Supersymmetric celestial OPEs and soft algebras from the ambitwistor string worldsheet

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Using the ambitwistor string, we complete the list of celestial operator product expansion (OPE) coefficients for supersymmetric theories. This uses the ambitwistor string worldsheet conformal field theory to dynamically generate the OPE coefficients for maximally supersymmetric gauge theory, as well as gravity and Einstein-Yang-Mills theories, including all helicity and orientation configurations. This extends previous purely bosonic results [T. Adamo *et al.*, arXiv:2111.02279] to include supersymmetry and provides explicit formulas which are, to the best of our knowledge, not in the literature. We also examine how the supersymmetric infinite dimensional soft algebras behave compared to the purely bosonic cases.

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I. INTRODUCTION

By Mellin transforming dynamical observables such as massless scattering amplitudes in a momentum basis, one obtains scattering amplitudes in a conformal primary basis. Since the action of the Lorentzian group on equal time slices of $\mathscr{I}$, i.e., the celestial sphere, is the Mobius transformation $\mathrm{PSL}(2,\mathbb{C})$, the Mellin transformed scattering amplitudes transform as correlation functions in conformal field theories on the celestial sphere [1–10].

A natural question one could ask is whether this is purely coincidental or the scattering amplitudes in the conformal primary basis can indeed be understood as correlators in a special conformal field theory (CFT). For any ordinary conformal field theory, the operator spectrum and operator product expansion (OPE) are two of the most important characteristics. The particle spectrum of the conjectured celestial CFT (CCFT) consists of conformal primaries. OPEs between such conformal primaries are universal features that capture the singular behavior when two operators are inserted close to each other, and the OPE coefficients are crucial to determine dynamical properties of CCFT. Some of the CCFT OPE coefficients have been computed through Mellin transform or symmetries in the literature [11–13].

Moreover, one also needs to consider how the properties of the momentum basis scattering amplitudes in the bulk are interpreted in the CCFT framework, if this correspondence were to be understood as a kind of holography. For example, in the case of soft theorems [15–17], it was observed that by reorganizing the CCFT OPEs, infinite towers of soft symmetries can be expressed in terms of infinite dimensional algebras. It is worth noting that the universality of soft theorems is only valid up to a certain order in soft expansion. Although one can associate such a tower of symmetries with higher soft limits of tree-level gravity amplitude, its physical meaning still requires clarification. For gravity and Einstein-Yang-Mills (EYM) theories, the governing algebra turns out to be the extension of the Viraroso algebra, the $w_{1+\infty}$ algebra [9,14,15]. Recent literature also witnessed efforts to explore the quantum extensions of this algebra, where self-dual Einstein gravity one-loop amplitudes were considered [16,17].

Besides the purely bosonic cases, supersymmetric celestial amplitudes have been explored in the literature using similar Mellin transform methods [12,18–20]. Although celestial OPE coefficients in supersymmetric theories have been explored [12,13], these studies have been restricted to minimal supersymmetry (SUSY) or pure $\mathcal{N}=4$ super Yang-Mills (SYM). To our knowledge, the full list of celestial OPE coefficients for all maximally supersymmetric four-dimensional (4D) theories has not yet been determined. Besides this, the dynamical origin of the SUSY celestial OPE coefficients is also unclear.

In [21–25], the authors made attempts to study the dynamical origin of celestial OPEs and holographic symmetries using twistor theory. In particular, by using the ambitwistor string worldsheet CFT to compute celestial OPEs, the authors of [25] have successfully generated and matched CCFT OPE coefficients with all $SL(2,\mathbb{R})$ descendants computed by Pate and collaborators [11].

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The results were presented in "master formulas" to include all incoming-outgoing/outgoing-outgoing orientation configurations, which contained the usual Euler-Beta functions in integral form. However, all the OPEs there only concerned gluons and gravitons. Hence the purpose of this article is to generalize this mechanism to supersymmetric cases and to calculate all possible celestial OPEs for maximally supersymmetric theories in 4D. In the end we provide a complete brochure of CCFT OPE coefficients in the form of master formulas using the worldsheet CFTs of the fully supersymmetric 4D ambitwistor strings. To make the discussion more self-contained, we include all subtleties in the purely bosonic cases as well as additional ones present in the supersymmetric cases. It is worth noting that other attempts to explore the dynamical origin of CCFT have been carried out in the context of string theory [26]; however, the formalism there requires additional artificial manipulations during the calculations, which contrast with the naturalness of the ambitwistor string worldsheet formalism.

The fully supersymmetric ambitwistor strings in 4D are the $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity (SUGRA) ambitwistor string theories [27,28]. Naturally one expects that these two worldsheet CFTs should generate all the $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA CCFT OPE coefficients; however, for the $\mathcal{N}=4$ EYM theory, there are no $\mathcal{N}=4$ EYM ambitwistor worldsheet theories. Nevertheless, we are still able to compute the OPE coefficients using the appropriate vertex operators from the SYM and SUGRA ambitwistor strings. Unlike twistor string theories that are chiral in the Grassman variables, the ambitwistor string breaks the manifest $SU(\mathcal{N})$ R-symmetry into $SU(\mathcal{N}/2)\times SU(\mathcal{N}/2)$ and splits the spectrum into two halves. One half lives on twistor space and the other half on dual twistor space, which manifests the ambidextrous nature of the theory.

As shown and stated in the purely bosonic calculations [25], the ambitwistor string exhibits properties that allow it to be a natural habitat for the interpretations of CCFT. For example, without Mellin transforming scattering amplitudes in the momentum basis, the OPEs between vertex operators dynamically generate the OPE coefficients in the conformal primary basis. Additionally, the worldsheet integrals of ambitwistor strings inherently localize on the boundary of the moduli space, constraining the computation precisely at the collinear region without any artificial manipulation. Besides these, the infinite tower of organizing principles for soft symmetries as well as the action of soft conformal primaries on hard ones can be obtained independently, without taking soft limits on the OPE coefficients. We shall see these traits present throughout the supersymmetric calculations as well.

Apart from the remarkable properties in the bosonic cases, the supersymmetric ambitwistor string also provides additional convenience. Because of the various helicities of particles in the spectrum, one needs to be extra cautious with the Mellin conformal scaling dimension. However, we shall see that in the worldsheet theory, the number of supersymmetry and the homogeneity of the vertex operator in twistor/dual-twistor space cohomology will resolve such subtleties automatically. As in the bosonic case, we are also able to compute the supersymmetric holographic symmetries and action of soft particles on hard ones without prior knowledge on the CCFT OPE coefficients; the w algebra does not differ from the purely bosonic case, still acting as a Poisson diffeomorphism on a plane in twistor space [24]. This also matches recent results in [15].

The paper is organized as follows: Sec. II introduces notations and basic knowledge of the ambitwistor string, Sec. III gives a detailed procedure to compute all like helicity OPEs, Sec. IV presents that for all mixed helicity OPEs, and Sec. V concludes with the holographic symmetries and soft-hard OPEs.

II. SETUP

A. Kinematics

The study of the OPE coefficients of CCFT involves examining celestial conformal primaries inserted on the celestial sphere. It was observed that the OPE limit of CCFT coincides with the collinear limit of the momentum basis scattering amplitude. To make this observation, we parametrize null four-momenta k^μ of massless particles in the following way using the stereographic coordinates $(z,\bar{z})$ on the celestial sphere:

$$p^\mu = \frac{\omega}{2}(1+|z|^2, -z-\bar{z}, -\mathrm{i}(z-\bar{z}), 1-|z|^2), \qquad (2.1)$$

where ω denotes the energy of the particle, which when Mellin transformed becomes the conformal scaling dimension Δ. Some simple algebra reveals $p_i^\mu(z_i,\bar{z}_i)\cdot p_{j,\mu}(z_j,\bar{z}_j)\propto |z_i-z_j|^2$, suggesting that when the two momenta become collinear as their corresponding celestial coordinates $(z_i,\bar{z}_i)\to(z_j,\bar{z}_j)$. The point we shall try to establish in this paper is that this clash of $(z_i,\bar{z}_i)\to(z_j,\bar{z}_j)$ naturally corresponds to the clash between insertions $\sigma_i\to\sigma_j$ of the vertex operator on the ambitwistor string worldsheet. Here we remark that the limit $(z_i,\bar{z}_i)\to(z_j,\bar{z}_j)$ on the celestial sphere could be separately considered as the holomorphic limit $z_{ij}=z_i-z_j\to 0$ and the antiholomorphic limit $\bar{z}_{ij}=\bar{z}_i-\bar{z}_j\to 0$. Since one could only consider such chiral treatment when z_i and $\bar{z}_i$ are independent, we complexify the celestial sphere to $S^2\times S^2$ or employ the (2,2) signature celestial torus [29].

Besides this, we also need to capture the orientation configuration of the particle using ε, namely $\varepsilon=1$ for outgoing particles and $\varepsilon=-1$ for incoming ones. This includes an additional parameter in our parametrization

$$p^\mu = \frac{\varepsilon\omega}{2}(1+z\bar{z}, -z-\bar{z}, -\mathrm{i}(z-\bar{z}), 1-z\bar{z}). \tag{2.2}$$

Using spinor helicity notations, we have the following identities for our null four-momenta:

$$p^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} p^\mu = k^\alpha \tilde{k}^{\dot{\alpha}} = \varepsilon\omega \begin{pmatrix} 1 \\ z \end{pmatrix} \begin{pmatrix} 1 & \bar{z} \end{pmatrix}, \tag{2.3}$$

which allows us to replace spinor helicity variables k^α and $\tilde{k}^{\dot{\alpha}}$ with holomorphic and antiholomorphic coordinates on the celestial sphere. More precisely, we have

$$k^\alpha = \sqrt{\omega}\begin{pmatrix} 1 \\ z \end{pmatrix}, \qquad \tilde{k}^{\dot{\alpha}} = \varepsilon\sqrt{\omega}\begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}. \tag{2.4}$$

Since the focus of this paper is to compute celestial OPEs for maximally supersymmetric theories, it is convenient to introduce Grassmann coordinates η^A and their complex conjugates $\tilde{\eta}_A$ with designated helicities $\pm\frac{1}{2}$ on the celestial sphere, where A labels R-symmetry and runs from $A = 1, 2, \ldots, \mathcal{N}$. With this, we could write down the supermultiplet containing all the particle content we wish to use [30]:

$$\begin{aligned}\mathcal{U}(k,\tilde{k},\eta^A) = {}& \mathcal{U}_0(k,\tilde{k}) + \eta^A \mathcal{U}_{1,A}(k,\tilde{k}) \\ &+ \frac{1}{2}\eta^A\eta^B \mathcal{U}_{2,AB}(k,\tilde{k}) + \cdots \\ &+ \frac{1}{(\mathcal{N}/2)!}\eta^{A_1}\cdots\eta^{A_{\mathcal{N}/2}} \mathcal{U}_{\frac{\mathcal{N}}{2},A_1\cdots A_{\mathcal{N}/2}}(k,\tilde{k}),\end{aligned} \tag{2.5}$$

where we stop at the $\mathcal{U}_{\mathcal{N}/2}$ term to include half of the particle content originating from the positive helicity multiplet. The other half comes from the negative helicity multiplet:

$$\begin{aligned}\bar{\mathcal{U}}(k,\tilde{k},\tilde{\eta}_A) = {}& \bar{\mathcal{U}}_0(k,\tilde{k}) + \tilde{\eta}_A \bar{\mathcal{U}}_1^A(k,\tilde{k}) + \frac{1}{2}\tilde{\eta}_A\tilde{\eta}_B \bar{\mathcal{U}}_2^{AB}(k,\tilde{k}) + \cdots \\ &+ \frac{1}{(\mathcal{N}/2)!}\tilde{\eta}_{A_1}\cdots\tilde{\eta}_{A_{\mathcal{N}/2}} \bar{\mathcal{U}}_{\frac{\mathcal{N}}{2}}^{A_1\cdots A_{\mathcal{N}/2}}(k,\tilde{k}).\end{aligned} \tag{2.6}$$

Note that this construction differs from the usual convention in the literature, where a single multiplet generates the entire spectrum. By splitting the spectrum on two multiplets, we have chosen to break the manifest $SU(\mathcal{N})$ R-symmetry and opted for $SU(\frac{\mathcal{N}}{2})\times SU(\frac{\mathcal{N}}{2})$, where $\mathcal{N}=4$ or 8 for our purpose. Although such a construction is unusual in the literature, the closest analog can be found in [31–33]. We shall come back to this point again when we introduce all the on-shell superfield ambitwistor vertex operators in the next subsection, where $(k, \tilde{k}, \tilde{\eta}_A)$ will be assigned as the supercoordinate system on the dual space.

Notice that as all particles have different helicities and descend by $\frac{1}{2}$ as one steps down the SUSY ladder, to balance the helicities of the terms in (2.5), η shall be designated to have helicity $\frac{1}{2}$. Formally, we could define the following helicity operator:

$$\mathfrak{h} \coloneqq \frac{1}{2}(-k^\alpha\partial_{k^\alpha} + \tilde{k}^{\dot{\alpha}}\partial_{\tilde{k}^{\dot{\alpha}}} + \eta^A\partial_{\eta^A}). \tag{2.7}$$

If the helicity of the first bosonic particle $\mathcal{U}_0$ in the multiplet has helicity h, the helicity operator $\mathfrak{h}$ assigns helicity h to the rest of the particles in the multiplet: $\mathfrak{h}\mathcal{U}(k,\tilde{k},\eta^A) = h\mathcal{U}(k,\tilde{k},\eta^A)$, where $\mathcal{U}(k,\tilde{k},\eta^A)$ represents any particle in the positive helicity multiplet. Similarly, a helicity operator can be defined for particles in the negative helicity multiplet,

$$\bar{\mathfrak{h}} \coloneqq \frac{1}{2}(-k^\alpha\partial_{k^\alpha} + \tilde{k}^{\dot{\alpha}}\partial_{\tilde{k}^{\dot{\alpha}}} + \tilde{\eta}_A\partial_{\tilde{\eta}_A}), \tag{2.8}$$

which assigns helicity $\bar{h}$ to each particle originating from the negative helicity multiplet: $\bar{\mathfrak{h}}\bar{\mathcal{U}}(k,\tilde{k},\tilde{\eta}_A) = \bar{h}\bar{\mathcal{U}}(k,\tilde{k},\tilde{\eta}_A)$.

B. Ambitwistor string

As mentioned before, we shall attempt to utilize the worldsheet CFT of the ambitwistor string to generate CCFT OPE coefficients. Hence we first introduce the tool of ambitwistor string here [27,28,34].

Ambitwistor strings are holomorphic maps from closed Riemann surfaces to the projective ambitwistor space $\mathbb{PA}$, i.e., the supersymmetric extension of the space of complex null geodesics considered up to scale [35,36]. In four dimensions, $\mathbb{PA}$ is parametrized by twistor and dual-twistor variables ambidextrously. Together with SUSY, we have

$$\mathcal{Z} = (\mu^{\dot{\alpha}}, \lambda_\alpha, \chi_A) \in \mathbb{PT}, \tag{2.9}$$

$$\mathcal{W} = (\tilde{\lambda}_{\dot{\alpha}}, \tilde{\mu}^\alpha, \tilde{\chi}^A) \in \mathbb{PT}^*, \tag{2.10}$$

where $\mathcal{Z}$ represents the homogeneous coordinate on $\mathbb{CP}^{3|\mathcal{N}}$ and $\mathbb{PT} = \{\mathcal{Z} \in \mathbb{CP}^{3|\mathcal{N}} | \lambda_\alpha \neq 0\}$. Similarly $\mathcal{W}$ denotes the homogeneous coordinate on dual twistor space $\mathbb{PT}^*$. χ_A and $\tilde{\chi}^A$ are fermionic, and A ranges from 1 to $\mathcal{N}$ label R-symmetry. Ambitwistor space can then be represented as a quadric

$$\mathbb{PA} = \{(\mathcal{Z},\mathcal{W}) \in \mathbb{PT}\times\mathbb{PT}^* | \mathcal{Z}\cdot\mathcal{W} = 0\}, \tag{2.11}$$

with $\mathcal{Z}\cdot\mathcal{W} = \tilde{\mu}^\alpha\lambda_\alpha + \mu^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}} + \chi_A\tilde{\chi}^A$. Geometrically, since ambitwistor space is the complexification of all null geodesics, $\mathbb{PA}$ can be represented by the complex null geodesics and their intersection with any Cauchy surface. In the case of 4D Minkowski spacetime, $\mathbb{PA}$ is equivalent to the cotangent bundle of complexified null infinity $\mathbb{PA} \cong \mathbb{P}(T^*\mathscr{I})$ [35]. Furthermore, there exist nonlocal relations between points in the supersymmetrized 4D Minkowski spacetime $(x,\theta,\tilde{\theta})$ and a quadric $(\lambda,\tilde{\lambda})\in\mathbb{CP}^1\times\mathbb{CP}^1$ in $\mathbb{PA}$:

$$\mu^{\dot{\alpha}}=\mathrm{i}(x^{\alpha\dot{\alpha}}+\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\lambda_{\alpha},\qquad \chi_A=\theta^{\alpha}_{A}\lambda_{\alpha}, \tag{2.12}$$

$$\tilde{\mu}^{\alpha}=-\mathrm{i}(x^{\alpha\dot{\alpha}}-\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}},\qquad \tilde{\chi}^A=\tilde{\theta}^{A\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}. \tag{2.13}$$

To define a worldsheet action governing holomorphic maps from the worldsheet to $\mathbb{PA}$, one uses the worldsheet spinors $\mathcal{Z},\mathcal{W}\in\Omega^0_\Sigma(K^{1/2}_\Sigma\times\mathbb{C}^{4|\mathcal{N}})$, and a $GL(1,\mathbb{C})$ Lagrange multiplier $a\in\Omega^{0,1}_\Sigma$ enforcing the target space to be on the quadric $\mathcal{Z}\cdot\mathcal{W}=0$ [37]:

$$S=\frac{1}{2\pi}\int_\Sigma \mathcal{W}\cdot\bar{\partial}\mathcal{Z}-\mathcal{Z}\cdot\bar{\partial}\mathcal{W}+a\mathcal{Z}\cdot\mathcal{W}+S_{\mathrm{matter}}, \tag{2.14}$$

where S_{matter} is determined by the theory one tries to describe using this action. For example, for Yang-Mills theory, S_{matter} will be the action for a worldsheet current algebra $j^{\mathsf{a}}\in\Omega^0_\Sigma(K_\Sigma\otimes\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$. Note that the worldsheet action is invariant under any holomorphic reparametrization as well as gauge transformation associated with $GL(1,\mathbb{C})$ Lagrange multiplier a. Gauge fixing this redundancy using the Becchi-Rouet-Stora-Tyutin quantization (BRST) procedure introduces Virasoro ghosts into the system.

First, we shall consider supersymmetric Yang-Mills theory. After gauge fixing and BRST quantization, the BRST cohomology contains vertex operators of the following form:

$$\begin{aligned}\mathcal{U}^{\mathsf{a}}_{+}(z,\bar{z},\eta)&=\int_\Sigma j^{\mathsf{a}}(\sigma)a(\mathcal{Z}),\mathcal{U}^{\mathsf{a}}_{-}(z,\bar{z},\tilde{\eta})\\&=\int_\Sigma j^{\mathsf{a}}(\sigma)\tilde{a}(\mathcal{W}),\end{aligned} \tag{2.15}$$

where σ_i represents a local coordinate on the worldsheet and j^{a} denotes the worldsheet current of conformal weight (1, 0). $a(\mathcal{Z})\in H^{0,1}(\mathbb{PT},\mathcal{O})$ and $\tilde{a}(\mathcal{W})\in H^{0,1}(\mathbb{PT}^*,\mathcal{O})$ denote positive and negative helicity gluon wave functions of homogeneity degree 0 on twistor and dual twistor space, respectively. To recover spacetime free fields, we use the supersymmetrized Penrose integral formula to transform the wave functions $a(\mathcal{Z})$ and $\tilde{a}(\mathcal{W})$ [38]:

$$\begin{aligned}\tilde{F}_{\dot{\alpha}\dot{\beta}}(x,\theta,\tilde{\theta})&=\int\langle\lambda\mathrm{d}\lambda\rangle\frac{\partial^2 a(\mathcal{Z})}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}\bigg|_{\mu^{\dot{\alpha}}=\mathrm{i}(x^{\alpha\dot{\alpha}}+\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\lambda_\alpha,\chi_A=\theta^\alpha_A\lambda_\alpha},\\F_{\alpha\beta}(x,\theta,\tilde{\theta})&=\int[\tilde{\lambda}\mathrm{d}\tilde{\lambda}]\frac{\partial^2\tilde{a}(\mathcal{W})}{\partial\tilde{\mu}^{\alpha}\partial\tilde{\mu}^{\beta}}\bigg|_{\tilde{\mu}^{\alpha}=-\mathrm{i}(x^{\alpha\dot{\alpha}}-\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}},\tilde{\chi}^A=\tilde{\theta}^{A\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}},\end{aligned} \tag{2.16}$$

where the explicit form of $a(\mathcal{Z})$ and $\tilde{a}(\mathcal{W})$ are representation agnostic. The expressions suggest that we are restricting ourselves to the supertwistor line enforced by the incidence relations in (2.12) and (2.13). However, these are not the only spacetime fields we could write down; as in [38], one could choose χ_A or $\tilde{\chi}^A$ to differentiate, which gives us four other spacetime objects:

$$\begin{aligned}\tilde{F}^A_{\dot{\alpha}}(x,\theta,\tilde{\theta})&=\int\langle\lambda\mathrm{d}\lambda\rangle\frac{\partial^2 a(\mathcal{Z})}{\partial\mu^{\dot{\alpha}}\partial\chi_A}\bigg|_{\mu^{\dot{\alpha}}=\mathrm{i}(x^{\alpha\dot{\alpha}}+\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\lambda_\alpha,\chi_A=\theta^\alpha_A\lambda_\alpha},\\F_{\alpha A}(x,\theta,\tilde{\theta})&=\int[\tilde{\lambda}\mathrm{d}\tilde{\lambda}]\frac{\partial^2\tilde{a}(\mathcal{W})}{\partial\tilde{\mu}^{\alpha}\partial\tilde{\chi}^{A}}\bigg|_{\tilde{\mu}^{\alpha}=-\mathrm{i}(x^{\alpha\dot{\alpha}}-\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}},\tilde{\chi}^A=\tilde{\theta}^{A\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}},\end{aligned} \tag{2.17}$$

$$\begin{aligned}\tilde{F}^{AB}(x,\theta,\tilde{\theta})&=\int\langle\lambda\mathrm{d}\lambda\rangle\frac{\partial^2 a(\mathcal{Z})}{\partial\chi_A\partial\chi_B}\bigg|_{\mu^{\dot{\alpha}}=\mathrm{i}(x^{\alpha\dot{\alpha}}+\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\lambda_\alpha,\chi_A=\theta^\alpha_A\lambda_\alpha},\\F_{AB}(x,\theta,\tilde{\theta})&=\int[\tilde{\lambda}\mathrm{d}\tilde{\lambda}]\frac{\partial^2\tilde{a}(\mathcal{W})}{\partial\tilde{\chi}^{A}\partial\tilde{\chi}^{B}}\bigg|_{\tilde{\mu}^{\alpha}=-\mathrm{i}(x^{\alpha\dot{\alpha}}-\mathrm{i}\theta^{\alpha}_{A}\tilde{\theta}^{A\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha}},\tilde{\chi}^A=\tilde{\theta}^{A\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}}.\end{aligned} \tag{2.18}$$

Together, they combine into the nonzero part of the curvature on spacetime:

$$\begin{aligned}\tilde{F}(x,\theta,\tilde{\theta})=&\tilde{F}_{\dot{\alpha}\dot{\beta}}\varepsilon_{\alpha\beta}\mathrm{d}x^{\alpha\dot{\alpha}}\wedge\mathrm{d}x^{\beta\dot{\beta}}+\tilde{F}^A_{\dot{\alpha}}\varepsilon_{\alpha\beta}\mathrm{d}x^{\alpha\dot{\alpha}}\wedge\mathrm{d}\theta^\beta_A\\&+\tilde{F}^{AB}\varepsilon_{\alpha\beta}\mathrm{d}\theta^\alpha_A\wedge\mathrm{d}\theta^\beta_B,\end{aligned} \tag{2.19}$$

$$\begin{aligned}F(x,\theta,\tilde{\theta})=&F_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\mathrm{d}x^{\alpha\dot{\alpha}}\wedge\mathrm{d}x^{\beta\dot{\beta}}+F_{\alpha A}\varepsilon_{\dot{\alpha}\dot{\beta}}\mathrm{d}x^{\alpha\dot{\alpha}}\wedge\mathrm{d}\tilde{\theta}^{A\dot{\beta}}\\&+F_{AB}\varepsilon_{\dot{\alpha}\dot{\beta}}\mathrm{d}\tilde{\theta}^{A\dot{\alpha}}\wedge\mathrm{d}\tilde{\theta}^{B\dot{\beta}}.\end{aligned} \tag{2.20}$$

The complete tree-level S-matrix of 4D SYM can be obtained by taking correlation functions of the vertex operators in (2.15). There are three worldsheet OPEs that need to be considered in the computations, namely the worldsheet current OPE, the $\mathcal{Z}-\mathcal{W}$ spinor OPE. First, the worldsheet current OPE follows:

$$j^{\mathsf{a}}(\sigma_i)j^{\mathsf{b}}(\sigma_j)\sim\frac{k\delta^{\mathsf{ab}}}{(\sigma_i-\sigma_j)^2}\mathrm{d}\sigma_i\mathrm{d}\sigma_j+\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_i-\sigma_j}\mathrm{d}\sigma_i, \tag{2.21}$$

where k is the level of the worldsheet current algebra, δ^{ab} is the Killing form of the Lie group, and f^{abc} is the structure constant. The double pole term here comes from gravitationally mediated multitrace interactions, and here we ignore such contributions and decouple gravitational degrees of freedom by setting $k\to 0$ [39–41]. From now on, the worldsheet current OPE takes the following simple form:

$$j^{\mathsf{a}}(\sigma_i)j^{\mathsf{b}}(\sigma_j)\sim\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_i-\sigma_j}\mathrm{d}\sigma_i. \tag{2.22}$$

Another OPE that needs to be accounted for is the $\mathcal{Z}-\mathcal{W}$ OPE between worldsheet spinors:

$$\mathcal{Z}^I(\sigma_i)\mathcal{W}_J(\sigma_j) \sim \frac{\delta^I_J\sqrt{\mathrm{d}\sigma_i\mathrm{d}\sigma_j}}{\sigma_i-\sigma_j}. \tag{2.23}$$

Instead of SYM, one could also consider supergravity described by our worldsheet action (2.14), which requires an additional fermionic $\rho-\tilde{\rho}$ system in the S_{matter} term,

$$\mathcal{V}_+(z,\bar{z},\eta) = \int_\Sigma \left[\tilde{\lambda},\frac{\partial h(\mathcal{Z})}{\partial\mu}\right] + \tilde{\rho}^{\dot{\alpha}}\rho^{\dot{\beta}}\frac{\partial^2 h(\mathcal{Z})}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}, \tag{2.24}$$

$$\mathcal{V}_-(z,\bar{z},\tilde{\eta}) = \int_\Sigma \left\langle \lambda,\frac{\partial\tilde{h}(\mathcal{W})}{\partial\tilde{\mu}}\right\rangle + \rho^{\alpha}\tilde{\rho}^{\beta}\frac{\partial^2\tilde{h}(\mathcal{W})}{\partial\tilde{\mu}^{\alpha}\partial\tilde{\mu}^{\beta}}, \tag{2.25}$$

where $h(\mathcal{Z})\in H^{0,1}(\mathbb{PT},\mathcal{O}(2))$ and $\tilde{h}(\mathcal{W})\in H^{0,1}(\mathbb{PT}^*,\mathcal{O}(2))$ are representatives of cohomology class of homogeneity degree 2 on twistor and dual twistor space, respectively. Just as in the SYM case, we use the supersymmetric version of the Penrose transform to obtain momentum eigenstates on spacetime [38]. The correlation functions of these vertex operators correctly produce the entire tree-level S-matrix of Einstein supergravity. Besides the $\mathcal{Z}-\mathcal{W}$ OPE in (2.23) we just introduced, an additional $\rho-\tilde{\rho}$ OPE needs to be accounted for in such computations:

$$\rho^I(\sigma_i)\tilde{\rho}_J(\sigma_j) \sim \frac{\delta^I_J\sqrt{\mathrm{d}\sigma_i\mathrm{d}\sigma_j}}{\sigma_i-\sigma_j}. \tag{2.26}$$

C. Vertex operators in conformal primary basis

Now as we mentioned before, the vertex operators of our worldsheet theory can be written in any basis; however, to make contact with the existing literature in the celestial holography community, we shall adopt the conformal primary basis here.

We shall start with the gluon wave functions here [22]:

$$a(\mathcal{Z}) = \int_{\mathbb{C}^*\times\mathbb{R}_+}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t^{2-\Delta}}\bar{\delta}^2(z-s\lambda(\sigma))e^{\mathrm{i}st\varepsilon[\mu(\sigma)\bar{z}]+\mathrm{i}s\sqrt{t}\chi_A(\sigma)\eta^A}, \tag{2.27}$$

$$\tilde{a}(\mathcal{W}) = \int_{\mathbb{C}^*\times\mathbb{R}_+}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t^{2-\Delta}}\bar{\delta}^2(\varepsilon\bar{z}-s\tilde{\lambda}(\sigma))e^{\mathrm{i}st\langle\tilde{\mu}(\sigma)z\rangle+\mathrm{i}s\sqrt{t}\tilde{\chi}^A(\sigma)\tilde{\eta}_A}, \tag{2.28}$$

where $z_\alpha=(1,z)$ and $\bar{z}_{\dot{\alpha}}=(1,\bar{z})$ as we have in the kinematics section. The holomorphic delta functions or the scattering equations are defined as follows [42,43]:

$$\bar{\delta}^2(z-s\lambda(\sigma)) \coloneqq \frac{1}{(2\pi\mathrm{i})^2}\bigwedge_{\alpha=0,1}\bar{\partial}\left(\frac{1}{z_\alpha-s\lambda_\alpha(\sigma)}\right). \tag{2.29}$$

One could check that it indeed acts as a delta function enforcing the content inside its brackets to vanish. Notice that the position of ε in the negative helicity wave function is slightly different compared to the one used in [25]. Instead of placing it on the exponential, it sits in front of $\bar{z}$. Rigorously speaking, this would contribute an overall sign factor in front when fermions are involved in the calculation. Since [25] only considered bosonic particles, this was not an issue there. However, for the supersymmetric theories we consider, it is important to place the orientation parameter ε in front of $\bar{z}$ at all times.

As the expansion suggested by Eq. (2.5), to extract individual vertex operators from the gluon wave functions (2.27) and (2.28), we take derivatives with respect to η or $\tilde{\eta}$ from the factors $e^{\mathrm{i}s\sqrt{t}\tilde{\chi}^A(\sigma)\eta_A}$ or $e^{\mathrm{i}s\sqrt{t}\chi_A(\sigma)\tilde{\eta}^A}$ in the vertex operators to obtain the desired number of η or $\tilde{\eta}$,

$$\mathcal{U}^{\mathsf{a}}_{+,\Delta}(z,\bar{z},\eta^A) = \mathcal{O}^{\mathsf{a}}_{+,\Delta}(z,\bar{z},h) + \eta^A\Gamma^{\mathsf{a}}_{+,\Delta,A}(z,\bar{z}) + \frac{1}{2}\eta^A\eta^B\Phi^{\mathsf{a}}_{\Delta,AB}(z,\bar{z}), \tag{2.30}$$

$$\mathcal{U}^{\mathsf{a}}_{-,\Delta}(z,\bar{z},\tilde{\eta}_A) = \mathcal{O}^{\mathsf{a}}_{-,\Delta}(z,\bar{z},h) + \tilde{\eta}_A\bar{\Gamma}^{\mathsf{a},A}_{-,\Delta}(z,\bar{z}) + \frac{1}{2}\tilde{\eta}_A\tilde{\eta}_B\Phi^{\mathsf{a},AB}_{\Delta}(z,\bar{z}). \tag{2.31}$$

We indeed see that all particles have the same conformal weight since the Grassman variables have weight 0. Because of the designated helicities of η and $\tilde{\eta}$ being $\pm\frac{1}{2}$, our particles also have the right helicity. However, we notice that, unlike usual SUSY expansions, negative helicity particles cannot be generated from the positive helicity multiplet nor vice versa. This suggests that the $SU(4)$ R-symmetry is not manifest in ambitwistor space $\mathbb{PA}$, but it splits between twistor and dual twistor space, reflecting the ambidextrous nature of our theory. Here we summarize the particle content in $\mathcal{N}=4$ SYM,

Particle	$\mathcal{O}^{\mathsf{a}}$	Γ^{a}_A	Φ^{a}_{AB}
Helicity	$+1$	$+\frac{1}{2}$	0

from the positive helicity supermultiplet (2.27), and

Particle	$\mathcal{O}^{\mathsf{a}}$	$\bar{\Gamma}^{\mathsf{a},A}$	$\Phi^{\mathsf{a},AB}$
Helicity	-1	$-\frac{1}{2}$	0

from the negative helicity supermultiplet (2.28). The first particles we extract are the spin 1 gluons of both $\pm$ helicities:

$$\mathcal{O}^{\mathsf{a},\varepsilon}_{+,\Delta}(z,\bar{z}) = \int_\Sigma j^{\mathsf{a}}(\sigma)\int_{\mathbb{C}^*\times\mathbb{R}_+}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t^{2-\Delta}}\bar{\delta}^2(z-s\lambda(\sigma)) \times \exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.32}$$

$$\mathcal{O}^{\mathsf{a},\varepsilon}_{-,\Delta}(z,\bar{z}) = \int_\Sigma j^{\mathsf{a}}(\sigma)\int_{\mathbb{C}^*\times\mathbb{R}_+}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t^{2-\Delta}}\bar{\delta}^2(\varepsilon\bar{z}-s\tilde{\lambda}(\sigma)) \times \exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle), \tag{2.33}$$

where we read off the term with zeroth power in η from both positive and negative helicity multiplets and set $\eta = \tilde{\eta} = 0$. Note that this is exactly the same vertex operator we used in [25] to compute bosonic OPEs. Similarly, if we extract the operators with first order in η or $\tilde{\eta}$ and set $\eta = \tilde{\eta} = 0$, we have the spin $\frac{1}{2}$ gluinos:

$$\Gamma^{\mathsf{a},\varepsilon}_{+,\Delta,A}(z,\bar{z}) = \int_\Sigma j^{\mathsf{a}}(\sigma) \int_{\mathbb{C}^*\times\mathbb{R}_+} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t^{\frac{3}{2}-\Delta}} \bar{\delta}^2(z - s\lambda(\sigma))(\mathrm{i}s\chi_A(\sigma)) \times \exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.34}$$

$$\bar{\Gamma}^{\mathsf{a},\varepsilon,A}_{-,\Delta}(z,\bar{z}) = \int_\Sigma j^{\mathsf{a}}(\sigma) \int_{\mathbb{C}^*\times\mathbb{R}_+} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t^{\frac{3}{2}-\Delta}} \bar{\delta}^2(\varepsilon\bar{z} - s\tilde{\lambda}(\sigma))(\mathrm{i}s\tilde{\chi}^A(\sigma)) \times \exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle). \tag{2.35}$$

Notice that the power of t differs from the gluon, this reflects the helicity of the particle we are writing down, and the Mellin conformal scaling dimension varies as the helicity of the particle varies. This is taken care of by the $\sqrt{t}$ factor on the exponential.

It is worth noting that the homogeneities of the corresponding vertex operators are not the same, since factors of s were brought down through differentiation. For example, the positive helicity gluinos here are of homogeneity -1 on twistor space and the negative helicity ones are of homogeneity -1 on dual twistor space.

The last particle in the $\mathcal{N} = 4$ SYM spectrum is the scalar, which just amounts to taking the term with second order in η or $\tilde{\eta}$ and setting $\eta = \tilde{\eta} = 0$,

$$\Phi^{\mathsf{a},\varepsilon}_{\Delta,AB}(z,\bar{z}) = \int_\Sigma j^{\mathsf{a}}(\sigma) \int \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t^{1-\Delta}} \bar{\delta}^2(z - s\lambda(\sigma)) \times (-s^2\chi_A(\sigma)\chi_B(\sigma)) \exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.36}$$

$$\Phi^{\mathsf{a},\varepsilon,AB}_{\Delta}(z,\bar{z}) = \int_\Sigma j^{\mathsf{a}}(\sigma) \int \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t^{1-\Delta}} \bar{\delta}^2(\varepsilon\bar{z} - s\tilde{\lambda}(\sigma)) \times (-s^2\tilde{\chi}^A(\sigma)\tilde{\chi}^B(\sigma)) \exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle). \tag{2.37}$$

Notice that there are two different representations of the scalar originating from either the positive or the negative helicity gluon wave function. This precisely reflects the fact that we are in ambitwistor space, where we have chosen to break the manifest $SU(4)$ symmetry of $\mathcal{N} = 4$ SYM into $SU(2) \times SU(2)$. Hence the two representations each takes half of the scalars. Equations (2.36) and (2.37) are related by $SU(4)$ transformation, which is not manifest in ambitwistor space. Later in the computations, we shall observe that to obtain vertex operators with correct homogeneity, one is forced to stay ambidextrous and use both of these representations of the scalar when needed.

Next we write down vertex operators for particles in the $\mathcal{N} = 8$ supergravity multiplet. First, the wave functions $h(\mathcal{Z})$ and $\tilde{h}(\mathcal{W})$ in (2.24) and (2.25) in the conformal primary basis can be written as

$$h(\mathcal{Z}) = \int_{\mathbb{C}^*\times\mathbb{R}_+} \frac{\mathrm{d}s}{s^3} \frac{\mathrm{d}t}{t^{3-\Delta}} \bar{\delta}^2(z - s\lambda) e^{\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}] + \mathrm{i}s\sqrt{t}\chi_A(\sigma)\tilde{\eta}^A}, \tag{2.38}$$

$$\tilde{h}(\mathcal{W}) = \int_{\mathbb{C}^*\times\mathbb{R}_+} \frac{\mathrm{d}s}{s^3} \frac{\mathrm{d}t}{t^{3-\Delta}} \bar{\delta}^2(\varepsilon\bar{z} - s\tilde{\lambda}(\sigma)) e^{\mathrm{i}st\langle\tilde{\mu}(\sigma)z\rangle + \mathrm{i}s\sqrt{t}\tilde{\chi}^A(\sigma)\eta_A}. \tag{2.39}$$

The power of s differs from the gluon multiplet as the cohomological homogeneity degree of the graviton wave function is 2 instead of 0. Notice that the position of ε in the negative helicity wave function also differs from that in [25] for reasons we explained in the gluon case. Following similar expansions as the SYM case, we have from the positive helicity graviton supermultiplet (2.38):

Particle	$\mathcal{G}$	Θ_A	V_{AB}	Ξ_{ABC}	Π_{ABCD}
Helicity	$+2$	$+\frac{3}{2}$	$+1$	$+\frac{1}{2}$	0

and from the negative helicity graviton supermultiplet (2.39):

Particle	$\mathcal{G}$	$\bar{\Theta}^A$	$\bar{V}^{AB}$	$\bar{\Xi}^{ABC}$	Π^{ABCD}
Helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0

After substituting $h(\mathcal{Z})$ and $\tilde{h}(\mathcal{W})$ in (2.24) and (2.25), extracting the terms with zeroth power in η or $\tilde{\eta}$, and setting $\eta = \tilde{\eta} = 0$, the spin 2 gravitons can be presented as follows:

$$\mathcal{G}^\varepsilon_{+,\Delta}(z,\bar{z}) = \varepsilon \int \frac{\mathrm{d}s}{s^2} \frac{\mathrm{d}t}{t^{2-\Delta}} (\mathrm{i}[\tilde{\lambda}(\sigma)\bar{z}] - \varepsilon st[\rho(\sigma)\bar{z}][\tilde{\rho}(\sigma)\bar{z}]) \times \bar{\delta}^2(z - s\lambda(\sigma)) \exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.40}$$

$$\mathcal{G}^\varepsilon_{-,\Delta}(z,\bar{z}) = \int \frac{\mathrm{d}s}{s^2} \frac{\mathrm{d}t}{t^{2-\Delta}} (\mathrm{i}\langle\lambda(\sigma)z\rangle - st\langle\tilde{\rho}(\sigma)z\rangle\langle\rho(\sigma)z\rangle) \times \bar{\delta}^2(\varepsilon\bar{z} - s\tilde{\lambda}(\sigma)) \exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle). \tag{2.41}$$

To go one step down the supermultiplets, we extract the vertex operator with one power of η or $\tilde{\eta}$ describing the spin $\pm\frac{3}{2}$ gravitinos, after setting $\eta = \tilde{\eta} = 0$,

$$\Theta^\varepsilon_{+,\Delta,A}(z,\bar{z}) = \varepsilon \int \frac{\mathrm{d}s}{s^2} \frac{\mathrm{d}t}{t^{\frac{3}{2}-\Delta}} (\mathrm{i}[\tilde{\lambda}(\sigma)\bar{z}] - \varepsilon st[\rho(\sigma)\bar{z}][\tilde{\rho}(\sigma)\bar{z}])(\mathrm{i}s\chi_A(\sigma)) \times \bar{\delta}^2(z - s\lambda(\sigma)) \exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.42}$$

$$\bar{\Theta}^{\varepsilon,A}_{-,\Delta}(z,\bar{z}) = \int \frac{\mathrm{d}s}{s^2} \frac{\mathrm{d}t}{t^{\frac{3}{2}-\Delta}} (\mathrm{i}\langle\lambda(\sigma)z\rangle - st\langle\tilde{\rho}(\sigma)z\rangle\langle\rho(\sigma)z\rangle)(\mathrm{i}s\tilde{\chi}^A(\sigma)) \times \bar{\delta}^2(\varepsilon\bar{z} - s\tilde{\lambda}(\sigma)) \exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle). \tag{2.43}$$

Similarly we could obtain the spin 1 gauge bosons

$$V^{\varepsilon}_{+,\Delta,AB}(z,\bar{z}) = \varepsilon\int\frac{\mathrm{d}s}{s^2}\frac{\mathrm{d}t}{t^{1-\Delta}}(\mathrm{i}[\tilde{\lambda}(\sigma)\bar{z}]-\varepsilon st[\rho(\sigma)\bar{z}][\tilde{\rho}(\sigma)\bar{z}])(-s^2\chi_A(\sigma)\chi_B(\sigma)) \times\bar{\delta}^2(z-s\lambda(\sigma))\exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.44}$$

$$\bar{V}^{\varepsilon,AB}_{-,\Delta}(z,\bar{z}) = \int\frac{\mathrm{d}s}{s^2}\frac{\mathrm{d}t}{t^{1-\Delta}}(\mathrm{i}\langle\lambda(\sigma)z\rangle - st\langle\tilde{\rho}(\sigma)z\rangle\langle\rho(\sigma)z\rangle)(-s^2\tilde{\chi}^A(\sigma)\tilde{\chi}^B(\sigma)) \times\bar{\delta}^2(\varepsilon\bar{z}-s\tilde{\lambda}(\sigma))\exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle), \tag{2.45}$$

and the spin $\frac{1}{2}$ gauginos

$$\Xi^{\varepsilon}_{+,\Delta,ABC}(z,\bar{z}) = \varepsilon\int\frac{\mathrm{d}s}{s^2}\frac{\mathrm{d}t}{t^{\frac{1}{2}-\Delta}}(\mathrm{i}[\tilde{\lambda}(\sigma)\bar{z}]-\varepsilon st[\rho(\sigma)\bar{z}][\tilde{\rho}(\sigma)\bar{z}]) \times(-\mathrm{i}s^3\chi_A(\sigma)\chi_B(\sigma)\chi_C(\sigma))\bar{\delta}^2(z-s\lambda(\sigma))\exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.46}$$

$$\Xi^{\varepsilon,ABC}_{-,\Delta}(z,\bar{z}) = \int\frac{\mathrm{d}s}{s^2}\frac{\mathrm{d}t}{t^{\frac{1}{2}-\Delta}}(\mathrm{i}\langle\lambda(\sigma)z\rangle - st\langle\tilde{\rho}(\sigma)z\rangle\langle\rho(\sigma)z\rangle) \times(-\mathrm{i}s^3\tilde{\chi}^A(\sigma)\tilde{\chi}^B(\sigma)\tilde{\chi}^C(\sigma))\bar{\delta}^2(\varepsilon\bar{z}-s\tilde{\lambda}(\sigma))\exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle), \tag{2.47}$$

as well as the scalars

$$\Pi^{\varepsilon}_{\Delta,ABCD}(z,\bar{z}) = \varepsilon\int\frac{\mathrm{d}s}{s^2}\frac{\mathrm{d}t}{t^{-\Delta}}(\mathrm{i}[\tilde{\lambda}(\sigma)\bar{z}]-\varepsilon st[\rho(\sigma)\bar{z}][\tilde{\rho}(\sigma)\bar{z}]) \times(s^4\chi_A(\sigma)\chi_B(\sigma)\chi_C(\sigma)\chi_D(\sigma))\bar{\delta}^2(z-s\lambda(\sigma)) \times\exp(\mathrm{i}\varepsilon ts[\mu(\sigma)\bar{z}]), \tag{2.48}$$

$$\Pi^{\varepsilon,ABCD}_{\Delta}(z,\bar{z}) = \int\frac{\mathrm{d}s}{s^2}\frac{\mathrm{d}t}{t^{-\Delta}}(\mathrm{i}\langle\lambda(\sigma)z\rangle - st\langle\tilde{\rho}(\sigma)z\rangle\langle\rho(\sigma)z\rangle) \times(s^4\tilde{\chi}^A(\sigma)\tilde{\chi}^B(\sigma)\tilde{\chi}^C(\sigma)\tilde{\chi}^D(\sigma))\bar{\delta}^2(\varepsilon\bar{z}-s\tilde{\lambda}(\sigma)) \times\exp(\mathrm{i}ts\langle\tilde{\mu}(\sigma)z\rangle). \tag{2.49}$$

Note that just as in SYM, we obtain two representations of the gravity scalar from two origins. For the same reason as in SYM, the $SU(8)$ symmetry is manifestly broken into $SU(4)\times SU(4)$ in ambitwistor space; hence (2.48) and (2.49) each represents half of the gravity scalars.

In the following two sections, we demonstrate the methodology of computing OPEs involving the gluinos, the gravitinos, and their bosonic superpartners in different helicity and orientation configurations. Since all other OPEs follow similar procedures, we simply list the results in the Appendix.

III. LIKE HELICITY OPEs

A. Gluino-gluino OPE

Here we spell out the calculation explicitly for the like helicity gluino-gluino OPE. The majority of the steps here will follow through from the like helicity gluon-gluon computation in [25], apart from the subtlety of the additional $\chi-\tilde{\chi}$ fermionic OPE,

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \int_{\Sigma_i\times\Sigma_j\times(\mathbb{C}^*)^2\times\mathbb{R}^2_+}\mathrm{d}\sigma_i\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_{ij}}\frac{\mathrm{d}s_i}{s_i}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}} \times(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}s_j\chi_B(\sigma_j))\bar{\delta}^2(z_i-s_i\lambda(\sigma_i))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j)) \times\exp(\mathrm{i}\varepsilon_it_is_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}\varepsilon_jt_js_j[\mu(\sigma_j)\bar{z}_j]), \tag{3.1}$$

where we have already performed the $j-j$ OPE given by (2.22). The immediate feature of this OPE one could observe is that in the limit $\sigma_i-\sigma_j=\sigma_{ij}\to 0$, the two delta functions simultaneously enforce $\langle z_iz_j\rangle = z_i - z_j\to 0$. This simply notes the fact that the collision of two vertex operator insertions on the ambitwistor string worldsheet coincides with the collision of insertion points on the celestial sphere. With this understanding in place, we could begin manipulating the expression to see the desired OPE appearing. In the following we simply do some of the integrals here to make the resulting vertex operator more manifest.

The first thing we need to do here is to perform the s_i integral against the first delta function. This sets $s_i = \langle\xi z_i\rangle/\langle\xi\lambda(\sigma_i)\rangle$ for arbitrary reference spinor $\xi_\alpha\neq z_{i\alpha}$. For simplicity we set $\xi_\alpha=\iota_\alpha=(0,1)$, for which $\langle\iota z_i\rangle=1$. This gives us

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim\int\mathrm{d}\sigma_i\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_{ij}}\mathrm{d}s_j\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}\bar{\delta}(\langle z_i\lambda(\sigma_i)\rangle) \times(-\chi_A(\sigma_i)\chi_B(\sigma_j))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j)) \times\exp\left(\mathrm{i}\varepsilon_it_i\frac{[\mu(\sigma_i)\bar{z}_i]}{\langle\iota\lambda(\sigma_i)\rangle}+\mathrm{i}\varepsilon_jt_js_j[\mu(\sigma_j)\bar{z}_j]\right), \tag{3.2}$$

Notice that the number of χ remaining and the color current $j^{\mathsf{a}}(\sigma_j)$ already indicate that what we end up with on the right-hand side should be the scalar in the gluon multiplet Φ^{a}_{AB}. The following computation will reveal whether the homogeneity of the resulting expression matches that of a scalar. Now we could use the definition of the holomorphic delta function to integrate by parts to trade $\bar{\delta}(\langle z_i\lambda(\sigma_i)\rangle)$ with the σ_{ij} pole, obtaining

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j) \sim f^{\mathsf{abc}}\int \mathrm{d}\sigma_i \frac{j^{\mathsf{c}}(\sigma_j)}{\langle z_i\lambda(\sigma_i)\rangle}\bar\delta(\sigma_{ij})\mathrm{d}s_j \frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}} \times(-\chi_A(\sigma_i)\chi_B(\sigma_j))\bar\delta^2(z_j-s_j\lambda(\sigma_j)) \times\exp\left(\mathrm{i}\varepsilon_i t_i\frac{[\mu(\sigma_i)\bar z_i]}{\langle\iota\lambda(\sigma_i)\rangle}+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar z_j]\right). \tag{3.3}$$

Then we perform the σ_i integral with respect to the delta function $\bar\delta(\sigma_{ij})$ to get

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j) \sim f^{\mathsf{abc}}\int_{\Sigma_j\times\mathbb{C}^*\times\mathbb{R}^2_+} \frac{j^{\mathsf{c}}(\sigma_j)}{\langle z_i\lambda(\sigma_j)\rangle}\mathrm{d}s_j \frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}} \times(-\chi_A(\sigma_j)\chi_B(\sigma_j))\bar\delta^2(z_j-s_j\lambda(\sigma_j)) \times\exp\left(\mathrm{i}\varepsilon_i t_i\frac{[\mu(\sigma_j)\bar z_i]}{\langle\iota\lambda(\sigma_j)\rangle}+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar z_j]\right). \tag{3.4}$$

The remaining delta function enforces the following relations:

$$\frac{\langle\iota\lambda(\sigma_j)\rangle}{\langle z_i\lambda(\sigma_j)\rangle}=\frac{\langle\iota z_j\rangle}{\langle z_i z_j\rangle}=\frac{1}{z_{ij}},\qquad \frac{1}{\langle\iota\lambda(\sigma_j)\rangle}=\frac{s_j}{\langle\iota z_j\rangle}=s_j, \tag{3.5}$$

to obtain our pole $\frac{1}{\langle z_i\lambda(\sigma_j)\rangle}=\frac{s_j}{z_{ij}}$. Substitute these in our expression, which reads

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\int j^{\mathsf{c}}(\sigma_j)\mathrm{d}s_j s_j \frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}(-\chi_A(\sigma_j)\chi_B(\sigma_j)) \times\bar\delta^2(z_j-s_j\lambda(\sigma_j))\exp(\mathrm{i}\varepsilon_i t_i s_j[\mu(\sigma_j)\bar z_i] +\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar z_j]). \tag{3.6}$$

Note that the scalar Φ^{a}_{AB} in the integral expression requires $\mathrm{d}s_j s_j$ to have homogeneity -2, which in our expression is given by the identities we just used.

To proceed from here, one needs to combine the exponential terms. To do this, we notice that the first term in the exponential requires an additional t_j factor to match the content of the second term; hence we rescale $t_i\mapsto t_i t_j$ to get

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\int j^{\mathsf{c}}(\sigma_j)\mathrm{d}s_j s_j \frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{2-\Delta_i-\Delta_j}}(-\chi_A(\sigma_j)\chi_B(\sigma_j)) \times\bar\delta^2(z_j-s_j\lambda(\sigma_j))\exp(\mathrm{i}\varepsilon_i t_i t_j s_j[\mu(\sigma_j)\bar z_i] +\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar z_j]). \tag{3.7}$$

From here, some algebra on the exponential leads us to

$$\mathrm{i}t_j s_j(\varepsilon_i t_i[\mu(\sigma_j)\bar z_i]+\varepsilon_j[\mu(\sigma_j)\bar z_j]) =\mathrm{i}s_j t_j\left(1+\frac{\varepsilon_i}{\varepsilon_j}t_i\right)\left(\frac{\varepsilon_i t_i}{1+\frac{\varepsilon_i}{\varepsilon_j}t_i}[\mu(\sigma_j)\bar z_{ij}]+\varepsilon_j[\mu(\sigma_j)\bar z_j]\right), \tag{3.8}$$

where $\bar z_{ij\dot\alpha}\coloneqq\bar z_{i\dot\alpha}-\bar z_{j\dot\alpha}$. Now we need to get rid of the dependence of t_i on the exponential, which could be achieved by rescaling $t_j\mapsto t_j|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|$,

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j)\sim\frac{f^{\mathsf{abc}}}{z_{ij}}\int j^{\mathsf{c}}(\sigma_j)\mathrm{d}s_j s_j\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1}}\frac{\mathrm{d}t_j}{t_j^{2-\Delta_i-\Delta_j}}(-\chi_A(\sigma_j)\chi_B(\sigma_j))\bar\delta^2(z_j-s_j\lambda(\sigma_j)) \times\exp\left[\mathrm{i}t_j s_j\left(\frac{\varepsilon_i t_i}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|}[\mu(\sigma_j)\bar z_{ij}]+\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)[\mu(\sigma_j)\bar z_j]\right)\right], \tag{3.9}$$

where sgn denotes the sign function. We notice that all t_i dependence on the exponential is now bundled together with a factor of $\bar z_{ij}$, which vanishes in the OPE limit. To make the form of the integral more transparent, we rearrange to get

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j)\sim\frac{f^{\mathsf{abc}}}{z_{ij}}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1}}\int j^{\mathsf{c}}(\sigma_j)\int_{\mathbb{C}^*\times\mathbb{R}_+}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_j}{t_j^{1-(\Delta_i+\Delta_j-1)}}(\mathrm{i}s_j\chi_A(\sigma_j))(\mathrm{i}s_j\chi_B(\sigma_j))\bar\delta^2(z_j-s_j\lambda(\sigma_j)) \times\exp\left[\mathrm{i}t_j s_j\left(\frac{\varepsilon_i t_i}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|}[\mu(\sigma_j)\bar z_{ij}]+\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)[\mu(\sigma_j)\bar z_j]\right)\right]. \tag{3.10}$$

Now we just Taylor expand in the first term on the exponential to get

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \times \int j^{\mathsf{c}}(\sigma_j)\int_{\mathbb{C}^*\times\mathbb{R}_+}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_j}{t_j^{1-(\Delta_i+\Delta_j-1)}}(\mathrm{i}s_j\chi_A(\sigma_j))(\mathrm{i}s_j\chi_B(\sigma_j))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j)) \times \exp[\mathrm{i}t_j s_j \mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)[\mu(\sigma_j)\bar{z}_j]]. \tag{3.11}$$

It is now evident that the last three integrals give us a scalar vertex operator $\Phi^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{\Delta_i+\Delta_j-1,AB}$ at $(z_j,\bar{z}_j)$. All together, we have a master formula including all $SL(2,\mathbb{R})$ descendants of the OPE with an arbitrary orientation configuration,

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1+m}} \times \left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \bar{\partial}_j^m \Phi^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j). \tag{3.12}$$

To see the Beta function coefficients as appearing in the literature, first we recall two different integral representations of the Euler Beta function:

$$B(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t=\int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{3.13}$$

Now we start with the case when both gluinos are incoming or outgoing, namely $\varepsilon_i=\varepsilon_j=\varepsilon$, and the t_i integral immediately gives

$$\Gamma^{\mathsf{a},\varepsilon}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{b},\varepsilon}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{\bar{z}_{ij}^m}{m!}B\left(\Delta_i+m-\frac{1}{2},\Delta_j-\frac{1}{2}\right) \times \bar{\partial}_j^m\Phi^{\mathsf{c},\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j). \tag{3.14}$$

We see that when $m=0$, our coefficient just gives the Beta function in the literature [13].

Now for the mixed incoming/outgoing case $\varepsilon_i=-\varepsilon_j=\varepsilon$. We will need to use the alternative expression of the Beta function to split the integral into

$$\frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_0^1\frac{\mathrm{d}t_i t_i^{\Delta_i+m-\frac{3}{2}}}{(1-t_i)^{\Delta_i+\Delta_j+m-1}}(-\bar{z}_{ij})^m\bar{\partial}_j^m\Phi^{\mathsf{c},-\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j) + \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{(-1)^{\Delta_i+\Delta_j}}{m!}\int_1^\infty\frac{\mathrm{d}t_i t_i^{\Delta_i+m-\frac{3}{2}}}{(1-t_i)^{\Delta_i+\Delta_j+m-1}}(\bar{z}_{ij})^m\bar{\partial}_j^m\Phi^{\mathsf{c},\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j), \tag{3.15}$$

where the first integral just straightforwardly gives

$$\frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}B\left(\Delta_i+m-\frac{1}{2},2-\Delta_i-\Delta_j-m\right)(-\bar{z}_{ij})^m\bar{\partial}_j^m\Phi^{\mathsf{c},-\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j). \tag{3.16}$$

For the second integral, we just need to reparametrize $t_i\mapsto\frac{1}{t_i}$ which gives

$$\frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{(-1)^{\Delta_i+\Delta_j}}{m!}\int_0^1\frac{\mathrm{d}t_i t_i^{\Delta_j-\frac{3}{2}}}{(1-t_i)^{\Delta_i+\Delta_j+m-1}}(\bar{z}_{ij})^m\bar{\partial}_j^m\Phi^{\mathsf{c},\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j) = \frac{-f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{(\bar{z}_{ij})^m}{m!}B\left(\Delta_j-\frac{1}{2},2-\Delta_i-\Delta_j-m\right)\bar{\partial}_j^m\Phi^{\mathsf{c},\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j), \tag{3.17}$$

where $(-1)^{\Delta_i+\Delta_j}$ flips the orientation of the gluino and gives an overall minus sign.

Combine everything to get

$$\Gamma^{\mathsf{a},\varepsilon}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{b},-\varepsilon}_{+,\Delta_j,B}(z_j,\bar z_j) \sim \frac{-f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{(\bar z_{ij})^m}{m!}\bar\partial_j^m\bigg[B\left(\Delta_j-\frac{1}{2},2-\Delta_i-\Delta_j-m\right) \times \Phi^{\mathsf{c},\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar z_j) - (-1)^m B\left(\Delta_i+m-\frac{1}{2},2-\Delta_i-\Delta_j-m\right) \times \Phi^{\mathsf{c},-\varepsilon}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar z_j)\bigg]. \tag{3.18}$$

Here we see that the mixed orientation OPE coefficients are also contained in the master formula. From now on we shall just write down master formulas containing both orientation configurations and all $SL(2,\mathbb{R})$ descendants for all the other cases, leaving the reader to work out the individual Beta functions.

Now that we have an explicit procedure to compute the like helicity gluino-gluino OPE, we could use this to do other OPEs with a slight change in coefficients, for example, the like helicity gluon-gluino OPE $\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar z_j)$:

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar z_j) \sim \int d\sigma_i\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_{ij}}\frac{ds_i}{s_i}\frac{ds_j}{s_j}\frac{dt_i}{t_i^{2-\Delta_i}}\frac{dt_j}{t_j^{\frac{3}{2}-\Delta_j}}(\mathrm{i}s_j\chi_A(\sigma_j)) \times \bar\delta^2(z_i-s_i\lambda(\sigma_i))\bar\delta^2(z_j-s_j\lambda(\sigma_j)) \times \exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar z_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar z_j]). \tag{3.19}$$

Here we shall give the master formula directly after manipulating the integrals following the same procedure as the gluino-gluino case

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar z_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar z_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{3}{2}+m}} \times\left(\frac{\varepsilon_i\bar z_{ij}}{\varepsilon_j}\right)^m\bar\partial_j^m\Gamma^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j-1,A}(z_j,\bar z_j), \tag{3.20}$$

where one could check the coefficients against the literature [12,13] for the $m=0$, $\varepsilon_i=\varepsilon_j=\varepsilon$ case.

B. Gravitino-gravitino OPE

Next up we consider the like helicity gravitino-gravitino OPE, where the difference compared with the gluino-gluino case is that we no longer have the worldsheet current OPE. Instead, we have $\mathcal{Z}-\mathcal{W}$ OPEs and $\rho-\tilde\rho$ OPEs given by (2.23) and (2.26). More specifically, we only use the bosonic part of the $\mathcal{Z}-\mathcal{W}$ OPE,

$$\mu^{\dot\alpha}(\sigma_i)\tilde\lambda_{\dot\beta}(\sigma_j)=\frac{\delta^{\dot\alpha}_{\dot\beta}\sqrt{d\sigma_i d\sigma_j}}{\sigma_i-\sigma_j}. \tag{3.21}$$

First, write down the OPE in integral form:

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Theta^{\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j)\sim\varepsilon_i\varepsilon_j\int\frac{ds_i}{s_i^2}\frac{ds_j}{s_j^2}\frac{dt_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{dt_j}{t_j^{\frac{3}{2}-\Delta_j}}d\sigma_i(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}s_j\chi_B(\sigma_j)) \times(\mathrm{i}[\tilde\lambda(\sigma_i)\bar z_i]-\varepsilon_i s_i t_i[\rho(\sigma_i)\bar z_i][\tilde\rho(\sigma_i)\bar z_i])(\mathrm{i}[\tilde\lambda(\sigma_j)\bar z_j]-\varepsilon_j s_j t_j[\rho(\sigma_j)\bar z_j][\tilde\rho(\sigma_j)\bar z_j]) \times\bar\delta^2(z_i-s_i\lambda(\sigma_i))\bar\delta^2(z_j-s_j\lambda(\sigma_j))\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar z_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar z_j]). \tag{3.22}$$

Now we take all possible Wick contractions to get a slightly more involved expression:

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Theta^{\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j)\sim\int_{\Sigma_i\times\Sigma_j}\int_{(\mathbb{C}^*)^2\times(\mathbb{R}_+)^2}\frac{dt_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{ds_i}{s_i^2}\frac{dt_j}{t_j^{2-\Delta_i-\Delta_j}}\frac{ds_j}{s_j^2}\varepsilon_i\varepsilon_j \times\bar\delta^2(z_i-s_i\lambda(\sigma_i))\bar\delta^2(z_j-s_j\lambda(\sigma_j))e^{it_is_it_j\varepsilon_i[\mu(\sigma_i)\bar z_i]+it_js_j\varepsilon_j[\mu(\sigma_j)\bar z_j]}(is_i\chi_A(\sigma_i))(is_j\chi_B(\sigma_j)) \times\bigg(\varepsilon_i\varepsilon_j t_i t_j^3 s_i s_j\frac{[\bar z_i\bar z_j]^2}{\sigma_{ij}^2}-\varepsilon_i\varepsilon_j t_i t_j^3 s_i s_j\frac{[\bar z_i\bar z_j]^2}{\sigma_{ij}^2}-i\varepsilon_i t_i t_j^2 s_i\frac{[\bar z_i\bar z_j][\tilde\lambda(\sigma_i)\bar z_i]}{\sigma_{ij}}-i\varepsilon_j t_j^2 s_j\frac{[\bar z_j\bar z_i][\tilde\lambda(\sigma_j)\bar z_j]}{\sigma_{ji}} +\varepsilon_i^2 t_i^2 t_j^3 s_i^2\frac{[\bar z_i\bar z_j][\tilde\rho(\sigma_i)\bar z_i][\rho(\sigma_i)\bar z_i]}{\sigma_{ij}}+\varepsilon_j^2 t_j^3 s_j^2\frac{[\bar z_i\bar z_j][\tilde\rho(\sigma_j)\bar z_j][\rho(\sigma_j)\bar z_j]}{\sigma_{ij}}+\varepsilon_j\varepsilon_i t_i t_j^3 s_i s_j \times\frac{[\bar z_i\bar z_j][\tilde\rho(\sigma_i)\bar z_i][\rho(\sigma_j)\bar z_j]}{\sigma_{ij}}+\varepsilon_j\varepsilon_i t_i t_j^3 s_i s_j\frac{[\bar z_j\bar z_i][\tilde\rho(\sigma_j)\bar z_j][\rho(\sigma_i)\bar z_i]}{\sigma_{ji}}\bigg), \tag{3.23}$$

where we have rescaled $t_i \mapsto t_i t_j$. One notices that double poles appeared but nicely cancel out leaving only simple poles in the OPE, which is rather miraculous in its own right since there were no extra constraints required; the structure of the ambitwistor string vertex operators dictate the simple pole.

Since the structures of the holomorphic functions are exactly the same as in the gluino-gluino case, the expression still localizes on $z_{ij} = 0$ or $\sigma_{ij} = 0$ on the ambitwistor string worldsheet. The roadmap for computing these integrals follows directly from the gluino-gluino case. Hence we perform the s_i integral first against the first holomorphic delta function, integrate by parts to extract residue at $\sigma_{ij} = 0$, and then use the identities enforced by the remaining delta function to write the pole in a desired form. Since we have already rescaled $t_i \mapsto t_i t_j$, we just need to rescale $t_j \mapsto \frac{t_j}{|1+\frac{\varepsilon_i t_i}{\varepsilon_j}|}$ and expand around $\bar{z}_{ij}$ to obtain the entire $SL(2,\mathbb{R})$ tower of all orientation configurations.

The master formula in this case reads

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Theta^{\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1+m}} \times\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m V^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,AB}(z_j,\bar{z}_j). \tag{3.24}$$

One could check against the literature for the case with $m=0, \varepsilon_i = \varepsilon_j$, where they match up to R-symmetry and the spin 1 graviphoton V, which the authors of [13] consider to be the $\mathcal{N}=1$ case.

Following very similar steps, one could go up the ladder in one of the supersymmetric multiplets and compute the like helicity graviton-gravitino OPE

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Theta^{\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar{z}_j) \sim \varepsilon_i\varepsilon_j\int\frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j^2}\frac{\mathrm{d}t_i}{t_i^{2-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}\mathrm{d}\sigma_i \times(\mathrm{i}s_j\chi_A(\sigma_j))(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i]-\varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i])(\mathrm{i}[\tilde{\lambda}(\sigma_j)\bar{z}_j]-\varepsilon_j s_j t_j[\rho(\sigma_j)\bar{z}_j][\tilde{\rho}(\sigma_j)\bar{z}_j]) \times\bar{\delta}^2(z_i-s_i\lambda(\sigma_i))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j))\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar{z}_j]). \tag{3.25}$$

Here we just give the master formula after evaluating the integrals

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Theta^{\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar{z}_j) \sim \frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{3}{2}+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m\Theta^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,A}(z_j,\bar{z}_j). \tag{3.26}$$

C. Gravitino-gluino OPE

In this subsection, we consider the mixing OPEs between super Yang-Mills and Einstein supergravity. First, write down the gravitino-gluino OPE

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \varepsilon_i\int\frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}j^{\mathsf{a}}(\sigma_j)(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}s_j\chi_B(\sigma_j)) \times(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i]-\varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i])\bar{\delta}^2(z_i-s_i\lambda(\sigma_i))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j)) \times\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar{z}_j]). \tag{3.27}$$

Once again, just as the two previous cases, there is no mixing between the two holomorphic delta functions, which suggests that the methodology for this should not differ too much from the other like helicity cases. However, we notice that the OPE we need to consider here is slightly different. The only possible contraction comes from $\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i]$ and the exponential $e^{i\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]}$, which just gives us

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \mathrm{i}\bar{z}_{ij}\int\frac{\mathrm{d}s_i}{s_i}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{\frac{1}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}\frac{j^{\mathsf{a}}(\sigma_j)}{\sigma_{ij}}(\mathrm{i}s_i\chi_A(\sigma_i)) \times(\mathrm{i}s_j\chi_B(\sigma_j))\bar{\delta}^2(z_i-s_i\lambda(\sigma_i))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j))\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar{z}_j]), \tag{3.28}$$

where $\bar{z}_{ij} \coloneqq [\bar{z}_i \bar{z}_j]$. We notice that the simple pole emerged in front of the expression, which allows us to integrate by parts to obtain $\bar{\delta}(\sigma_{ij})$ just as before. Here we simply follow exactly the same steps as in Sec. III A to obtain the final expression containing all $SL(2,\mathbb{R})$ descendants:

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j) \sim \frac{-\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+m}} \times \left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \bar{\partial}_j^m \Phi^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{\Delta_i+\Delta_j,AB}(z_j,\bar{z}_j). \tag{3.29}$$

Similarly, we could go back up the SUSY hierarchy in either multiplet and compute the graviton-gluino OPE and the gravitino-gluon OPE, namely $\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar{z}_j)$ and $\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\mathcal{O}^{\mathsf{a},\varepsilon_j}_{+,\Delta_j}(z_j,\bar{z}_j)$. Here we just give their corresponding master formulas

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar{z}_j) \sim \frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{1}{2}+m}} \times \left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \bar{\partial}_j^m \Gamma^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,A}(z_j,\bar{z}_j), \tag{3.30}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\mathcal{O}^{\mathsf{a},\varepsilon_j}_{+,\Delta_j}(z_j,\bar{z}_j) \sim \frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{1}{2}+m}} \times \left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \bar{\partial}_j^m \Gamma^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,A}(z_j,\bar{z}_j). \tag{3.31}$$

One can check against the literature [13] for the case $m=0, \varepsilon_i=\varepsilon_j$.

We note here that since the highest possible number of supersymmetry one could have in Einstein-Yang-Mills theories in 4D is 4, we would only be able to compute OPEs between any particle from the gluon multiplet and the graviton, the gravitino, and the vector in the graviton supermultiplet. Indeed, if one attempts to compute, for example, the OPE between a gluon and the scalar in the graviton multiplet, the homogeneity mismatches with the number of supersymmetry in the resulting vertex operator.

IV. MIXED HELICITY OPEs

In all the like helicity computations, we demonstrated how to extract a single vertex operator out of the integrals and obtain the desired Euler Beta functions by rescaling certain parameters. One would naively expect a similar procedure to work for the mixed helicity OPEs. However, it turns out that the most fundamental observation we made for the like helicity OPEs does not hold anymore, namely the holomorphic delta functions enforcing worldsheet OPEs and celestial OPEs to coincide. The new scattering equations here complexify dramatically, which quickly hinders the steps we developed for the like helicity cases. This is because the new scattering equations now localize the computation in a region in the moduli space where the vertex operators are ill-defined. Hence we would perform certain reparametrization on the affine coordinates s_i and s_j to move to an appropriate patch in the moduli space, where the majority of the steps we developed for the like helicity cases would follow through.

The majority of the calculations in this section follow directly from the mixed helicity section of [25], with an additional subtlety in the fermionic $\chi-\tilde{\chi}$ OPE. Moreover, we notice that in the negative helicity operators we introduced in Sec. II C, the positions of the orientation parameter ϵ differ from the ones in [25]. To avoid the sign ambiguity that could occur, we adopt the more rigorous positioning of ε, which makes a slight adaptation to the pure bosonic calculations.

A. Gluino-gluino OPE

The key difference between mixed helicity OPEs and the like helicity ones is the structure of the holomorphic delta functions. Here we zoom in on the general structure of all mixed helicity OPEs:

$$\bar{\delta}^2(z_i - s_i\lambda(\sigma_i))\mathrm{e}^{\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]}\bar{\delta}^2(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j))\mathrm{e}^{\mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle}. \tag{4.1}$$

The first thing we notice is that, due to the $\mathcal{Z}-\mathcal{W}$ OPE (2.23), one could have contractions between $\lambda(\sigma_i)$ in the first delta function and $\tilde{\mu}(\sigma_j)$ on the exponential. Similarly there could be contractions between $\tilde{\lambda}(\sigma_j)$ and $\mu(\sigma_i)$. To compute such OPEs, we half-Fourier transform the holomorphic delta functions,

$$\bar{\delta}^2(z_i - s_i\lambda(\sigma_i)) = \int_{\mathbb{C}^2}\frac{\mathrm{d}^2\mathsf{m}}{(2\pi)^2}\mathrm{e}^{\mathrm{i}\langle\mathsf{m}z_i\rangle - \mathrm{i}s_i\langle\mathsf{m}\lambda(\sigma_i)\rangle},$$
$$\bar{\delta}^2(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j)) = \int_{\mathbb{C}^2}\frac{\mathrm{d}^2\tilde{\mathsf{m}}}{(2\pi)^2}\mathrm{e}^{\mathrm{i}\varepsilon_j[\tilde{\mathsf{m}}\bar{z}_j] - \mathrm{i}s_j[\tilde{\mathsf{m}}\tilde{\lambda}(\sigma_j)]}, \tag{4.2}$$

and use the following rule from Polchinski [44]:

$$\mathrm{e}^{\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]}\mathrm{e}^{-\mathrm{i}s_j[\tilde{\mathsf{m}}\tilde{\lambda}(\sigma_j)]} \sim \exp\left(\mathrm{i}\frac{\varepsilon_i t_i s_i s_j}{\sigma_{ij}}[\tilde{\mathsf{m}}\bar{z}_i]\right) :\mathrm{e}^{\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]}\mathrm{e}^{-\mathrm{i}s_j[\tilde{\mathsf{m}}\tilde{\lambda}(\sigma_j)]}:, \tag{4.3}$$

where $:(\cdots):$ indicates normal ordering.

After properly attending the OPEs above, one could write the mixed helicity gluino-gluino OPE as follows:

$$\Gamma^{\mathrm{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathrm{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)$$
$$\sim -\int \mathrm{d}\sigma_i \frac{\delta_A{}^B f^{\mathrm{abc}} j^{\mathrm{c}}(\sigma_j)}{\sigma_{ij}^2}\mathrm{d}s_i\mathrm{d}s_j\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}$$
$$\times\bar\delta^2\left(z_i-s_i\lambda(\sigma_i)-\frac{t_js_is_jz_j}{\sigma_{ij}}\right)\bar\delta^2\left(\varepsilon_j\bar z_j-s_j\tilde\lambda(\sigma_j)+\frac{\varepsilon_it_is_is_j\bar z_i}{\sigma_{ij}}\right)$$
$$\times\exp(\mathrm{i}\varepsilon_it_is_i[\mu(\sigma_i)\bar z_i]+\mathrm{i}t_js_j\langle\tilde\mu(\sigma_j)z_j\rangle), \tag{4.4}$$

where we have performed the $\chi_A(\sigma_i)\tilde\chi^B(\sigma_j)$ OPE sitting in front of the expression according to the fermionic part of the $\mathcal{Z}-\mathcal{W}$ OPE (2.23):

$$\chi_A(\sigma_i)\tilde\chi^B(\sigma_j)\sim\frac{\delta_A{}^B\sqrt{\mathrm{d}\sigma_i\mathrm{d}\sigma_j}}{\sigma_i-\sigma_j}. \tag{4.5}$$

At first sight, this expression seems hopeless as we have a double pole σ_{ij}^2 appearing. However, as we mentioned earlier, this expression is ill-defined on the current affine coordinate patch, and it requires certain appropriately chosen rescaling to appear regular. Apart from the double pole, we also observe that the third term in each delta function becomes singular when $\sigma_{ij}\to 0$, unless either s_i or s_j goes to 0 at the same rate.

Now that we have identified the standing difficulties with the current parametrization of the expression, we notice that a viable rescaling could be either $s_i\mapsto s_i\sigma_{ij}$ or $s_j\mapsto s_j\sigma_{ij}$. These two rescalings will correspond to the holomorphic and antiholomorphic limits on the celestial sphere, respectively, as we mentioned in Sec. II A. Also notice that either of these rescalings would bring an overall factor of σ_{ij}, which reduces the power of the σ_{ij} pole to 1. Hence we solved both problems simultaneously using these rescalings. Without loss of generality, we shall begin with the holomorphic rescaling $s_i\mapsto s_i\sigma_{ij}$,

$$\Gamma^{\mathrm{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathrm{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)$$
$$\sim -\int \mathrm{d}\sigma_i \frac{\delta_A{}^B f^{\mathrm{abc}} j^{\mathrm{c}}(\sigma_j)}{\sigma_{ij}}\mathrm{d}s_i\mathrm{d}s_j\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}$$
$$\times\bar\delta^2(z_i-s_i\sigma_{ij}\lambda(\sigma_i)-t_js_is_jz_j)\bar\delta^2(\varepsilon_j\bar z_j-s_j\tilde\lambda(\sigma_j)+\varepsilon_it_is_is_j\bar z_i)$$
$$\times\exp(\mathrm{i}\varepsilon_it_is_i\sigma_{ij}[\mu(\sigma_i)\bar z_i]+\mathrm{i}t_js_j\langle\tilde\mu(\sigma_j)z_j\rangle). \tag{4.6}$$

Following the roadmap described in the like helicity cases, first we shall perform the s_i integral using the first delta function and get

$$\Gamma^{\mathrm{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathrm{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)$$
$$\sim -\int \mathrm{d}\sigma_i \frac{\delta_A{}^B f^{\mathrm{abc}} j^{\mathrm{c}}(\sigma_j)}{\sigma_{ij}}\mathrm{d}s_j\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}$$
$$\times\bar\delta(\sigma_{ij}\langle z_i\lambda(\sigma_i)\rangle+t_js_j\langle z_iz_j\rangle)\bar\delta^2(\varepsilon_j\bar z_j-s_j\tilde\lambda(\sigma_j)+\varepsilon_it_is_i^*s_j\bar z_i)$$
$$\times\exp(\mathrm{i}\varepsilon_it_is_i^*\sigma_{ij}[\mu(\sigma_i)\bar z_i]+\mathrm{i}t_js_j\langle\tilde\mu(\sigma_j)z_j\rangle), \tag{4.7}$$

where $s_i^*=\frac{1}{\sigma_{ij}\langle\iota\lambda(\sigma_i)\rangle+t_js_j}$. Now we rescale $t_i\mapsto t_it_j$ and integrate by parts to extract residue at $\sigma_{ij}=0$,

$$\Gamma^{\mathrm{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathrm{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)$$
$$\sim -\int\frac{\delta_A{}^B f^{\mathrm{abc}} j^{\mathrm{c}}(\sigma_j)}{z_{ij}}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{3-\Delta_i-\Delta_j}}$$
$$\times\bar\delta^2(\varepsilon_j\bar z_j-s_j\tilde\lambda(\sigma_j)+\varepsilon_it_i\bar z_i)\exp(\mathrm{i}t_js_j\langle\tilde\mu(\sigma_j)z_j\rangle). \tag{4.8}$$

Notice that we did not need the identities enforced by the remaining delta function as in the like helicity cases; the z_{ij} pole came out automatically. Another feature of the mixed helicity configuration is that the holomorphic part of the exponential was eliminated straight away, and instead the complexity has been shifted to the holomorphic delta function. Now notice that there is still dependence on t_i inside the remaining delta function, and to get rid of it, we rewrite the delta function through some algebra,

$$\varepsilon_j\bar z_j-s_j\tilde\lambda(\sigma_j)+\varepsilon_it_i\bar z_i$$
$$=\varepsilon_j\left(1+\frac{\varepsilon_i}{\varepsilon_j}t_i\right)\bar z_j-s_j\tilde\lambda(\sigma_j)+\varepsilon_it_i\bar z_{ij}. \tag{4.9}$$

Now we just need to rescale $s_j\mapsto s_j(1+\frac{\varepsilon_i}{\varepsilon_j}t_i)$ to move all t_i dependence inside the delta function to the $\bar z_{ij}$ term which we shall Taylor expand around 0. However, to ensure our exponential to be invariant, we also need to rescale $t_j\mapsto\frac{t_j}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|}$. After these two rescalings we have

$$\Gamma^{\mathrm{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathrm{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)$$
$$\sim-\frac{\delta_A{}^B f^{\mathrm{abc}}}{z_{ij}}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_it_i^{\Delta_i-\frac{3}{2}}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j}}\int_{\Sigma_j\times\mathbb{C}^*\times\mathbb{R}_+}j^{\mathrm{c}}(\sigma_j)$$
$$\times\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_j}{t_j^{3-\Delta_i-\Delta_j}}\bar\delta^2\left(\varepsilon_j\bar z_j-s_j\tilde\lambda(\sigma_j)+\frac{\varepsilon_i\varepsilon_jt_i}{\varepsilon_j+\varepsilon_it_i}\bar z_{ij}\right)$$
$$\times\exp(\mathrm{i}\,\mathrm{sgn}(\varepsilon_j+\varepsilon_it_i)t_js_j\langle\tilde\mu(\sigma_j)z_j\rangle), \tag{4.10}$$

where we recognize the last three integrals represent a negative helicity gluon vertex operator. In the end to get all the $SL(2,\mathbb{R})$ descendants to appear in our expression, we simply need to Taylor expand the remaining delta function around $\bar z_{ij}=0$ to obtain the master formula in the mixed helicity case:

$$\Gamma^{\mathrm{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathrm{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)$$
$$\sim\frac{\delta_A{}^B f^{\mathrm{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_it_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j}}$$
$$\times\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_it_i}\right)^m\bar\partial_j^m\mathcal{O}^{\mathrm{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_it_i)}_{-,\Delta_i+\Delta_j-1}(z_j,\bar z_j). \tag{4.11}$$

Similarly, if we rescale $s_j \mapsto s_j\sigma_{ij}$, we would obtain a positive helicity gluon in the end. Combining the holomorphic part and the antiholomorphic part we have

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \frac{\delta_A{}^B f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j}} \times\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m \bar{\partial}_j^m \mathcal{O}^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j) + \frac{\delta_A{}^B f^{\mathsf{abc}}}{\bar{z}_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_j t_j^{\Delta_j-\frac{3}{2}+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{\Delta_i+\Delta_j}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m \times \partial_i^m \mathcal{O}^{\mathsf{c},\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_i+\Delta_j-1}(z_i,\bar{z}_i), \tag{4.12}$$

which essentially flips holomorphicity and interchanges i and j.

Using practically the same procedure, we could also compute the mixed helicity gluon-gluino OPE:

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{b},\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \int \mathrm{d}\sigma_i \frac{f^{\mathsf{abc}} j^{\mathsf{c}}(\sigma_j)}{\sigma_{ij}}\frac{\mathrm{d}s_i}{s_i}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{2-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}(\mathrm{i}s_j\tilde{\chi}^A(\sigma_j)) \times\bar{\delta}^2\left(z_i - s_i\lambda(\sigma_i) - \frac{t_j s_i s_j z_j}{\sigma_{ij}}\right)\bar{\delta}^2\left(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j)+\frac{\varepsilon_i t_i s_i s_j \bar{z}_i}{\sigma_{ij}}\right) \times\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle). \tag{4.13}$$

Note that in this case we could only rescale s_i as rescaling s_j cancels the σ_{ij} pole and the OPE becomes nonsingular. Here we write down the master formula obtained:

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{b},\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{-\frac{1}{2}+\Delta_i+\Delta_j}} \times\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m\bar{\Gamma}^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),A}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j). \tag{4.14}$$

One could check against the literature [12,13] for the case $m=0, \varepsilon_i=\varepsilon_j$.

B. Gravitino-gravitino OPE

Now we compute the mixed helicity gravitino-gravitino OPE. After treating all the $\mathcal{Z}-\mathcal{W}$ OPEs, we have the same holomorphic delta functions as before:

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Theta}^{\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j)\sim\varepsilon_i\int\frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j^2}\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}\mathrm{d}\sigma_i(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i]-\varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i]) \times(\mathrm{i}\langle\lambda(\sigma_j)z_j\rangle - s_j t_j\langle\tilde{\rho}(\sigma_j)z_j\rangle\langle\rho(\sigma_j)z_j\rangle)(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}s_j\tilde{\chi}^B(\sigma_j))\bar{\delta}^2\left(z_i-s_i\lambda(\sigma_i)-\frac{t_j s_i s_j z_j}{\sigma_{ij}}\right) \times\bar{\delta}^2\left(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j)+\frac{\varepsilon_i t_i s_i s_j\bar{z}_i}{\sigma_{ij}}\right)\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle). \tag{4.15}$$

However, notice that when we perform the $\chi-\tilde{\chi}$ OPE $(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}s_j\tilde{\chi}^B(\sigma_j))$, we will have a simple σ_{ij} pole, following the strategy developed in Sec. IV A, and we need to rescale either $s_i\mapsto s_i\sigma_{ij}$ or $s_j\mapsto s_j\sigma_{ij}$ to make the holomorphic delta function appear normal as in the like helicity cases again. However, to keep the simple pole in front, we need to ensure that the rescaling does not change the power of our pole. After combining all s_i and s_j in our expression, we see that the s integrals read $\int_{(\mathbb{C}^*)^2}\frac{\mathrm{d}s_i}{s_i}\frac{\mathrm{d}s_j}{s_j}$, which certainly does not affect the pole when rescaled. From here we simply follow the computation in Sec. IV A, which should give us the holomorphic part and the antiholomorphic part at the same time when we choose to rescale s_i or s_j.

The master formula we obtain for this can be written as

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Theta}^{\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j) \sim\frac{\mathrm{i}\varepsilon_i\delta_A{}^B\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{2+\Delta_i+\Delta_j}} \times\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m\mathcal{G}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j) +\frac{\mathrm{i}\varepsilon_i\delta_A{}^B z_{ji}}{\varepsilon_j\bar{z}_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_j t_j^{\Delta_j-\frac{3}{2}+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{2+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m \times\partial_i^m\mathcal{G}^{\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_j+\Delta_i}(z_i,\bar{z}_i). \tag{4.16}$$

Analogously, we could also compute the graviton-gravitino mixed helicity OPE:

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Theta}^{\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \varepsilon_i \int \frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j^2}\frac{\mathrm{d}t_i}{t_i^{2-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}\mathrm{d}\sigma_i(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i] - \varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i])$$
$$\times(\mathrm{i}\langle\lambda(\sigma_j)z_j\rangle - s_j t_j\langle\tilde{\rho}(\sigma_j)z_j\rangle\langle\rho(\sigma_j)z_j\rangle)(\mathrm{i}s_j\tilde{\chi}^A(\sigma_j))\bar{\delta}^2\left(z_i - s_i\lambda(\sigma_i) - \frac{t_j s_i s_j z_j}{\sigma_{ij}}\right)$$
$$\times\bar{\delta}^2\left(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j) + \frac{\varepsilon_i t_i s_i s_j\bar{z}_i}{\sigma_{ij}}\right)\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i] + \mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle). \tag{4.17}$$

Notice that there is no longer any σ_{ij} pole generated by the $\chi - \tilde{\chi}$ OPE. However, when we attempt to regularize the holomorphic delta functions by rescaling $s_i \mapsto s_i\sigma_{ij}$, we see that the homogeneity of s_i here generates a $\frac{1}{\sigma_{ij}}$ pole for us for free. Note that one could only rescale s_i here since rescaling s_j would lead to us having a nonsingular OPE. Here we just give the resulting master formula

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Theta}^{\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \frac{\mathrm{i}\varepsilon_i\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\frac{3}{2}+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m\bar{\Theta}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),A}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j). \tag{4.18}$$

One could check against the literature [13] for the case $m = 0, \varepsilon_i = \varepsilon_j$ except for the R-symmetry factor.

C. Gravitino-gluino OPE

Now we have both mixed helicity OPEs for SYM and SUGRA, and just as in the like helicity section, we compute the mixed helicity EYM OPEs. We begin with the gravitino-gluino mixed helicity OPE

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{a},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \varepsilon_i\int\frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}j^{\mathsf{a}}(\sigma_j)\mathrm{d}\sigma_i$$
$$\times(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}s_j\tilde{\chi}^B(\sigma_j))(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i] - \varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i])\bar{\delta}^2\left(z_i - s_i\lambda(\sigma_i) - \frac{t_j s_i s_j z_j}{\sigma_{ij}}\right)$$
$$\times\bar{\delta}^2\left(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j) + \frac{\varepsilon_i t_i s_i s_j\bar{z}_i}{\sigma_{ij}}\right)\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i] + \mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle). \tag{4.19}$$

Here we have the $\chi - \tilde{\chi}$ OPE giving us a simple pole; however, we notice that we could only rescale s_i here to maintain the simple σ_{ij} pole. Hence one ends up with only the antiholomorphic part. To get the holomorphic part describing a positive helicity gluon, we will need the opposite helicity configuration $\bar{\Theta}^{\varepsilon_i,A}_{-,\Delta_i}(z_i,\bar{z}_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j)$. The master formula we obtain is

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{a},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \frac{-\mathrm{i}\varepsilon_i\delta_A{}^B\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{1+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m\mathcal{O}^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j). \tag{4.20}$$

Now we could follow the same steps to compute two other OPEs, namely the graviton-gluino OPE and the gravitino-gluon OPE. First, the graviton-gluino OPE

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{a},\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \varepsilon_i\int\frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{2-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{\frac{3}{2}-\Delta_j}}j^{\mathsf{a}}(\sigma_j)\mathrm{d}\sigma_i$$
$$\times(\mathrm{i}s_j\tilde{\chi}^A(\sigma_j))(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i] - \varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i])\bar{\delta}^2\left(z_i - s_i\lambda(\sigma_i) - \frac{t_j s_i s_j z_j}{\sigma_{ij}}\right)$$
$$\times\bar{\delta}^2\left(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j) + \frac{\varepsilon_i t_i s_i s_j\bar{z}_i}{\sigma_{ij}}\right)\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i] + \mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle). \tag{4.21}$$

Observe that we could only rescale s_i here as rescaling s_j gives nonsingular OPE. The master formula can be written as

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{a},\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j)\sim\frac{\mathrm{i}\varepsilon_i\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\frac{1}{2}+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m\bar{\Gamma}^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j),\tag{4.22}$$

with the gravitino-gluon OPE

$$\begin{aligned}\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\mathcal{O}^{\mathsf{a},\varepsilon_j}_{-,\Delta_j}(z_j,\bar{z}_j)\sim\varepsilon_i\int\frac{\mathrm{d}s_i}{s_i^2}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{\frac{3}{2}-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{2-\Delta_j}}j^{\mathsf{a}}(\sigma_j)\mathrm{d}\sigma_i\\ \times(\mathrm{i}s_i\chi_A(\sigma_i))(\mathrm{i}[\tilde{\lambda}(\sigma_i)\bar{z}_i]-\varepsilon_i s_i t_i[\rho(\sigma_i)\bar{z}_i][\tilde{\rho}(\sigma_i)\bar{z}_i])\bar{\delta}^2\left(z_i-s_i\lambda(\sigma_i)-\frac{t_j s_i s_j z_j}{\sigma_{ij}}\right)\\ \times\bar{\delta}^2\left(\varepsilon_j\bar{z}_j-s_j\tilde{\lambda}(\sigma_j)+\frac{\varepsilon_i t_i s_i s_j\bar{z}_i}{\sigma_{ij}}\right)\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle).\end{aligned}\tag{4.23}$$

Notice that no matter whether we rescale s_i or s_j, the expression refuses to give us any σ_{ij} pole. This means that the OPE between a gravitino and a gluon is always regular, which agrees with the statement in the literature [13]. This comes from the fact that there is no Lorentz invariant 3-vertex for such a configuration. This can also be checked by BCFW methods in [9].

D. Scalar OPE

Here we address the discussion we had in Sec. II about splitting our spectrum between twistor space and dual twistor space. We mentioned that the only ambiguity is with the scalars in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA, where half of the scalars originate from the positive helicity multiplet and half from the negative one. We shall see that both of the representations are needed to compute the scalar-scalar OPEs. To keep things simple and illustrate our point, we demonstrate the calculation for the gluon scalars. First, take two scalars originated from the positive helicity multiplet:

$$\begin{aligned}&\Phi^{\mathsf{a},\varepsilon_i}_{\Delta_i,AB}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j}_{\Delta_j,CD}(z_j,\bar{z}_j)\\&\sim\int\mathrm{d}\sigma_i\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_{ij}}\frac{\mathrm{d}s_i}{s_i}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{1-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{1-\Delta_j}}\\&\quad\times(-s_i^2\chi_A(\sigma_i)\chi_B(\sigma_i))(-s_j^2\chi_C(\sigma_j)\chi_D(\sigma_j))\bar{\delta}^2(z_i-s_i\lambda(\sigma_i))\\&\quad\times\bar{\delta}^2(z_j-s_j\lambda(\sigma_j))\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar{z}_j]).\end{aligned}\tag{4.24}$$

From the number of fermionic indices on the right-hand side, it is straightforward to deduce whether the resulting vertex operator should carry $4R$-symmetry indices with homogeneity -4 on twistor space. However, there is no such particle present in the spectrum. If one were to ignore this and proceed with the computation naively as in Sec. III A, one would end up with

$$\begin{aligned}\Phi^{\mathsf{a},\varepsilon_i}_{\Delta_i,AB}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j}_{\Delta_j,CD}(z_j,\bar{z}_j)\sim\int\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{z_{ij}}\mathrm{d}s_j s_j\frac{\mathrm{d}t_i}{t_i^{1-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{1-\Delta_j}}(\chi_A(\sigma_j)\chi_B(\sigma_j)\chi_C(\sigma_j)\chi_D(\sigma_j))\bar{\delta}^2(z_j-s_j\lambda(\sigma_j))\\ \times\exp(\mathrm{i}\varepsilon_i t_i s_j[\mu(\sigma_j)\bar{z}_i]+\mathrm{i}\varepsilon_j t_j s_j[\mu(\sigma_j)\bar{z}_j]),\end{aligned}\tag{4.25}$$

where the resulting expression has homogeneity -2 on twistor space, which disagrees with the number of χ it contains. As the OPE between two scalars of negative helicity origin is very similar, we proceed to consider the remaining option, namely the scalar-scalar OPE with opposite helicity origin:

$$\begin{aligned}\Phi^{\mathsf{a},\varepsilon_i}_{\Delta_i,AB}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j,CD}_{\Delta_j}(z_j,\bar{z}_j)\sim\int\frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)}{\sigma_{ij}}\frac{\mathrm{d}s_i}{s_i}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{1-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{1-\Delta_j}}\\ \times(-s_i^2\chi_A(\sigma_i)\chi_B(\sigma_i))(-s_j^2\tilde{\chi}^C(\sigma_j)\tilde{\chi}^D(\sigma_j))\bar{\delta}^2\left(z_i-s_i\lambda(\sigma_i)-\frac{t_j s_i s_j z_j}{\sigma_{ij}}\right)\\ \times\bar{\delta}^2\left(\varepsilon_j\bar{z}_j-s_j\tilde{\lambda}(\sigma_j)+\frac{\varepsilon_i t_i s_i s_j\bar{z}_i}{\sigma_{ij}}\right)\exp(\mathrm{i}\varepsilon_i t_i s_i[\mu(\sigma_i)\bar{z}_i]+\mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle).\end{aligned}\tag{4.26}$$

First, the order of σ_{ij} pole we have after computing the $\chi-\tilde{\chi}$ OPEs is 3; however, notice that by rescaling $s_i \mapsto s_i\sigma_{ij}$ or $s_j \mapsto s_j\sigma_{ij}$, the pole becomes $\frac{1}{\sigma_{ij}}$. Proceed with rescaling $s_i \mapsto s_i\sigma_{ij}$, and we see that after performing the s_i integral and integrating by parts

$$\Phi^{\mathsf{a},\varepsilon_i}_{\Delta_i,AB}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j,CD}_{\Delta_j}(z_j,\bar{z}_j) \sim \int \frac{f^{\mathsf{abc}}j^{\mathsf{c}}(\sigma_j)\varepsilon_{AB}{}^{CD}}{z_{ij}}\frac{\mathrm{d}s_j}{s_j}\frac{\mathrm{d}t_i}{t_i^{1-\Delta_i}}\frac{\mathrm{d}t_j}{t_j^{3-\Delta_j}} \times \bar{\delta}^2\left(\varepsilon_j\bar{z}_j - s_j\tilde{\lambda}(\sigma_j)+\frac{\varepsilon_i t_i\bar{z}_i}{t_j}\right)\exp(\mathrm{i}t_j s_j\langle\tilde{\mu}(\sigma_j)z_j\rangle), \tag{4.27}$$

where the s_i integral provided us with the number of s_j needed to obtain homogeneity 0, which agrees perfectly with the number of χ remaining. Indeed, we see that a negative helicity gluon vertex operator is a few steps away from the expression on the right-hand side. If we were to proceed with rescaling on s_j instead of s_i, a positive helicity gluon vertex operator will appear on the right-hand side. The OPE for the $\mathcal{N}=8$ scalar-scalar is very similar, and we summarize the master formulas in Eqs. (A5) and (A21) in the Appendix.

V. SUPERSYMMETRIC HOLOGRAPHIC SYMMETRY

Following the steps in Sec. 5 in [25], one could obtain the soft algebra of gluino-gluon, gravitino-graviton, graviton-gluino, and gravitino-gluon by shifting and relabeling the indices. We shall see that the algebras remain invariant as the purely nonsupersymmetric cases, which agrees with the recent discovery in the literature [15].

A. SYM soft symmetries

Since we have explicit representations of the vertex operators of all particle content in our framework, to see soft symmetries one just needs to take residues at certain values of the conformal scaling dimension Δ and then perform the OPE. For demonstration purposes, we start with the like helicity gluon-gluino both outgoing scenario. It turns out that the most convenient way of expressing our vertex operator here is the integrated form of (2.27),

$$\Gamma^{\mathsf{a}}_{+,\Delta,A}(z,\bar{z}) = \mathrm{i}\int j^{\mathsf{a}}(\sigma)\chi_A(\sigma)\frac{\langle\iota\lambda(\sigma)\rangle^{\Delta-\frac{1}{2}}}{\langle\iota z\rangle^{\Delta-\frac{1}{2}}}\bar{\delta}(\langle\lambda(\sigma)z\rangle) \times\frac{(-\mathrm{i})^{\frac{1}{2}-\Delta}\Gamma(\Delta-\frac{1}{2})}{[\mu(\sigma)\bar{z}]^{\Delta-\frac{1}{2}}}, \tag{5.1}$$

where the s-integral and the t-integral have been performed. The soft gluinos are defined to be the residues at half integer values $\Delta = k+\frac{1}{2}$, where $k\in\{0,-1,-2,\ldots\}$,

$$L^{\mathsf{a}}_{+,k+\frac{1}{2},A}(z,\bar{z}) \coloneqq \mathrm{Res}_{\Delta=k+\frac{1}{2}}\Gamma^{\mathsf{a}}_{+,\Delta,A}(z,\bar{z}) = \frac{1}{2\pi}\oint\frac{(-\mathrm{i})^{-k}j^{\mathsf{a}}(\sigma)}{(-k)!}\frac{[\mu(\sigma)\bar{z}]^{-k}\langle\iota\lambda(\sigma)\rangle^{k}}{\langle\lambda(\sigma)z\rangle}\chi_A(\sigma). \tag{5.2}$$

Notice that the soft gluino vertex operator here is similar compared to the soft gluon vertex operator; hence by relabeling $k=3-2p$ and binomial expanding $[\mu(\sigma)\bar{z}] = \mu^{\dot{0}}+\bar{z}\mu^{\dot{1}}$ in $\bar{z}$,

$$L^{\mathsf{a}}_{+,\frac{7}{2}-2p,A}(z,\bar{z}) = \sum_{m=\frac{3}{2}-p}^{p-\frac{3}{2}}\frac{\bar{z}^{p-m-\frac{3}{2}}S^{\mathsf{a},p}_{m,A}(z)}{\Gamma(p-m-\frac{1}{2})\Gamma(p+m-\frac{1}{2})}, \tag{5.3}$$

where $g^p_m(\sigma) = (\mu^{\dot{0}})^{p+m-\frac{3}{2}}(\mu^{\dot{1}})^{p-m-\frac{3}{2}}$, p runs from $\frac{3}{2},2,\frac{5}{2},\ldots$, and

$$S^{\mathsf{a},p}_{m,A}(z) = \frac{\mathrm{i}^{2p-2}}{2\pi\mathrm{i}}\oint\frac{j^{\mathsf{a}}(\sigma)g^p_m(\sigma)}{\langle\iota\lambda(\sigma)\rangle^{2p-3}\langle\lambda(\sigma)z\rangle}\chi_A(\sigma). \tag{5.4}$$

Here we also take the expression for soft gluons from [25],

$$R^{\mathsf{a}}_{+,3-2q}(z,\bar{z}) = \sum_{n=1-q}^{q-1}\frac{\bar{z}^{q-n-1}S^{\mathsf{a},q}_n(z)}{\Gamma(q-n)\Gamma(q+n)}, \tag{5.5}$$

where

$$S^{\mathsf{a},q}_n(z) = \frac{\mathrm{i}^{2q-2}}{2\pi\mathrm{i}}\oint\frac{j^{\mathsf{a}}(\sigma)\tilde{g}^q_n(\sigma)}{\langle\iota\lambda(\sigma)\rangle^{2q-3}\langle\lambda(\sigma)z\rangle}, \tag{5.6}$$

with $\tilde{g}^q_n(\sigma) = (\mu^{\dot{0}})^{q+n-1}(\mu^{\dot{1}})^{q-n-1}$ and q runs from $1,\frac{3}{2},2,\ldots$. Using techniques to compute like helicity gluino-gluino OPEs in Sec. III A, we have for gluon-gluino

$$S^{\mathsf{a},p}_m(z_i)S^{\mathsf{b},q}_{n,A}(z_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}S^{\mathsf{c},p+q-1}_{m+n,A}(z_j). \tag{5.7}$$

B. SUGRA soft symmetries

Next up we consider the like helicity outgoing-outgoing graviton-gravitino soft OPE. To do this, first we write the outgoing positive helicity gravitino vertex operator in the following integrated form:

$$\Theta_{+,\Delta,A}(z,\bar{z}) = \mathrm{i}\int\left(\tilde{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\alpha}}}+\tilde{\rho}^{\dot{\alpha}}\rho^{\dot{\beta}}\frac{\partial^2}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}\right)\chi_A(\sigma)\frac{\langle\iota\lambda(\sigma)\rangle^{\Delta+\frac{1}{2}}}{\langle\iota z\rangle^{\Delta+\frac{1}{2}}} \times\bar{\delta}(\langle\lambda(\sigma)z\rangle)\frac{(-\mathrm{i})^{-\frac{1}{2}-\Delta}\Gamma(\Delta-\frac{3}{2})}{[\mu(\sigma)\bar{z}]^{\Delta-\frac{3}{2}}}. \tag{5.8}$$

The soft gravitinos $I_{+,k+\frac{1}{2},A}(z,\bar{z})$ are defined to be the residues at half integer values $\Delta = k+\frac{1}{2}$, where

$k \in \{1, 0, -1, \ldots\}$. Although this is not the conventional way to label the indices, it is of importance during the computation, and we shall see that we could unwind such strange labeling toward the end of the calculation,

$$\begin{aligned} I_{+,k+\frac{1}{2},A}(z,\bar{z}) &:= \mathrm{Res}_{\Delta=k+\frac{1}{2}}\Theta_{+,\Delta,A}(z,\bar{z}) \\ &= \frac{1}{2\pi}\oint\left(\tilde{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\alpha}}}+\tilde{\rho}^{\dot{\alpha}}\rho^{\dot{\beta}}\frac{\partial^2}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}\right) \\ &\quad\times\frac{\mathrm{i}^{-1-k}[\mu(\sigma)\bar{z}]^{1-k}\langle\iota\lambda(\sigma)\rangle^{k+1}}{(1-k)!\langle\lambda(\sigma)z\rangle}\chi_A(\sigma). \end{aligned} \tag{5.9}$$

This is similar compared to the soft graviton vertex operator excluding supersymmetry and weight in $\lambda(\sigma)$. To expand $I_{+,k+\frac{1}{2},A}(z,\bar{z})$ in soft modes, we relabel $k = 4 - 2p$ and binomial expand $[\mu(\sigma)\bar{z}] = \mu^{\dot{0}} + \bar{z}\mu^{\dot{1}}$ in $\bar{z}$,

$$I_{+,\frac{9}{2}-2p,A}(z,\bar{z}) = \sum_{m=\frac{3}{2}-p}^{p-\frac{3}{2}} \frac{\bar{z}^{p-m-\frac{3}{2}} w^p_{m,A}(z)}{\Gamma(p-m-\frac{1}{2})\Gamma(p+m-\frac{1}{2})}, \tag{5.10}$$

where $g^p_m(\sigma) = (\mu^{\dot{0}})^{p+m-\frac{3}{2}}(\mu^{\dot{1}})^{p-m-\frac{3}{2}}$ just as in the SYM case with $p \in \{\frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ and the soft modes $w^p_{m,A}(z)$ are defined as

$$\begin{aligned} w^p_{m,A}(z) &= \frac{\mathrm{i}^{2p}}{2\pi\mathrm{i}}\oint\left(\tilde{\lambda}^{\dot{\alpha}}\frac{\partial g^p_m(\sigma)}{\partial\mu^{\dot{\alpha}}}+\tilde{\rho}^{\dot{\alpha}}\rho^{\dot{\beta}}\frac{\partial^2 g^p_m(\sigma)}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}\right) \\ &\quad\times\frac{\chi_A(\sigma)}{\langle\iota\lambda(\sigma)\rangle^{2p-5}\langle\lambda(\sigma)z\rangle}. \end{aligned} \tag{5.11}$$

Together with soft graviton modes from [25],

$$\begin{aligned} w^q_n(z) &= \frac{\mathrm{i}^{2q}}{2\pi\mathrm{i}}\oint\left(\tilde{\lambda}^{\dot{\alpha}}\frac{\partial \tilde{g}^q_n(\sigma)}{\partial\mu^{\dot{\alpha}}}+\tilde{\rho}^{\dot{\alpha}}\rho^{\dot{\beta}}\frac{\partial^2 \tilde{g}^q_n(\sigma)}{\partial\mu^{\dot{\alpha}}\partial\mu^{\dot{\beta}}}\right) \\ &\quad\times\frac{1}{\langle\iota\lambda(\sigma)\rangle^{2q-5}\langle\lambda(\sigma)z\rangle}, \end{aligned} \tag{5.12}$$

where $\tilde{g}^q_n(\sigma) = (\mu^{\dot{0}})^{q+n-1}(\mu^{\dot{1}})^{q-n-1}$ and q runs from $1, \frac{3}{2}, 2, \ldots$. We obtain the soft graviton-gravitino OPE

$$w^p_{m,A}(z_i)w^q_n(z_j) \sim \frac{2(m(q-1)-n(p-\frac{3}{2}))}{z_{ij}} w^{p+q-2}_{m+n,A}(z_j), \tag{5.13}$$

where we have used the fact that

$$\begin{aligned} \{g^p_m, \tilde{g}^q_n\} &:= \varepsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial g^p_m}{\partial\mu^{\dot{\alpha}}}\frac{\partial \tilde{g}^q_n}{\partial\mu^{\dot{\beta}}} \\ &= 2\left(m(q-1)-n\left(p-\frac{3}{2}\right)\right)g^{p+q-2}_{m+n}. \end{aligned} \tag{5.14}$$

Because of the convention we have chosen, the index p begins at $\frac{3}{2}$ instead of 1. To make algebra look just as the usual infinite dimensional symmetry algebra introduced in [14], we simply relabel p by $p + \frac{1}{2}$. The algebra we obtain is

$$w^p_{m,A}(z_i)w^q_n(z_j) \sim \frac{2(m(q-1)-n(p-1))}{z_{ij}} w^{p+q-2}_{m+n,A}(z_j). \tag{5.15}$$

Notice that the soft expansion and binomial expansion we consider here do not differ from the pure bosonic case, only carrying an extra factor of $\chi_A(\sigma)$ which is not present in the bosonic case. Hence the w algebra here is still the diffeomorphism of the $\mu^{\dot{\alpha}}$ plane. However, one could consider doing the soft and binomial expansion on the entire supermultiplet and then take the OPE, in which case the fermionic coordinate $\chi_A(\sigma)$ on twistor space will also need to be expanded. Then we see that the algebra we obtain is a SUSY extension of the diffeomorphism of the $\mu^{\dot{\alpha}}$ plane and the diffeomorphism of the $\mu^{\dot{\alpha}} - \chi$ hypersurface.

It is straightforward to consider the supersymmetric soft Einstein-Yang-Mills OPEs, namely graviton-gluino and gluon-gravitino. To do this we simply take binomial expansions of the corresponding soft particles and take their OPE. We still consider the outgoing-outgoing like helicity configuration. Here we just present the results, which stay invariant as the purely bosonic soft gluon-graviton algebra,

$$w^p_m(z_i)S^q_{n,A}(z_j) \sim \frac{2(m(q-1)-n(p-1))}{z_{ij}} S^{p+q-2}_{m+n,A}(z_j) \tag{5.16}$$

for the graviton-gluino and

$$w^p_{m,A}(z_i)S^q_n(z_j) \sim \frac{2(m(q-1)-n(p-1))}{z_{ij}} S^{p+q-2}_{m+n,A}(z_j) \tag{5.17}$$

for the gravitino-gluon.

C. Soft-hard OPE

So far we have seen hard-hard and soft-soft OPEs in maximally supersymmetric theories, and it is worthwhile to consider the action of a soft particle acting on a hard one. One essentially just follows the steps in Sec. V in [25]. Here we simply state the results. A soft gluino acting on a hard gluon yields

$$\begin{aligned} &L^{\mathsf{a}}_{+,k-\frac{1}{2},A}(z_i,\bar{z}_i)\mathcal{O}^{\mathsf{b},\varepsilon}_{J,\Delta}(z_j,\bar{z}_j) \\ &\quad\sim\frac{\varepsilon^{k-1}}{(1-k)!}\frac{f^{\mathsf{abc}}}{z_{ij}}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j-2\bar{h}+r) \\ &\quad\times\Gamma^{\mathsf{c},\varepsilon}_{J-\mathrm{sgn}(J)/2,\Delta+k-\frac{3}{2},A}(z_j,\bar{z}_j), \end{aligned} \tag{5.18}$$

where $k \in \{1, 0, -1, \ldots\}$, $J = \pm 1$ denotes helicity of the gluon, $\mathrm{sgn}(J) = \pm 1$ is the sign of J, and $\bar{h} = (\Delta - J)/2$.

If the order is reversed, a soft gluon acting as a hard gluino gives us

$$R^{\mathsf{a}}_{+,k}(z_i, \bar{z}_i)\Gamma^{\mathsf{b},\varepsilon}_{J,\Delta,A}(z_j, \bar{z}_j) \sim \frac{\varepsilon^{k-1}}{(1-k)!}\frac{f^{\mathsf{abc}}}{z_{ij}}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} + r)\Gamma^{\mathsf{c},\varepsilon}_{J,\Delta+k-1,A}(z_j, \bar{z}_j), \tag{5.19}$$

where now $k \in \{1, 0, -1, \ldots\}$, $J = \pm\frac{1}{2}$ denotes the helicity of the gluino, and $\bar{h} = (\Delta - J)/2$.

For the gravitino-graviton soft-hard OPE, we have for a soft gravitino acting on a hard graviton

$$I_{+,k-\frac{1}{2},A}(z_i, \bar{z}_i)\mathcal{G}^{\varepsilon}_{J,\Delta}(z_j, \bar{z}_j) \sim \frac{-\varepsilon^k}{(1-k)!}\frac{\bar{z}_{ij}}{z_{ij}}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} - 1 + r) \times \Theta^{\varepsilon}_{J-\mathrm{sgn}(J)/2,\Delta+k-\frac{1}{2},A}(z_j, \bar{z}_j), \tag{5.20}$$

where $k \in \{2, 1, 0, \ldots\}$, $J = \pm 2$ denotes the helicity of the graviton, and $\bar{h} = (\Delta - J)/2$.

The OPE with the reversed order, namely a soft graviton acting on a hard gravitino reads

$$H_{+,k}(z_i, \bar{z}_i)\Theta^{\varepsilon}_{J,\Delta,A}(z_j, \bar{z}_j) \sim \frac{-\varepsilon^k}{(1-k)!}\frac{\bar{z}_{ij}}{z_{ij}}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} - 1 + r)\Theta^{\varepsilon}_{J,\Delta+k,A}(z_j, \bar{z}_j), \tag{5.21}$$

where $k \in \{2, 1, 0, \ldots\}$, $J = \pm\frac{3}{2}$ now denotes the helicity of the gravitino, and $\bar{h} = (\Delta - J)/2$.

Similarly, we could consider the action of a soft gravitino on a hard gluon:

$$I_{+,k-\frac{1}{2},A}(z_i, \bar{z}_i)\mathcal{O}^{\mathsf{a},\varepsilon}_{J,\Delta}(z_j, \bar{z}_j) \sim \frac{-\varepsilon^k}{(1-k)!}\frac{\bar{z}_{ij}}{z_{ij}}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} - 1 + r) \times \Gamma^{\mathsf{a},\varepsilon}_{J-\mathrm{sgn}(J)/2,\Delta+k-\frac{1}{2},A}(z_j, \bar{z}_j), \tag{5.22}$$

where $k \in \{2, 1, 0, \ldots\}$, $J = \pm 1$ denotes the helicity of the gluon, and $\bar{h} = (\Delta - J)/2$.

The other super EYM OPE we could consider is a soft graviton acting on a hard gluino:

$$H_{+,k}(z_i, \bar{z}_i)\Gamma^{\mathsf{a},\varepsilon}_{J,\Delta,A}(z_j, \bar{z}_j) \sim \frac{-\varepsilon^k}{(1-k)!}\frac{\bar{z}_{ij}}{z_{ij}}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} - 1 + r)\Gamma^{\mathsf{a},\varepsilon}_{J,\Delta+k,A}(z_j, \bar{z}_j), \tag{5.23}$$

where $k \in \{2, 1, 0, \ldots\}$, $J = \pm\frac{1}{2}$ denotes the helicity of the gluino, and $\bar{h} = (\Delta - J)/2$.

One could also expand the product acting on the vertex operators into a sum, where it essentially follows the identity

$$\frac{-\varepsilon^k}{(1-k)!}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} - 1 + r)\mathcal{U}_A(z_j, \bar{z}_j) = \sum_{l=0}^{1-k}\frac{(-1)^{k+l}\varepsilon^k}{l!(1-k-l)!}\frac{\Gamma(2\bar{h}+1)}{\Gamma(2\bar{h}+k+l)}\bar{z}^l_{ij}\bar{\partial}^l_j\mathcal{U}_A(z_j, \bar{z}_j), \tag{5.24}$$

$$\frac{\varepsilon^{k-1}}{(1-k)!}\prod_{r=1}^{1-k}(\bar{z}_{ij}\bar{\partial}_j - 2\bar{h} + r)\mathcal{U}_A(z_j, \bar{z}_j) = \sum_{l=0}^{1-k}\frac{(-1)^{k+l-1}\varepsilon^{k-1}}{l!(1-k-l)!}\frac{\Gamma(2\bar{h})}{\Gamma(2\bar{h}+k+l-1)}\bar{z}^l_{ij}\bar{\partial}^l_j\mathcal{U}_A(z_j, \bar{z}_j), \tag{5.25}$$

where $\mathcal{U}_A$ denotes either a gluino or a gravitino. Substituting these identities in our expressions, we see that they match results from [9,10].

VI. CONCLUSION

The maximally supersymmetric ambitwistor string worldsheet theory provides an explicit realization for the putative celestial CFT on the celestial sphere for maximally supersymmetric Yang-Mills, Einstein gravity, and Einstein-Yang-Mills theories. In [25], the bosonic version has already been worked out in detail. After adding supersymmetry to the entire construction, the identification of the worldsheet OPE limit and the momentum space collinear limit enforced by the scattering equation is still valid. Beyond this, the present paper demonstrated how homogeneity on twistor space elegantly organizes the supermultiplets and automatically implements physical constraints.

For future development, it would be interesting to further explore the approach of using worldsheet theories to compute celestial OPEs. At the moment, explorations in the area of asymptotically flat holography has been focusing on 4D. One could use the Ramond-Neveu-Schwarz formalism (RNS) ambitwistor string which exists in an arbitrary dimension to compute analogous OPEs, and it would be interesting to see whether the flat holography constructions are unique to 4D. Similarly, one could speculate how the infinite dimensional soft algebras will change in higher dimensions.

Besides this, it would also be worthwhile to demonstrate the effectiveness of the 4D ambitwistor string worldsheet theory in calculating celestial OPEs by computing higher order subleading OPEs. For this purpose, simple off-shell OPE calculations would not be enough, and some information about the helicity configuration of the correlator would have to be invoked.

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APPENDIX: SUMMARY OF ALL OPEs

From here on we list all singular collinear OPEs computable within our framework in the form of master equations. The roadmap from the master equation to obtain the Euler Beta functions for different orientation configurations is laid out in Sec. III A. The literature on this [12,13] have focused on the leading order and incoming-incoming or outgoing-outgoing configuration where $\varepsilon_i = \varepsilon_j$. One could easily extract the Euler Beta function coefficients from the following master formulas and check against the existing ones in the literature, where we see that they match up to R-symmetry and scalars. For the rest of the OPEs, to the best of our knowledge, we believe this is the first time they have been written down, which we list here for future references.

1. $\mathcal{N} = 4$ SYM

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,B} \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \bar{\partial}_j^m \Phi^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{\Delta_i+\Delta_j-1,AB}(z_j,\bar{z}_j), \tag{A1}$$

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Gamma^{\mathsf{b},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{3}{2}+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m \bar{\partial}_j^m \Gamma^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j-1,A}(z_j,\bar{z}_j), \tag{A2}$$

$$\begin{aligned}\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{b},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j) \sim{}& \frac{\delta_A{}^B f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m \bar{\partial}_j^m \mathcal{O}^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j)\\ &+\frac{\delta_A{}^B f^{\mathsf{abc}}}{\bar{z}_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_j t_j^{\Delta_j-\frac{3}{2}+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{\Delta_i+\Delta_j}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m \partial_i^m \mathcal{O}^{\mathsf{c},\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_i+\Delta_j-1}(z_i,\bar{z}_i),\end{aligned} \tag{A3}$$

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Gamma}^{\mathsf{b},\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{-\frac{1}{2}+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m \bar{\partial}_j^m \bar{\Gamma}^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),A}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j), \tag{A4}$$

$$\begin{aligned}\Phi^{\mathsf{a},\varepsilon_i}_{\Delta_i,AB}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j,CD}_{\Delta_j}(z_j,\bar{z}_j) \sim{}& \frac{\varepsilon_{AB}{}^{CD} f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-1+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m \bar{\partial}_j^m \mathcal{O}^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j)\\ &+\frac{\varepsilon_{AB}{}^{CD} f^{\mathsf{abc}}}{\bar{z}_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_j t_j^{\Delta_j-1+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{\Delta_i+\Delta_j}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m \partial_i^m \mathcal{O}^{\mathsf{c},\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_i+\Delta_j-1}(z_i,\bar{z}_i),\end{aligned} \tag{A5}$$

$$\mathcal{O}^{\mathsf{a},\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j,AB}_{\Delta_j}(z_j,\bar{z}_j) \sim \frac{f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m \bar{\partial}_j^m \Phi^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),AB}_{\Delta_i+\Delta_j-1}(z_j,\bar{z}_j), \tag{A6}$$

$$\Gamma^{\mathsf{a},\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Phi^{\mathsf{b},\varepsilon_j,BC}_{\Delta_j}(z_j,\bar{z}_j) \sim \frac{\delta_A{}^{[B} f^{\mathsf{abc}}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{1}{2}}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m \bar{\partial}_j^m \bar{\Gamma}^{\mathsf{c},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),C]}_{-,\Delta_i+\Delta_j-1}(z_j,\bar{z}_j). \tag{A7}$$

2. $\mathcal{N}=8$ SUGRA

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Theta^{\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar{z}_j)\sim\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m V^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,AB}(z_j,\bar{z}_j), \tag{A8}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\Theta^{\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar{z}_j)\sim\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{3}{2}+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m \Theta^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,A}(z_j,\bar{z}_j), \tag{A9}$$

$$\begin{aligned}\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{\Theta}^{\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar{z}_j)\sim&\frac{\mathrm{i}\varepsilon_i\delta_A{}^B\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{2+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m \mathcal{G}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j)\\&+\frac{\mathrm{i}\varepsilon_i\delta_A{}^B z_{ji}}{\varepsilon_j\bar{z}_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_j t_j^{\Delta_j-\frac{3}{2}+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{2+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m\partial_i^m \mathcal{G}^{\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_j+\Delta_i}(z_i,\bar{z}_i),\end{aligned} \tag{A10}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{\Theta}^{\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar{z}_j)\sim\frac{\mathrm{i}\varepsilon_i\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\frac{3}{2}+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m \bar{\Theta}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),A}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j), \tag{A11}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)\bar{V}^{\varepsilon_j,AB}_{-,\Delta_j}(z_j,\bar{z}_j)\sim\frac{\mathrm{i}\varepsilon_i\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+1}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m \bar{V}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),AB}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j), \tag{A12}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar{z}_i)V^{\varepsilon_j}_{+,\Delta_j,AB}(z_j,\bar{z}_j)\sim\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-1}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m V^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,AB}(z_j,\bar{z}_j), \tag{A13}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\bar{V}^{\varepsilon_j,BC}_{-,\Delta_j}(z_j,\bar{z}_j)\sim\frac{-\delta_A{}^{[B}\varepsilon_i\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+\frac{3}{2}}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m \bar{\Theta}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),C]}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j), \tag{A14}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)V^{\varepsilon_j}_{+,\Delta_j,BC}(z_j,\bar{z}_j)\sim\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{1}{2}}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m \Xi^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,ABC}(z_j,\bar{z}_j), \tag{A15}$$

$$\begin{aligned}V^{\varepsilon_i}_{+,\Delta_i,AB}(z_i,\bar{z}_i)\bar{V}^{\varepsilon_j,CD}_{-,\Delta_j}(z_j,\bar{z}_j)\sim&\frac{\mathrm{i}\delta_{[A}{}^{[C}\delta_{B]}{}^{D]}\varepsilon_i\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-1+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+2}}\left(\frac{\varepsilon_i\varepsilon_j\bar{z}_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar{\partial}_j^m \mathcal{G}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar{z}_j)\\&+\frac{\mathrm{i}\delta_{[A}{}^{[C}\delta_{B]}{}^{D]}\varepsilon_i z_{ji}}{\varepsilon_j\bar{z}_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_j t_j^{\Delta_j-1+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{\Delta_i+\Delta_j+2}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m\partial_j^m \mathcal{G}^{\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_i+\Delta_j}(z_i,\bar{z}_i),\end{aligned} \tag{A16}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar{z}_i)\Xi^{\varepsilon_j}_{+,\Delta_j,BCD}(z_j,\bar{z}_j)\sim\frac{\varepsilon_i\varepsilon\bar{z}_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+m}}\left(\frac{\varepsilon_i\bar{z}_{ij}}{\varepsilon_j}\right)^m\bar{\partial}_j^m \Pi^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{\Delta_i+\Delta_j-1,A[BCD]}(z_j,\bar{z}_j), \tag{A17}$$

$$\bar{\Theta}^{\varepsilon_i,A}_{-,\Delta_i}(z_i,\bar{z}_i)\Xi^{\varepsilon_j}_{+,\Delta_j,BCD}(z_j,\bar{z}_j)\sim\frac{-\mathrm{i}\delta_{[B}{}^A z_{ij}}{\bar{z}_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+1}}\left(\frac{\varepsilon_i\varepsilon_j z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\partial_j^m V^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,CD]}(z_j,\bar{z}_j), \tag{A18}$$

$$\bar{\Theta}^{\varepsilon_i,A}_{-,\Delta_i}(z_i,\bar{z}_i)\Pi^{\varepsilon_j}_{\Delta_j,BCDE}(z_j,\bar{z}_j)\sim\frac{-\mathrm{i}\delta_{[B}{}^A z_{ij}}{\bar{z}_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{\mathrm{d}t_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+\frac{1}{2}}}\left(\frac{\varepsilon_i\varepsilon_j z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\partial_j^m \Xi^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,CDE]}(z_j,\bar{z}_j), \tag{A19}$$

$$\Xi^{\varepsilon_i,ABC}_{-,\Delta_i}(z_i,\bar z_i)\Pi^{\varepsilon_j}_{\Delta_j,DEFG}(z_j,\bar z_j)\sim\frac{\delta_{[D}{}^{[A}\delta_E{}^B\delta_F{}^{C]}z_{ij}}{\bar z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-\frac{1}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+\frac{3}{2}}}\left(\frac{\varepsilon_i\varepsilon_j z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\partial_j^m\Theta^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,G]}(z_j,\bar z_j), \tag{A20}$$

$$\begin{aligned}\Pi^{\varepsilon_i,ABCD}_{\Delta_i}(z_i,\bar z_i)\Pi^{\varepsilon_j}_{\Delta_j,EFGH}(z_j,\bar z_j)\sim&\frac{i\varepsilon_{EFGH}{}^{ABCD}\varepsilon_j\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+2}}\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar\partial_j^m\mathcal{G}^{\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar z_j)\\&+\frac{i\varepsilon_{EFGH}{}^{ABCD}z_{ji}}{\bar z_{ji}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_j t_j^{\Delta_j+m}}{|1+\frac{\varepsilon_j}{\varepsilon_i}t_j|^{\Delta_i+\Delta_j+2}}\left(\frac{\varepsilon_j\varepsilon_i z_{ji}}{\varepsilon_i+\varepsilon_j t_j}\right)^m\partial_i^m\mathcal{G}^{\mathrm{sgn}(\varepsilon_i+\varepsilon_j t_j)}_{+,\Delta_i+\Delta_j}(z_i,\bar z_i),\end{aligned} \tag{A21}$$

3. Supersymmetric Einstein-Yang-Mills

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,B}(z_j,\bar z_j)\sim\frac{-\varepsilon_i\varepsilon_j\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+m}}\left(\frac{\varepsilon_i\bar z_{ij}}{\varepsilon_j}\right)^m\bar\partial_j^m\Phi^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{\Delta_i+\Delta_j,AB}(z_j,\bar z_j), \tag{A22}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar z_i)\Gamma^{\mathsf{a},\varepsilon_j}_{+,\Delta_j,A}(z_j,\bar z_j)\sim\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{1}{2}+m}}\left(\frac{\varepsilon_i\bar z_{ij}}{\varepsilon_j}\right)^m\bar\partial_j^m\Gamma^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,A}(z_j,\bar z_j), \tag{A23}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\mathcal{O}^{\mathsf{a},\varepsilon_j}_{+,\Delta_j}(z_j,\bar z_j)\sim\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j-\frac{1}{2}+m}}\left(\frac{\varepsilon_i\bar z_{ij}}{\varepsilon_j}\right)^m\bar\partial_j^m\Gamma^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{+,\Delta_i+\Delta_j,A}(z_j,\bar z_j), \tag{A24}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\bar\Gamma^{\mathsf{a},\varepsilon_j,B}_{-,\Delta_j}(z_j,\bar z_j)\sim\frac{-i\varepsilon_i\delta_A{}^B\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{1+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar\partial_j^m\mathcal{O}^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar z_j), \tag{A25}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar z_i)\bar\Gamma^{\mathsf{a},\varepsilon_j,A}_{-,\Delta_j}(z_j,\bar z_j)\sim\frac{i\varepsilon_i\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\frac{1}{2}+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar\partial_j^m\bar\Gamma^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),A}_{-,\Delta_i+\Delta_j}(z_j,\bar z_j), \tag{A26}$$

$$V^{\varepsilon_i}_{+,\Delta_i,AB}(z_i,\bar z_i)\Phi^{\mathsf{a},\varepsilon_j,CD}_{\Delta_j}(z_j,\bar z_j)\sim\frac{i\varepsilon_{AB}{}^{CD}\varepsilon_i\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-1+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{1+\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar\partial_j^m\mathcal{O}^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i)}_{-,\Delta_i+\Delta_j}(z_j,\bar z_j), \tag{A27}$$

$$\mathcal{G}^{\varepsilon_i}_{+,\Delta_i}(z_i,\bar z_i)\Phi^{\mathsf{a},\varepsilon_j,AB}_{\Delta_j}(z_j,\bar z_j)\sim\frac{i\varepsilon_i\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-2+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j}}\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar\partial_j^m\Phi^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),AB}_{\Delta_i+\Delta_j}(z_j,\bar z_j), \tag{A28}$$

$$\Theta^{\varepsilon_i}_{+,\Delta_i,A}(z_i,\bar z_i)\Phi^{\mathsf{a},\varepsilon_j,BC}_{\Delta_j}(z_j,\bar z_j)\sim\frac{-i\delta_A{}^{[B}\varepsilon_i\bar z_{ij}}{z_{ij}}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{R}_+}\frac{dt_i t_i^{\Delta_i-\frac{3}{2}+m}}{|1+\frac{\varepsilon_i}{\varepsilon_j}t_i|^{\Delta_i+\Delta_j+\frac{1}{2}}}\left(\frac{\varepsilon_i\varepsilon_j\bar z_{ij}}{\varepsilon_j+\varepsilon_i t_i}\right)^m\bar\partial_j^m\bar\Gamma^{\mathsf{a},\mathrm{sgn}(\varepsilon_j+\varepsilon_i t_i),C]}_{-,\Delta_i+\Delta_j}(z_j,\bar z_j). \tag{A29}$$

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