Letter

Growth of the $\frac{1}{16}$-BPS index in 4d $\mathcal{N}=4$ supersymmetric Yang-Mills theory

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We study the microcanonical superconformal index of $\frac{1}{16}$-supersymmetric operators in 4d $\mathcal{N}=4$ $U(N)$ super Yang-Mills (SYM) theory. We show, numerically for $N \leq 10$, that the large-charge asymptotics are consistent with the entropy of supersymmetric black holes in the dual anti de Sitter space. We then prove, using representation theory, that the index agrees precisely with the multigraviton index, when the charge is less than $2(N+1)$, and begins to deviate for larger values of charge. Thus the $U(N)$ SYM index interpolates between multigraviton values at small charges and black hole growth at large charges.

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I. INTRODUCTION

According to the AdS/CFT correspondence, a black hole in asymptotically anti de Sitter space should be interpreted in the dual conformal field theory as an ensemble of states with the same conserved charges as that of the black hole. A concrete setup to verify this expectation in the context of the prototype duality relating four-dimensional maximally supersymmetric $U(N)$ super Yang-Mills ($\mathcal{N}=4$ SYM) theory and five-dimensional supergravity in AdS_5 space was proposed in [1–3]. Consider $\frac{1}{16}$-supersymmetric, or Bogomol'nyi-Prasad-Sommerfield (BPS), states in the SYM theory on $S^3 \times R^t$, i.e., states annihilated by one supercharge $\mathcal{Q}$ of the theory and its Hermitian conjugate. The AdS/CFT correspondence predicts that at small values of charges the number of such states should equal that of an ensemble of supergravitons, while at large charges the statistical entropy of the ensemble of such states should agree with the thermodynamic entropy of $\frac{1}{16}$-BPS black holes in the dual AdS_5 theory. In this paper we show how the superconformal index counting $\frac{1}{16}$-BPS states in SYM indeed interpolates between these two types of behaviors.

The scale of the gravitational theory, as set by the radius of curvature of the asymptotic AdS_5 in five-dimensional Planck units, corresponds to $N^{1/4}$ in the SYM theory. The BPS black hole solution is then specified by four independent charges—three R-charges labeling the representations of $SU(4)_R$ and two angular momenta on S^3, constrained by one relation. In the $\mathcal{N}=4$ SYM theory, $\frac{1}{16}$-BPS states are labeled by the charges that commute with $\mathcal{Q}$, these are the two angular momenta and two of the three R-charges.[1] The essential part of the problem can be formulated in any $\mathcal{N}=1$ superconformal field theory with a gravity dual, where one has a single R-charge Q and the two angular momenta. In fact, in the simplest (and first to be discovered) BPS black hole solution [6], only one combination J of the two angular momenta is nonzero. This is the simplest setting within which one can study the problem of microscopic entropy of supersymmetric AdS_5 black holes. In this setting the supercharge $\mathcal{Q}$ can be chosen such that the combination of charges that commutes with it is $2J+Q$. For $\mathcal{N}=4$ SYM written in this $\mathcal{N}=1$ language, the quantity $n=3(2J+Q)$ is quantized so as to be an integer, this is the situation we discuss in this paper.

The regime of validity of the black hole solution is $N \to \infty$ (classical gravity theory) and n/N^2 finite (large horizon area). The thermodynamic entropy of these black holes in supergravity [6] is given by

$$S_{\mathrm{BH}} \equiv \frac{1}{4} A_{\mathrm{H}} = N^2\, s(n/N^2), \tag{1}$$

where

$$s(\nu) = \frac{\pi}{2 \cdot 3^{1/6}} \nu^{2/3} + O(\nu^{1/3}), \qquad \nu \to \infty. \tag{2}$$

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[1]The nonlinear relation between the five charges in the gravitational theory is not directly visible in the SYM theory, and this is part of what makes this problem subtle. Unraveling this issue was an important part of the recent progress [4,5].

On the microscopic side, one considers the superconformal index $d_N(n)$ defined as the trace of $(-1)^F$ over the subspace of the Hilbert space of SYM with charge n,[2]

$$\mathcal{I}_N(\mathrm{x}) = \mathrm{Tr}_{\mathcal{H}_{\rm phys}}(-1)^F \mathrm{e}^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}}\mathrm{x}^n \equiv \sum_n d_N(n)\mathrm{x}^n. \quad (3)$$

The main question is whether its asymptotic behavior agrees with the thermodynamic entropy of the black hole in the *large-N limit*, i.e.,

$$\frac{1}{N^2}\log d_N(n) \overset{?}{\to} s(n/N^2), \quad (4)$$

as $n, N \to \infty$, n/N^2 fixed and finite.

The authors of [7] (based on earlier work in [8,9]) showed how to construct the complete set of gauge-invariant operators of the SYM theory in the regime of small charges $\frac{n}{N^2} \to 0$ as $N \to \infty$, where one can ignore all trace relations among matrices. They showed, further, that this set agrees exactly with the set of multigraviton states in the gravitational theory. The initial failure to find the exponential growth of states at large charges led to the question of whether the $\mathcal{Q}$-cohomology contains any other states even at finite N. It is notable that the number of multigraviton BPS states does not depend on N, so that their corresponding growth as $N \to \infty$ is much smaller than that of the black hole entropy. Therefore, if the growth of the index agrees with the black hole entropy, there must be new states in the cohomology.

Recent work [4,5,10–21] has shown that the answer to the question posed in (4) is yes. All the approaches start by considering the index function, which can be calculated as an integral over $N \times N$ unitary matrices [22,3],

$$\mathcal{I}_N(\mathrm{x}) = \int DU \, \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} i_{\rm s}(\mathrm{x}^j)\, \mathrm{Tr}\, U^j\, \mathrm{Tr}(U^\dagger)^j\right), \quad (5)$$

where $i_{\rm s}(\mathrm{x})$ is the index trace as in (3) but taken over all single "letters" of the gauge theory. Then, one estimates the growth of states by using analytic properties of the relevant special functions in (5). One important feature of all these analyses is that one needs to consider complex values of chemical potentials in order to see the growth.

In this paper we consider the problem from the microcanonical point of view by studying the matrix integral (5) for finite values of N. Firstly, this leads to a direct (and nontrivial) numerical verification, in terms of the microcanonical integers $d_N(n)$, of the recently obtained results in the canonical ensemble quoted above. This result does not rely on any particular value of the chemical potential and, in particular, unlike the canonical ensemble, one does not need to complexify the chemical potentials in order to see the growth. Secondly, we present an argument based on representation theory that explains that the values of the index correspond precisely to the index of multigravitons for states with low-lying charge $n \leq 2N + 1$. For higher values of charge n new states do begin to contribute, so that as n reaches νN^2 there is an exponential growth of states. The index $d_N(n)$ thus interpolates from the graviton behavior at small charge to the black hole behavior at large charge, which is the finite-N, microcanonical manifestation of the Hawking-Page transition.

II. THE INDEX AT INFINITE N AND AT FINITE N

Why is the superconformal index expected to have an exponential growth of states equal to that of a black hole? The cleanest reasoning is a formal argument involving the Euclidean functional integral. The AdS/CFT conjecture asserts the equality of the functional integrals of the boundary SYM theory and the AdS space. In the CFT the superconformal index can be interpreted as such a functional integral with periodic boundary conditions by the usual procedure. However, it is not clear whether the Euclidean BPS black hole solution in AdS_5 contributes to the corresponding AdS functional integral—even at the classical level where the functional integral can be approximated by the exponential of the on-shell action.

The issue is one of regulating the infrared behavior of the BPS black hole solution. This problem was addressed in [5] where it was shown, by considering a supersymmetric deformation of the BPS black hole (BH) away from extremality, that there is a regulator consistent with supersymmetry, and that the regulated on-shell action equals the BPS BH entropy in the limit. One thus reaches the conclusion that the BH contributes to the functional integral and, therefore, the index should grow at least as fast as the exponential of the BPS BH entropy at large N.[3,4]

The superconformal index is protected upon change of coupling, for the same reason that the Witten index is protected [29]. Therefore, assuming that there are no states coming in from infinity, we can calculate the index as the trace (3) over the $\mathcal{Q}$-cohomology at weak or even zero coupling.

An important development was the calculation of the complete $\mathcal{Q}$-cohomology at weak coupling at $N = \infty$ (i.e., ignoring all trace relations) in [7] following the earlier work

[2]We recall that only $\mathcal{Q}$-invariant states contribute to (3), and it is therefore independent of β.

[3]There are further possible corrections to this statement coming from (a) quantum corrections to the BH entropy, (b) the possible existence of other saddles in the AdS functional integral, and (c) the possibility of wall crossing when one flows from weak to strong coupling, (see [23–26] for a discussion of these issues in BHs in asymptotically flat space), none of which we will discuss here.

[4]It would be interesting to have a more precise statement about *equality* of the index and the black hole entropy along the lines of [27,28].

of [8,9]. It was shown there that the single-trace $\mathcal{Q}$-cohomology is in one-to-one correspondence with the free supergraviton states in the dual AdS_5 [30]. The value of the trace (3) over this infinite-N $\mathcal{Q}$-cohomology is

$$i_{\text{grav}}(\mathrm{x}) = \frac{3\mathrm{x}^2}{1-\mathrm{x}^2} - \frac{2\mathrm{x}^3}{1-\mathrm{x}^3}. \tag{6}$$

The SYM operators making up the infinite-N $\mathcal{Q}$ cohomology also exist at finite N. Following [7–9], we call them "graviton operators in the SYM" and we use the notation $i_{\text{grav}}(\mathrm{x})$ for their index.

The recent progress in this problem relies on a careful study of the finite-N index (5). The Hamiltonian calculation to reach this integral expression goes as follows [3,22]. One first calculates the "single-letter index," namely the trace (3) taken over all operators made up of the elementary fields of the theory and derivatives. In this calculation one has to be careful about subtracting the constraints arising from the equations of motion. The only fields (or constraints) contributing to the index are those which are annihilated by the supercharge $\mathcal{Q}$. The result of this calculation is

$$i_s(\mathrm{x}) = \frac{3\mathrm{x}^2 - 3\mathrm{x}^4 - 2\mathrm{x}^3 + 2\mathrm{x}^6}{(1-\mathrm{x}^3)^2} = 3\mathrm{x}^2 - 2\mathrm{x}^3 - 3\mathrm{x}^4 + \cdots$$
$$= 1 - \frac{(1-\mathrm{x}^2)^3}{(1-\mathrm{x}^3)^2}. \tag{7}$$

One then projects the single-letter operators counted by (7) to the gauge-invariant subspace. Upon doing so by integrating over the gauge group using the Haar measure, we reach the matrix model (5).

The single graviton operators in SYM and the single-letter index at finite N are building blocks from which we calculate multigraviton and multitrace indices, respectively, by the operation of plethystic exponentiation [31]. At $N = \infty$ one first projects to the space of gauge-invariant single traces to obtain (6) and then exponentiates to produce multitraces. This quantity has the multi-graviton-like small growth. In the finite-N formula (5), in contrast, one first exponentiates the states charged under the gauge group and then projects on to gauge-invariant states. We proceed to show that the integral (5), as a function of charge, interpolates between the multigraviton answer and the black hole answer.

III. LARGE CHARGE OPERATORS FORM THE BLACK HOLE

The semiclassical entropy of the AdS_5 BH in supergravity is given by the real part of the Legendre transform of its regularized on-shell action [5] (with $\nu = n/N^2$),

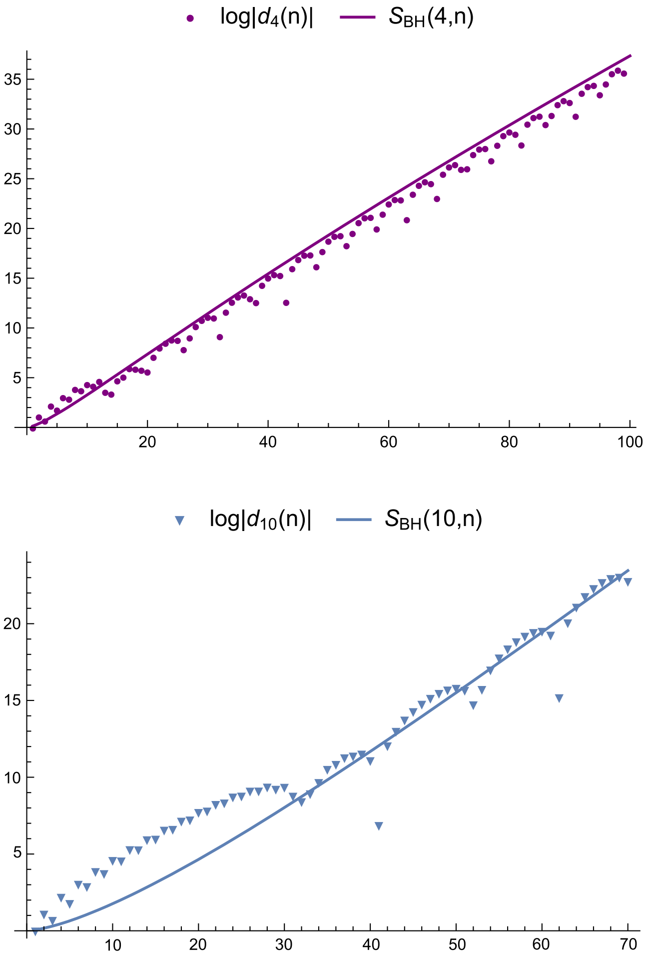

FIG. 1. Comparison between microscopic entropy $\log|d_N(n)|$ and BH entropy $S_{\text{BH}}(N,n)$ for $N = 4$, 10.

$$S_{\text{BH}} = \text{Re}\int d\tau\, \exp(N^2\mathcal{E}(\tau)),$$
$$\mathcal{E}(\tau) = -\frac{2\pi i\tau}{3}\nu - \frac{\pi i(2\tau+1)^3}{27\tau^2}. \tag{8}$$

Denoting by $\tau_* = \tau_*(\nu)$, the solution of the extremization equation $\mathcal{E}'(\tau_*) = 0$, we obtain at leading order in the large-N saddle-point expansion,

$$S_{\text{BH}}(N,n) = N^2 s(n/N^2), \qquad s(\nu) = \text{Re}\mathcal{E}(\tau_*(\nu)). \tag{9}$$

The asymptotic expression (2) is obtained by further expanding in large ν.

In the canonical ensemble, the analytic calculation of $\log\mathcal{I}_N(\tau)$ was done using two different methods. The first method uses a rewriting of the integrand of (5) in terms of the elliptic gamma function and estimating its behavior [15,16,21,32,33]. The second method is to extend the integrand of (5) to a doubly periodic function on the complex plane and then take the $\tau \to 0$ limit (see Sec. 4.5 in

[20]). Both these methods lead to agreement with the on-shell action $\mathcal{E}(\tau)$ in the Cardy-like limit $\tau \to 0$,

Here we study the microcanonical data $d_N(n)$. We computed these numbers by using the elliptic gamma function representation for $N = 2, 3, 4$.[5] For higher values of N we use the formula (18). In this method, the computational bottleneck is to produce the characters of the permutation group S_d, which leads to the charge cutoff $n \le 2d$.[6] In Fig. 1 we present the comparison between the microscopic $\log|d_N(n)|$ and the expression (8) for the BH entropy for $N = 4$, 10.[7]

There are many points to note here. Firstly, we find agreement as $n \to \infty$, as expected from the above discussion. For small charges, until $2N$, the microscopic degeneracies deviate from the BH curve. Instead, they follow the graviton curve for these small charges as explained below. After charge $2N$, the degeneracies latch on to the BH curve very soon [within $O(N)$] and exhibit regular bumps of size N. In particular, the good agreement of the microscopics with the BH when n and N are small is remarkable. Although our numerics are not precise enough to declare such a conclusion, they suggest the conjecture that the simple expression (8) governs the perturbative entropy even at finite N.

IV. SMALL CHARGE OPERATORS ARE GRAVITONS

The index function of multigraviton states is the plethystic exponential of the single-graviton index $i_{\rm grav}$ given in (6),

$$\mathcal{I}_{\text{multi-grav}}(\mathrm{x}) = \sum_n d_{\rm grav}(n)\mathrm{x}^n \coloneqq \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} i_{\rm grav}(x^k)\right) = \prod_{n=1}^{\infty} \frac{(1-\mathrm{x}^{3n})^2}{(1-\mathrm{x}^{2n})^3} = \frac{\eta(\tau)^2}{\eta(\frac{2\tau}{3})^3}. \tag{10}$$

We see from this expression that the multigraviton index is equivalent to a gas of three real bosonic oscillators of frequencies $2n$ and one complex fermionic oscillator of frequencies $3n$, $n = 1, 2, \ldots$. Using the standard modular properties of the Dedekind η function to estimate the growth of states, we obtain

$$\log\, d_{\rm grav}(n) \overset{n\to\infty}{\to} \frac{\pi}{3}\sqrt{5n}, \tag{11}$$

which is equivalent to an effective central charge of $\frac{5}{6}$. In Fig. 2 we show the growth of $\log d_{\rm grav}(n)$ in comparison to the BH entropy.

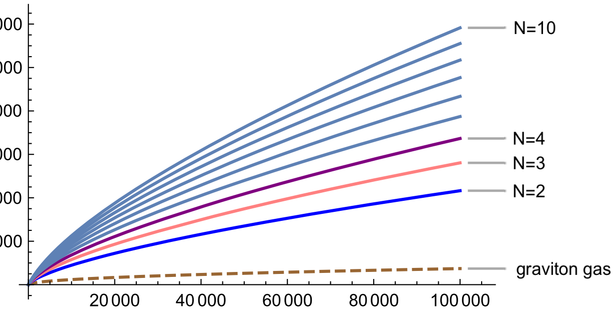

FIG. 2. The gravitational black hole entropy $S_{\rm BH} = N^2 s(n/N^2) = a_2(Nn)^{2/3} + O(n^{1/3})$ for $N = 2, 3, \ldots, 10$ and the logarithm of the index of the graviton gas (dashed line).

We now discuss the gauge theory index over the whole range of charges. The pattern is as follows.[8] For $\frac{1}{2}n < N+1$, the gauge theory index d_N and the multi-graviton index $d_{\rm grav}$ agree exactly, as illustrated in Table I and as we prove below. As we increase the charge n, d_N falls behind for a small interval before picking up and dominating $d_{\rm grav}(n)$ at large n, as shown in Fig. 3. For very large charges, d_N agrees with the BH partition function, as discussed in the previous section.

Our goal now is to prove that, when $n \le 2N+1$, $d_N(n)$ as defined in (3), (5) agrees with $d_{\rm grav}(n)$ defined as the coefficient of the multigraviton index (10). Using the relation (7), we write the multigraviton index (10) as

$$\mathcal{I}_{\text{multi-grav}}(\mathrm{x}) = \sum_n d_{\rm grav}(n)\mathrm{x}^n = \prod_{k=1}^{\infty} \frac{1}{1 - i_{\rm s}(\mathrm{x}^k)}. \tag{12}$$

Expanding both the expressions (5), (12) in terms of products of $i_{\rm s}(\mathrm{x}^k)$ over different k, and recalling from (7) that the index $i_{\rm s}(\mathrm{x})$ starts with the power x^2, we see that this agreement is equivalent to the following assertion. For $\sum_{j=1}^m j\,k_j \le N$,

$$\int DU \prod_{j=1}^{m} \frac{1}{k_j!\,j^{k_j}} (\mathrm{Tr}\,U^j\,\mathrm{Tr}\,U^{\dagger j})^{k_j} = 1, \tag{13}$$

[5] All these calculations were performed using PARI/GP [34] on a MacBookPro 2017. The time taken to calculate d_N increases rapidly with N—after initialization, and putting a cutoff at $n = 100$, it took 5 ms for $N = 2$ and 26 min for $N = 4$.

[6] The time taken to calculate the character tables from $d = 1$ to 20 was 4 seconds, while the final case dealt with here ($d = 35$) alone took 20 hours. All the character tables were computed using GAP [35]. Having obtained the characters, calculating the coefficients d_N is quite fast, e.g., the case $N = 10$, $n \le 70$ took 14 min using PARI/GP.

[7] The numbers $d_N(n)$ are integers in absolute value but typically have a nonzero phase. This phase is the reflection of the complexification of the chemical potential in the canonical ensemble.

[8] It seems to be important for the small charge observations that we are considering $U(N)$ and not $SU(N)$. The multi-graviton index under discussion thus includes the singleton sector [36,37].

TABLE I. The $U(N)$ SYM index $d_N(n)$ equals $d_{\rm grav}(n)$ for $\frac{1}{2}n < N+1$ and then starts to differ.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
d_2	3	−2	9	−6	11	−6	9	14	−21	36	−17	−18	114	−194	258	−168
d_3	3	−2	9	−6	21	−18	33	−22	36	6	−19	90	−99	138	−9	−210
d_4	3	−2	9	−6	21	−18	48	−42	78	−66	107	−36	30	114	−165	390
d_5	3	−2	9	−6	21	−18	48	−42	99	−96	172	−156	252	−160	195	48
$d_{\rm grav}$	3	−2	9	−6	21	−18	48	−42	99	−96	200	−198	381	−396	711	−750

which we now prove using some simple concepts from representation theory of $U(N)$. The ideas below have appeared in closely related contexts in [38,39].

The basic idea is to expand the traces of powers of the gauge field in terms of the $U(N)$ group characters, which is precisely the content of the Frobenius character formula [40,41]. Recall that the representations of $U(n)$ and those of the symmetric group S_n are both labeled by partitions $\boldsymbol{\lambda}$ of n. We denote the corresponding characters as $\tilde{\chi}_{\boldsymbol{\lambda}}$ and $\chi^{\boldsymbol{\lambda}}$, respectively. Now note that the gauge theory operator in (13) is uniquely associated with a cycle shape P through the following bijection:

$$P \equiv \prod_{j=1}^{m} (j)^{k_j} \leftrightarrow \prod_{j=1}^{m} (\mathrm{Tr}\, U^j)^{k_j} \equiv \mathcal{O}_P(U). \tag{14}$$

We think of P as a partition of the integer $|P| \coloneqq \sum_{j=1}^{m} j k_j$, labeling a conjugacy class in $S_{|P|}$. Here $|P|$ is called the weight of the partition P, and the number $\ell(P) \coloneqq \sum_{j=1}^{m} k_j$ is called the length or the number of parts of the partition P. The Frobenius formula for $U(N)$ states that

$$\mathcal{O}_P(U) = \sum_{\ell(\boldsymbol{\lambda}) \le N} \tilde{\chi}_{\boldsymbol{\lambda}}(U) \chi^{\boldsymbol{\lambda}}(P). \tag{15}$$

Using the formula (15) for $\mathcal{O}_P(U)$ and using the first orthogonality relation of the group characters of $U(N)$, we obtain

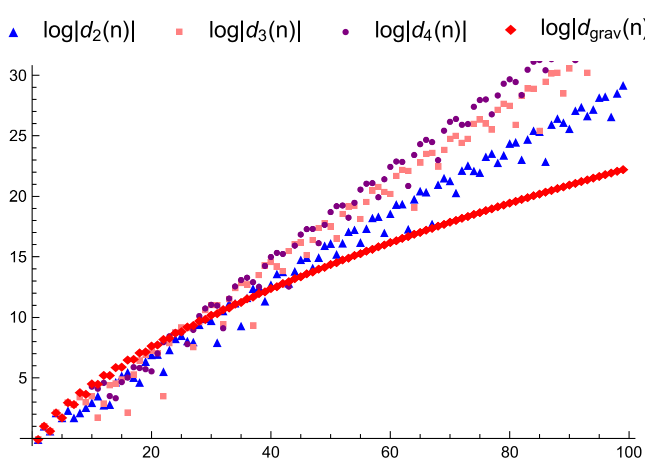

FIG. 3. Microscopic data $d_N(n)$ for $N = 2, 3, 4$ vs $d_{\rm grav}$.

$$\int DU\, \mathcal{O}_P(U)\, \mathcal{O}_P(U^\dagger) = \sum_{\ell(\boldsymbol{\lambda}) \le N} \chi^{\boldsymbol{\lambda}}(P)^2. \tag{16}$$

Here we have used the fact that the characters of the symmetric group are real (in fact, integers). When $|P| \le N$, any partition of $|P|$ cannot have more than N parts, so that the sum over $\boldsymbol{\lambda}$ on the right-hand side of (16) runs over *all* partitions of $|P|$. In this case we can use the second orthogonality relation of the characters of the symmetric group, i.e.,

$$\sum_{\boldsymbol{\lambda}} \chi^{\boldsymbol{\lambda}}(P)^2 = \prod_{j=1}^{m} k_j! j^{k_j} \equiv z_P. \tag{17}$$

Upon putting together Eqs. (16) and (17), we obtain the assertion (13). It is important in this argument that $|P| \le N$; this condition guarantees that the power of x in the index (5) is less than $2(N+1)$. As long as this holds, the coefficient d_N is independent of N and agrees with $d_{\rm grav}$—which is manifestly independent of N.

In fact the Frobenius relation can also be used to write down an explicit formula for the index (5). Upon expanding the exponential in (5) and using the formula (16), we obtain

$$\mathcal{I}_N(\mathrm{x}) = \sum_P i_{\rm s}(\mathrm{x})_P \frac{1}{z_P} \sum_{\ell(\boldsymbol{\lambda}) \le N} \chi^{\boldsymbol{\lambda}}(P)^2, \tag{18}$$

where $i_{\rm s}(\mathrm{x})_P \coloneqq \prod_{j=1}^{m} i_{\rm s}(\mathrm{x}^j)^{k_j}$.

We end with some brief comments.

(1) We propose that Eqs. (3), (5), and (10), (12) should be interpreted as the *black hole transform* of the single-graviton index (6). The input single-graviton index can be calculated using only the global symmetries of the AdS theory, and the transform d_N informs us about large BH solutions. Using the relation of the single-letter trace to the graviton index, this transform is interpreted as a holographic relation between the single-letter trace and the black hole.

(2) Is there a theory on the gravitational side which directly captures the dynamics of $\frac{1}{16}$-BPS states, akin to a topological theory? The mathematical context used here—the Frobenius-Schur duality, which relates the representations of $U(N)$ and those of the

symmetric group—has been used fruitfully in the past to relate matrix models appearing in gauge theories to string theories [42–46]. However, the appearance of black holes—which we clearly see in the matrix model here—is not seen in the usual topological versions of AdS/CFT dualities [47,48]. This should be related to the fact that in this paper we consider energies which scale as N^2, while the usual treatments took $N \to \infty$ strictly.

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