Letter

Generalization of Weinberg's compositeness relations

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We generalize the time-honored Weinberg's compositeness relations by including the range corrections through considering a general form factor. In Weinberg's derivation, he considered the effective range expansion up to $\mathcal{O}(p^2)$ and made two additional approximations: neglecting the nonpole term in the Low equation and approximating the form factor by a constant. We lift the second approximation and work out an analytic expression for the form factor. For a positive effective range, the form factor is of a single-pole form. An integral representation of the compositeness is obtained and is expected to have a smaller uncertainty than that derived from Weinberg's relations. We also establish an exact relation between the wave function of a bound state and the phase of the scattering amplitude neglecting the nonpole term. The deuteron is analyzed as an example, and the formalism can be applied to other cases where range corrections are important.

I. INTRODUCTION

Deciding whether a particle is composite or elementary via low-energy scattering observables had been considered impossible until Weinberg proposed his relations [1] that connect the scattering length a and effective range r with the compositeness of an S-wave shallow bound state. For instance, the deuteron may be a superposition of a composite (molecular) state of two nucleons and a compact component (called an "elementary particle" in the original publication [1] to be distinct from the two-nucleon composite component). The probability of finding the deuteron in the molecular state is called compositeness X. The relations read [1]

$$a = -\frac{2X_W}{1 + X_W} R + O(m_\pi^{-1}),$$ (1)

$$r = -\frac{1 - X_W}{X_W} R + O(m_\pi^{-1}),$$ (2)

where $R = 1/\sqrt{2\mu|E_B|}$, μ is the reduced mass, E_B is the binding energy, which is $E_B = -2.224575(9)$ MeV [2] for the deuteron case, and the $O(m_\pi^{-1})$ terms are due to neglecting the finite-range interactions. We denote the compositeness from the above equations as X_W in order to distinguish it from the one to be derived later. Had the deuteron an appreciable compact component, Weinberg's relations would predict a large and negative r and a small a, which clearly contradict the measured values [3]

$$a = -5.419(7) \text{ fm}, \quad r = 1.766(8) \text{ fm}.$$ (3)

So one can conclude that the deuteron is mostly a composite system of two nucleons. The compositeness X, being a probability, should be in the range [0, 1]. However, the X_W value may go beyond that range. For instance, the above relations lead to a value of 1.68 for the deuteron, indicating a sizable correction. Besides the deuteron, Weinberg's relations and their extensions [4–26] have been widely used to study many near-threshold hadrons (see Ref. [27] for a review).

On the other hand, the wave function of the deuteron has been studied by hundreds of works (see Ref. [28] for a review). Mostly constrained by the scattering phase shift, various models can predict wave functions with similar properties at long distances, suggesting a potential model-independent relationship between the two terms. In fact,

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such a relation exists formally for purely local potentials in the context of the inverse scattering problem (see, e.g., Ref. [29]). In the derivation of Weinberg's relations, the form factor of the deuteron, which encodes its coupling to the proton and neutron, is approximated by a constant (see below), which greatly limits the behavior of the wave function and, therefore, precludes a detailed study on it.

In this Letter, we will generalize the compositeness relations by lifting the constant-form-factor approximation completely. Consequently, the uncertainty of compositeness is considerably reduced, and a relation between the wave function and the scattering phase shift follows. The obtained compositeness for a bound state will always be in the range [0, 1]. We will also derive an analytic expression for the form factor considering the effective range expansion (ERE).

II. DERIVATION

We restrict our discussion to the near-threshold region and thus consider only the S-wave interaction with a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$. Its half-shell T matrix

$$T_{p,k} := \langle p | \hat{T}(h_k + i\varepsilon) | k \rangle, \qquad (4)$$

with p the momentum of one constituent particle in the center-of-mass frame of the two-body system, the normalization $\langle p | k \rangle = (2\pi)^3 \delta(p-k)/p^2$, and the kinetic energy $h_k = k^2/(2\mu)$, is constrained by the Low equation [1]

$$T_{p,k} = V_{p,k} + \frac{g(p)g^*(k)}{h_k - E_B} + \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{T_{p,q} T_{k,q}^*}{h_k + i\varepsilon - h_q}, \qquad (5)$$

where $V_{p,k} := \langle p | \hat{V} | k \rangle$, $E_B < 0$ is the binding energy of the bound state $|B\rangle$, and $g(p) := \langle p | \hat{V} | B \rangle$ is the form factor.

For a shallow bound state with $|E_B| \ll h_\Lambda := \Lambda^2/(2\mu)$, where Λ is a hard momentum scale, the second term on the right-hand side of Eq. (5) is enhanced by $(h_k - E_B)^{-1}$ in the low-momentum regime, where $k \ll \Lambda$ compared with the nonpole term $V_{p,k}$. Then Weinberg made two approximations. The first is to ignore the nonpole term $V_{p,k}$ in the Low equation. The second is to replace the form factor $g(p)$ by a constant g, amounting to keeping only the leading-order term in a nonrelativistic expansion. In the following, we adopt only the first one, i.e., neglecting the nonpole term, which should work well in the near-threshold region dominated by the pole. Note that a nonpole term of the form that can be generated by redefining $g(k)$ is already accounted for by the second term of Eq. (5).

With a separable ansatz,

$$T_{p,k} = t_k g(p) g^*(k), \qquad (6)$$

the Low equation can be solved (details can be found in the Supplemental Material [30]),

$$T_{p,k} = \frac{1}{1 - F(h_k)} \frac{g(p)g^*(k)}{h_k - E_B}, \qquad (7)$$

with

$$F(W) := \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{(W - E_B)|g(q)|^2}{(h_q - E_B)^2 (W - h_q)}. \qquad (8)$$

The solution is obtained without the so-called Castillejo-Dalitz-Dyson zeros [31] as Weinberg did (for a discussion of the impact of such zeros on the compositeness, see Refs. [6,7,32]). Interestingly, the compositeness X shows up as

$$F(\infty) = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|\langle q | \hat{V} | B \rangle|^2}{(h_q - E_B)^2} = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} |\langle q | B \rangle|^2 = X, \qquad (9)$$

where the Schrödinger equation $\langle q | (h_q + \hat{V}) | B \rangle = E_B \langle q | B \rangle$ has been used. The integral in Eq. (8) can be solved in a closed form if $g(q)$ is approximated by a constant, as done in Ref. [1].

One can define, with the convention $\delta_B(0) = 0$,

$$\delta_B(E = h_p) := \arg T_{p,p} = -\arg(1 - F(E + i\varepsilon)), \qquad (10)$$

where δ_B means the phase of the on-shell T matrix $T_{p,p} = t_p |g(p)|^2$ and differs from the full phase shift δ by having neglected the nonpole term in Eq. (5) [for convenience, we take the convention $\delta(0) = 0$, the same as that for δ_B]. Nevertheless, we have $\delta_B \approx \delta$ in the low-momentum regime where the near-threshold pole dominates and the full on-shell T matrix ($\propto e^{i\delta} \sin \delta$) is well approximated. Furthermore, one can show

$$-\pi \le \delta_B \le 0 \qquad (11)$$

by noting $F(0) \le 0$ and $\mathrm{Im} F(E + i\varepsilon) \le 0$ for $E \ge 0$, and $\delta_B(\infty) = 0$ unless $X = 1$.

One can work out a dispersive representation of $F(W)$ (see the Supplemental Material for the derivations [30]),

$$F(W) = 1 - \exp\left(\frac{W - E_B}{\pi} \int_0^\infty dE \frac{-\delta_B(E)}{(E - W)(E - E_B)}\right), \qquad (12)$$

which, combined with Eq. (9), leads to an integral representation of X in terms of the low-energy observable δ_B, which is the phase shift under the approximation neglecting $V_{p,k}$,

$$X = 1 - \exp\left(\frac{1}{\pi} \int_0^\infty dE \frac{\delta_B(E)}{E - E_B}\right). \qquad (13)$$

The above formula presents a generalization of Weinberg's treatment by lifting the leading-order nonrelativistic approximation to the form factor [an expression of the compositeness in terms of the T matrix given by Eq. (5) can be found in Refs. [8,11]]. It ensures that X computed in this way cannot be larger than 1; furthermore, since $\delta_B \le 0$, we also have $X \ge 0$. Thus, $X \in [0,1]$ is ensured, in contrast to that computed from Eqs. (1) and (2).

The form factor can also be constructed from δ_B by noticing

$$\mathrm{Im}F(h_p + i\varepsilon) = -\frac{\pi p \mu}{(2\pi)^3} \frac{|g(p)|^2}{h_p - E_B}. \qquad (14)$$

Working out the imaginary part of Eq. (12) and comparing it with Eq. (14), one finds

$$|g(p)|^2 = -\frac{(2\pi)^3}{\pi p \mu}(h_p - E_B)\sin\delta_B(E)$$
$$\times \exp\left[\frac{h_p - E_B}{\pi} \fint_0^\infty dE \frac{-\delta_B(E)}{(E - h_p)(E - E_B)}\right], \quad (15)$$

where $\fint$ denotes the principal value integral.

The form factor $g(p)$ is related to the radial wave function in the momentum space $\tilde{u}(p)$ and that in the position space $u(r)$ as follows:

$$\tilde{u}(p) = \frac{p}{(2\pi)^{3/2}} \frac{g(p)}{h_p - E_B}, \qquad (16)$$

$$u(r) = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty dp\, \tilde{u}(p)\sin(pr). \qquad (17)$$

These wave functions are normalized as

$$\int_0^\infty dp |\tilde{u}(p)|^2 = \int_0^\infty dr |u(r)|^2 = X. \qquad (18)$$

We note that Eq. (15) does not determine the phase of $g(p)$. However, if the system respects time reversal symmetry and the bound state is not degenerated, $u(r)$, $\tilde{u}(p)$, and $g(p)$ can all be made real.

Finally, with the convention in the current Letter, we can write the ERE,

$$p\cot\delta_B \approx -\frac{8\pi^2}{\mu}\mathrm{Re}T^{-1}(h_p) = \frac{1}{a} + \frac{r}{2}p^2 + \mathcal{O}(p^4), \qquad (19)$$

where the $\approx$ is used to remind us that the nonpole term $V_{p,k}$ has been neglected from the Low equation.

III. DISCUSSION

Equation (13) is a generalization of Weinberg's relations. It is expected to have a smaller uncertainty, of $\mathcal{O}(\Lambda^{-2})$, than

that of Weinberg's relations, i.e., $\mathcal{O}(\Lambda^{-1})$ with $\Lambda \sim m_\pi$ for the deuteron case (for discussions of the uncertainty of Weinberg's relations, see, e.g., Refs. [21,24]). Here the small dimensionless quantity for estimating uncertainties needs to be understood as $1/(R\Lambda)$, and we neglect R for simplicity. Typically, momenta appear in a square form in the potential. Therefore, we expect the ignored $V_{p,k}$ only brings an uncertainty of $\mathcal{O}(\Lambda^{-2})$. The reason that Weinberg's relations have an uncertainty of $\mathcal{O}(\Lambda^{-1})$ is that the constant-form-factor approximation $g(p) = g_0$ has been applied to the estimation of X. Although this approximation neglects the $\mathcal{O}(\Lambda^{-2})$ terms of $g^2(p)$, whose Λ dependence reads $g^2(p) = g_0^2 + \frac{p^2}{\Lambda^2}\tilde{g}^2(\frac{p^2}{\Lambda^2})$, the resulting uncertainty is

$$\Delta X = \frac{1}{\Lambda^2}\int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{q^2 \tilde{g}^2(\frac{q^2}{\Lambda^2})}{(h_q - E_B)^2} \sim \mathcal{O}(\Lambda^{-1}), \quad (20)$$

where the integral is linearly divergent when $\Lambda \to \infty$ and needs to be cut at the hard scale Λ. As the derivation of Eq. (13) does not rely on this approximation, we expect the uncertainty is of $\mathcal{O}((R\Lambda)^{-2})$.

In fact, when we take the ERE up to $\mathcal{O}(p^2)$ for δ_B as given in Eq. (19), an analytic model-independent, based on the separable ansatz in Eq. (6), expression (the form factor was modeled in a Gaussian form in, e.g., Ref. [33] and with a hard cutoff in, e.g., Ref. [5]) can be worked out for the form factor $g^2(p)$ from Eq. (15),

$$g^2(p) = \frac{8\pi^2}{\mu^2 R} \times \begin{cases} t + \mathcal{O}(p^4) & t \in [0,1] \\ \frac{a^2}{R^2}\frac{1}{1+(a+R)^2 p^2} + \mathcal{O}(p^4) & t > 1 \end{cases}, \quad (21)$$

where $t := 1/\sqrt{1 + 2r/a} = -a/(a + 2R)$, which is just X_W using Eqs. (1) and (2), and we have expressed r in terms of $r = 2R(a + R)/a$ in the second line. The above expression is exact for ERE up to $\mathcal{O}(p^2)$, and $\mathcal{O}(p^4)$ therein denotes that higher-order terms have been neglected in the ERE.

One sees that the form factor is a constant for $X_W \in [0,1]$, i.e.,

$$a \in [-R, 0], \quad r \le 0, \qquad (22)$$

and the relation between X_W and the coupling constant in Ref. [1] is reproduced in this case.

However, when $X_W > 1$, which corresponds to $a \in (-2R, -R)$ and $r \in (0, R)$, $g^2(p)$ contains $\mathcal{O}(p^2)$ terms and is of a single-pole form. Since the form factor enters the T matrix through the Low equation (5), such $\mathcal{O}(p^2)$ terms would contribute to the effective range and need to be taken into account consistently up to $\mathcal{O}(p^2)$. However, it was neglected in the original treatment of Ref. [1]; then for $r > 0$ the value of X_W solved using a constant form factor

(or coupling) is larger than 1, as is the case for the deuteron ($X_W = 1.68$), and loses its direct interpretation as a probability. On the contrary, the compositeness given in Eq. (13), which is the exact solution of the Low equation with the nonpole term $V_{p,k}$ neglected, will be exactly $X = 1$ because $\delta_B(\infty) = -\pi$, as can be seen from Eq. (19). Thus, we conclude that the compositeness of the deuteron is 1. The same also applies to the $D_{s0}^*(2317)$ and $D_{s1}(2460)$ as computed by the RQCD Collaboration using lattice quantum chromodynamics in Ref. [34], where the isoscalar DK and D^*K effective ranges are positive and the authors obtained larger-than-1 values for the compositeness using Eqs. (1) and (2). For comparison, the DK effective range obtained by the Hadron Spectrum Collaboration is negative, and the extracted compositeness close to 1 using the same relations is valid [35]. From the above discussion, considering the binding energy ranging from 22 to 60 MeV at the two pion masses therein [35], the systematic uncertainty of X from using Weinberg's relations should be about $(R\Lambda)^{-2} \in [0.03, 0.07]$, which is much less than $(R\Lambda)^{-1} \in [0.16, 0.27]$ that one would naively assume. Here $\Lambda \sim m_\rho$ is estimated from the lightest meson exchanged between $D^{(*)}K$.

Since Wigner's causality inequality constrains the effective range r to be negative semidefinite for a zero-range interaction (see, e.g., Refs. [26,36]), a positive r implies a sizable range correction. This is reflected by the p^2 dependence in Eq. (21). For the deuteron case, one has $|a + R| \approx 1.1$ fm, at the order of the inverse of the pion mass.

There is a crucial difference between a negative and a positive r (see also Refs. [10,13,26] for related discussions). The T matrix has two poles in the complex momentum plane using ERE up to a nonvanishing effective range term, which are located at $p_- = i/R$ and $p_+ = -i/(R + a)$. Here we have chosen to express the quantities in terms of a and R, and the effective range is $r = 2R(R + a)/a$. a must be negative in order to have a bound state pole. For $a \in (-R, 0)$, p_- is a bound state pole, and p_+ is a remote virtual state pole; in this case, $r < 0$, and, correspondingly, the form factor is just a constant up to $\mathcal{O}(p^2)$. However, for $a \in (-2R, -R)$, p_+ becomes a bound state pole as well; in this case, $r \in (0, R)$. Although this pole is spurious, it effectively resums range corrections (see Ref. [37] for discussions of range corrections in two-photon decays of hadronic molecules), and the form factor in Eq. (21) is indeed proportional to $1/(p^2 - p_+^2)$. For $a < -2R$, the roles of p_+ and p_- are interchanged, and the above discussions still apply. For the fine-tuning case $a \approx -2R$, the poles are close to each other and the Low equation needs to be modified.

The result of Eq. (13) for X has an uncertainty because δ_B is obtained by neglecting the nonpole term $V_{p,k}$, which could have a sizable contribution in the high-momentum range. Thus, we introduce

$$\delta_B(E, \underline{\Lambda}) = \begin{cases} \delta_B(E) & E \le h_\Lambda \\ 0 & E > h_\Lambda \end{cases},$$

$$\delta_B(E, \bar{\Lambda}) = \begin{cases} \delta_B(E) & E \le h_\Lambda \\ -\pi & E > h_\Lambda \end{cases}, \quad (23)$$

with $h_\Lambda = \Lambda^2/(2\mu)$, so that

$$-\pi \le \delta_B(E, \bar{\Lambda}) \le \delta_B(E) \le \delta_B(E, \underline{\Lambda}) \le 0. \quad (24)$$

The corresponding compositenesses, form factors, and wave functions will be denoted in a similar way. For the compositeness, one has

$$X(\underline{\Lambda}) = 1 - \exp\left(\frac{1}{\pi} \int_0^{h_\Lambda} dE \frac{\delta_B(E)}{E - E_B}\right) \le X \le 1 = X(\bar{\Lambda}). \quad (25)$$

For the form factor, one can show

$$g^2(p, \underline{\Lambda}) \le g^2(p) \le g^2(p, \bar{\Lambda}), \quad \text{when } p \le \Lambda, \quad (26)$$

$$g^2(p, \underline{\Lambda}) = g^2(p, \bar{\Lambda}) = 0, \quad \text{when } p > \Lambda. \quad (27)$$

Thus, with a larger Λ, one has tighter bounds on X, which, on the other hand, bear a larger uncertainty. For the wave function in the momentum space, there are similar relations. For the position space, however, there are not.

For a shallow bound state, the $p/(h_p - E_B)$ factor in Eq. (16) peaks sharply around $p = \sqrt{2\mu|E_B|}$. Once Λ is chosen beyond the peaking range, $\tilde{u}(p)$ will be largely determined by the long-distance physics. When we go to the position space, the short-distance (small-r) part of $u(r)$ receives little contribution from the small-p part of $\tilde{u}(p)$ as a consequence of the uncertainty principle. To be more concrete, because of the $\sin(pr)$ term in Eq. (17), $\tilde{u}(p)$ contributes to $u(r)$ only when $p \gtrsim \pi/(2r)$. This reflects the inability to probe the short-distance structure of a shallow bound state using the information of low-energy scattering. For the deuteron, for instance, it is impossible to distinguish the compact nucleon-nucleon component, i.e., the component of the small-r part of $u(r)$, from a possible elementary-particle core using only the low-energy nucleon-nucleon scattering.

IV. ANALYSIS OF DEUTERON

As discussed before, our prediction for the compositeness of the deuteron is simply 100%. The uncertainty of this value is determined by how well δ_B approximates the genuine phase shift. As argued above, around Eq. (20), for the deuteron, while Weinberg's relations would predict $X = 1.68 + \mathcal{O}((Rm_\pi)^{-1} \simeq 0.3)$, we have $X = 1 + \mathcal{O}(0.3^2 = 0.09)$.
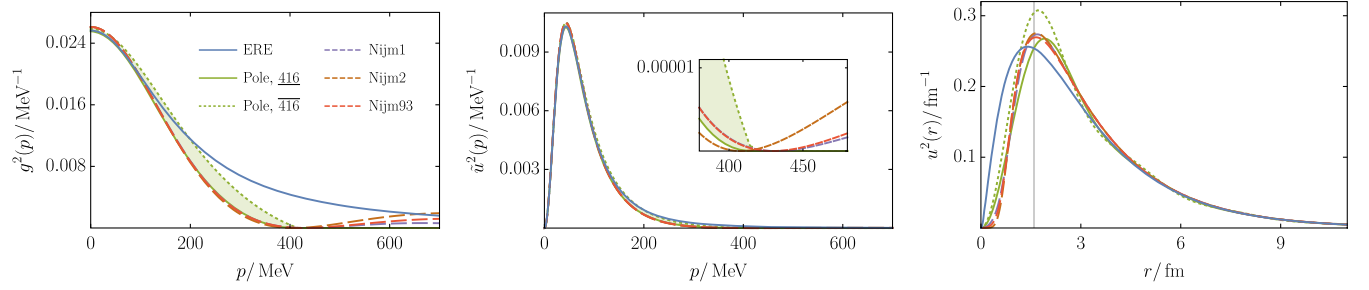
FIG. 1. The form factors (left) and wave functions in the momentum space (middle) and position space (right) for various cases: obtained from $\delta_B = \delta_{\mathrm{ERE}}$ (blue), obtained from $\delta_B = \delta_{\mathrm{Pole}}$ from Eq. (28) with a cutoff 416 MeV (green), and using three Nijmegen models (dashed). The shaded area indicates that they are the bounds for $g^2(p)$ and $\tilde{u}^2(p)$. The vertical gray line in the right plot denotes 1.6 fm.

We plot the form factor and wave functions of the deuteron in Fig. 1, where we also include the results from three famous Nijmegen models [38,39] [the references provide wave functions both in position and momentum spaces directly, and the form factors are obtained from Eq. (16)]. The three Nijmegen models give $X = (94.246$–$94.365)\%$ (the remaining $1 - X$ part comes from the D-wave component). It is interesting to find that the ERE phase shift produces a form factor that drops quickly, as given in the second line of Eq. (21), showing the importance of the range corrections neglected in Weinberg's relations. Compared to Nijmegen models, the discrepancy of the form factor from ERE becomes larger at higher momenta unsurprisingly. However, thanks to the $p/(h_p - E_B)$ factor in Eq. (16), the wave function $\tilde{u}^2(p)$ drops faster than $g^2(p)$, so the discrepancy becomes hardly visible for $\tilde{u}^2(p)$ even in the high-momentum range. For the wave function in position space, the discrepancy is mainly in the small-r range as expected. The deuteron wave function has been computed using many other methods; e.g., Ref. [28] compared wave functions from the Nijmegen group potentials [38] and Argonne v18 potential [40], Ref. [41] compared their results with the results from the Idaho (500) N^{3}LO potential of Ref. [42], the N^{3}LO (550/600) potential of Ref. [43], and the CD-Bonn potential [44], and Ref. [45] discussed wave functions in an EFT context. Most of these wave functions share a quite similar large-r behavior and differ mainly in the $r \lesssim 2$ fm range.

We can even go beyond the ERE by using a parametrization fitted to the experimental phase shifts up to a higher momentum. We adopt the following parametrization used in Ref. [46], where the authors refer to it as the pole approximation:

$$p \cot \delta_{\mathrm{Pole}} = \frac{1}{a} + \frac{r}{2}p^2 + \frac{v_2 p^4}{1 - D p^2},\qquad(28)$$

with $a = -5.4030$ fm, $r = 1.7494$ fm, $v_2 = 0.163$ fm^3, and $D = 0.225526$ fm^2. When $p > D^{-1/2} = 416$ MeV,

$\delta_{\mathrm{Pole}} < -\pi$, violating the restriction (11). So we have to introduce a cutoff $\Lambda = 416$ MeV, with which $X(\underline{\Lambda}) = 92.8\%$, consistent with $X = 1 + \mathcal{O}(0.09)$ given above. We also note that, even with a much smaller Λ, e.g., one would have $X \geq X(\underline{m_\pi}) = 62\%$ is already larger than 50%, which reliably concludes the deuteron to be mostly composite. The corresponding form factors and wave functions are included in Fig. 1. It is clear from the $g^2(p)$ plots that δ_{Pole} is better than δ_{ERE} at approximating Nijmegen models.

Although it is hard to quantify the uncertainty of neglecting the nonpole term $V_{p,k}$ from the Low equation, such an approximation, in fact, works rather well up to a relatively high momentum for the deuteron case, as can be seen from Fig. 2. Here, the δ_B curves for the Nijmegen models and for the ERE are shown together with the empirical phase shift δ_{Pole}. It is clear that δ_B from the Nijmegen models approximate δ_{Pole} very well even up to the cutoff 416 MeV. Note that, although the nonpole term is neglected from the Low equation, the resulting T matrix is more than simply a pole term.
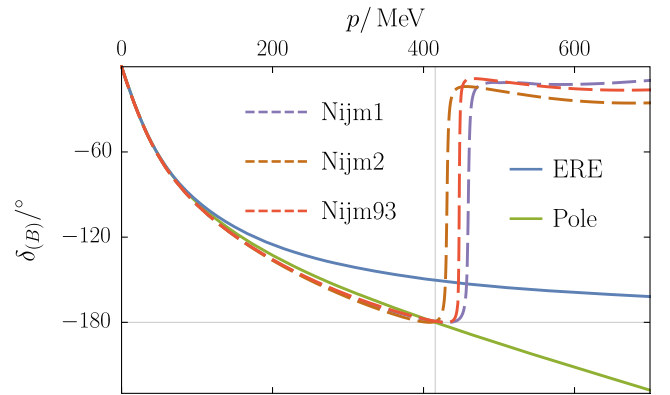

FIG. 2. δ_B from ERE and the Nijmegen models and the phase shift from an empirical parametrization used in Ref. [46] (denoted by "Pole"). The vertical gray line denotes $D^{-1/2} = 416$ MeV.

V. SUMMARY

In this Letter, we have generalized Weinberg's compositeness relations with a general form factor (and thus the range corrections are included), which also builds an exact relation between the phase shift δ_B, with the nonpole term neglected from the Low equation, and the bound state wave function. For a shallow bound state, δ_B can approximate well the full phase shift in the low-momentum range, which goes up to more than 400 MeV for the deuteron case. The compositeness derived from Eq. (13) is strictly within [0, 1] for a bound state even when the range corrections are important. It is expected to have a significantly smaller uncertainty than that from Weinberg's relations. An analytic expression for the form factor is obtained considering the ERE up to $\mathcal{O}(p^2)$. It is a constant for $a \in [-R, 0]$, and $r \leq 0$, verifying Weinberg's approximation. However, if the effective range is positive, the form factor contains $\mathcal{O}(p^2)$ contributions in a single-pole form, and thus Weinberg's treatment is not self-consistent, as it keeps only part of the $\mathcal{O}(p^2)$ contributions.

We then analyzed the deuteron as an example. The range corrections are important in this famous case, as reflected in the strongly momentum-dependent form factor derived here.

Given that many near-threshold states were observed in particular in the past two decades, the formalism will find its applications in such systems. Of particular importance is that the uncertainty of the extracted compositeness will be substantially reduced compared to that from Weinberg's relations. The formalism can also be applied to other composite systems with short-range interactions beyond hadron physics.

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