de Sitter black holes as constrained states in the Euclidean path integral

Patrick Draper[1] and Szilard Farkas[2]

[1]*Department of Physics, University of Illinois, Urbana, Illinois 61801, USA*
[2]*65th Street, New York City, New York, USA*

Schwarzschild–de Sitter black holes have two horizons that are at different temperatures for generic values of the black hole mass. Since the horizons are out of equilibrium, the solutions do not admit a smooth Euclidean continuation, and it is not immediately clear what role they play in the gravitational path integral. We show that Euclidean Schwarzschild–de Sitter is a genuine saddle point of a certain constrained path integral, providing a consistent Euclidean computation of the probability $\sim e^{-(S_{dS}-S_{SdS})}$ to find a black hole in the de Sitter bath.

I. INTRODUCTION

A surprising property of de Sitter space is that local excitation lowers the total entropy. In the case of black holes, the sum of the cosmological and black hole horizon areas in Schwarzschild–de Sitter (SdS) solutions decreases as the black hole mass increases. On these grounds, it has been suggested that local excitations should be thought of as constrained states of horizon degrees of freedom in empty de Sitter [1,2] (see also more recently Refs. [3–7]. Here, we show that the constrained state idea is also useful to describe the contribution of black holes to the bulk Euclidean path integral.

In static coordinates, the Euclidean continuation of SdS is

$$ds^2 = f(\rho)dt^2 + f(\rho)^{-1}d\rho^2 + \rho^2 d\Omega^2$$
$$f(\rho) = 1 - 2M/\rho - (\rho/L)^2. \tag{1}$$

L is the de Sitter radius, and M is the black hole mass parameter. The black hole and cosmological horizons $r_{b,c}$ are located at the zeros of f, and this metric has an irremovable conical singularity at one of them. More precisely, each horizon has a potential conical singularity, and typically only one of the two can be removed by a choice of periodicity of t. For this reason, Euclidean SdS does not provide a useful saddle point of the ordinary path integral, except in the Nariai limit, where the temperatures coincide and the solution is smooth [8].

However, Euclidean SdS does provide a genuine saddle point of a constrained path integral. The constraint is the specification of boundary data on a sphere that uniquely determines the mass of the enclosed black hole. This constraint can easily be imposed if the path integral is decomposed into path integrals over fields in the two regions separated by the sphere, similarly to the factorization of transition amplitudes into regions separated by a time slice. In this paper, we study this factorization and use it to show that black holes contribute to the de Sitter partition function with the probability weight $e^{S_{SdS}-S_{dS}}$. This result can be anticipated on other grounds, but our goal is to explain how it arises from the semiclassical approximation to the Euclidean gravitational path integral.

Previous work has also interpreted SdS in terms of constrained path integrals [9]. Reference [9] placed one of the conical singularities on a spatial slice continued to Lorentzian signature. Our treatment differs by working purely in the Euclidean signature, with a fixed mass constraint applied geometrically at a finite radius (which would become a timelike surface after continuation). Our constrained instanton exhibits a discontinuous lapse function instead of a conical singularity; however, the semiclassical actions obtained by the two methods agree. For other uses of constrained instantons in gravitational path integrals, see Refs. [10,11].

This paper is a companion to Ref. [12] in which we discuss some other properties of Euclidean SdS in four dimensions, focusing on cavity partition functions and equilibrium thermodynamics. Recent interesting work related to this includes Ref. [13], which considers similar physics in two dimensions and in finite causal diamonds [14,15], and Refs. [16,17], which address the stability problem for cavity black holes with zero or negative cosmological constant.

Recently, we became aware of an analysis of the same problem by Morvan *et al.*, which subsequently appeared in Ref. [18]. The results in Ref. [18] are in agreement with

ours, although the methods differ in the details; our focus is on the factorization properties of path integrals and the geometrical imposition of the constraint.

II. FACTORIZATION AND BOUNDARY TERMS IN THE PATH INTEGRAL

Before examining SdS specifically, let us take a step back and review some properties of the gravitational path integral, particularly the factorization properties with fixed boundary data in the Hamiltonian formalism. In writing the path integral, we can use either the Euclidean Lagrangian action $I_{\mathcal{L}}$, containing both the Einstein-Hilbert (EH) action and the Gibbons-Hawking-York (GHY) boundary term or the Euclidean Arnowitt-Deser-Misner (ADM) action with certain boundary terms B_N and B_{N_a}:

$$I_{\mathcal{L}} = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}(R-2\Lambda) - \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{\gamma}K, \quad (2)$$

$$I_{\mathrm{ADM}} = \frac{1}{16\pi}\int_{t_1}^{t_2} dt\int_{\Sigma_t} d^3x(\pi^{ab}\partial_t h_{ab} - N\mathcal{H} - N_a\mathcal{H}^a) + B_N + B_{N_a}. \quad (3)$$

For our purposes, it will be sufficient to focus on manifolds of fixed topology. The path integral in the Hamiltonian formalism is

$$Z = \int Dh_{ab}D\pi^{ab}DNDN_a e^{-I_{\mathrm{ADM}}[h_{ab},\pi^{ab},N,N_a]}. \quad (4)$$

The Hamiltonian and momentum constraints are[1]

$$\mathcal{H} = {}^{(3)}R + \frac{1}{h}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right), \quad (5)$$

$$\mathcal{H}^b = -2\sqrt{h}D_a\left(\frac{\pi^{ab}}{\sqrt{h}}\right), \quad (6)$$

where D_a is the covariant derivative compatible with the metric h_{ab} on the slice Σ_t induced by the spacetime metric g_{ab}.

We will be interested in the path integral over metrics on $\mathcal{M} = S^4$, for which the saddle point is empty Euclidean de Sitter, and we denote the partition function by Z_{dS}, as well as in path integrals on subregions of S^4. In the former case, there are no boundaries, and so boundary terms play no role, but in the latter case, there will be boundaries that partition the manifold. The Euclidean time foliation is also periodic, and therefore we omit the limits on the t integrals going forward.

Different choices for the boundary terms B_N and B_{N_a} correspond not only to different classical variational problems but also to different ways in which the full path integral

can be split into path integrals over fields in different parts of $\mathcal{M}$. We discuss first the standard choice of boundary terms, related to the canonical ensemble, and then an alternate choice, related to the microcanonical ensemble.

A. Canonical boundary data

The boundary terms B_N and B_{N_a} are usually chosen so that I_{ADM} has a well-posed variational problem with fixed h_{ab}, N, and N_a at the boundary. They can be derived by requiring that their variations cancel the surface terms generated by the variations of the bulk term. Let us introduce a radial coordinate ρ, analogous to the one that appears in the static coordinate representation of de Sitter, although we make no assumptions about metric symmetries at this stage. Using the Gauss-Codazzi relations, we can write the spatial curvature scalar ${}^{(3)}R$ as

$${}^{(3)}R = {}^{(2)}R + k^2 - k_{ab}k^{ab} - 2D_a(r^a k) + 2D_a(r^b D_b r^a), \quad (7)$$

where r^a is the unit normal to surfaces of fixed ρ, s_{ab} and ${}^{(2)}R$ are the induced metric and curvature scalar, $k_{ab} = s_a^c D_c r_b$ is the extrinsic curvature, and $k = k_a^a$. Let $\mathcal{T}$ be a surface defined by $\rho = r$, and consider the action on one side of $\mathcal{T}$. Using Eq. (7), the integral of $N\mathcal{H}$ can be written as the bulk integral of an expression containing at most first-order radial derivatives of N and h_{ab}, plus a boundary term

$$2\int dt\int_{S_t} d^2x\sqrt{s}(Nk - r_a r^b D_b r^a), \quad (8)$$

where S_t is the intersection of $\mathcal{T}$ and Σ_t. Since $r^a r_a = 1$, the second term in the integrand vanishes. So, we can cancel the contribution of (8) to the action by the choice

$$B_N = -\frac{1}{8\pi}\int dt\int_{S_t} d^2x\sqrt{s}Nk \quad \text{(canonical)}. \quad (9)$$

Similarly, the sum of the momentum constraint term in I_{ADM} and the boundary term

$$B_{N_a} = -\frac{1}{8\pi}\int dt\int_{S_t} d^2x\sqrt{s}N_a\frac{\pi^{ab}}{\sqrt{h}}r_b \quad (10)$$

is proportional to the bulk integral of $\sqrt{s}\pi^{ab}D_a N_b$. Therefore, the boundary terms (9) and (10) allow us to write I_{ADM} as the bulk integral of a function of the field variables and their derivatives, with at most first-order radial derivatives. A related property is that the variational problem of I_{ADM} with these boundary terms is well posed with fixed h_{ab}, N, and N_a on $\mathcal{T}$.

Now, let us consider both sides of $\mathcal{T}$. $\mathcal{T}$ divides $\mathcal{M}$ into two regions, $\mathcal{M}_1$ and $\mathcal{M}_2$. We would like to write the path integral over fields defined on $\mathcal{M}$ as a product of path integrals over fields on $\mathcal{M}_1$ and $\mathcal{M}_2$, with a path integral

---

[1]Note that the relative sign of the curvature and momentum terms differs between Euclidean and Lorentzian signature.

over data at the interface. The result, given below in Eq. (13), is in some sense what we might expect, but in fact, it is intricately tied to the choice of the boundary terms. Since we will change the boundary terms in the next section, we start by elaborating on this connection.

One way to approach the problem of factorization is to consider the domain of the action functional appearing in the path integral.[2] Of course, the action is well defined—i.e., without specifying additional UV data—for smooth metrics, but it is also well defined in this sense for a larger space of field variables. We have some freedom in this extension.

We focus on the continuity properties of the fields at $\mathcal{T}$. If we take h_{ab}, N, and N_a to be continuous (in ρ) at $\mathcal{T}$, then it is not necessary for k and π^{ab} to be continuous (in ρ) at $\mathcal{T}$ in order for the ADM action to be well defined. There may be Dirac delta functions in the bulk integrand, but they are multiplied by continuous functions of ρ and so are integrable without regularization.

Furthermore, we have seen that the action can be written as a bulk integral with at most first-order radial derivatives of the field variables (in fact no derivatives of π^{ab}). In this form, there will be no Dirac delta functions in the action density in $\mathcal{M}$ even if k and π^{ab} are discontinuous, merely products of discontinuous functions. The action on all of $\mathcal{M}$ can be written in this way, as can the action on each of the subregions $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, in this form, the action functional manifestly satisfies the additivity property

$$I_{\text{ADM}}(\mathcal{M}) = I_{\text{ADM}}(\mathcal{M}_1) + I_{\text{ADM}}(\mathcal{M}_2). \tag{11}$$

This decomposition is immediate when using the "purely bulk" form of the action in the subregions, and therefore it also holds in the equivalent bulk + boundary form, for boundaries placed just on either side of $\mathcal{T}$.

Let us illustrate these comments more explicitly with the example of discontinuous k. Using Eq. (7), the ADM action of a metric on all of $\mathcal{M}$ contains terms that can be expressed in different ways:

$$
\begin{aligned}
I_{\text{ADM}} &\supset +\frac{1}{8\pi} \int_{\mathcal{M}} N D_a(r^a k) \\
&= -\frac{1}{8\pi} \int_{\mathcal{M}} (D_a N) r^a k \\
&= -\frac{1}{8\pi} \int_{\mathcal{M}_1} (D_a N) r^a k - \frac{1}{8\pi} \int_{\mathcal{M}_2} (D_a N) r^a k \\
&= +\frac{1}{8\pi} \int_{\mathcal{M}_1} N D_a(r^a k) + \frac{1}{8\pi} \int_{\mathcal{M}_2} N D_a(r^a k) \\
&\quad - \frac{1}{8\pi} \int_{\mathcal{T}^-} d^3x \sqrt{s} N k - \frac{1}{8\pi} \int_{\mathcal{T}^+} d^3x \sqrt{s} N k.
\end{aligned} \tag{12}
$$

There are no boundary terms in the first line because $\mathcal{M}$ does not have a boundary. In this line, the integrand contains radial derivatives of k and so exhibits a delta function singularity if k has a discontinuity in ρ at $\mathcal{T}$. However, it is multiplied by N, which is continuous at the singularity, and so the integral remains well defined. In the second line, both $\partial_\rho N$ and k can be discontinuous in ρ at $\mathcal{T}$, but again the action functional is well defined, and moreover there is no delta function singularity. This makes the split into subregions trivial in the third line. In the fourth line, we return to the original form of the action in the subregions, which has boundary terms on both sides of $\mathcal{T}$, denoted $\mathcal{T}^{\pm}$. The boundaries excise the delta function singularity, and its contribution is reproduced by the boundary terms.

With Eq. (11), it is clear how the path integral decomposes. Variables that are required to be continuous in ρ at $\mathcal{T}$ are shared between the two regions and therefore must be integrated over only once. We can write the factorization as

$$Z_{\text{dS}} = \int_{\mathcal{T}} Ds_{ab} DN DN_a Z(\mathcal{M}_1 | s_{ab}, N, N_a) Z(s_{ab}, N, N_a | \mathcal{M}_2), \tag{13}$$

where $\mathcal{T}$ in the subscript of the integral is a mnemonic that integration variables are defined on the boundary $\mathcal{T}$. We indicated explicitly the dependence of the path integrals in $\mathcal{M}_1$ and $\mathcal{M}_2$ on the boundary data at $\mathcal{T}$, using notation suggestive of the insertion of a complete set of states in quantum mechanics. Since $Z(\mathcal{M}_1)$ and $Z(\mathcal{M}_2)$ are otherwise independent, it is clear that Z_{dS} receives contributions from metrics on $\mathcal{M}$ in which some properties can be discontinuous—for example, $h_{\rho\rho}$ may be discontinuous in ρ at $\mathcal{T}$, leading to discontinuous k, as discussed above. In the next section, we will consider a different but analogous factorization.

The boundary data over which we integrate in Eq. (13) includes only the metric s_{ab} induced on $S_t = \Sigma_t \cap \mathcal{T}$, even though the discussion above suggests that the entire spatial metric h_{ab} needs to be continuous at $\mathcal{T}$ in order for Eq. (11) to hold. The difference arises because of a choice of gauge, which we now discuss in some detail.

In the Hamiltonian formalism, it is convenient to impose the gauge condition

$$g_{\rho\alpha} = 0 \quad \text{for } \alpha \neq \rho \text{ at } \mathcal{T}, \tag{14}$$

and then there is no integration with respect to the boundary value of these components (i.e., with respect to $h_{\rho\alpha}$, $\alpha \neq \rho$, and N_ρ). This condition is not restrictive; since it is imposed only on the boundary, the stationarity of the action still provides the full set of field equations in the bulk. Also, for any solution, the gauge condition (14) can always be satisfied by an appropriate foliation in the neighborhood of $\mathcal{T}$. We will assume that this gauge condition is satisfied.

---

[2]The importance of boundary terms that allow us to write the action in a purely bulk form satisfying Eq. (11) was already noted in the context of the usual factorization of transition amplitudes [19].

This leaves only the question of the continuity properties of $h_{\rho\rho}$.

To address the question of $h_{\rho\rho}$, it is convenient to look first at the Lagrangian form of the action. As is well known, the variational problem of $I_{\mathcal{L}}$ is well posed even if only the induced metric on the boundary is fixed. Therefore, in the Lagrangian formalism, $g_{\rho\alpha}$ does not need to be continuous in ρ at $\mathcal{T}$, and the factorization property analogous to Eq. (13) has an integral only over the induced metric on $\mathcal{T}$. To see that the same is true in the Hamiltonian formalism, first note that the terms containing the momentum in the purely bulk form of the ADM action (terms proportional to $D_{(a}N_{b)}$, $\partial_t h_{ab}$, and π_{ab}) do not contain radial derivatives of h_{ab} at $\mathcal{T}$. This is because the gauge condition Eq. (14) sets $N^\rho = 0$ on $\mathcal{T}$. Another useful consequence of (14) is that it ensures that I_{ADM} with the boundary terms (9) and (10) is equal to $I_{\mathcal{L}}$ after π^{ab} is eliminated by Hamilton's equation. It follows that $h_{\rho\rho}$ does not need to be continuous for the factorization in the Hamiltonian formalism with boundary terms (9) and (10) if Eq. (14) is satisfied.

This is the reason why we integrate only over s_{ab}, N, and N_a in Eq. (13). (Our notation for the path integral measure is schematic and does not indicate explicitly in DN_a that N_ρ is not integrated over.) The path integral over the boundary values of the remaining field variables [i.e., $h_{\rho\rho}$ and π^{ab} if Eq. (14) holds] can be performed independently in the two regions. Therefore, these integrations are automatically included in the definition of $Z(\mathcal{M}_1)$ and $Z(\mathcal{M}_2)$.

B. Microcanonical boundary data

The factorization in (13) is not the only interesting case. Now, we discuss a different decomposition which is connected to a different set of boundary terms. First, note that Eq. (7) contains only first-order radial derivatives of the metric, apart from the term proportional to $D_a(r^a k)$.[3] Instead of the boundary term in (9), let us take

$$B_N = 0 \qquad \text{(microcanonical)}. \qquad (15)$$

With this choice, the term $-2ND_a(r^a k)$ in the action density is no longer effectively integrated by parts. We still choose B_{N_a} as in Eq. (10) [this is why "canonical" appears in Eq. (9) but not in Eq. (10)], and then we only need to integrate by parts in the momentum constraint term to cancel out B_{N_a} and write I_{ADM} as a purely bulk integral.

What are the factorization properties of the path integral with these boundary terms? Again, let us appeal to the action functional. Take s_{ab}, k, and N_a to be continuous in ρ at $\mathcal{T}$. Then, in the gauge (14), the action is well defined

without additional UV data, even if N and $h_{\rho\rho}$ are discontinuous.[4] The additivity property (11) is manifest already in the ordinary form of the ADM action,

$$I_{\text{ADM}} \supset \frac{1}{8\pi} \int_{\mathcal{M}} ND_a(r^a k)$$

$$= \frac{1}{8\pi} \int_{\mathcal{M}_1} ND_a(r^a k) + \frac{1}{8\pi} \int_{\mathcal{M}_2} ND_a(r^a k), \quad (16)$$

and it implies the following factorization of the path integral:

$$Z_{\text{dS}} = \int_{\mathcal{T}} Ds_{ab} Dk DN_a Z(\mathcal{M}_1|s_{ab}, k, N_a) Z(s_{ab}, k, N_a|\mathcal{M}_2). \qquad (17)$$

This holds already in the ordinary form of the ADM action, Eq. (3), with the new choice of boundary term $B_N = 0$ on either side of $\mathcal{T}$. [The measure in (17) is somewhat schematic, but it is not important for our purposes.]

Let us pause and make two comments:

(i) Equations (17) and (13) may be thought of as giving two different "complete set of states" insertions into the same partition function Z_{dS}.

(ii) The temporal foliation of $\mathcal{T}$ need not have a continuation into the bulk of both $\mathcal{M}_1$ and $\mathcal{M}_2$ that is nonsingular everywhere. That will be the case for Euclidean SdS, and we will deal with this mild complication below. Although we used the Hamiltonian formalism in the derivation above, once a foliation is defined on $\mathcal{T}$, we can specify the boundary data with which $Z(\mathcal{M}_1|s_{ab}, k, N_a)$ and $Z(s_{ab}, k, N_a|\mathcal{M}_2)$ are well defined. Then, they can be computed using the Lagrangian formulation with action $I_{\mathcal{L}}$, for which no bulk foliation is required, plus appropriate boundary terms.

---

[3]The radial derivatives in the last term in Eq. (7) are only first order because $r^a r_a = 1$ implies that $A^a = r^b D_b r^a$ is orthogonal to r^a, so its divergence $D_a A^a$ contains derivatives of A^a only in directions tangent to the constant ρ surfaces.

[4]$h_{\rho\rho}$ is not fixed on $\mathcal{T}$ in the canonical factorization, and therefore it can be discontinuous in the path integration. In the microcanonical factorization, with the purely bulk form of the action, essentially we require continuity of k, relax continuity of N, and swap a $(D_a N)r^a k$ term in the action for an $ND_a(r^a k)$ term. By inspecting these terms, one can see that this swap does not introduce a singularity even if $h_{\rho\rho}$ is discontinuous as long as k and s_{ab} are continuous. (k does not depend on $\partial_\rho h_{\rho\rho}$, so it is nonsingular even if $h_{\rho\rho}$ is discontinuous; furthermore, it can still be continuous even if $h_{\rho\rho}$ is not because k also depends on $\partial_\rho s_{ab}$, which is unfixed.) All the other terms in the action are the same as in the canonical factorization. It follows that $h_{\rho\rho}$ is not fixed on $\mathcal{T}$ in the microcanonical factorization and can be integrated over independently in the two regions. However, we will see $h_{\rho\rho}$ is actually continuous in the saddle point solution. Physically, this is because a discontinuous $h_{\rho\rho}$ implies the presence of a shell of matter, which gravitates and pulls the cosmological horizon inward, lowering the total entropy [12].

III. SdS AS A CONSTRAINED STATE

Now, let us discuss constrained states. The restriction to a black hole of mass M is obtained formally by inserting a delta functional at $\mathcal{T}$, concentrated on the boundary data (s_{ab}^M, k^M, N_a^M) of the black hole. This gives

$$Z_{\text{SdS}}^M = Z(\mathcal{M}_1 | s_{ab}^M, k^M, N_a^M) Z(s_{ab}^M, k^M, N_a^M | \mathcal{M}_2). \quad (18)$$

Before we proceed, let us elaborate on the motivation for fixing k instead of N. A priori, we could do either, and no particular justification is required for choosing one or the other. However, there is a reason to preferentially fix k. With the canonical factorization and a shared fixed boundary metric, we obtain two products of canonical partition functions. However, the canonical ensemble is not well defined for spherical cavities at positive cosmological constant. The lowest-action solution in each region consistent with the boundary conditions contains a cosmological horizon and possesses a negative heat capacity [12]. In contrast, the cavity solutions with fixed k have a consistent microcanonical thermodynamic interpretation [12]. This is one reason we proceed with the microcanonical factorization.

As in the more general discussion of Sec. II, we adopt a coordinate system where $\mathcal{T}$ lies at a fixed radial coordinate $\rho = r$. On $\mathcal{T}$, we set $N_a = N_a^M = 0$, we fix $s_{ab} = s_{ab}^M$ to the ordinary round metric $r^2 d\Omega_2^2$, and we take $k = k^M = 2\sqrt{f(r)}/r$ with f given in Eq. (1).

The semiclassical approximation can now be applied to each region separately. Euclidean SdS is a solution in each region, and now *both* conical singularities can be removed by an appropriate rescaling of the lapse function. Let the periodicity of the time coordinate be such that the metric (1) is smooth for $\rho < r$; i.e., $t \sim t + 1/T_b$, where the black hole temperature $T_b(M)$ is determined by the mass M consistent with the fixed value of k. The corresponding cosmological horizon temperature T_c is likewise determined by M. Then, the smooth semiclassical solution for $\rho > r$ and the same temporal periodicity is

$$ds^2 = f(\rho) \left(\frac{T_b}{T_c} \right)^2 dt^2 + f(\rho)^{-1} d\rho^2 + \rho^2 d\Omega^2. \quad (19)$$

There is no conical singularity at the cosmological horizon $\rho = r_c$. The lapse is not continuous at $\mathcal{T}$, but it does not need to be.

The momenta vanish in SdS, and so one might conclude that the semiclassical action (3) vanishes. This is not quite correct for reasons described in Refs. [20,21]. As alluded to in the bullet points above, the problem is the time foliation used in the ADM action above cannot be extended everywhere into the bulk; the foliation breaks down at the two horizons. The SdS solutions are smooth at these points, but the ADM action cannot be evaluated. Fortunately, there is a

trick whereby we can still take advantage of the simplicity of the ADM form in the evaluation of the action on a stationary solution. We start with the EH action in the entire spacetime, which has no boundaries. The EH form is valid everywhere. We draw infinitesimal boundaries $\mathcal{T}_b$ and $\mathcal{T}_c$ around the black hole and cosmological horizons. If Eq. (14) holds, the EH action between $\mathcal{T}_b$ and $\mathcal{T}_c$ is equal to the ADM action with the canonical boundary terms (9) and (10) minus the GHY terms at $\mathcal{T}_b$ and $\mathcal{T}_c$. (This follows from the equality of $I_{\mathcal{L}}$ and the canonical ADM action on a solution.) As a mnemonic, we may think of this trick as removing the coordinate singularities of the ADM formalism by switching to the Lagrangian formalism very close to the singularities, with suitable boundary terms at the interfaces.

In this form, the on-shell action is particularly simple to evaluate. In the limit that the boundaries $\mathcal{T}_{b,c}$ are taken infinitesimally close to the horizons, all of the bulk contributions to the SdS action vanish. The bulk ADM contributions in Eq. (3) vanish because of the constraints and $\pi^{ab} = 0$, and the EH contributions inside the infinitesimal boundaries $\mathcal{T}_{b,c}$ vanish because the curvature is finite and integrated over a region of zero volume in the limit. The B_{N_a} boundary terms near the horizons are also zero on SdS, and the canonical Hamiltonian boundary terms at each horizon vanish because $Nk \to 0$ at those points. Thus, the entirety of the classical action is due to GHY terms on the infinitesimal boundaries $\mathcal{T}_{b,c}$:

$$\begin{aligned} I_{\text{tot}} &= I_{\text{GHY}}(\mathcal{T}_b) + I_{\text{GHY}}(\mathcal{T}_c) \\ &= -(A_b + A_c)/4. \end{aligned} \quad (20)$$

Happily, the result does not depend on the radius r where we placed the constraint; the total action is just minus the total entropy.[5] Regardless of where the constraint is placed, the boundary data correspond to the same physical state in Lorentzian signature—the unique Lorentzian SdS solution with that value of M—and so the r-independence of Eq. (20) means that Z_{SdS}^M is a well-defined property of this state. More generally, it is not necessary to construct it with a spherically symmetric constraint. $\mathcal{T}$ can be any static surface enclosing the black hole, and for suitable boundary data, we will still obtain the same Z_{SdS}^M.

As a formal prescription for performing a continuation of the full discontinuous Euclidean solution to Lorentzian signature, we can proceed as follows. First, we change the t coordinate in one region so that the lapse is continuous but the coordinate periodicity is not. Then, in each region,

---

[5]It is also possible to obtain the same result for the action by regulating the conical singularities by an arbitrary smooth metric, then taking the limit [22]. This fact follows from the Gauss-Bonnet theorem, but it does not appear to shed light on the role of SdS as a constrained state in the semiclassical approximation.

we perform the naive continuation, which erases the discontinuity in the periodicity by decompactifying t in both regions. The result is the unique SdS solution of the given mass, which is smooth apart from the black hole singularity.

IV. CONCLUSIONS

Thus, the saddle point of the constrained path integral computes the probability of finding a black hole of mass M in the de Sitter ensemble,

$$P = Z_{\text{SdS}}^M / Z_{\text{dS}} \sim e^{S_b + S_c - \pi L^2}, \qquad (21)$$

normalizing to the unconstrained partition function. This interpretation offers a resolution to the semiclassical meaning of Euclidean SdS. The usual conical singularity can be exchanged for a discontinuity in the local temperature at some radius, encoding the absence of thermal equilibrium. The discontinuous geometry is a stationary point of the action with the constraint, and it yields the expected result that fluctuating a black hole into existence is exponentially rare in the entropy deficit.

ACKNOWLEDGMENTS

[1] T. Banks, B. Fiol, and A. Morisse, Towards a quantum theory of de Sitter space, J. High Energy Phys. 12 (2006) 004.

[2] T. Banks and W. Fischler, Holographic theory of accelerated observers, the S-matrix, and the emergence of effective field theory, arXiv:1301.5924.

[3] C. V. Johnson, de Sitter black holes, Schottky peaks, and continuous heat engines, arXiv:1907.05883.

[4] C. V. Johnson, Specific heats and Schottky peaks for black holes in extended thermodynamics, Classical Quantum Gravity 37, 054003 (2020).

[5] J. Dinsmore, P. Draper, D. Kastor, Y. Qiu, and J. Traschen, Schottky anomaly of de Sitter black holes, Classical Quantum Gravity 37, 054001 (2020).

[6] T. Banks and W. Fischler, Holographic space-time, Newton's law, and the dynamics of horizons, arXiv:2003.03637.

[7] L. Susskind, Black holes hint towards de Sitter-matrix theory, arXiv:2109.01322.

[8] P. H. Ginsparg and M. J. Perry, Semiclassical perdurance of de Sitter space, Nucl. Phys. **B222**, 245 (1983).

[9] R. Bousso and S. W. Hawking, Primordial black holes: Pair creation, Lorentzian condition, and evaporation, Int. J. Theor. Phys. **38**, 1227 (1999).

[10] J. Cotler and K. Jensen, Gravitational constrained instantons, Phys. Rev. D **104**, L081501 (2021).

[11] J. Cotler and K. Jensen, Wormholes and black hole microstates in AdS/CFT, J. High Energy Phys. 09 (2021) 001.

[12] P. Draper and S. Farkas, Euclidean de Sitter black holes and microcanonical equilibrium, arXiv:2203.01871.

[13] A. Svesko, E. Verheijden, E. P. Verlinde, and M. R. Visser, Quasi-local energy and microcanonical entropy in two-dimensional nearly de sitter gravity, 2022, arXiv:2203.00700.

[14] T. Jacobson and M. Visser, Gravitational thermodynamics of causal diamonds in (A)dS, SciPost Phys. **7**, 079 (2019).

[15] T. Banks, P. Draper, and S. Farkas, Path integrals for causal diamonds and the covariant entropy principle, Phys. Rev. D **103**, 106022 (2021).

[16] D. Marolf and J. E. Santos, The canonical ensemble reloaded: The complex-stability of Euclidean quantum gravity for black holes in a box, arXiv:2202.11786.

[17] D. Marolf and J. E. Santos, Stability of the microcanonical ensemble in Euclidean quantum gravity, arXiv:2202.12360.

[18] E. K. Morvan, J. P. van der Schaar, and M. R. Visser, On the Euclidean action of de Sitter black holes and constrained instantons, arXiv:2203.06155.

[19] S. W. Hawking, The path-integral approach to quantum gravity, in *General Relativity: An Einstein Centenary Survey* (Cambridge University Press, Cambridge, UK, 1979), pp. 746–789.

[20] M. Banados, C. Teitelboim, and J. Zanelli, Black Hole Entropy and the Dimensional Continuation of the Gauss-Bonnet Theorem, Phys. Rev. Lett. **72**, 957 (1994).

[21] C. Teitelboim, Gravitational thermodynamics of Schwarzschild-de Sitter space, in *Meeting on Strings and Gravity: Tying the Forces Together* (De Boeck, Bruxelles, 2001), pp. 291–299.

[22] R. Gregory, I. G. Moss, and B. Withers, Black holes as bubble nucleation sites, J. High Energy Phys. 03 (2014) 081.