R^2,\mu_F^2,z))\\ &-\ln|\hat{F}_c(-q^2)|^2\delta(1-z)).\end{aligned} \tag{12}$$

The fact that $\mathcal{S}_c$ and $\hat{F}_c$ are RG invariant with respect to μ_R and μ_F implies that the derivative with respect to q^2 of Δ_c in the first term in (12) has to be a function of only q^2 and z. While the first term is finite, the second term will be proportional to singular K_c and finite G_c of the kernel $\Gamma_{\hat{F},c}$. This allows us to decompose the kernel $\Gamma_{\mathcal{S},c}$ into a singular $\bar{K}_c$ and finite $\bar{G}_c$ pieces to all orders in perturbation theory and write (11) as $d\mathcal{S}_c(q^2,z)/d\ln(q^2) = \mathcal{S}_c(q^2,z)\otimes(\bar{K}_c(\mu_s^2,z)+\bar{G}_c(q^2,\mu_s^2,z))/2$. We find that $\bar{K}_c$ can depend only on μ_s and process independent anomalous dimension A^c as it is proportional to the K_c of the FF. However, $\bar{G}_c(q^2,\mu_s^2,z)$ will contain the process dependent parts. Here, the scale μ_s is an arbitrary scale. The fact that $\bar{K}_c+\bar{G}_c$ decomposition is valid to all orders in perturbation theory implies that the $\mathcal{S}_c$ is factorizable; i.e., we can write $\mathcal{S}_c(q^2,z) = Z_c(q^2,\mu_s^2,z)\otimes\mathcal{S}_{c,\mathrm{fin}}(q^2,\mu_s^2,z)$ and identify the IR singular $\bar{K}_c = d\ln Z_c/d\ln(q^2)$ and IR finite $\bar{G}_c = d\ln\mathcal{S}_{c,fin}/d\ln(q^2)$. $\mathcal{S}_{c,\mathrm{fin}}$ is IR finite. Z_c is IR singular, and the fact that it depends on $\bar{K}_c(\mu_s^2)$ implies that we can fix only the structure of $\ln(q^2)$ terms in Z_c. However, the complete singular structure of Z_c and its dependence on μ_s and q^2 can be obtained by solving the renormalization group

$$\mu_s^2\frac{dZ_c(\mu_s^2,q^2,z)}{d\mu_s^2} = \gamma_{\mathcal{S},c}(\mu_s^2,q^2,z)\otimes Z_c(\mu_s^2,q^2,z), \tag{13}$$

where $\gamma_{\mathcal{S},c}$ takes the remarkable structure $\xi_1(\mu_s^2,z)\times\ln(q^2/\mu_s^2)+\xi_2(\mu_s^2,z)$ to all orders in perturbation theory. This structure follows from the fact that Z_c has to contain right infrared poles to cancel against those from FF and AP kernels, leaving Δ_c finite. We find that

$$\gamma_{\mathcal{S},c} = \left(A^c(\mu_s^2)\ln\left(\frac{q^2}{\mu_s^2}\right)-\frac{f^c(\mu_s^2)}{2}\right)\delta(1-z)+P'_{cc}(\mu_s),$$

where

$$P'_{cc} = \frac{2A^c(\mu_s^2)}{(1-z)_+}+2C^c(\mu_s^2)\ln(1-z)+2D^c(\mu_s^2). \tag{14}$$

Here $A^c, (D^c, C^c)$ and f^c are cusp, collinear, and soft anomalous dimensions, respectively. The solution to $\mathcal{S}_c$ takes the form

$$\begin{aligned}\mathcal{S}_c(q^2,z,\epsilon) &= \mathcal{C}\exp\left(\int_0^{q^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\mathcal{S}_c}(\lambda^2,z,\epsilon)\right)\\ &= \mathcal{C}\exp(2\Phi_c(q^2,z,\epsilon))\end{aligned} \tag{15}$$

with $\mathcal{S}_c(q^2=0,z,\epsilon) = \delta(1-z)$.

V. TRANSCENDENTALITY PRINCIPLE

In the following, we study the logarithmic structure of Φ_c using the available fixed order results and propose an all order generalization based on their remarkable transcendentality structure. The $\Gamma_{\mathcal{S},c}$ can be determined using Δ_c and $\hat{F}_c$ which are known to third order in a_s and to desired accuracy in ϵ for DY ($c = q$), Higgs boson production in gluon fusion ($c = g$) and in bottom quark annihilation ($c = b$) (see [53–65] and [2–5]). In Δ_cs, the explicit results to third order in a_s show a certain universal structure for leading SV distribution as well as NSV logarithm; for example, at order a_s^i, both of them have degree $2i$ independent of c. Similarly, in FFs, computed in dimensional regularization, if we assign n_ϵ weight for ϵ^{-n_ϵ} and n_ζ for $\ln^{n_\zeta}(1-z)$, then the highest weight at every order in ϵ shows uniform transcendentality $\omega = n_\epsilon + n_\zeta$. Hence, the explicit results for $\Gamma_{\mathcal{S}_c}$ obtained from Δ_c and $\hat{F}_c$ in dimensional regularization also reveal the rich structure for SV distributions and NSV logarithms through transcendental weight.

We now turn to $\mathcal{S}_c$. Note that $\mathcal{S}_c$ is UV finite, and hence a simple dimensional analysis implies that the $\Gamma_{\mathcal{S},c}$ can be expanded in powers of $\hat{a}_s(q^2/\mu^2)^{\epsilon/2}$. The fact that Δ_c is finite implies the soft-collinear function $\mathcal{S}_c$ has to contain right soft and collinear singularities to cancel against those from FF and the AP kernels. These singularities appear as poles in ϵ resulting from the Feynman loop and phase space integrals. In [13,55], the all order structure of the SV part of $\Gamma_{\mathcal{S},c}$ or equivalently the SV part of Φ_c was determined. Here, we generalize this to include the NSV part by modifying $\Gamma_{\mathcal{S},c}$ in such a way that it contains additional collinear sensitive terms that cancel collinear singularities from the NSV part of AP kernels giving rise to the right NSV part of Δ_c. Keeping RG invariance intact, we write

$$\Phi^c(\hat{a}_s, q^2, \mu^2, z, \epsilon) = \Phi_A^c + \Phi_B^c = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \left(\frac{i\epsilon}{1-z}\right) \times \left(\hat{\phi}_c^{SV,(i)}(\epsilon) + (1-z)\hat{\varphi}_c^{(i)}(z,\epsilon)\right), \quad (16)$$

where $S_\epsilon = \exp(\frac{\epsilon}{2}[\gamma_E - \ln(4\pi)])$ with γ_E being the Euler Mascheroni constant. The term $q^2(1-z)^2$ inside the parentheses is the scale corresponding to soft gluon emissions. Note that we have normalized the second term by this soft scale. The functions $\hat{\phi}_c^{SV,(i)}(\epsilon)$ and $\hat{\varphi}_c^{(i)}(z,\epsilon)$ contain poles in ϵ. The first term Φ_A^c containing $(1-z)^{i\epsilon}/(1-z)\hat{\phi}_c^{SV,(i)}(\epsilon)$ is sufficient to obtain the right distributions $\mathcal{D}_j$ and $\delta(1-z)$ in Δ_c, and they constitute to the SV contributions to CF (see [13,14]). The NSV terms $\ln^k(1-z), k = 0, \ldots,$ in Δ_c, on the other hand, are generated from the first as well as the second term Φ_B^c containing $(1-z)^{i\epsilon}\hat{\varphi}_c^{(i)}(z,\epsilon)$. Note that in Φ_A^c, the entire z dependence factors out leaving only $\hat{\phi}_c^{SV,(i)}(\epsilon)$ at every order. This happens because the soft gluons factorize at a single scale, namely $q^2(1-z)^2$ at every order in a_s. Consequently, the entire series containing soft gluon contributions can be summed up to obtain exponential solution $\exp(2\Phi_A^c)$. Explicit computation of the exponent Φ_B^c demonstrates a peculiar dependence on the scaling variable z through $\hat{\varphi}_c^{(i)}(z,\epsilon)$ at every order in $\hat{a}_s$, given an accuracy in ϵ. In Φ_B^c, we find that the highest power of $\ln(1-z)$ is controlled by the order of a_s and the accuracy in ϵ. In particular, if we assign n_ϵ weight for ϵ^{-n_ϵ} and n_L for $\ln^{n_L}(1-z)$, then the highest weight at every order in a_s shows uniform transcendentality $\omega = n_\epsilon + n_L$. For example, at the order a_s, we find $\omega = 1$ irrespective of the accuracy in ϵ, at a_s^2, $\omega = 2$, and so on. If we generalize this uniform transcendentality to all orders, the highest power of $\ln(1-z)$ turns out to be $i + j$,

$$\Phi_B^c = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2}{\mu^2}\right)^{i\frac{\epsilon}{2}} S_\epsilon^i \sum_{j=-i}^{\infty} \sum_{k=0}^{i+j} \hat{\Phi}_k^{c,(i,j)} \epsilon^j \ln^k(1-z). \quad (17)$$

Because of this structure, we find in the successive orders of a_s in Δ_c that there is an increment of two in the power of leading $\ln(1-z)$ terms.

VI. MULTISCALE STRUCTURE

In [66] the CFs were computed up to third order in a_s using the method of threshold expansion in dimensional regularization. Interestingly, for the diagonal channel, $\hat{\sigma}_{gg}$, the results show a remarkable structure in terms of z and ϵ. One finds that $\hat{\sigma}_{gg}$ factorizes into terms of the form $(1-z)^\epsilon$ and functions that depend only on ϵ. Generalization to ith order in a_s gives factorization of the form $\sum_{\eta=2}^{2i}(1-z)^{\eta\epsilon/2}\chi_i^\eta(\epsilon)$. The factor $(1-z)^{\eta\epsilon/2}$ results from soft and collinear configurations of partons at the corresponding soft and collinear scales given by $(q^2(1-z))^{\eta\epsilon/2}$. This allows us to sum up the $\ln(1-z)$ terms in (17) to obtain

$$\Phi_B^c = \sum_{i=1}^{\infty} \hat{a}_s^i \sum_{\eta=2}^{2i} \left(\frac{q^2(1-z)^{\frac{\eta}{i}}}{\mu^2}\right)^{\frac{i\epsilon}{2}} S_\epsilon^i \tilde{\varphi}_{c,\eta}^{(i)}(\epsilon). \quad (18)$$

The form of the above solution inspired from the structure of fixed order results obtained in [66] explicitly reveals the presence of multiple scales. One finds that every collinear parton gives $(1-z)^{\epsilon/2}$ and soft parton gives $(1-z)^\epsilon$, while pure virtual contributions to born amplitude give $\delta(1-z)$ and the hard part from the real emissions gives terms proportional to $(1-z)^\eta, \eta \geq 0$. At given order a_s, we can determine the values of η by counting the allowed soft and collinear configurations in that order. The values of η extracted from results known to third order can be used to extrapolate to obtain the upper limit on η at ith order in a_s and it turns out to be $2i$. The coefficients of the scales $\chi_i^\eta(\epsilon)$ can be expanded in powers of ϵ. The singularity structure in ϵ is completely determined by the finiteness of the mass factorized result. Note that the multiscale structure of the solution is peculiar to the NSV part of the solution.

VII. INTEGRAL REPRESENTATION

Having studied the general structure of Φ_B^c, our next task is to sum up the series to obtain a compact integral representation similar to the SV case. We use Φ_B^c given in (16) to obtain

$$\Phi_B^c = \int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} L^c(a_s(\lambda^2), z) + \varphi_{f,c}(a_s(q^2(1-z)^2), z, \epsilon) + \varphi_{s,c}(a_s(\mu_F^2), z, \epsilon). \quad (19)$$

Here, the first two terms are finite as $\epsilon \to 0$ while $\varphi_{s,c}$ is divergent. Since Φ_B^c is RG invariant, $\varphi_{s,c}$ satisfies the RG equation:

$$\mu_F^2 \frac{d}{d\mu_F^2} \varphi_{s,c}(a_s(\mu_F^2), z) = L^c(a_s(\mu_F^2), z). \quad (20)$$

Further, the fact that Δ_c in (8) is finite at every order in a_s in the limit $\epsilon \to 0$ allows us to determine the coefficients L^c in terms of the NSV coefficients C^c and D^c in AP splitting kernels. We find, at each order in perturbative expansion,

$$L^c(a_s(\mu_F^2), z) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) L_i^c(z)$$
$$\text{with} \quad L_i^c(z) = C_i^c \ln(1-z) + D_i^c, \quad (21)$$

where the coefficients C_i^c and D_i^c are related to those of cusp A_i^c and collinear B_i^c anomalous dimensions (see [55,56,61,67–71] and for beyond three loops, see [64,69,70,72]). The finite part $\varphi_{f,c}$ can be expanded in powers of a_s:

$$\varphi_{f,c}(\lambda^2, z) = \sum_{i=1}^{\infty} a_s^i(\lambda^2) \sum_{k=0}^{i} \varphi_{c,i}^{(k)} \ln^k(1-z), \quad (22)$$

where the highest power of $\ln(1-z)$ is in accordance with the same in (17). Defining $\{\mu_i\} = \mu_R, \mu_F$, we get

$$\Delta_c(q^2, \{\mu_i^2\}, z) = C_0^c(q^2, \{\mu_i^2\}) \mathcal{C} \exp(2\Psi^c(q^2, \mu_F^2, z)), \quad (23)$$

where

$$\Psi^c(q^2, \mu_F^2, z) = \frac{1}{2} \int_{\mu_F^2}^{q^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} P'_{cc}(a_s(\lambda^2), z) + \mathcal{Q}^c(a_s(q^2(1-z)^2), z),$$
$$\text{with} \quad \mathcal{Q}^c(a_s, z) = \left(\frac{1}{1-z} \bar{G}_{SV}^c(a_s)\right)_+ + \varphi_{f,c}(a_s, z). \quad (24)$$

The coefficient C_0^c is a z independent coefficient and is expanded in powers of $a_s(\mu_R^2)$ as $C_0^c(q^2, \mu_R^2, \mu_F^2) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2) C_{0i}^c(q^2, \mu_R^2, \mu_F^2)$. An elaborate discussion on Φ^c can be found in the longer version of the paper [42]. The integral representation given in (23) is suitable for obtaining certain SV and NSV terms to all orders, which subsequently lead to a framework to resum the diagonal NSV terms [42].

VIII. ALL ORDER PREDICTIONS

Given Ψ^c at order a_s, expanding the exponential in powers of a_s we obtain the leading SV terms $(\mathcal{D}_3, \mathcal{D}_2)$, $(\mathcal{D}_5, \mathcal{D}_4), \ldots, (\mathcal{D}_{2i-1}, \mathcal{D}_{2i-2})$ and the leading NSV terms $\ln^3(1-z), \ln^5(1-z), \ldots, \ln^{2i-1}(1-z)$ at $a_s^2, a_s^3, \ldots, a_s^i$, respectively, for all i. Since C_1^c is identically zero, $\ln^{2i}(1-z)$ terms do not contribute for all i. At this stage, we can ask whether these predictions will be affected if we include a second order result for Ψ^c. Since the power of the leading logarithm at ε^j accuracy is $2+j$ and hence at ε^0 order the highest logarithm is $\log^2(1-z)$, we observe that the second order result for Ψ^c will only contribute to subleading logarithms at a_s^2, not to leading ones. A similar prediction at third order will also be unaffected by the third order result for Ψ^c and so on. Now from Ψ^c to order a_s^2, we can predict the tower consisting of $(\mathcal{D}_3, \mathcal{D}_2), (\mathcal{D}_5, \mathcal{D}_4), \ldots, (\mathcal{D}_{2i-3}, \mathcal{D}_{2i-4})$ and of $L_z^4, L_z^6, \ldots, L_z^{2i-2}$ at $a_s^3, a_s^4, \ldots, a_s^i$, respectively, for all i. Note that even though the L_z^4 term is absent at the second order in Ψ^c at the accuracy ε^0, we can predict this term simply because of convolutions between $\mathcal{D}_l$ and L_z^m from first and second order terms in Ψ^c. Generalizing this, if we know Ψ^c up to nth order, we can predict $(\mathcal{D}_{2i-2n+1}, \mathcal{D}_{2i-2n})$ and L_z^{2i-n} at every order in a_s^i for all i.

IX. CONCLUSIONS

In this article, we have set up a formalism to sum up both SV and NSV logarithms of diagonal CFs of inclusive production of a colorless state in hadron colliders to all orders in perturbative QCD. The simple factorization structure helped us to set up a Sudakov type integro-differential equation with respect to q^2 for the soft-collinear function. The latter implies a remarkable factorization of the IR singular part in the soft-collinear function to all orders in perturbation theory. Its solution admits an exponential structure, and thanks to the uniform transcendentality structure for the leading logarithms of CF$_d$'s, we could parametrize the z dependence of the solution at every order in a_s given an accuracy in ϵ. The resulting integral representation for the solution allows us to predict certain SV and NSV terms to all orders from the knowledge of previous order information, and in addition, it will be useful for resummation studies in Mellin-N space. Our result will be useful for phenomenological studies for processes such as Drell-Yan and Higgs boson productions at the LHC.

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