Quantum theory of Weyl-invariant scalar-tensor gravity

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We perform a manifestly covariant quantization of a Weyl-invariant (i.e., a locally scale-invariant) scalar-tensor gravity in the extended de Donder gauge condition (or harmonic gauge condition) for general coordinate invariance and a new scalar gauge for Weyl invariance within the framework of the BRST formalism. We show that choral symmetry, which is a Poincaré-like $IOSp(8|8)$ supersymmetry in the case of Einstein gravity, is extended to a Poincaré-like $IOSp(10|10)$ supersymmetry. We point out that there is a gravitational conformal symmetry in quantum gravity and account for how conventional conformal symmetry in a flat Minkowski space-time is related to the gravitational conformal symmetry. Moreover, we examine the mechanism of the spontaneous symmetry breaking of the choral symmetry, and show that the gravitational conformal symmetry is spontaneously broken to the Poincaré symmetry and the corresponding massless Nambu-Goldstone bosons are the graviton and the dilaton. We also prove the unitarity of the physical S matrix on the basis of the BRST quartet mechanism.

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I. INTRODUCTION

There is no question that symmetry plays the central role in both elementary particle physics and quantum gravity. For instance, in Yang-Mills theory it has been found that we have a non-Abelian gauge symmetry and that this symmetry gives rise to physically significant effects, such as asymptotic freedom and quark confinement.

It is well known that there are two kinds of symmetries in nature: global symmetry and gauge symmetry. In order to understand nature more deeply, it is necessary to understand the meaning of both symmetries. The meaning of global symmetry is clear in the sense that it operates on physical observables in a direct manner and shows the real symmetry of a physical system. On the other hand, the meaning of gauge symmetry is more elusive than that of global symmetry since it does not operate on physical observables directly. To treat gauge symmetry properly in quantum field theory, it is essential to fix the gauge symmetry by a suitable gauge condition, and consequently physical observables are defined as BRST-invariant operators. Thus, it is sometimes said that gauge symmetry is a redundancy in the mathematical description of a physical system rather than a property of the system itself.

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Another important property of symmetries is that many global symmetries are not exact, but rather only approximate, whereas gauge symmetry is exact. For instance, there is a clear prediction of violation of baryon and lepton numbers by a quantum anomaly in the standard model. This fact is also supported by the theory of quantum gravity. In particular, when a black hole evaporates at the quantum level the baryon and lepton numbers are not conserved, whereas gauge quantum numbers such as electric and magnetic charges are precisely conserved since they are measured by the flux integrals at infinity.

Thus, if a certain global symmetry plays a critical role in physics, it must be promoted to a gauge symmetry. This statement holds in particular when constructing theories involving quantum gravity. In our previous work [1], we presented a quantum theory of globally scale-invariant gravity with a real scalar field, which is equivalent to the well-known Brans-Dicke gravity [2], by constructing its manifestly covariant BRST formalism. Since many studies of Brans-Dicke gravity have been limited to a classical analysis, our theory has provided us with some useful information on the quantum aspects of Brans-Dicke gravity. Indeed, based on this quantum gravity we have elucidated a mechanism of how scale invariance is spontaneously broken, and consequently how a massless "dilaton" emerges thanks to the Nambu-Goldstone theorem in quantum gravity [1,3]. Then, it is natural to generalize our formulation to the case of locally scale-invariant (or, equivalently, Weyl-invariant) scalar-tensor gravity and ask if we can get some useful knowledge about the quantum aspects of the theory.

In this article, we perform a manifestly covariant BRST quantization of Weyl-invariant scalar-tensor gravity with a real scalar field in addition to the metric tensor field, investigate the remaining global symmetries and their spontaneous symmetry breaking, prove the unitarity of the S matrix, and elucidate that there exists a gravitational analog of conformal symmetry in our theory. Long ago, in pioneering work by Nakanishi [4,5], on the basis of the Einstein-Hilbert action in the de Donder gauge (harmonic gauge) for a general coordinate transformation (GCT), it was shown that there remains a huge residual symmetry—which is a Poincaré-like $ISOp(8|8)$ supersymmetry called "choral symmetry"—in addition to the BRST symmetry and $GL(4)$ symmetry, etc. In our present formulation, adopting the extended de Donder gauge condition for the GCT and a new scalar gauge condition for the Weyl transformation, the choral symmetry is extended to a Poincaré-like $ISOp(10|10)$ supersymmetry, which includes the scale symmetry and gravitational special conformal symmetry. It is of interest that, as in a flat Minkowski space-time, both the scale symmetry and special conformal symmetry are spontaneously broken, and not only is the dilation a Nambu-Goldstone boson for the scale symmetry, but its derivative also provides a Nambu-Goldstone boson for the special conformal transformation.

The paper is organized as follows. In Sec. II we discuss a general gravitational theory for which there are two local symmetries: the general coordinate invariance and the Weyl symmetry. We point out that in such a theory we must choose a gauge-fixing condition for the GCT carefully in such a way that it does not violate the Weyl symmetry, and similarly a gauge-fixing condition for the Weyl transformation should be selected in order not to break the GCT. In Sec. III, beginning with Weyl-invariant scalar-tensor gravity [6], we fix the GCT and the Weyl transformation by the extended de Donder gauge and the new scalar gauge conditions, and construct a gauge-fixed, BRST-invariant quantum Lagrangian. In Sec. IV we calculate various equal-time (anti)commutation relations (ETCRs) among the fundamental fields, in particular, the Nakanishi-Lautrup auxiliary field and the Faddeev-Popov (FP) ghosts. In Sec. V we derive the ETCRs involving the gravitational field. In Sec. VI we prove the unitarity of the physical S matrix by means of the BRST quartet mechanism. In Sec. VII we show that there is a choral symmetry —which is an $IOSp(10|10)$ supersymmetry—in our theory. In Sec. VIII we point out the existence of a gravitational conformal symmetry even in quantum gravity, and we investigate its spontaneous symmetry breaking in Sec. IX. The final section is devoted to a discussion.

Two appendices are included for technical details. In Appendix A a derivation of the equation for the b_ρ field is given, and in Appendix B we account for the relationship between the gravitational conformal symmetry and conventional conformal symmetry.

II. CONSISTENCY BETWEEN TWO BRST SYMMETRIES

We wish to perform a manifestly covariant BRST quantization of a gravitational theory that is invariant under both a GCT and Weyl transformation or, equivalently, a local scale transformation. To take a more general theory into consideration, without specifying the concrete expression of the gravitational Lagrangian density, we start with the classical Lagrangian density[1]

$$\mathcal{L}_c = \mathcal{L}_c(g_{\mu\nu}, \phi), \tag{2.1}$$

which includes the metric tensor field $g_{\mu\nu}$ and a scalar field ϕ as dynamical variables.[2] We assume that $\mathcal{L}_c$ does not involve more than first-order derivatives of the metric and matter fields.

We have a physical situation in mind where we fix the general coordinate symmetry and Weyl symmetry by suitable gauge conditions. It is a familiar fact that after introducing the gauge conditions, instead of two local gauge symmetries, we are left with two kinds of global symmetries, which are called the BRST symmetries. The BRST transformation δ_B corresponding to the GCT is defined as

$$\begin{aligned}
\delta_B g_{\mu\nu} &= -(\nabla_\mu c_\nu + \nabla_\nu c_\mu) \\
&= -(c^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu c^\alpha g_{\alpha\nu} + \partial_\nu c^\alpha g_{\mu\alpha}), \\
\delta_B \tilde{g}^{\mu\nu} &= h(\nabla^\mu c^\nu + \nabla^\nu c^\mu - g^{\mu\nu}\nabla_\rho c^\rho), \\
\delta_B \phi &= -c^\lambda \partial_\lambda \phi, \qquad \delta_B c^\rho = -c^\lambda \partial_\lambda c^\rho, \\
\delta_B \bar{c}_\rho &= iB_\rho, \qquad \delta_B B_\rho = 0,
\end{aligned} \tag{2.2}$$

where c^ρ and $\bar{c}_\rho$ are the FP ghost and antighost, respectively, B_ρ is the Nakanishi-Lautrup (NL) field, and we have defined $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} \equiv h g^{\mu\nu}$. For later convenience, in place of the NL field B_ρ we introduce a new NL field defined as

$$b_\rho = B_\rho - ic^\lambda \partial_\lambda \bar{c}_\rho, \tag{2.3}$$

and its BRST transformation reads

$$\delta_B b_\rho = -c^\lambda \partial_\lambda b_\rho. \tag{2.4}$$

[1] We follow the notation and conventions of Ref. [7]. Lowercase greek $(\mu, \nu, \cdots)$ and Latin letters $(i, j, \cdots)$ are used for space-time and spatial indices, respectively; for instance, $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. The Riemann curvature tensor and Ricci tensor are defined by $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\sigma\mu}$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, respectively. The Minkowski metric tensor is denoted by $\eta_{\mu\nu}$; $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = -1$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$.

[2] It is straightforward to add the other fields, such as gauge fields and spinors.

The other BRST transformation $\bar{\delta}_B$ corresponding to the Weyl transformation is defined as

$$\bar{\delta}_B g_{\mu\nu} = 2c g_{\mu\nu}, \qquad \bar{\delta}_B \tilde{g}^{\mu\nu} = 2c\tilde{g}_{\mu\nu}, \\ \bar{\delta}_B \phi = -c\phi, \qquad \bar{\delta}_B \bar{c} = iB, \qquad \bar{\delta}_B c = \bar{\delta}_B B = 0, \quad (2.5)$$

where c and $\bar{c}$ are the FP ghost and antighost, respectively, and B is the NL field. Note that the two BRST transformations are nilpotent, i.e.,

$$\delta_B^2 = \bar{\delta}_B^2 = 0. \quad (2.6)$$

To complete the two BRST transformations, we have to fix not only the GCT BRST transformation δ_B on $c, \bar{c}$, and B, but also the Weyl BRST transformation δ_B on $c^\rho, \bar{c}_\rho$, and b_ρ. It is easy to determine the former BRST transformation since the fields $c, \bar{c}$, and B are all scalar fields, so their BRST transformations should take the form

$$\delta_B B = -c^\lambda \partial_\lambda B, \quad \delta_B c = -c^\lambda \partial_\lambda c, \quad \delta_B \bar{c} = -c^\lambda \partial_\lambda \bar{c}. \quad (2.7)$$

On the other hand, there is an ambiguity in fixing the latter BRST transformation, but we would like to propose a recipe for achieving this goal. The recipe is to just assume that the two BRST transformations anticommute with each other, that is,

$$\{\delta_B, \bar{\delta}_B\} \equiv \delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B = 0, \quad (2.8)$$

which requires us to take

$$\bar{\delta}_B b_\rho = \bar{\delta}_B c^\rho = \bar{\delta}_B \bar{c}_\rho = 0. \quad (2.9)$$

Now we would like to explain an important point that is occasionally missed in the theoretical physics literature, when two BRST transformations coexist in a theory. Suppose we fix the GCT by a gauge condition $F^\alpha(g_{\mu\nu}, \phi) = 0$ and the Weyl transformation by a gauge condition $F(g_{\mu\nu}, \phi) = 0$. Then, the gauge-fixed and BRST-invariant Lagrangian density is given by

$$\mathcal{L}_q = \mathcal{L}_c + \delta_B(\bar{c}_\alpha F^\alpha) + \bar{\delta}_B(\bar{c}F), \quad (2.10)$$

where the first term is the classical Lagrangian density (2.1). In this situation, a natural question arises about the gauge-fixing conditions: can we take any gauge-fixing conditions if they fix gauge symmetries anyway? If not, what gauge conditions are suitable for F^α and F?

In order to answer these questions, let us take the two BRST transformations separately and check whether the quantum Lagrangian density (2.10) is really invariant under the BRST transformations up to surface terms. First, taking the Weyl BRST transformation leads to

$$\bar{\delta}_B \mathcal{L}_q = \bar{\delta}_B \delta_B(\bar{c}_\alpha F^\alpha) = -\delta_B \bar{\delta}_B(\bar{c}_\alpha F^\alpha) \\ = -\delta_B[(\bar{\delta}_B \bar{c}_\alpha)F^\alpha - \bar{c}_\alpha \bar{\delta}_B F^\alpha], \quad (2.11)$$

where we have used $\bar{\delta}_B \mathcal{L}_c = 0$ and Eqs. (2.6) and (2.8). This equation clearly shows that the conditions

$$\bar{\delta}_B \bar{c}_\alpha = 0, \qquad \bar{\delta}_B F^\alpha = 0 \quad (2.12)$$

are sufficient conditions such that the Lagrangian density (2.10) is invariant under the Weyl BRST transformation.

It is of interest to notice that the former condition in Eq. (2.12) leads to two remaining equations in Eq. (2.9). To see this fact, let us take the GCT BRST transformation of the former equation as

$$0 = \delta_B \bar{\delta}_B \bar{c}_\alpha = -\bar{\delta}_B \delta_B \bar{c}_\alpha = -i\bar{\delta}_B B_\alpha \\ = -i[\bar{\delta}_B b_\alpha + i(\bar{\delta}_B c^\lambda)\partial_\lambda \bar{c}_\alpha], \quad (2.13)$$

which implies $\bar{\delta}_B b_\alpha = \bar{\delta}_B c^\lambda = 0$, which coincide with the remaining two equations in Eq. (2.9).

On the other hand, the latter condition in Eq. (2.12) gives rise to important information on the gauge condition for the GCT: the gauge-fixing condition for the GCT must be invariant under the Weyl transformation. Thus, for instance, the conventional de Donder gauge condition (or harmonic gauge condition),

$$\partial_\mu \tilde{g}^{\mu\nu} = 0, \quad (2.14)$$

is not suitable when there is Weyl invariance.[3]

Next, let us apply the GCT BRST transformation to $\mathcal{L}_q$. To do this, since the Lagrangian density is in general a quantity with density, it is more convenient to write it as $\mathcal{L}_q \equiv \sqrt{-g}\mathcal{L}'_q$ and $F = \sqrt{-g}F'$, where F and F' are scalars. Then, taking the GCT BRST variation leads to

$$\delta_B \mathcal{L}_q = \delta_B(\sqrt{-g}\mathcal{L}'_q) = \delta_B \bar{\delta}_B(\sqrt{-g}\bar{c}F') = -\bar{\delta}_B \delta_B(\sqrt{-g}\bar{c}F') \\ = -\bar{\delta}_B[-\sqrt{-g}\nabla_\rho c^\rho \bar{c}F' + \sqrt{-g}(-c^\rho \partial_\rho \bar{c})F' \\ - \sqrt{-g}\bar{c}(-c^\rho \partial_\rho F')] \\ = \partial_\rho \bar{\delta}_B(c^\rho \bar{c}F), \quad (2.15)$$

which means that $\mathcal{L}_q$ is indeed invariant under the GCT BRST transformation up to a surface term. In obtaining this result, we have assumed that

$$\delta_B F' = -c^\rho \partial_\rho F', \quad (2.16)$$

[3] In two space-time dimensions the de Donder condition is Weyl invariant, so it can be used as the gauge-fixing condition for the GCT.

which is nothing but the requirement that the quantity F' should be a scalar under the GCT. Thus, only a scalar function F' (or, equivalently, a scalar density F) makes sense as a gauge-fixing condition for Weyl invariance. Of course, this scalar function must break Weyl invariance. As suitable gauge-fixing conditions, in this paper we choose $F^\nu = \partial_\mu(\tilde{g}^{\mu\nu}\phi^2)$ and $F = \partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi)$.

III. QUANTUM WEYL-INVARIANT SCALAR-TENSOR GRAVITY

In this section, as a classical Lagrangian[4] we take a Weyl-invariant scalar-tensor gravity whose Lagrangian is of the form [6]

$$\mathcal{L}_c = \sqrt{-g}\left(\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \tag{3.1}$$

where ϕ is a real scalar field with a ghost-like kinetic term, and R is the scalar curvature. In addition to the invariance under the GCT, this Lagrangian is also invariant under the Weyl transformation (or the local scale transformation), defined as

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \qquad \phi \to \phi' = \Omega^{-1}(x)\phi. \tag{3.2}$$

Recall that in order to prove the invariance, we need to use the following transformation of the scalar curvature under Eq. (3.2):

$$R \to R' = \Omega^{-2}(R - 6\Omega^{-1}\Box\Omega), \tag{3.3}$$

where $\Box\Omega \equiv h^{-1}\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu\Omega)$.

As explained in the previous section, we have to pay attention to what gauge-fixing conditions should be chosen for the GCT and Weyl transformation in a consistent manner. For instance, taking the de Donder condition as a gauge condition for the GCT is not allowed since it breaks Weyl symmetry in four space-time dimensions. There are several interesting choices of suitable gauge conditions for the GCT, but we shall refer to only two representative examples. The first gauge condition for the GCT is a Weyl-invariant version of the de Donder gauge:

$$\partial_\mu((-g)^{\frac{1}{4}}g^{\mu\nu}) = 0. \tag{3.4}$$

This gauge choice is invariant under the Weyl transformation (3.2) and is physically interesting in the sense that it makes use of only the metric tensor field. However, some fields such as the Nakanishi-Lautrup field become not a normal vector field but rather a vector field with density, which makes several formulas ugly, so we do not adopt Eq. (3.4) as a gauge condition for the GCT. The second gauge condition, which we will use in this article and call the "extended de Donder gauge," is given by

$$\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0, \tag{3.5}$$

which is also invariant under the Weyl transformation (3.2).

Next, let us consider a gauge-fixing condition for the Weyl transformation. From the consistency discussed in Sec. II, an appropriate gauge condition must obey the condition that it is invariant under the GCT, that is, a scalar quantity. Since there are many scalars constructed from the real scalar field ϕ and the Riemannian tensors, we might be left in the dark on this issue. However, surprisingly enough, if we impose the requirement that the FP ghost's Lagrangian should have a Weyl-invariant metric $\tilde{g}^{\mu\nu}\phi^2$ instead of the standard metric $\tilde{g}^{\mu\nu}$, a suitable gauge condition for the GCT can be uniquely chosen. Such a gauge condition, which we call the "scalar gauge condition," reads

$$\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi) = 0, \tag{3.6}$$

which can be alternatively written as

$$\Box\phi^2 = 0. \tag{3.7}$$

Incidentally, the unitary gauge $\phi = \text{const}$ is often taken to show that the Weyl-invariant scalar-tensor gravity (3.1) is equivalent to the Einstein-Hilbert term, but this gauge choice is not as interesting since no conformal symmetry remains.

After taking the extended de Donder gauge condition (3.5) for the GCT and the scalar gauge condition (3.6) for the Weyl transformation, the gauge-fixed and BRST-invariant quantum Lagrangian is given by

$$\begin{aligned}\mathcal{L}_q &= \mathcal{L}_c + \mathcal{L}_{\text{GF+FP}} + \bar{\mathcal{L}}_{\text{GF+FP}} \\ &= \mathcal{L}_c + i\delta_B(\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\nu) + i\bar{\delta}_B[\bar{c}\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi)] \\ &= \sqrt{-g}\left(\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \\ &\quad - \tilde{g}^{\mu\nu}\phi^2(\partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda) \\ &\quad + \tilde{g}^{\mu\nu}\phi\partial_\mu B\partial_\nu\phi - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}\partial_\nu c, \end{aligned} \tag{3.8}$$

where surface terms are dropped. Note that the last term, which is the FP ghost's term for the Weyl transformation, certainly involves the Weyl-invariant metric $\tilde{g}^{\mu\nu}\phi^2$. Let us rewrite this Lagrangian concisely as

$$\mathcal{L}_q = \sqrt{-g}\frac{1}{12}\phi^2 R - \frac{1}{2}\tilde{g}^{\mu\nu}E_{\mu\nu}, \tag{3.9}$$

[4] For simplicity, we henceforth call a Lagrangian density a Lagrangian.

where we have defined

$$E_{\mu\nu} \equiv -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \phi^2(\partial_\mu b_\nu + i\partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda) - \phi\partial_\mu B\partial_\nu\phi + i\phi^2\partial_\mu\bar{c}\partial_\nu c + (\mu \leftrightarrow \nu). \tag{3.10}$$

Moreover, it is sometimes more convenient to introduce the dilaton $\sigma(x)$ by defining

$$\phi(x) \equiv e^{\sigma(x)} \tag{3.11}$$

and rewriting Eq. (3.9) in the form

$$\mathcal{L}_q = e^{2\sigma(x)}\left(\sqrt{-g}\frac{1}{12}R - \frac{1}{2}\tilde{g}^{\mu\nu}\hat{E}_{\mu\nu}\right), \tag{3.12}$$

where we have defined

$$\hat{E}_{\mu\nu} \equiv -\frac{1}{2}\partial_\mu\sigma\partial_\nu\sigma + \partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda - \partial_\mu B\partial_\nu\sigma + i\partial_\mu\bar{c}\partial_\nu c + (\mu \leftrightarrow \nu). \tag{3.13}$$

Note that the relation between $E_{\mu\nu}$ and $\hat{E}_{\mu\nu}$ is given by

$$E_{\mu\nu} = \phi^2\hat{E}_{\mu\nu} = e^{2\sigma}\hat{E}_{\mu\nu}. \tag{3.14}$$

From the Lagrangian $\mathcal{L}_q$, it is straightforward to derive the field equations by taking the variation with respect to $g_{\mu\nu}$, ϕ (or σ), b_ν, B, c^ρ, $\bar{c}_\rho$, c, and $\bar{c}$ in order:

$$\begin{aligned}
&\frac{1}{12}\phi^2 G_{\mu\nu} - \frac{1}{12}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\phi^2 - \frac{1}{2}\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) = 0,\\
&\frac{1}{6}\phi^2 R - E - 2g^{\mu\nu}\phi\partial_\mu B\partial_\nu\phi - \phi^2\Box B = 0,\\
&\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0, \quad \partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi) = 0,\\
&g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho = g^{\mu\nu}\partial_\mu\partial_\nu c^\rho = g^{\mu\nu}\partial_\mu\partial_\nu\bar{c} = g^{\mu\nu}\partial_\mu\partial_\nu c = 0,
\end{aligned} \tag{3.15}$$

where we have defined the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and $E \equiv g^{\mu\nu}E_{\mu\nu}$. The two gauge-fixing conditions in Eq. (3.15) lead to a very simple equation for the dilaton:

$$g^{\mu\nu}\partial_\mu\partial_\nu\sigma = 0. \tag{3.16}$$

It is worth noticing that it is not the scalar field ϕ but rather the dilaton σ that satisfies this type of equation. Furthermore, the trace part of the Einstein equation, i.e., the first field equation in Eq. (3.15) and the field equation for ϕ also give us the equation for B:

$$g^{\mu\nu}\partial_\mu\partial_\nu B = 0. \tag{3.17}$$

Finally, using the field equations obtained thus far, after some calculations, we can also derive the equation for b_ρ[5]:

$$g^{\mu\nu}\partial_\mu\partial_\nu b_\rho = 0. \tag{3.18}$$

In other words, by setting $X^M = \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c}\}$, it turns out that X^M obeys the very simple equation

$$g^{\mu\nu}\partial_\mu\partial_\nu X^M = 0. \tag{3.19}$$

This fact, together with the gauge condition $\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0$, produces the two kinds of conserved currents:

$$\begin{aligned}
\mathcal{P}^{\mu M} &\equiv \tilde{g}^{\mu\nu}\phi^2\partial_\nu X^M = \tilde{g}^{\mu\nu}\phi^2(1\overset{\leftrightarrow}{\partial}_\nu X^M),\\
\mathcal{M}^{\mu MN} &\equiv \tilde{g}^{\mu\nu}\phi^2(X^M\overset{\leftrightarrow}{\partial}_\nu Y^N),
\end{aligned} \tag{3.20}$$

where we have defined $X^M\overset{\leftrightarrow}{\partial}_\mu Y^N \equiv X^M\partial_\mu Y^N - (\partial_\mu X^M)Y^N$.

IV. CANONICAL QUANTIZATION AND EQUAL-TIME COMMUTATION RELATIONS

In this section, after introducing the canonical commutation relations, we will evaluate various ETCRs among fundamental variables. To simplify various expressions, we obey the following abbreviations adopted in Ref. [5]:

$$\begin{aligned}
&[A, B'] = [A(x), B(x')]|_{x^0 = x'^0}, \qquad \delta^3 = \delta(\vec{x} - \vec{x}'),\\
&\tilde{f} = \frac{1}{\tilde{g}^{00}} = \frac{1}{\sqrt{-g}g^{00}} = \frac{1}{hg^{00}},
\end{aligned} \tag{4.1}$$

where we assume that $\tilde{g}^{00}$ is invertible.

Now let us set up the canonical (anti)commutation relations:

$$\begin{aligned}
&[g_{\mu\nu}, \pi_g^{\rho\lambda\prime}] = i\frac{1}{2}(\delta_\mu^\rho\delta_\nu^\lambda + \delta_\mu^\lambda\delta_\nu^\rho)\delta^3, \quad [\phi, \pi_\phi'] = +i\delta^3,\\
&[B, \pi_B'] = +i\delta^3, \{c^\sigma, \pi_{c\lambda}'\} = \{\bar{c}_\lambda, \pi_{\bar{c}}^{\sigma\prime}\} = +i\delta_\lambda^\sigma\delta^3,\\
&\{c, \pi_c'\} = \{\bar{c}, \pi_{\bar{c}}'\} = +i\delta^3,
\end{aligned} \tag{4.2}$$

where the other (anti)commutation relations vanish. Here the canonical variables are $g_{\mu\nu}, \phi, B, c^\rho, \bar{c}_\rho, c, \bar{c}$ and the corresponding canonical conjugate momenta are $\pi_g^{\mu\nu}, \pi_\phi, \pi_B, \pi_{c\rho}, \pi_{\bar{c}}^\rho, \pi_c, \pi_{\bar{c}}$, respectively, and the b_μ field is regarded as not a canonical variable but rather a conjugate momentum of $\tilde{g}^{0\mu}$.

To remove second-order derivatives of the metric involved in R, we perform an integration by parts once and rewrite the Lagrangian (3.8) as

[5]The details of the calculation are presented in Appendix A.

$$\mathcal{L}_q = -\frac{1}{12}\tilde{g}^{\mu\nu}\phi^2(\Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\alpha} - \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\sigma\nu}) - \frac{1}{6}\phi\partial_\mu\phi(\tilde{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \tilde{g}^{\mu\nu}\Gamma^\alpha_{\nu\alpha}) + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \partial_\mu(\tilde{g}^{\mu\nu}\phi^2)b_\nu - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\rho\partial_\nu c^\rho + \tilde{g}^{\mu\nu}\partial_\mu B\phi\partial_\nu\phi - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}\partial_\nu c + \partial_\mu\mathcal{V}^\mu, \tag{4.3}$$

where the surface term $\mathcal{V}^\mu$ is defined as

$$\mathcal{V}^\mu = \frac{1}{12}\phi^2(\tilde{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \tilde{g}^{\mu\nu}\Gamma^\alpha_{\nu\alpha}) - \tilde{g}^{\mu\nu}\phi^2 b_\nu. \tag{4.4}$$

Using this Lagrangian, the concrete expressions for canonical conjugate momenta become

$$\begin{aligned}
\pi_g^{\mu\nu} &= \frac{\partial\mathcal{L}_q}{\partial\dot{g}_{\mu\nu}} \\
&= -\frac{1}{24}\sqrt{-g}\phi^2\Big[-g^{0\lambda}g^{\mu\nu}g^{\sigma\tau} - g^{0\tau}g^{\mu\lambda}g^{\nu\sigma} - g^{0\sigma}g^{\mu\tau}g^{\nu\lambda} \\
&\quad + g^{0\lambda}g^{\mu\tau}g^{\nu\sigma} + g^{0\tau}g^{\mu\nu}g^{\lambda\sigma} + \frac{1}{2}(g^{0\mu}g^{\nu\lambda} + g^{0\nu}g^{\mu\lambda})g^{\sigma\tau}\Big]\partial_\lambda g_{\sigma\tau} \\
&\quad - \frac{1}{6}\sqrt{-g}\left[\frac{1}{2}(g^{0\mu}g^{\rho\nu} + g^{0\nu}g^{\rho\mu}) - g^{\mu\nu}g^{\rho 0}\right]\phi\partial_\rho\phi \\
&\quad - \frac{1}{2}\sqrt{-g}(g^{0\mu}g^{\nu\rho} + g^{0\nu}g^{\mu\rho} - g^{0\rho}g^{\mu\nu})\phi^2 b_\rho, \\
\pi_\phi &= \frac{\partial\mathcal{L}_q}{\partial\dot{\phi}} = \tilde{g}^{0\mu}\partial_\mu\phi + 2\tilde{g}^{0\mu}\phi b_\mu \\
&\quad + \frac{1}{6}\phi(-\tilde{g}^{\alpha\beta}\Gamma^0_{\alpha\beta} + \tilde{g}^{0\alpha}\Gamma^\beta_{\alpha\beta}) + \tilde{g}^{0\mu}\partial_\mu B\phi, \\
\pi_B &= \frac{\partial\mathcal{L}_q}{\partial\dot{B}} = \tilde{g}^{0\mu}\phi\partial_\mu\phi, \\
\pi_{c\sigma} &= \frac{\partial\mathcal{L}_q}{\partial\dot{c}^\sigma} = -i\tilde{g}^{0\mu}\phi^2\partial_\mu\bar{c}_\sigma, \\
\pi_{\bar{c}}^\sigma &= \frac{\partial\mathcal{L}_q}{\partial\dot{\bar{c}}_\sigma} = i\tilde{g}^{0\mu}\phi^2\partial_\mu c^\sigma, \\
\pi_c &= \frac{\partial\mathcal{L}_q}{\partial\dot{c}} = -i\tilde{g}^{0\mu}\phi^2\partial_\mu\bar{c}, \\
\pi_{\bar{c}} &= \frac{\partial\mathcal{L}_q}{\partial\dot{\bar{c}}} = i\tilde{g}^{0\mu}\phi^2\partial_\mu c,
\end{aligned} \tag{4.5}$$

where we have defined the time derivative such that $\dot{g}_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial t} \equiv \partial_0 g_{\mu\nu}$, and differentiation of ghosts is taken from the right.

From now on, we would like to evaluate various nontrivial ETCRs in order. Let us first work with the ETCR in Eq. (4.2):

$$[\pi_g^{\alpha 0}, g'_{\mu\nu}] = -i\frac{1}{2}(\delta^\alpha_\mu\delta^0_\nu + \delta^0_\mu\delta^\alpha_\nu)\delta^3. \tag{4.6}$$

The canonical conjugate momentum $\pi_g^{\alpha 0}$ has the structure

$$\pi_g^{\alpha 0} = A^\alpha + B^{\alpha\beta}\partial_\beta\phi + C^{\alpha\beta}b_\beta, \tag{4.7}$$

where $A^\alpha, B^{\alpha\beta}$, and $C^{\alpha\beta} \equiv -\frac{1}{2}\tilde{g}^{00}g^{\alpha\beta}\phi^2$ have no $\dot{g}_{\mu\nu}$, and $B^{\alpha\beta}\partial_\beta\phi$ does not have $\dot{\phi}$ since $\pi_g^{\alpha 0}$ does not include the dynamics of the metric and the scalar fields. Then, we find that Eq. (4.6) produces

$$[g_{\mu\nu}, b'_\rho] = -i\tilde{f}\phi^{-2}(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})\delta^3. \tag{4.8}$$

From this ETCR, we can easily derive ETCRs:

$$\begin{aligned}
[g^{\mu\nu}, b'_\rho] &= i\tilde{f}\phi^{-2}(g^{\mu 0}\delta^\nu_\rho + g^{\nu 0}\delta^\mu_\rho)\delta^3, \\
[\tilde{g}^{\mu\nu}, b'_\rho] &= i\tilde{f}\phi^{-2}(\tilde{g}^{\mu 0}\delta^\nu_\rho + \tilde{g}^{\nu 0}\delta^\mu_\rho - \tilde{g}^{\mu\nu}\delta^0_\rho)\delta^3.
\end{aligned} \tag{4.9}$$

Here we have used the fact that, since a commutator works as a derivation, we can have the formulas

$$\begin{aligned}
[g^{\mu\nu}, \Phi'] &= -g^{\mu\alpha}g^{\nu\beta}[g_{\alpha\beta}, \Phi'], \\
[\tilde{g}^{\mu\nu}, \Phi'] &= -\left(\tilde{g}^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}\tilde{g}^{\mu\nu}g^{\alpha\beta}\right)[g_{\alpha\beta}, \Phi'],
\end{aligned} \tag{4.10}$$

where Φ is a generic field. Similarly, the ETCR $[\pi_g^{\alpha 0}, \phi'] = 0$ yields

$$[\phi, b'_\rho] = 0. \tag{4.11}$$

The ETCR $[\pi_g^{\alpha 0}, B'] = 0$ yields

$$[B, b'_\rho] = 0. \tag{4.12}$$

Moreover, the ETCRs $[\pi_B, \phi'] = 0$ and $[\pi_B, B'] = -i\delta^3$ produce, respectively,

$$[\dot{\phi}, \phi'] = 0, \qquad [\dot{\phi}, B'] = -i\tilde{f}\phi^{-1}\delta^3. \tag{4.13}$$

As for the ETCRs involving FP ghosts, let us first consider the anti-ETCRs, $\{\pi_{c\lambda}, c^{\sigma\prime}\} = \{\pi_{\bar{c}}^\sigma, \bar{c}'_\lambda\} = i\delta^\sigma_\lambda\delta^3$. These anti-ETCRs lead to the same anti-ETCR,

$$\{\dot{\bar{c}}_\lambda, c^{\sigma\prime}\} = -\{\dot{c}^\sigma, \bar{c}'_\lambda\} = -\tilde{f}\phi^{-2}\delta^\sigma_\lambda\delta^3, \tag{4.14}$$

where we have used a useful identity for generic variables Φ and Ψ,

$$[\Phi, \dot{\Psi}'] = \partial_0[\Phi, \Psi'] - [\dot{\Phi}, \Psi'], \tag{4.15}$$

which holds for the anticommutation relation as well. In a similar way, the anti-ETCRs $\{\pi_c, c'\} = \{\pi_{\bar{c}}, \bar{c}'\} = i\delta^3$ yield

$$\{\dot{\bar{c}}, c'\} = -\{\dot{c}, \bar{c}'\} = -\tilde{f}\phi^{-2}\delta^3. \tag{4.16}$$

Moreover, $[\pi_g^{\alpha 0}, c^{\sigma\prime}] = [\pi_g^{\alpha 0}, \bar{c}'_\lambda] = 0$ give us the ETCRs

$$[b_\rho, c^{\sigma\prime}] = [b_\rho, \bar{c}'_\lambda] = 0, \tag{4.17}$$

and similarly, $[\pi_g^{\alpha 0}, c'] = [\pi_g^{\alpha 0}, \bar{c}'] = 0$ produce

$$[b_\rho, c'] = [b_\rho, \bar{c}'] = 0. \tag{4.18}$$

To calculate the ETCRs between B and the FP ghosts, it is necessary to utilize the ETCRs $[B, \pi'_{c\lambda}] = [B, \pi_{\bar{c}}^{\sigma\prime}] = [B, \pi'_c] = [B, \pi'_{\bar{c}}] = 0$, and consequently we have

$$[B, \dot{\bar{c}}'_\lambda] = [B, \dot{c}^{\sigma\prime}] = [B, \dot{\bar{c}}'] = [B, \dot{c}'] = 0. \tag{4.19}$$

Furthermore, taking the Weyl BRST transformation of the third ETCR reads[6]

$$0 = \{[i\bar{Q}_B, B], \dot{\bar{c}}'\} + [B, \{i\bar{Q}_B, \dot{\bar{c}}'\}] = [B, i\dot{B}'], \tag{4.20}$$

where we have used the Weyl BRST transformation (2.5). As a result, we have the ETCR

$$[B, \dot{B}'] = 0. \tag{4.21}$$

Next, from $[\pi_B, \bar{c}'_\lambda] = [\pi_B, c^{\sigma\prime}] = 0$, we find

$$[\dot{\phi}, \bar{c}'_\lambda] = [\dot{\phi}, c^{\sigma\prime}] = 0. \tag{4.22}$$

Similarly, from $[\pi_B, \bar{c}'] = [\pi_B, c'] = 0$, we have

$$[\dot{\phi}, \bar{c}'] = [\dot{\phi}, c'] = 0. \tag{4.23}$$

Using the field equation for $\bar{c}_\lambda$ in Eq. (3.15), i.e., $g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\lambda = 0$, the ETCR $[\phi, \bar{c}'_\lambda] = 0$, and Eqs. (4.15) and (4.22), it is easy to derive the equations

$$[\phi, \ddot{\bar{c}}'_\lambda] = [\dot{\phi}, \dot{\bar{c}}'_\lambda] = [\ddot{\phi}, \bar{c}'_\lambda] = 0. \tag{4.24}$$

[6]We define the BRST transformation for the Weyl transformation as $\bar{\delta}_B\Phi \equiv [i\bar{Q}_B, \Phi\}$, where Φ is a generic field and $[,\}$ denotes the graded bracket. Of course, in the case of the GCT BRST transformation, it is replaced by $\delta_B\Phi \equiv [iQ_B, \Phi\}$.

Similar equations also hold when $\bar{c}'_\lambda$ is replaced with $c^{\sigma\prime}$, c', or $\bar{c}'$.

Now, using the equations obtained above, we are ready to evaluate the type of ETCRs $[\dot{\Phi}, b'_\rho]$, where Φ is a generic field. First, let us focus on $[\dot{\phi}, b'_\rho]$. To do this, we start with $[\dot{\phi}, \bar{c}'_\rho] = 0$ in Eq. (4.22) and take its BRST variation for the GCT as follows:

$$\begin{aligned} 0 &= \{iQ_B, [\dot{\phi}, \bar{c}'_\rho]\} \\ &= \{[iQ_B, \dot{\phi}], \bar{c}'_\rho\} + [\dot{\phi}, \{iQ_B, \bar{c}'_\rho\}] \\ &= \{-\partial_0(c^\lambda\partial_\lambda\phi), \bar{c}'_\rho\} + [\dot{\phi}, i(b'_\rho + ic^{\lambda\prime}\partial_\lambda\bar{c}'_\rho)] \\ &= -\{\dot{c}^\lambda, \bar{c}'_\rho\}\partial_\lambda\phi + i[\dot{\phi}, b'_\rho]. \end{aligned} \tag{4.25}$$

Using Eq. (4.14), we obtain

$$[\dot{\phi}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\phi\delta^3. \tag{4.26}$$

It turns out that the ETCRs $[\pi_{c\lambda}, \pi_g^{\alpha 0\prime}] = [\pi_{\bar{c}}^\sigma, \pi_g^{\alpha 0\prime}] = 0$ give rise to

$$[\dot{\bar{c}}_\lambda, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\bar{c}_\lambda\delta^3, \quad [\dot{c}^\sigma, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho c^\sigma\delta^3. \tag{4.27}$$

Similarly, the ETCRs $[\pi_c, \pi_g^{\alpha 0\prime}] = [\pi_{\bar{c}}, \pi_g^{\alpha 0\prime}] = 0$ give us

$$[\dot{\bar{c}}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\bar{c}\delta^3, \quad [\dot{c}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho c\delta^3. \tag{4.28}$$

In order to evaluate $[\dot{B}, b'_\rho]$, we make use of $[\pi_c, b'_\rho] = 0$, which can be easily proved. Taking its BRST transformation for the Weyl transformation leads to the equation

$$[\{i\bar{Q}_B, \pi_c\}, b'_\rho] = 0, \tag{4.29}$$

where $[i\bar{Q}_B, b'_\rho] = 0$ was used. We can show that $\{i\bar{Q}_B, \pi_c\} = \tilde{g}^{0\mu}\phi^2\partial_\mu B$, so by using Eqs. (4.9) and (4.11) we can calculate

$$[\dot{B}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho B\delta^3. \tag{4.30}$$

Finally, the ETCR $[\dot{g}_{\mu\nu}, b'_\rho]$ (or, equivalently, $[g_{\mu\nu}, \dot{b}'_\rho]$) can be obtained by using the method developed in our previous article [1]. We only report the result here, which is written as

$$\begin{aligned} [\dot{g}_{\mu\nu}, b'_\rho] = &-i\{\tilde{f}\phi^{-2}(\partial_\rho g_{\mu\nu} + \delta^0_\mu\dot{g}_{\rho\nu} + \delta^0_\nu\dot{g}_{\rho\mu})\delta^3 \\ &+ [(\delta^k_\mu - 2\delta^0_\mu\tilde{f}\tilde{g}^{0k})g_{\rho\nu} + (\mu \leftrightarrow \nu)]\partial_k(\tilde{f}\phi^{-2}\delta^3)\}, \end{aligned} \tag{4.31}$$

or, equivalently,

$$[g_{\mu\nu}, \dot{b}'_\rho] = i\{[\tilde{f}\phi^{-2}\partial_\rho g_{\mu\nu} - \partial_0(\tilde{f}\phi^{-2})(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})]\delta^3 + [(\delta^k_\mu - 2\delta^0_\mu \tilde{f}\tilde{g}^{0k})g_{\rho\nu} + (\mu \leftrightarrow \nu)]\partial_k(\tilde{f}\phi^{-2}\delta^3)\}. \tag{4.32}$$

Following our previous calculation [1], we can prove that

$$[b_\mu, b'_\nu] = 0, \quad [b_\mu, \dot{b}'_\nu] = i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3. \tag{4.33}$$

V. EQUAL-TIME COMMUTATION RELATIONS IN THE GRAVITATIONAL SECTOR

The remaining nontrivial ETCRs are related to the time derivative of the metric field, i.e., the ETCRs $[\dot{g}_{\mu\nu}, \Phi']$, where Φ is a generic field. In this section, we will evaluate such ETCRs.

First of all, let us start with the ETCR $[\pi_\phi, g'_{\mu\nu}] = 0$. From the expression for π_ϕ in Eq. (4.5), this ETCR can be described as

$$\tilde{g}^{00}[\dot{\phi}, g'_{\mu\nu}] + \frac{1}{6}\phi(\tilde{g}^{00}g^{\rho\sigma} - \tilde{g}^{0\rho}g^{0\sigma})[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] + \tilde{g}^{00}\phi[\dot{B}, g'_{\mu\nu}] = -4i\tilde{f}\phi^{-1}\sqrt{-g}\delta^0_\mu\delta^0_\nu\delta^3. \tag{5.1}$$

Next, the ETCR $[\pi_\phi, \phi'] = -i\delta^3$ produces the equation

$$(\tilde{g}^{00}g^{\rho\sigma} - \tilde{g}^{0\rho}g^{0\sigma})[\dot{g}_{\rho\sigma}, \phi'] = 0. \tag{5.2}$$

Moreover, the ETCR $[\pi_\phi, B'] = 0$ reads

$$(\tilde{g}^{00}g^{\rho\sigma} - \tilde{g}^{0\rho}g^{0\sigma})[\dot{g}_{\rho\sigma}, B'] = 6i\phi^{-2}\delta^3. \tag{5.3}$$

The extended de Donder gauge, $\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0$, can be rewritten as

$$\mathcal{D}^{\lambda\rho\sigma}\dot{g}_{\rho\sigma} + 4\phi^{-1}g^{\lambda\rho}\partial_\rho\phi = (2g^{\lambda\rho}g^{\sigma k} - g^{\rho\sigma}g^{\lambda k})\partial_k g_{\rho\sigma}, \tag{5.4}$$

where $\mathcal{D}^{\lambda\rho\sigma} \equiv g^{0\lambda}g^{\rho\sigma} - 2g^{\lambda\rho}g^{0\sigma}$. Since the rhs of Eq. (5.4) is independent of $\dot{g}_{\mu\nu}$, it commutes with $g_{\mu\nu}$, ϕ, or B. Thus, we have three identities:

$$\mathcal{D}^{\lambda\rho\sigma}[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] + 4\phi^{-1}g^{\lambda 0}[\dot{\phi}, g'_{\mu\nu}] = 0, \tag{5.5}$$

$$\mathcal{D}^{\lambda\rho\sigma}[\dot{g}_{\rho\sigma}, \phi'] = 0, \tag{5.6}$$

$$\mathcal{D}^{\lambda\rho\sigma}[\dot{g}_{\rho\sigma}, B'] = 4i\tilde{f}\phi^{-2}g^{\lambda 0}\delta^3. \tag{5.7}$$

In Eqs. (5.6) and (5.7), we have used Eq. (4.13).

Putting $\lambda = 0$ in Eq. (5.6) and using Eq. (5.2), we have

$$g^{\rho\sigma}[\dot{g}_{\rho\sigma}, \phi'] = g^{0\rho}g^{0\sigma}[\dot{g}_{\rho\sigma}, \phi'] = 0. \tag{5.8}$$

In general, from the argument of symmetry, $[\dot{g}_{\rho\sigma}, \phi']$ must be of the form

$$[\dot{g}_{\rho\sigma}, \phi'] = a_1(g_{\rho\sigma} + a_2\delta^0_\rho\delta^0_\sigma)\delta^3, \tag{5.9}$$

where a_1 and a_2 are constants. Equation (5.8) then requires us to take $a_1 = a_2 = 0$. Thus, we have

$$[\dot{g}_{\rho\sigma}, \phi'] = 0. \tag{5.10}$$

Next, in a similar manner, we can set

$$[\dot{g}_{\rho\sigma}, B'] = b_1(g_{\rho\sigma} + b_2\delta^0_\rho\delta^0_\sigma)\delta^3, \tag{5.11}$$

where b_1 and b_2 are constants. From Eq. (5.3), b_1 is determined to be $2i\tilde{f}\phi^{-2}$, and then Eq. (5.7) requires b_2 to be vanishing, so we can obtain

$$[\dot{g}_{\rho\sigma}, B'] = 2i\tilde{f}\phi^{-2}g_{\rho\sigma}\delta^3. \tag{5.12}$$

Finally, we wish to evaluate $[\dot{g}_{\rho\sigma}, g'_{\mu\nu}]$, for which we need to perform some calculations. Before doing so, let us rewrite Eq. (5.1) by means of Eqs. (5.10) and (5.12) in the form

$$(\tilde{g}^{00}g^{\rho\sigma} - \tilde{g}^{0\rho}g^{0\sigma})[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] = -12i\phi^{-2}\left(g_{\mu\nu} + \frac{2}{g^{00}}\delta^0_\mu\delta^0_\nu\right)\delta^3. \tag{5.13}$$

Similarly, Eq. (5.5) reduces to

$$(g^{0\lambda}g^{\rho\sigma} - 2g^{\lambda\rho}g^{0\sigma})[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] = 0. \tag{5.14}$$

We are now ready to evaluate the ETCR $[\dot{g}_{\rho\sigma}, g'_{\mu\nu}]$. This ETCR has a symmetry under the simultaneous exchange of $(\mu\nu) \leftrightarrow (\rho\sigma)$ and primed $\leftrightarrow$ unprimed, in addition to the usual symmetry $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$. Then, we can write down its general expression as

$$[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] = \{c_1 g_{\rho\sigma}g_{\mu\nu} + c_2(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\mu}) + h\tilde{f}[c_3(\delta^0_\rho\delta^0_\sigma g_{\mu\nu} + \delta^0_\mu\delta^0_\nu g_{\rho\sigma}) + c_4(\delta^0_\rho\delta^0_\mu g_{\sigma\nu} + \delta^0_\rho\delta^0_\nu g_{\sigma\mu} + \delta^0_\sigma\delta^0_\mu g_{\rho\nu} + \delta^0_\sigma\delta^0_\nu g_{\rho\mu})] + (h\tilde{f})^2 c_5\delta^0_\rho\delta^0_\sigma\delta^0_\mu\delta^0_\nu\}\delta^3, \tag{5.15}$$

where $c_i (i = 1, \ldots, 5)$ are some coefficients. Imposing Eq. (5.14) on Eq. (5.15) leads to relations among the coefficients:

$$c_3 = 2(c_1 + c_2), \quad c_4 = -c_2, \quad c_5 = 4(c_1 + c_2). \tag{5.16}$$

Furthermore, imposing Eq. (5.13), we can determine c_2, c_3, c_4, and c_5 via c_1 as

$$c_3 = -c_1 - 12i\tilde{f}\phi^{-2}, \qquad c_4 = -c_2 = \frac{3}{2}c_1 + 6i\tilde{f}\phi^{-2},$$
$$c_5 = -2c_1 - 24i\tilde{f}\phi^{-2}. \tag{5.17}$$

In order to fix the coefficient c_1, we need to calculate the ETCR $[\dot{g}_{kl}, g'_{mn}]$ explicitly in terms of $[\pi_g^{kl}, g'_{mn}] = -i\frac{1}{2}(\delta_m^k\delta_n^l + \delta_m^l\delta_n^k)\delta^3$ in Eq. (4.2) and the concrete expression for π_g^{kl} in Eq. (4.5). To do this, from Eq. (4.5), let us write

$$\pi_g^{kl} = \hat{A}^{kl} + \hat{B}^{kl\rho}b_\rho + \hat{C}^{klmn}\dot{g}_{mn} + \hat{D}^{kl}\dot{\phi}. \tag{5.18}$$

Here $\hat{A}^{kl}$, $\hat{B}^{kl\rho}$, $\hat{C}^{klmn}$, and $\hat{D}^{kl}$ commute with g_{mn}, and $\hat{C}^{klmn}$ and $\hat{D}^{kl}$ are defined as[7]

$$\hat{C}^{klmn} = \frac{1}{24}h\phi^2K^{klmn}, \quad \hat{D}^{kl} = \frac{1}{6}\phi(\tilde{g}^{00}g^{kl} - \tilde{g}^{0k}g^{0l}), \tag{5.19}$$

where the definition of K^{klmn} and its property are given by

$$K^{klmn} = \begin{vmatrix} g^{00} & g^{0l} & g^{0n} \\ g^{k0} & g^{kl} & g^{kn} \\ g^{m0} & g^{ml} & g^{mn} \end{vmatrix},$$
$$K^{klmn}\frac{1}{2}(g^{00})^{-1}(g_{ij}g_{mn} - g_{im}g_{jn} - g_{\rm in}g_{jm}) = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k). \tag{5.20}$$

From Eq. (5.18), we can calculate

$$[\dot{g}_{kl}, g'_{mn}] = \hat{C}^{-1}_{klpq}([\pi_g^{pq}, g'_{mn}] - \hat{B}^{pq\rho}[b_\rho, g'_{mn}] - \hat{D}^{pq}[\dot{\phi}, g'_{mn}])$$
$$= -i\frac{1}{2}\hat{C}^{-1}_{klpq}(\delta_m^p\delta_n^q + \delta_m^q\delta_n^p)\delta^3, \tag{5.21}$$

where we have used Eqs. (4.2), (4.8), and (5.10). Since we can calculate

$$\hat{C}^{-1}_{klpq} = 12\tilde{f}\phi^{-2}(g_{kl}g_{pq} - g_{kp}g_{lq} - g_{kq}g_{lp}), \tag{5.22}$$

we can eventually arrive at the result

$$[\dot{g}_{kl}, g'_{mn}] = -12i\tilde{f}\phi^{-2}(g_{kl}g_{mn} - g_{km}g_{ln} - g_{kn}g_{lm})\delta^3. \tag{5.23}$$

Meanwhile, from Eq. (5.15) we have the ETCR

$$[\dot{g}_{kl}, g'_{mn}] = [c_1g_{kl}g_{mn} + c_2(g_{km}g_{ln} + g_{kn}g_{lm})]\delta^3. \tag{5.24}$$

[7]It turns out that the concrete expressions for $\hat{A}^{kl}$ and $\hat{B}^{kl\rho}$ are irrelevant to the calculation of $[\dot{g}_{kl}, g'_{mn}]$.

Hence, comparing Eq. (5.23) with Eq. (5.24), we can obtain

$$c_1 = -12i\tilde{f}\phi^{-2}, \qquad c_2 = 12i\tilde{f}\phi^{-2}. \tag{5.25}$$

Note that these values satisfy the relation in Eq. (5.17), $-c_2 = \frac{3}{2}c_1 + 6i\tilde{f}\phi^{-2}$, which gives us a nontrivial verification of our result. In this way, we have succeeded in getting the following ETCR:

$$[\dot{g}_{\rho\sigma}, g'_{\mu\nu}] = -12i\tilde{f}\phi^{-2}[g_{\rho\sigma}g_{\mu\nu} - g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} + h\tilde{f}(\delta_\rho^0\delta_\mu^0g_{\sigma\nu}$$
$$+ \delta_\rho^0\delta_\nu^0g_{\sigma\mu} + \delta_\sigma^0\delta_\mu^0g_{\rho\nu} + \delta_\sigma^0\delta_\nu^0g_{\rho\mu})]\delta^3. \tag{5.26}$$

VI. UNITARITY OF THE PHYSICAL S MATRIX

As in the conventional BRST formalism, the physical state $|\text{phys}\rangle$ is defined by imposing two subsidiary conditions [8]:

$$Q_B|\text{phys}\rangle = \bar{Q}_B|\text{phys}\rangle = 0. \tag{6.1}$$

It is then well known that the physical S matrix is unitary under the assumption that all of the BRST singlet states have positive norm. In this section, we would like to prove the unitarity of the physical S matrix. Since there is a ghostlike scalar field ϕ as well as timelike and longitudinal components of the metric field in our formalism, this is not a trivial problem.

In analyzing the unitarity, it is enough to take account of asymptotic fields of all of the fundamental fields and the free part of the Lagrangian. Let us first assume the asymptotic fields as

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}, \qquad \phi = \phi_0 + \tilde{\phi}, \qquad b_\mu = \beta_\mu, \qquad B = \beta,$$
$$c^\mu = \gamma^\mu, \qquad \bar{c}_\mu = \bar{\gamma}_\mu, \qquad c = \gamma, \qquad \bar{c} = \bar{\gamma}, \tag{6.2}$$

where $\eta_{\mu\nu}(=\eta^{\mu\nu})$ is the flat Minkowski metric with mostly positive signature, and ϕ_0 is a constant. In this section, the Minkowski metric is used to lower or raise the Lorentz indices. Using these asymptotic fields, the free part of the Lagrangian reads

$$\mathcal{L}_q = \frac{1}{12}\phi_0^2\left(\frac{1}{4}\varphi_{\mu\nu}\Box\varphi^{\mu\nu} - \frac{1}{4}\varphi\Box\varphi - \frac{1}{2}\varphi^{\mu\nu}\partial_\mu\partial_\rho\varphi_\nu{}^\rho + \frac{1}{2}\varphi^{\mu\nu}\partial_\mu\partial_\nu\varphi\right)$$
$$+ \frac{1}{6}\phi_0\tilde{\phi}(-\Box\varphi + \partial_\mu\partial_\nu\varphi^{\mu\nu}) + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - i\phi_0^2\partial_\mu\bar{\gamma}_\rho\partial^\mu\gamma^\rho$$
$$- \left(2\eta^{\mu\nu}\phi_0\tilde{\phi} - \phi_0^2\varphi^{\mu\nu} + \frac{1}{2}\phi_0^2\eta^{\mu\nu}\varphi\right)\partial_\mu\beta_\nu$$
$$+ \phi_0\partial_\mu\beta\partial^\mu\tilde{\phi} - i\phi_0^2\partial_\mu\bar{\gamma}\partial^\mu\gamma, \tag{6.3}$$

where $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\varphi \equiv \eta^{\mu\nu}\varphi_{\mu\nu}$. Based on this Lagrangian, it is easy to derive the linearized field equations:

$$\frac{1}{12}\phi_0\left(\frac{1}{2}\Box\varphi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box\varphi - \partial_\rho\partial_{(\mu}\varphi_{\nu)}{}^\rho + \frac{1}{2}\partial_\mu\partial_\nu\varphi + \frac{1}{2}\eta_{\mu\nu}\partial_\rho\partial_\sigma\varphi^{\rho\sigma}\right) + \frac{1}{6}(-\eta_{\mu\nu}\Box + \partial_\mu\partial_\nu)\tilde{\phi} + \phi_0\partial_{(\mu}\beta_{\nu)} - \frac{1}{2}\phi_0\eta_{\mu\nu}\partial_\rho\beta^\rho = 0, \tag{6.4}$$

$$\frac{1}{6}(\Box\varphi - \partial_\mu\partial_\nu\varphi^{\mu\nu}) + 2\partial_\rho\beta^\rho + \Box\beta = 0, \tag{6.5}$$

$$\partial_\mu\tilde{\phi} - \frac{1}{2}\phi_0\left(\partial^\nu\varphi_{\mu\nu} - \frac{1}{2}\partial_\mu\varphi\right) = 0, \tag{6.6}$$

$$\Box\tilde{\phi} = \Box\gamma^\mu = \Box\bar{\gamma}_\mu = \Box\gamma = \Box\bar{\gamma} = 0. \tag{6.7}$$

Here we have introduced the symmetrization notation $A_{(\mu}B_{\nu)} \equiv \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$. Now, by applying ∂^μ to Eq. (6.6) and using Eq. (6.7), we can obtain

$$\partial_\mu\partial_\nu\varphi^{\mu\nu} - \frac{1}{2}\Box\varphi = 0. \tag{6.8}$$

Next, taking the trace of Eq. (6.4) with the help of Eqs. (6.7) and (6.8) leads to

$$\Box\varphi + 24\partial_\rho\beta^\rho = 0. \tag{6.9}$$

Then, with the help of Eqs. (6.8) and (6.9), Eq. (6.5) can be rewritten as

$$\Box\beta = 0. \tag{6.10}$$

Moreover, applying ∂^μ to Eq. (6.4) yields

$$\Box\beta_\mu = 0. \tag{6.11}$$

Finally, using various equations obtained thus far, Eq. (6.4) is reduced to the form

$$\Box\varphi_{\mu\nu} + 24\partial_{(\mu}\beta_{\nu)} = 0, \tag{6.12}$$

which means that the field $\varphi_{\mu\nu}$ is a dipole field:

$$\Box^2\varphi_{\mu\nu} = 0. \tag{6.13}$$

On the other hand, the other fields are all simple pole fields:

$$\Box\tilde{\phi} = \Box\beta_\mu = \Box\beta = \Box\gamma^\mu = \Box\bar{\gamma}_\mu = \Box\gamma = \Box\bar{\gamma} = 0. \tag{6.14}$$

Note that Eq. (6.14) corresponds to Eq. (3.19) in a curved space-time.

Following the standard technique, we can calculate the four-dimensional (anti)commutation relations (4D CRs) between asymptotic fields. The point is that the simple pole fields, for instance, the Nakanishi-Lautrup field $\beta(x)$, can be expressed in terms of the invariant delta function $D(x)$ as

$$\beta_\mu(x) = \int d^3z D(x-z)\overleftrightarrow{\partial}{}_0^z\beta_\mu(z), \tag{6.15}$$

whereas the dipole field $\varphi_{\mu\nu}(x)$ can be written as

$$\begin{aligned}\varphi_{\mu\nu}(x) &= \int d^3z[D(x-z)\overleftrightarrow{\partial}{}_0^z\varphi_{\mu\nu}(z) + E(x-z)\overleftrightarrow{\partial}{}_0^z\Box\varphi_{\mu\nu}(z)] \\ &= \int d^3z[D(x-z)\overleftrightarrow{\partial}{}_0^z\varphi_{\mu\nu}(z) - 24E(x-z)\overleftrightarrow{\partial}{}_0^z\partial_{(\mu}\beta_{\nu)}(z)],\end{aligned} \tag{6.16}$$

where in the last equality we have used Eq. (6.12). Here the invariant delta function $D(x)$ for massless simple pole fields and its properties are described as

$$D(x) = -\frac{i}{(2\pi)^3}\int d^4k\epsilon(k_0)\delta(k^2)e^{ikx}, \qquad \Box D(x) = 0,$$
$$D(-x) = -D(x), \qquad D(0,\vec{x}) = 0, \qquad \partial_0 D(0,\vec{x}) = \delta^3(x), \tag{6.17}$$

where $\epsilon(k_0) \equiv \frac{k_0}{|k_0|}$. Similarly, the invariant delta function $E(x)$ for massless dipole fields and its properties are given by

$$E(x) = -\frac{i}{(2\pi)^3}\int d^4k\epsilon(k_0)\delta'(k^2)e^{ikx}, \quad \Box E(x) = D(x),$$
$$E(-x) = -E(x), \quad E(0,\vec{x}) = \partial_0 E(0,\vec{x}) = \partial_0^2 E(0,\vec{x}) = 0,$$
$$\partial_0^3 E(0,\vec{x}) = -\delta^3(x). \tag{6.18}$$

It is easy to show that the rhs of Eqs. (6.15) and (6.16) are independent of z^0. Thus, for instance, when we evaluate the four-dimensional commutation relation $[\varphi_{\mu\nu}(x), \varphi_{\sigma\tau}(y)]$, we can put $z^0 = y^0$ and use the three-dimensional commutation relations among asymptotic fields. The resultant 4D CRs are summarized as

$$\begin{aligned}[\varphi_{\mu\nu}(x), \varphi_{\sigma\tau}(y)] = 12i\phi_0^{-2}[&(\eta_{\mu\nu}\eta_{\sigma\tau} - \eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma})D(x-y) \\ &+ (\eta_{\mu\sigma}\partial_\nu\partial_\tau + \eta_{\nu\sigma}\partial_\mu\partial_\tau + \eta_{\mu\tau}\partial_\nu\partial_\sigma \\ &+ \eta_{\nu\tau}\partial_\mu\partial_\sigma)E(x-y)],\end{aligned} \tag{6.19}$$

$$[\varphi_{\mu\nu}(x), \beta_\rho(y)] = i\phi_0^{-2}(\eta_{\mu\rho}\partial_\nu + \eta_{\nu\rho}\partial_\mu)D(x-y), \tag{6.20}$$

$$[\varphi_{\mu\nu}(x), \beta(y)] = -2i\phi_0^{-1}\eta_{\mu\nu}D(x-y), \tag{6.21}$$

$$[\tilde{\phi}(x), \beta(y)] = i\phi_0^{-1} D(x-y), \tag{6.22}$$

$$\{\gamma^\sigma(x), \bar{\gamma}_\tau(y)\} = -\phi_0^{-2}\delta_\tau^\sigma D(x-y), \tag{6.23}$$

$$\{\gamma(x), \bar{\gamma}(y)\} = -\phi_0^{-2} D(x-y). \tag{6.24}$$

The other 4D CRs vanish identically.

Now we would like to discuss the issue of the unitarity of the theory in hand. To do this, it is convenient to perform the Fourier transformation of Eqs. (6.19)–(6.24). However, for the dipole field we cannot use the three-dimensional Fourier expansion to define the creation and annihilation operators. We therefore make use of the four-dimensional Fourier expansion [5][8]:

$$\varphi_{\mu\nu}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4p\,\theta(p_0)[\varphi_{\mu\nu}(p)e^{ipx} + \varphi_{\mu\nu}^\dagger(p)e^{-ipx}], \tag{6.25}$$

where $\theta(p_0)$ is the step function. For any simple pole fields, we adopt the same Fourier expansion; for instance,

$$\beta_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4p\,\theta(p_0)[\beta_\mu(p)e^{ipx} + \beta_\mu^\dagger(p)e^{-ipx}]. \tag{6.26}$$

Incidentally, for a generic simple pole field Φ, the three-dimensional Fourier expansion is defined as

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3p\frac{1}{\sqrt{2|\vec{p}|}} \times [\Phi(\vec{p})e^{-i|\vec{p}|x_0 + i\vec{p}\cdot\vec{x}} + \Phi^\dagger(\vec{p})e^{i|\vec{p}|x_0 - i\vec{p}\cdot\vec{x}}], \tag{6.27}$$

whereas the four-dimensional Fourier expansion reads

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^4p\,\theta(p_0)[\Phi(p)e^{ipx} + \Phi^\dagger(p)(p)e^{-ipx}]. \tag{6.28}$$

Thus, the annihilation operator $\Phi(p)$ in the four-dimensional Fourier expansion is related to the annihilation operator $\Phi(\vec{p})$ in the three-dimensional Fourier expansion as

$$\Phi(p) = \theta(p_0)\delta(p^2)\sqrt{2|\vec{p}|}\Phi(\vec{p}). \tag{6.29}$$

[8]The Fourier transform of a field is denoted by the same field except for the argument x or p, for simplicity.

Based on these Fourier expansions, we can calculate the Fourier transforms of Eqs. (6.19)–(6.24):

$$\begin{aligned}[\varphi_{\mu\nu}(p), \varphi_{\sigma\tau}^\dagger(q)] = 12\phi_0^{-2}\theta(p_0)\delta^4(p-q) &\times [\delta(p^2)(\eta_{\mu\nu}\eta_{\sigma\tau} - \eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma}) \\ &- 3\delta'(p^2)(\eta_{\mu\sigma}p_\nu p_\tau + \eta_{\nu\sigma}p_\mu p_\tau \\ &+ \eta_{\mu\tau}p_\nu p_\sigma + \eta_{\nu\tau}p_\mu p_\sigma)],\end{aligned} \tag{6.30}$$

$$[\varphi_{\mu\nu}(p), \beta_\rho^\dagger(q)] = i\phi_0^{-2}(\eta_{\mu\rho}p_\nu + \eta_{\nu\rho}p_\mu) \times \theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.31}$$

$$[\varphi_{\mu\nu}(p), \beta^\dagger(q)] = -2\phi_0^{-1}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.32}$$

$$[\tilde{\phi}(p), \beta^\dagger(q)] = \phi_0^{-1}\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.33}$$

$$\{\gamma^\sigma(p), \bar{\gamma}_\tau^\dagger(q)\} = i\phi_0^{-2}\delta_\tau^\sigma\theta(p_0)\delta(p^2)\delta^4(p-q), \tag{6.34}$$

$$\{\gamma(p), \bar{\gamma}^\dagger(q)\} = i\phi_0^{-2}\theta(p_0)\delta(p^2)\delta^4(p-q). \tag{6.35}$$

Next, let us turn our attention to the linearized field equations. In the Fourier transformation, Eq. (6.6) takes the form

$$p^\nu\varphi_{\mu\nu} - \frac{1}{2}p_\mu\varphi = 2\phi_0^{-1}p_\mu\tilde{\phi}. \tag{6.36}$$

If we fix the degree of freedom associated with $\tilde{\phi}$ (which will be discussed later), this equation gives us four independent relations in ten components of $\varphi_{\mu\nu}(p)$, thereby reducing the number of independent components of $\varphi_{\mu\nu}(p)$ to six. To deal with six independent components of $\varphi_{\mu\nu}(p)$, it is convenient to take a specific Lorentz frame such that $p_1 = p_2 = 0$ and $p_3 > 0$, and choose the six components as follows:

$$\begin{aligned}&\varphi_1(p) = \frac{1}{2}[\varphi_{11}(p) - \varphi_{22}(p)], \quad \varphi_2(p) = \varphi_{12}(p), \\ &\omega_0(p) = -\frac{1}{2p_0}\varphi_{00}(p), \quad \omega_I(p) = -\frac{1}{p_0}\varphi_{0I}(p), \\ &\omega_3(p) = -\frac{1}{2p_3}\varphi_{33}(p),\end{aligned} \tag{6.37}$$

where the index I takes the transverse components $I = 1, 2$.

In this respect, it is worthwhile to consider the GCT BRST transformation for these components. First, let us write down the GCT BRST transformation for the Fourier expansion of the asymptotic fields, which reads

$$\begin{aligned}&\delta_B\varphi_{\mu\nu}(p) = -i[p_\mu\gamma_\nu(p) + p_\nu\gamma_\mu(p)], \\ &\delta_B\gamma^\mu(p) = 0, \qquad \delta_B\bar{\gamma}_\mu(p) = i\beta_\mu(p), \\ &\delta_B\tilde{\phi}(p) = \delta_B\beta_\mu(p) = \delta_B\beta(p) = \delta_B\gamma(p) = \delta_B\bar{\gamma}(p) = 0.\end{aligned} \tag{6.38}$$

Using this BRST transformation, the GCT BRST transformation for the components in Eq. (6.37) takes the form

$$\delta_B \varphi_I(p) = 0, \qquad \delta_B \omega_\mu(p) = i\gamma_\mu(p),$$
$$\delta_B \bar{\gamma}_\mu(p) = i\beta_\mu(p), \qquad \delta_B \gamma_\mu(p) = \delta_B \beta_\mu(p) = 0, \quad (6.39)$$

where $p_1 = p_2 = 0$ was used. This BRST transformation implies that $\varphi_I(p)$ could be the physical observable, while the set of fields $\{\omega_\mu(p), \beta_\mu(p), \gamma_\mu(p), \bar{\gamma}_\mu(p)\}$ might belong to the BRST quartet, which are dropped from the physical state by the Kugo-Ojima subsidiary condition, $Q_B|\mathrm{phys}\rangle = 0$ [8]. However, note that $\beta_\mu(p), \gamma_\mu(p)$, and $\bar{\gamma}_\mu(p)$ are simple pole fields obeying $p^2\beta_\mu(p) = p^2\gamma_\mu(p) = p^2\bar{\gamma}_\mu(p) = 0$, but $\varphi_{\mu\nu}(p)$ is a dipole field satisfying $(p^2)^2\varphi_{\mu\nu}(p) = 0$, so that a naive Kugo-Ojima quartet mechanism does not work.

To clarify the BRST quartet mechanism, let us calculate their 4D CRs. From Eqs. (6.30)–(6.35) and the definition (6.37), it is straightforward to derive the following 4D CRs:

$$[\varphi_I(p), \varphi_J^\dagger(q)] = -12\phi_0^{-2}\delta_{IJ}\theta(p_0)\delta(p^2)\delta^4(p-q), \quad (6.40)$$

$$[\varphi_I(p), \omega_\mu^\dagger(q)] = [\varphi_I(p), \beta_\mu^\dagger(q)] = [\beta_\mu(p), \beta_\nu^\dagger(q)] = 0, \quad (6.41)$$

$$[\omega_\mu(p), \beta_\nu^\dagger(q)] = -i\phi_0^{-2}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p-q), \quad (6.42)$$

$$\{\gamma_\mu(p), \bar{\gamma}_\nu^\dagger(q)\} = i\phi_0^{-2}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p-q). \quad (6.43)$$

In addition, we have a rather complicated expression for $[\omega_\mu(p), \omega_\nu^\dagger(q)]$ because $\varphi_{\mu\nu}(p)$ is a dipole field, but luckily this expression is not necessary for our aim [8]. It is known how to extract a simple pole field from a dipole field, which amounts to using an operator defined by [8]

$$\mathcal{D}_p = \frac{1}{2|\vec{p}|^2} p_0 \frac{\partial}{\partial p_0} + c, \quad (6.44)$$

where c is a constant. Using this operator, we can define a simple pole field $\hat{\varphi}_{\mu\nu}(p)$ from the dipole field $\varphi_{\mu\nu}(p)$, which obeys $(p^2)^2\varphi_{\mu\nu}(p) = 0$, as

$$\hat{\varphi}_{\mu\nu}(p) \equiv \varphi_{\mu\nu}(p) - \mathcal{D}_p p^2 \varphi_{\mu\nu}(p)$$
$$= \varphi_{\mu\nu}(p) - 24i\mathcal{D}_p p_{(\mu}\beta_{\nu)}(p), \quad (6.45)$$

where in the last equality we have used the Fourier transform of the linearized field equation (6.12). It is then easy to verify the equation

$$p^2 \hat{\varphi}_{\mu\nu}(p) = 0. \quad (6.46)$$

Then, in Eq. (6.37) we replace $\varphi_{\mu\nu}$ of ω_μ with $\hat{\varphi}_{\mu\nu}$, and we redefine ω_μ as $\hat{\omega}_\mu$:

$$\hat{\omega}_0(p) = -\frac{1}{2p_0}\hat{\varphi}_{00}(p), \qquad \hat{\omega}_I(p) = -\frac{1}{p_0}\hat{\varphi}_{0I}(p),$$
$$\hat{\omega}_3(p) = -\frac{1}{2p_3}\hat{\varphi}_{33}(p). \quad (6.47)$$

The key point is that with this redefinition from ω_μ to $\hat{\omega}_\mu$, the BRST transformation and the 4D CRs remain unchanged owing to $\delta_B\beta_\mu = 0$ and $[\beta_\mu(p), \beta_\nu^\dagger(q)] = [\varphi_I(p), \beta_\mu^\dagger(q)] = 0$, that is,

$$\delta_B \hat{\omega}_\mu(p) = i\gamma_\mu(p), \quad [\hat{\omega}_\mu(p), \beta_\nu^\dagger(q)] = [\omega_\mu(p), \beta_\nu^\dagger(q)],$$
$$[\varphi_I(p), \hat{\omega}_\mu^\dagger(q)] = [\varphi_I(p), \omega_\mu^\dagger(q)]. \quad (6.48)$$

Now it turns out that all of the fields $\{\varphi_I, \hat{\omega}_\mu, \beta_\mu, \gamma_\mu, \bar{\gamma}_\mu\}$ are simple pole fields.[9] Since all of the fields become simple pole fields, we can obtain the standard creation and annihilation operators in the three-dimensional Fourier expansion from those in the four-dimensional one through Eq. (6.29). As a result, the three-dimensional (anti)commutation relations, which are denoted as $[\Phi(\vec{p}), \Phi^\dagger(\vec{q})\}$ with $\Phi(\vec{p}) \equiv \{\varphi_I(\vec{p}), \hat{\omega}_\mu(\vec{p}), \beta_\mu(\vec{p}), \gamma_\mu(\vec{p}), \bar{\gamma}_\mu(\vec{p})\}$, are given by[10]

$$[\Phi(\vec{p}), \Phi^\dagger(\vec{q})\} = \left(\begin{array}{c|cc|cc} -12\phi_0^{-2}\delta_{IJ} & & & & \\ \hline & [\hat{\omega}_\mu(\vec{p}), \hat{\omega}_\nu^\dagger(\vec{q})] & -i\phi_0^{-2}\eta_{\mu\nu} & & \\ & i\phi_0^{-2}\eta_{\mu\nu} & 0 & & \\ \hline & & & & i\phi_0^{-2}\eta_{\mu\nu} \\ & & & -i\phi_0^{-2}\eta_{\mu\nu} & \end{array}\right)\delta(\vec{p}-\vec{q}). \quad (6.49)$$

The (anti)commutation relations (6.49) have in essence the same structure as those of the Yang-Mills theory [8]. Hence, we find that φ_I could be the physical observable, while the set of fields $\{\hat{\omega}_\mu, \beta_\mu, \gamma_\mu, \bar{\gamma}_\mu\}$ belongs to the BRST quartet.

[9] Without the redefinition, $\varphi_I(p)$ is already a simple pole field, as can be seen in Eq. (6.40).

[10] The bracket $[A, B\}$ is the graded commutation relation denoting either a commutator or anticommutator, according to the Grassmann-even or -odd character of A and B, i.e., $[A, B\} = AB - (-)^{|A||B|}BA$.

Next, let us move on to another BRST transformation: the BRST transformation for the Weyl transformation. The Weyl BRST transformation for the asymptotic fields is of the form

$$\bar{\delta}_B \varphi_{\mu\nu} = 2c\eta_{\mu\nu}, \quad \bar{\delta}_B \tilde{\phi} = -\phi_0 \gamma, \quad \bar{\delta}_B \gamma = 0, \quad \bar{\delta}_B \bar{\gamma} = i\beta, \\ \bar{\delta}_B \beta = \bar{\delta}_B \beta_\mu = \bar{\delta}_B \gamma_\mu = \bar{\delta}_B \bar{\gamma}_\mu = 0. \tag{6.50}$$

The Weyl BRST transformation of φ_I is vanishing,

$$\bar{\delta}_B \varphi_I = 0, \tag{6.51}$$

which means that together with $\delta_B \varphi_I = 0$, φ_I is truly the physical observable. The four-dimensional (anti)commutation relations among the fields $\{\tilde{\phi}, \beta, \gamma, \bar{\gamma}\}$ read

$$[\tilde{\phi}(p), \tilde{\phi}^\dagger(q)] = 0, \\ [\tilde{\phi}(p), \beta^\dagger(q)] = \phi_0^{-1} \theta(p_0) \delta(p^2) \delta^4(p-q), \\ \{\gamma(p), \bar{\gamma}^\dagger(q)\} = i\phi_0^{-2} \theta(p_0) \delta(p^2) \delta^4(p-q). \tag{6.52}$$

As can also be seen in these 4D CRs, all of the fields $\{\varphi_I, \tilde{\phi}, \beta, \gamma, \bar{\gamma}\}$ are massless simple pole fields. Via Eq. (6.29), the three-dimensional (anti)commutation relations $[\Phi(\vec{p}), \Phi^\dagger(\vec{q})\}$ with $\Phi(\vec{p}) \equiv \{\varphi_I(\vec{p}), \tilde{\phi}(\vec{p}), \beta(\vec{p}), \gamma(\vec{p}), \bar{\gamma}(\vec{p})\}$ are of the form

$$[\Phi(\vec{p}), \Phi^\dagger(\vec{q})\} = \left(\begin{array}{c|cc|cc} -12\phi_0^{-2}\delta_{IJ} & & & & \\ \hline & 0 & \phi_0^{-1} & & \\ & \phi_0^{-1} & 0 & & \\ \hline & & & & i\phi_0^{-2} \\ & & & -i\phi_0^{-2} & \end{array}\right) \delta(\vec{p} - \vec{q}). \tag{6.53}$$

Thus, φ_I is the physical observable while the set of fields $\{\tilde{\phi}, \beta, \gamma, \bar{\gamma}\}$ consists of the BRST quartet and is the unphysical mode by the Kugo-Ojima subsidiary condition 8]]. Here it is worth mentioning that the ghost-like scalar field ϕ belongs to the unphysical mode, so together with the result obtained in the analysis of the GCT BRST cohomology the physical S matrix is found to be unitary.

VII. CHORAL SYMMETRY

As mentioned in Sec. III, a set of fields (including the space-time coordinates x^μ) $X^M \equiv \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c}\}$ obeys a very simple equation:

$$g^{\mu\nu} \partial_\mu \partial_\nu X^M = 0. \tag{7.1}$$

This equation holds if and only if we adopt the extended de Donder gauge and the new scalar gauge as gauge-fixing conditions for the GCT and the Weyl transformation, respectively. The existence of this simple equation suggests that there could be many of conserved currents defined in Eq. (3.20). In this section, we show explicitly that there exist such currents and we have a huge global symmetry called choral symmetry, which is the $IOSp(10|10)$ symmetry in the present theory.

Let us start with the Lagrangian (3.12), which can be cast in the form

$$\mathcal{L}_q = \tilde{g}^{\mu\nu} \phi^2 \left(\frac{1}{12} R_{\mu\nu} - \frac{1}{2} \hat{E}_{\mu\nu} \right). \tag{7.2}$$

Here we note that $\tilde{g}^{\mu\nu}\phi^2$ is a Weyl-invariant metric and the Ricci tensor is invariant under only a global scale transformation. We can further rewrite it in the form

$$\mathcal{L}_q = \tilde{g}^{\mu\nu} \phi^2 \left(\frac{1}{12} R_{\mu\nu} - \frac{1}{2} \eta_{NM} \partial_\mu X^M \partial_\nu X^N \right) \\ = \tilde{g}^{\mu\nu} \phi^2 \left(\frac{1}{12} R_{\mu\nu} - \frac{1}{2} \partial_\mu X^M \tilde{\eta}_{MN} \partial_\nu X^N \right), \tag{7.3}$$

where we have introduced an $IOSp(10|10)$ metric $\eta_{NM} = \eta^T_{MN} \equiv \tilde{\eta}_{MN}$ defined as [9]

$$\eta_{NM} = \tilde{\eta}_{MN} = \left(\begin{array}{cc|cc|cc|cc} & \delta_\mu^\nu & & & & & & \\ \delta_\nu^\mu & & & & & & & \\ \hline & & -1 & -1 & & & & \\ & & -1 & 0 & & & & \\ \hline & & & & & -i\delta_\mu^\nu & & \\ & & & & i\delta_\nu^\mu & & & \\ \hline & & & & & & & -i \\ & & & & & & i & \end{array}\right). \tag{7.4}$$

Let us note that this $IOSp(10|10)$ metric η_{NM}, which is a c-number quantity, has the symmetry property that

$$\eta_{MN} = (-)^{|M|\cdot|N|} \eta_{NM} = (-)^{|M|} \eta_{NM} = (-)^{|N|} \eta_{NM}, \tag{7.5}$$

where the statistics index $|M|$ is 0 or 1 when X^M is Grassmann-even or Grassmann-odd, respectively. This property comes from the fact that η_{MN} is "diagonal" in the sense that its off-diagonal, Grassmann-even, and Grassmann-odd (and vice versa) matrix elements vanish, i.e., $\eta_{MN} = 0$ when $|M| \neq |N|$, thereby being $|M| = |N| = |M| \cdot |N|$ in front of η_{MN} [9].

Now that the quantum Lagrangian (7.3) is expressed in a manifestly $IOSp(10|10)$-invariant form except for the Weyl-invariant metric $\tilde{g}^{\mu\nu}\phi^2$ (which will be discussed later), there could exist an $IOSp(10|10)$ symmetry as a global symmetry in our theory. Let us show this fact first. The infinitesimal OSp rotation is defined by

$$\delta X^M = \eta^{ML}\varepsilon_{LN}X^N \equiv \varepsilon^M{}_N X^N, \tag{7.6}$$

where η^{MN} is the inverse matrix of η_{MN}, and the infinitesimal parameter ε_{MN} has the following properties:

$$\varepsilon_{MN} = (-)^{1+|M|\cdot|N|}\varepsilon_{NM}, \quad \varepsilon_{MN}X^L = (-)^{|L|(|M|+|N|)}X^L\varepsilon_{MN}. \tag{7.7}$$

Moreover, in order to find the conserved current, we assume that the infinitesimal parameter ε_{MN} depends on the space-time coordinates x^μ, i.e., $\varepsilon_{MN} = \varepsilon_{MN}(x^\mu)$.

Assuming for a while that the metric $\tilde{g}^{\mu\nu}\phi^2$ and $R_{\mu\nu}$ are invariant, the infinitesimal variation of the quantum Lagrangian (7.3) under the OSp rotation (7.6) is given by

$$\delta\mathcal{L}_q = -\tilde{g}^{\mu\nu}\phi^2(\partial_\mu\varepsilon_{NM}X^M\partial_\nu X^N + \varepsilon_{NM}\partial_\mu X^M\partial_\nu X^N). \tag{7.8}$$

It is easy to prove that the second term on the rhs vanishes owing to the first property in Eq. (7.7). Thus, $\mathcal{L}_q$ is invariant under the infinitesimal OSp rotation. The conserved current is then calculated as

$$\begin{aligned}\delta\mathcal{L}_q &= -\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}X^M\partial_\nu X^N \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}[X^M\partial_\nu X^N - (-)^{|M|\cdot|N|}X^N\partial_\nu X^M] \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}(X^M\partial_\nu X^N - \partial_\nu X^M X^N) \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}X^M\overleftrightarrow{\partial}_\nu X^N \\ &\equiv -\frac{1}{2}\partial_\mu\varepsilon_{NM}\mathcal{M}^{\mu MN},\end{aligned} \tag{7.9}$$

from which the conserved current $\mathcal{M}^{\mu MN}$ for the OSp rotation takes the form

$$\mathcal{M}^{\mu MN} = \tilde{g}^{\mu\nu}\phi^2 X^M\overleftrightarrow{\partial}_\nu X^N. \tag{7.10}$$

In a similar way, we can derive the conserved current for the infinitesimal translation

$$\delta X^M = \varepsilon^M, \tag{7.11}$$

where ε^M is the infinitesimal parameter, and assume that it is a local one when deriving the corresponding conserved current. Indeed, assuming again that the metric $\tilde{g}^{\mu\nu}\phi^2$ and $R_{\mu\nu}$ are invariant under the translation, we can show that $\mathcal{L}_q$ is invariant under an infinitesimal translation,

$$\begin{aligned}\delta\mathcal{L}_q &= -\tilde{g}^{\mu\nu}\phi^2\eta_{NM}\partial_\mu\varepsilon^M\partial_\nu X^N \\ &= -\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_N\partial_\nu X^N \\ &\equiv -\partial_\mu\varepsilon_M\mathcal{P}^{\mu M},\end{aligned} \tag{7.12}$$

which implies that the conserved current $\mathcal{P}^{\mu M}$ for the translation reads

$$\mathcal{P}^{\mu M} = \tilde{g}^{\mu\nu}\phi^2\partial_\nu X^M = \tilde{g}^{\mu\nu}\phi^2(1\overleftrightarrow{\partial}_\nu X^M). \tag{7.13}$$

The above proofs only make sense under the assumption that the metric $\tilde{g}^{\mu\nu}\phi^2$ and $R_{\mu\nu}$ are invariant under the $IOSp(10|10)$ symmetry. So the problem reduces to a question: is this assumption correct? The answer is obviously "no," but the noninvariant terms can be compensated by a suitable Weyl transformation. To show this fact, let us consider only the case of the infinitesimal OSp rotation since we can treat the case of the translation in a perfectly similar manner. Under the infinitesimal OSp rotation (7.6), the dilaton $\sigma(x)$, which is defined as $\phi = e^\sigma$, transforms as

$$\delta\sigma = \eta^{\sigma L}\varepsilon_{LN}X^N = -\varepsilon_{BN}X^N, \tag{7.14}$$

where we have used Eq. (7.4) and

$$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \tag{7.15}$$

where we recall that the matrix η^{ML} is the inverse matrix of η_{ML}. As for the scalar field $\phi(x)$, this transformation for the dilaton can be interpreted as a Weyl transformation:

$$\phi \to \phi' = e^{\epsilon(x)}\phi, \tag{7.16}$$

where the infinitesimal parameter is defined as $\epsilon(x) = -\varepsilon_{BN}X^N$. This Weyl transformation induces the Weyl transformation for the metric tensor field at the same time:

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{-2\epsilon(x)}g_{\mu\nu}. \tag{7.17}$$

Let us recall that the metric $\tilde{g}^{\mu\nu}\phi^2$ is the Weyl-invariant metric, and thus it is invariant under the Weyl transformation (7.16) and (7.17). This implies that $\tilde{g}^{\mu\nu}\phi^2$ is essentially invariant under the OSp rotation if an appropriate Weyl transformation is achieved.

What about $R_{\mu\nu}$? Even if $R_{\mu\nu}$ is not invariant under the Weyl transformation in itself, this object comes from the classical Lagrangian of the Weyl-invariant scalar-tensor gravity in Eq. (3.1), so together with the metric tensor and the scalar field it essentially becomes invariant under the Weyl transformation (7.16) and (7.17). Thus, in this sense, $R_{\mu\nu}$ is also invariant under the OSp rotation. In any case, it is worth stressing that in the present formulation, the choral symmetry $IOSp(10|10)$ is not only a symmetry of the FP ghosts and the Nakanishi-Lautrup fields, but is also closely related to the classical fields $g_{\mu\nu}$ and ϕ which lie in the classical Lagrangian.

An important remark is relevant to the expression of the conserved currents (7.10) and (7.13). To make the quantum Lagrangian $\mathcal{L}_q$ invariant under the choral symmetry $IOSp(10|10)$, it is necessary to perform the Weyl transformation (7.16) and (7.17). Then, it is natural to ask if, because of this associated Weyl transformation, the expression for the currents would be modified or not. Here a miracle happens. As shown in Refs. [10,11], the current for the Weyl transformation identically vanishes in Weyl-invariant scalar-tensor gravity. Thus, although we make the Weyl transformation (7.16) and (7.17), the conserved currents (7.10) and (7.13) are unchanged.

From the conserved currents (7.10) and (7.13), the corresponding conserved charges become

$$\begin{aligned} M^{MN} &\equiv \int d^3x \mathcal{M}^{0MN} = \int d^3x \tilde{g}^{0\nu} \phi^2 X^M \overset{\leftrightarrow}{\partial}_\nu X^N, \\ P^M &\equiv \int d^3x \mathcal{P}^{0M} = \int d^3x \tilde{g}^{0\nu} \phi^2 \partial_\nu X^M. \end{aligned} \tag{7.18}$$

It then turns out that, using various ETCRs obtained so far, the $IOSp(10|10)$ generators $\{M^{MN}, P^M\}$ generate an $IOSp(10|10)$ algebra:

$$\begin{aligned} &[P^M, P^N\} = 0, \\ &[M^{MN}, P^R\} = i[P^M \tilde{\eta}^{NR} - (-)^{|N||R|} P^N \tilde{\eta}^{MR}], \\ &[M^{MN}, M^{RS}\} = i[M^{MS} \tilde{\eta}^{NR} - (-)^{|N||R|} M^{MR} \tilde{\eta}^{NS} \\ &\qquad - (-)^{|N||R|} M^{NS} \tilde{\eta}^{MR} \\ &\qquad + (-)^{|M||R| + |N||S|} M^{NR} \tilde{\eta}^{MS}]. \end{aligned} \tag{7.19}$$

As a final remark, it is worth pointing out that all of the global symmetries in the present theory are expressed in terms of the generators of the choral symmetry. For instance, the BRST charges for the GCT and Weyl transformation are expressed, respectively, as

$$\begin{aligned} Q_B &\equiv M(b_\rho, c^\rho) = \int d^3x \tilde{g}^{0\nu} \phi^2 b_\rho \overset{\leftrightarrow}{\partial}_\nu c^\rho, \\ \bar{Q}_B &\equiv M(B, c) = \int d^3x \tilde{g}^{0\nu} \phi^2 B \overset{\leftrightarrow}{\partial}_\nu c. \end{aligned} \tag{7.20}$$

VIII. GRAVITATIONAL CONFORMAL SYMMETRY

Even though we already fixed the Weyl symmetry by the scalar gauge condition (3.6), we still have its linearized, residual symmetries. In order to look for the residual symmetries, it is convenient to take the extended de Donder gauge (3.5) into consideration simultaneously.[11] With the help of the extended de Donder gauge (3.5), the scalar gauge condition (3.6) can be rewritten as

$$0 = \partial_\mu (\tilde{g}^{\mu\nu} \phi \partial_\nu \phi) = \partial_\mu (\tilde{g}^{\mu\nu} \phi^2 \partial_\nu \sigma) = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu \sigma, \tag{8.1}$$

where we have used the relation between the scalar field and dilaton, $\phi = e^\sigma$. Under the Weyl transformation (3.2) with $\Omega(x) \equiv e^{\Lambda(x)}$, the dilaton σ transforms as

$$\sigma \to \sigma' = \sigma - \log \Omega = \sigma - \Lambda, \tag{8.2}$$

where we have used the Weyl transformation (3.2) for the scalar field. Since $\tilde{g}^{\mu\nu} \phi^2$ is a Weyl-invariant quantity, the Weyl transformation changes Eq. (8.1) to

$$0 = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu \sigma \to 0 = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu \sigma' = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu (\sigma - \Lambda). \tag{8.3}$$

This equation shows that when we use the extended de Donder gauge, the scalar gauge condition is still invariant under the Weyl transformation as long as

$$g^{\mu\nu} \partial_\mu \partial_\nu \Lambda = 0 \tag{8.4}$$

is satisfied, thereby implying the existence of the residual symmetries [12–14]. Selecting the coefficients appropriately for later convenience, the solution to Eq. (8.4) is given by

$$\Lambda = \lambda - 2k_\mu x^\mu, \tag{8.5}$$

where λ and k_μ are constants.[12]

We can also verify the invariance of the quantum Lagrangian under the residual symmetries more directly. To do this, let us assume that Λ (or, equivalently, λ and k_μ) are the infinitesimal parameters. It then turns out that the quantum Lagrangian (3.12) is invariant under the residual symmetries,

$$\begin{aligned} &\delta g_{\mu\nu} = 2(\lambda - 2k_\rho x^\rho) g_{\mu\nu}, \\ &\delta\sigma = -(\lambda - 2k_\rho x^\rho), \qquad \delta b_\mu = 2k_\mu B, \end{aligned} \tag{8.6}$$

[11]The same strategy was adopted in different theories in Refs. [12–14].

[12]It is shown in Appendix B that the transformations associated with the parameters λ and k_μ correspond to dilatation and the special conformal transformation, respectively, in a flat Minkowski background.

where the other fields are unchanged. The generators corresponding to the transformation parameters λ and k_μ are constructed out of those of the choral symmetry as, respectively,

$$D_0 \equiv -P(B) = -\int d^3x \tilde{g}^{0\nu}\phi^2 \partial_\nu B,$$
$$K^\mu \equiv 2M^\mu(x, B) = 2\int d^3x \tilde{g}^{0\nu}\phi^2 x^\mu \overleftrightarrow{\partial}_\nu B. \quad (8.7)$$

In addition to the generators D_0 and K^μ, one can construct the translation generator P_μ and $GL(4)$ generator $G^\mu{}_\nu$ from those of the choral symmetry $IOSp(10|10)$ as

$$P_\mu \equiv P_\mu(b) = \int d^3x \tilde{g}^{0\nu}\phi^2 \partial_\nu b_\mu,$$
$$G^\mu{}_\nu \equiv M^\mu{}_\nu(x, b) - iM^\mu{}_\nu(c^\tau, \bar{c}_\tau)$$
$$= \int d^3x \tilde{g}^{0\lambda}\phi^2 (x^\mu \overleftrightarrow{\partial}_\lambda b_\nu - ic^\mu \overleftrightarrow{\partial}_\lambda \bar{c}_\nu). \quad (8.8)$$

Now we would like to show that in our theory there is a gravitational conformal algebra that is slightly different from the conformal algebra in a flat Minkowski space-time. To this aim, let us consider a set of generators $\{P_\mu, G^\mu{}_\nu, K^\mu, D_0\}$. From these generators, we wish to construct the generator D for a scale transformation. Recall that in conformal field theory in four-dimensional Minkowski space-time the dilatation generator obeys the following algebra for a local operator $O_i(x)$ of conformal dimension Δ_i [15,16][13]:

$$[iD, O_i(x)] = x^\mu \partial_\mu O_i(x) + \Delta_i O_i(x). \quad (8.9)$$

Since the scalar field $\phi(x)$ has conformal dimension 1, it must satisfy the equation

$$[iD, \phi(x)] = x^\mu \partial_\mu \phi(x) + \phi(x). \quad (8.10)$$

To be consistent with this equation, we shall make a generator for the scale transformation. From the definitions (8.7) and (8.8), we find

$$[iG^\mu{}_\nu, \phi(x)] = x^\mu \partial_\mu \phi(x), \qquad [iD_0, \phi(x)] = -\phi(x). \quad (8.11)$$

[13]For clarity, we will call a global scale transformation in a flat Minkowski space-time "dilatation." Dilatation is usually interpreted as a subgroup of the general coordinate transformation in a such way that the space-time coordinates are transformed as $x^\mu \to \Omega x^\mu$ in the flat space-time, where Ω is a constant scale factor, whereas the global scale transformation is a rescaling of all lengths by the same Ω as $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. The two viewpoints are completely equivalent since all of the lengths are defined via the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$.

The following linear combination of $G^\mu{}_\nu$ and D_0 does the job:

$$D \equiv G^\mu{}_\mu - D_0. \quad (8.12)$$

As a consistency check, it is valuable to see how this operator D acts on the metric field, whose result reads

$$[iD, g_{\sigma\tau}] = [iG^\mu{}_\mu, g_{\sigma\tau}] - [iD_0, g_{\sigma\tau}]$$
$$= (x^\mu \partial_\mu g_{\sigma\tau} + 2g_{\sigma\tau}) - 2g_{\sigma\tau} = x^\mu \partial_\mu g_{\sigma\tau}, \quad (8.13)$$

which implies that the metric field has conformal dimension 0, as desired, and this result will be used later when discussing spontaneous symmetry breaking.

Next, let us calculate an algebra among the generators $\{P_\mu, G^\mu{}_\nu, K^\mu, D\}$. After some calculations, we find that the algebra closes and takes the form

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, G^\rho{}_\sigma] = iP_\sigma \delta^\rho_\mu,$$
$$[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu,$$
$$[P_\mu, D] = iP_\mu, \quad [G^\mu{}_\nu, G^\rho{}_\sigma] = i(G^\mu{}_\sigma \delta^\rho_\nu - G^\rho{}_\nu \delta^\mu_\sigma),$$
$$[G^\mu{}_\nu, K^\rho] = iK^\mu \delta^\rho_\nu, \quad [G^\mu{}_\nu, D] = [K^\mu, K^\nu] = 0,$$
$$[K^\mu, D] = -iK^\mu, \quad [D, D] = 0. \quad (8.14)$$

To extract the gravitational conformal algebra in quantum gravity, it is necessary to introduce the "Lorentz" generator, which can be constructed from the $GL(4)$ generator as

$$M_{\mu\nu} \equiv -\eta_{\mu\rho} G^\rho{}_\nu + \eta_{\nu\rho} G^\rho{}_\mu. \quad (8.15)$$

In terms of the generator $M_{\mu\nu}$, the algebra (8.14) can be cast in the form

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, M_{\rho\sigma}] = i(P_\rho \eta_{\mu\sigma} - P_\sigma \eta_{\mu\rho}),$$
$$[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu, \quad [P_\mu, D] = iP_\mu,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\sigma}\eta_{\mu\rho} + M_{\rho\mu}\eta_{\sigma\nu} - M_{\rho\nu}\eta_{\sigma\mu}),$$
$$[M_{\mu\nu}, K^\rho] = i(-K_\mu \delta^\rho_\nu + K_\nu \delta^\rho_\mu), \quad [M_{\mu\nu}, D] = [K^\mu, K^\nu] = 0,$$
$$[K^\mu, D] = -iK^\mu, \quad [D, D] = 0, \quad (8.16)$$

where we have defined $K_\mu \equiv \eta_{\mu\nu} K^\nu$. It is of interest that the algebra (8.16) in quantum gravity, which we call "gravitational conformal algebra," formally resembles conformal algebra in flat Minkowski space-time except for the expression for $[P_\mu, K^\nu]$.[14] This difference reflects from the difference of the definition of conformal dimension in both gravity and conformal field theory, for which the metric tensor field $g_{\mu\nu}$ has 2 in gravity as seen in Eq. (3.2) while it has 0 in conformal field theory as seen in Eq. (8.13).

[14]In the case of conformal algebra in flat space-time, $[P_\mu, K^\nu] = -2i(\delta^\nu_\mu D + M_\mu{}^\nu)$.

IX. SPONTANEOUS BREAKING OF SYMMETRIES

In the theory in hand, there are huge global symmetries, which are $IOSp(10|10)$ supersymmetry, so it is valuable to investigate which symmetries are spontaneously broken or survive even in the quantum regime. In this section, we postulate the existence of a unique vacuum $|0\rangle$, which is normalized to unity:

$$\langle 0|0\rangle = 1. \tag{9.1}$$

Furthermore, we assume that the vacuum is translation invariant,

$$P_\mu|0\rangle = 0, \tag{9.2}$$

and the vacuum expectation values (VEVs) of the metric tensor $g_{\mu\nu}$ and scalar field ϕ are, respectively, the Minkowski metric $\eta_{\mu\nu}$ and a nonzero constant $\phi_0 \neq 0$:

$$\langle 0|g_{\mu\nu}|0\rangle = \eta_{\mu\nu}, \qquad \langle 0|\phi|0\rangle = \phi_0. \tag{9.3}$$

By a straightforward calculation, we can obtain the following VEVs:

$$\begin{aligned}
&\langle 0|[iP^\mu(x), b_\rho]|0\rangle = -\delta^\mu_\rho, \qquad \langle 0|\{iP^\mu(c^\tau), \bar{c}_\rho\}|0\rangle = i\delta^\mu_\rho,\\
&\langle 0|\{iP_\mu(\bar{c}_\tau), c^\rho\}|0\rangle = -i\delta^\rho_\mu,\\
&\langle 0|[iM^{\mu\nu}(x,x), \frac{1}{2}(\partial_\lambda b_\rho - \partial_\rho b_\lambda)]|0\rangle = -(\delta^\mu_\lambda\delta^\nu_\rho - \delta^\nu_\lambda\delta^\mu_\rho),\\
&\langle 0|\{iM^{\mu\nu}(x, c^\tau), \partial_\lambda\bar{c}_\rho\}|0\rangle = i\delta^\mu_\lambda\delta^\nu_\rho, \qquad \langle 0|\{iM^\mu{}_\nu(x, \bar{c}_\tau), \partial_\lambda c^\rho\}|0\rangle = -i\delta^\mu_\lambda\delta^\rho_\nu,\\
&\langle 0|[iP(\sigma), B]|0\rangle = 1, \qquad \langle 0|\{iP(c), \bar{c}\}|0\rangle = i, \qquad \langle 0|\{iP(\bar{c}), c\}|0\rangle = -i,\\
&\langle 0|\{iM(\sigma, c), \bar{c}\}|0\rangle = i\sigma_0, \qquad \langle 0|\{iM(\sigma, \bar{c}), c\}|0\rangle = -i\sigma_0,
\end{aligned} \tag{9.4}$$

where $\langle 0|\sigma(x)|0\rangle \equiv \sigma_0$. Equation (9.4) shows that the symmetries generated by the conserved charges

$$\{P^\mu(x), P^\mu(c^\tau), P_\mu(\bar{c}_\tau), M^{\mu\nu}(x,x), M^{\mu\nu}(x,c^\tau), M^\mu{}_\nu(x,\bar{c}_\tau), P(\sigma), P(c), P(\bar{c}), M(\sigma,c), M(\sigma,\bar{c})\}$$

are necessarily broken spontaneously, and therefore $b_\mu, c^\mu, \bar{c}_\mu, B, c$, and $\bar{c}$ acquire massless Nambu-Goldstone modes. Note that the exact masslessness of the dilaton σ cannot be proved in this way.

Next, on the basis of gravitational conformal symmetry, we will show that $GL(4)$, special conformal symmetry, and scale symmetry are spontaneously broken down to Poincaré symmetry. We find that the VEV of a commutator between the $GL(4)$ generator and the metric field reads

$$\langle 0|[iG^\mu{}_\nu, g_{\sigma\tau}]|0\rangle = \delta^\mu_\sigma\eta_{\nu\tau} + \delta^\mu_\tau\eta_{\nu\sigma}. \tag{9.5}$$

Thus, the Lorentz generator, which is defined in Eq. (8.15), has a vanishing VEV:

$$\langle 0|[iM_{\mu\nu}, g_{\sigma\tau}]|0\rangle = 0. \tag{9.6}$$

On the other hand, the symmetric part, which is defined as $\bar{M}_{\mu\nu} \equiv \eta_{\mu\rho}G^\rho{}_\nu + \eta_{\nu\rho}G^\rho{}_\mu$, has a nonvanishing VEV:

$$\langle 0|[i\bar{M}_{\mu\nu}, g_{\sigma\tau}]|0\rangle = 2(\eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma}). \tag{9.7}$$

Thus, $GL(4)$ symmetry is spontaneously broken to Lorentz symmetry, where the corresponding Nambu-Goldstone boson with ten independent components is nothing but the massless graviton [17]. Here it is interesting that in a sector of the scalar field, $GL(4)$ symmetry and of course Lorentz symmetry do not give rise to a symmetry breaking, as can be seen in the commutators

$$\langle 0|[iG^\mu{}_\nu, \phi]|0\rangle = \langle 0|[iM_{\mu\nu}, \phi]|0\rangle = \langle 0|[i\bar{M}_{\mu\nu}, \phi]|0\rangle = 0. \tag{9.8}$$

Now we wish to clarify how the scale symmetry and special conformal symmetry are spontaneously broken and what the corresponding Nambu-Goldstone bosons are. As for the scale symmetry, it is not the gravitational field but rather the dilaton that gives rise to spontaneous symmetry breaking. Indeed, Eq. (8.13) gives us

$$\langle 0|[iD, g_{\sigma\tau}]|0\rangle = 0. \tag{9.9}$$

On the other hand, for the dilaton, from Eq. (8.10) we have

$$\langle 0|[iD, \sigma]|0\rangle = 1, \tag{9.10}$$

which elucidates the spontaneous symmetry breaking of scale symmetry, whose Nambu-Goldstone boson is just the massless dilaton $\sigma(x)$.

Regarding the special conformal symmetry, we find

$$\langle 0|[iK^\mu, \partial_\nu \sigma]|0\rangle = 2\delta^\mu_\nu. \tag{9.11}$$

This equation means that the special conformal symmetry is certainly broken spontaneously and its Nambu-Goldstone boson is the derivative of the dilaton. This interpretation can also be verified from the gravitational conformal algebra. In the algebra (8.16), we have a commutator between P_μ and K^ν:

$$[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu. \tag{9.12}$$

Let us consider the Jacobi identity,

$$[[P_\mu, K^\nu], \sigma] + [[K^\nu, \sigma], P_\mu] + [[\sigma, P_\mu], K^\nu] = 0. \tag{9.13}$$

Using the translational invariance of the vacuum in Eq. (9.2) and the equation

$$[P_\mu, \sigma] = -i\partial_\mu \sigma, \tag{9.14}$$

and taking the VEV of the Jacobi identity (9.13), we can obtain the VEV

$$\langle 0|[K^\nu, \partial_\mu \sigma]|0\rangle = -2\delta^\nu_\mu \langle 0|[G^\rho{}_\rho - D, \sigma]|0\rangle = -2i\delta^\nu_\mu, \tag{9.15}$$

which coincides with Eq. (9.11), as promised. In other words, $GL(4)$ symmetry is spontaneously broken to Poincaré symmetry, whose Nambu-Goldstone boson is the graviton, and scale symmetry and special conformal symmetry are also spontaneously broken, and their corresponding Nambu-Goldstone bosons are the dilaton and derivative of the dilaton, respectively. It is of interest that the Nambu-Goldstone boson associated with special conformal symmetry is not an independent field in quantum gravity, as it is in conformal field theory [18].

X. CONCLUSION

In this article we have performed a manifestly covariant quantization and constructed a quantum theory of Weyl-invariant scalar-tensor gravity within the framework of the BRST formalism. In the past, Nakanishi developed a similar quantum gravitational theory of Einstein's general relativity [4,5], and the present work provides its natural generalization in the sense that Weyl symmetry is treated on the same footing as general coordinate symmetry.

Since Weyl-invariant scalar-tensor gravity has been known to be equivalent to general relativity in unitary gauge where the scalar field is gauge fixed to be a constant, it is natural to expect that our present theory shares several characteristic features with Nakanishi's quantum gravity. In particular, both theories have a huge global symmetry called "choral symmetry," but our choral symmetry $ISOp(10|10)$ is larger than that of Nakanishi's theory, which is $ISOp(8|8)$, owing to the presence of Weyl symmetry in our formulation. Compared with the case of general relativity, one peculiar feature of our choral symmetry is that choral symmetry needs Weyl symmetry in proving its invariance of the quantum Lagrangian so that it is closely related to a gravitational sector while in the case of general relativity the choral symmetry is isolated from classical Lagrangian and comes from purely the Lagrangian involving the Nakanishi-Lautrup field and the FP ghosts.

It is worth mentioning that in our quantum gravity there is a gravitational conformal algebra which is relevant to conventional conformal algebra in a flat Minkowski spacetime. According to the Zumino theorem [19], theories that are invariant under the GCT and Weyl transformation have conformal invariance in a flat Minkowski background at the classical level. The present study supports the conjecture that the Zumino theorem could be valid even in quantum gravity.

Last but not least, we should comment on the Weyl anomaly. In this respect, let us recall that in the manifestly scale-invariant regularization method [20–25], the scale invariance is free of scale anomalies. Though a completely satisfying formalism is still missing, we believe that in the Weyl-invariant regularization method, the Weyl invariance would also be kept at the operator level without the Weyl anomaly, and is spontaneously broken when considering states in the Hilbert space.

There is a lot of work to be done in future. First of all, we should develop manifestly Weyl-invariant regularization methods by introducing an additional scalar field that plays the role of the renormalization mass scale μ. Second, we should prove the quantum Zumino theorem in the case that the classical Lagrangian is an arbitrary Lagrangian that is invariant under the Weyl transformation. Third, we should add the Lagrangian of conformal gravity, that is, $\mathcal{L} \sim \sqrt{-g} C^2_{\mu\nu\rho\sigma}$ with conformal tensor $C_{\mu\nu\rho\sigma}$, and investigate if a similar analysis to the present work could be done. Finally, it is known that Weyl-invariant scalar-tensor gravity reduces to Weyl transverse gravity when the longitudinal general coordinate transformation is gauge fixed [26–29]. Weyl transverse gravity possesses Weyl symmetry, to which we could apply the present formulation and investigate various quantum aspects. We hope to return to these problems in the near future.

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APPENDIX A: DERIVATION OF EQ. (3.18)

In this appendix we present a derivation of Eq. (3.18). First of all, let us notice that the scalar gauge condition (3.6) is equivalent to the equation

$$\Box\phi^2 = 0. \tag{A1}$$

Then, the Einstein equation in Eq. (3.15) reads

$$G_{\mu\nu} - \phi^{-2}\nabla_\mu\nabla_\nu\phi^2 - 6\phi^{-2}\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) = 0. \tag{A2}$$

With the help of Eq. (A1), the trace part of this equation becomes

$$R = 6\phi^{-2}E. \tag{A3}$$

Inserting Eq. (A3) into Eq. (A2) leads to

$$R_{\mu\nu} = \phi^{-2}(\nabla_\mu\nabla_\nu\phi^2 + 6E_{\mu\nu}). \tag{A4}$$

Next, applying the covariant derivative ∇^μ to Eq. (A2) and using the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, we have

$$2\nabla^\mu\phi\nabla_\mu\nabla_\nu\phi^2 - \phi R_\nu{}^\mu\nabla_\mu\phi^2 + 12\nabla^\mu\phi\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) - 6\phi\nabla^\mu\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) = 0, \tag{A5}$$

where Eq. (A1) was used. Substituting Eq. (A4) into Eq. (A5) produces

$$\nabla^\mu\left(E_{\mu\nu} - \frac{1}{2}g_{\mu\nu}E\right) + \phi^{-1}\nabla_\nu\phi E = 0. \tag{A6}$$

At this point, we make use of an identity that holds for any symmetric tensor $S^{\mu\nu} = S^{\nu\mu}$ [1]:

$$\nabla_\nu S^\nu{}_\mu = h^{-1}\partial_\nu(hS^\nu{}_\mu) + \frac{1}{2}S_{\alpha\beta}\partial_\mu g^{\alpha\beta}. \tag{A7}$$

Identifying $S^{\mu\nu}$ with $E^{\mu\nu}$ and using Eq. (3.14), we obtain

$$g^{\rho\nu}\partial_\rho\hat{E}_{\mu\nu} - \frac{1}{2}g^{\alpha\beta}\partial_\mu\hat{E}_{\alpha\beta} = 0, \tag{A8}$$

where we used the extended de Donder gauge condition (3.5). Finally, when we calculate the lhs of Eq. (A8) using the definition of $\hat{E}_{\mu\nu}$ in Eq. (3.13), we can arrive at the desired Eq. (3.18).

APPENDIX B: RESIDUAL SYMMETRY AND CONFORMAL SYMMETRY

In this appendix we would like to explain that the residual symmetries found in Eq. (8.5) in a curved space-time reduce to a dilatational invariance and special conformal invariance in a flat Minkowski space-time.

Before doing so, let us first recall that a conformal transformation [15,16] can be defined as a general coordinate transformation that can be undone by a Weyl transformation when the space-time metric is the flat Minkowski one. With this definition, the conformal transformation is described by the equation

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = 2\Lambda(x)\eta_{\mu\nu}, \tag{B1}$$

where $\Lambda(x)$ is the infinitesimal transformation parameter of the Weyl transformation, i.e., $\Omega(x) \equiv e^{\Lambda(x)} \approx 1 + \Lambda(x)$.

Taking the trace of Eq. (B1) enables us to determine $\Lambda(x)$,

$$\Lambda = \frac{1}{4}\partial^\rho\epsilon_\rho. \tag{B2}$$

Inserting this Λ into Eq. (B1) yields

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{1}{2}\partial^\rho\epsilon_\rho\eta_{\mu\nu}, \tag{B3}$$

which is often called the "conformal Killing equation" in the Minkowski space-time. It is worth stressing that Eq. (B3) implies the following fact: the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ is invariant in the space of metric functions under a suitable combination of the general coordinate transformation and Weyl transformation in such a way that

$$\delta(\epsilon_\mu) = \delta_{\mathrm{GCT}}(\epsilon_\mu) - \delta_W\left(\Lambda = \frac{1}{4}\partial^\rho\epsilon_\rho\right) \tag{B4}$$

when the vector field $\epsilon_\mu(x)$ obeys the conformal Killing equation (B3). To put it differently, the characteristic feature of the theory under consideration is that the Lagrangian (3.1) possesses conformal symmetry with 15 global parameters, which is a subgroup of the general coordinate transformation and Weyl transformation.

Multiplying it by $\partial^\mu\partial^\nu$, we obtain

$$\Box\partial^\mu\epsilon_\mu = 0. \tag{B5}$$

Moreover, multiplying Eq. (B3) by $\partial^\mu\partial_\lambda$ and then symmetrizing the indices λ and ν leads to the equation

$$\partial_\nu\partial_\lambda\partial^\mu\epsilon_\mu = 0, \tag{B6}$$

where we have used Eqs. (B3) and (B5). It turns out that a general solution to Eq. (B6) reads

$$\epsilon^\mu = a^\mu + \omega^{\mu\nu}x_\nu + \lambda x^\mu + k^\mu x^2 - 2x^\mu k_\rho x^\rho, \tag{B7}$$

where $a^\mu, \omega^{\mu\nu} = -\omega^{\nu\mu}, \lambda$, and k^μ are all constant parameters that correspond to translation, Lorentz transformation, dilatation, and special conformal transformation, respectively.

At this point, it is useful to verify what form the infinitesimal parameter Λ generated by the "conformal Killing vector" ϵ^μ in Eq. (B7) takes. Actually, substituting Eq. (B7) into Eq. (B2), we have

$$\Lambda = \lambda - 2k_\mu x^\mu. \tag{B8}$$

This is nothing but the zero-mode solutions in Eq. (8.5). This result implies that finding the residual symmetries (8.5) amounts to solving the conformal Killing equation in a flat Minkowski space-time.

To summarize, we have explicitly shown that in our quantum gravity, Weyl symmetry—together with general coordinate invariance—generates conformal symmetry in a flat Minkowski background. This result is a quantum-mechanical generalization of the well-known Zumino theorem [19] which insists that theories that are invariant under both the general coordinate transformation and Weyl transformation (or local scale transformation) possess conformal symmetry in ae flat Minkowski background. Even if we used the Weyl-invariant classical Lagrangian (3.1), we think that the result obtained here would hold for any theories that are invariant under the GCT and Weyl transformation if we adopt the extended de Donder gauge and scalar gauge for these invariances.

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