Neveu-Schwarz-Ramond open superstring in the proper-time gauge: Free field theory

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We study the Neveu-Schwarz-Ramond (NSR) open superstring theory in the proper-time gauge. The string field action is obtained by evaluating the Polyakov string path integral. In this study, we focus on the open-string free-field action, which corresponds to the string path integral on a strip. Depending on the periodicity of the fermion fields, the open superstring has two sectors: The Neveu-Schwarz (NS) and Ramond (R) sectors. We can impose the gauge conditions to fix the (super) reparametrization invariance on the two-dimensional metric and its superpartner on the string world sheet to secure the covariance, in contrast to the light cone gauge condition. Accordingly, the proper-time emerges in the NS sector and both proper-time and its superpartner appear in the R-sector. Integration leads to free-string field actions in both sectors.

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I. INTRODUCTION

Quantum field theory describes the dynamics of nature in terms of the quantum fields of point particles. Since its introduction by Dirac [1], it has been the language in which one has attempted to understand fundamental forces. Quantum field theory flourished as quantum electrodynamics and quantum chromodynamics, which describes the strong interaction, and electroweak theory in the 1970s, which combines electromagnetism and weak interaction in a unified scheme.

Currently, we are confronting a new challenge of constructing quantum field theories of strings and replacing point particles with strings as fundamental objects. For the superstring, two approaches to construct quantum field theory are available: the light cone field theory of the NSR superstring [2–7] and Witten's superstring field theory based on the extended BRST symmetry [8]. Each approach helps us to understand the superstring dynamics and mechanism of the superstring field theory to a great extent; however, both approaches have limitations. The light cone superstring theory is not manifestly covariant and is plagued with various divergences [9–13]. Witten's superstring field theory also suffers from divergence owing to the midpoint contact interaction [14].

In this study, we propose a new covariant approach based on the Polyakov string path integral [15]. Because the Polyakov string path integral is well defined and finite, we expect that the divergence issues associated with the other two approaches may be resolved using this approach. When we evaluate the Polyakov string path integral on a strip of string worksheet, we can obtain a covariant field theoretical propagator if we impose the gauge condition to fix the reparametrization invariance. This approach has been applied to the bosonic string theory [16–22]. An important advantage of this approach is that it is easy to include higher interaction terms, and we can easily evaluate three -and four-string interactions to confirm local gauge invariance. The interacting NSR superstring in the proper-time gauge may be free of the notorious picture changing problem of conformal field theory formulation, if properly developed. As I put as the subtitle of the paper, "Free Field Theory," this paper will serve as a preliminary to a work [23] in this direction.

As a first step toward constructing a covariant interacting superstring theory, in the present study, we will focus on the free-field action of NSR superstrings. To obtain the free-field action, we evaluate the Polyakov string path integral on a strip with two spatial boundaries. By applying canonical quantization, we determine that the Hamiltonian only comprises constraints. To secure the covariance, we impose the gauge condition on the world sheet metric and its superpartner. Depending on the periodicity of the fermion fields, the NSR superstring has two sectors: periodic in the NS sector and antiperiodic in the Ramond sector. We demonstrate that the NS sector has a bosonic modular parameter that becomes the proper time, and the R-sector has both bosonic modular and fermionic supermodular parameters. After integrating both the modular and supermodular parameters, we obtain a Dirac propagator in the Ramond sector. In the NS sector, we obtain a Klein-Gordon

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type propagator, integrating the modular parameter (proper time).

II. CANONICAL QUANTIZATION OF NSR SUPERSTRING

The reparametrization-invariant and local supersymmetric actions for the NSR superstring are given by:

$$I = \int d^2\xi L,$$
$$L = \sqrt{-h}\left\{-\frac{1}{2}h^{\alpha\beta}\frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X_\mu}{\partial \xi^\beta} - \frac{i}{2}\bar{\psi}^\mu \gamma^a \frac{\partial \psi_\mu}{\partial \xi^\alpha} + \frac{1}{2}F^\mu F_\mu + \frac{1}{2}\bar{\chi}_\alpha \gamma^\beta \frac{\partial X^\mu}{\partial \xi^\beta}\gamma^\alpha \psi_\mu + \frac{1}{16}\bar{\psi}^\mu \psi_\mu \bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \chi_\beta\right\} \quad (1)$$

An auxiliary field F^μ is introduced to render the local supersymmetric algebra closed off-shell. The action is invariant under the reparametrization given by

$$\delta\xi = \zeta^a, \quad \delta X^\mu = \zeta^\alpha \partial_\alpha X^\mu, \quad \delta\psi^\mu = \zeta^\alpha \partial_\alpha \psi^\mu,$$
$$\delta e_\alpha{}^a = \zeta^\beta \partial_\beta e_\alpha{}^a + e_\beta{}^a \partial_\alpha \zeta^\beta, \quad \delta\xi_\alpha = \zeta^\beta \partial_\beta \chi_a + \chi_\beta \partial_\alpha \zeta^\beta \quad (2)$$

and local supersymmetric transformation given by

$$\delta X^\mu = i\epsilon\psi^\mu, \qquad \delta\psi^\mu = \gamma^\alpha\left(\partial_\alpha X^\mu - \frac{1}{2}\bar{\chi}_a \psi^\mu\right)\epsilon,$$
$$\delta e_\alpha{}^a = \epsilon\gamma^\alpha \chi_a, \qquad \delta\chi_a = 2iD_\alpha \epsilon \quad (3)$$

where the covariant derivative D_α is defined by

$$D_\alpha e = \partial_\alpha \epsilon - \frac{1}{2}\omega_\alpha \gamma^5 \epsilon, \quad (4)$$

with the connection

$$\omega_a = -\frac{1}{\epsilon}e_\alpha{}^a \epsilon^{\beta\gamma}\partial_\beta e_\gamma{}^b \eta_{ab} + \frac{1}{2}\bar{\chi}_\alpha \gamma^5 \gamma^\beta \chi_\beta. \quad (5)$$

In addition to the local supersymmetric transformation and reparametrization, the action is laso-invariant under conformal transformation.

$$\delta X^\mu = 0, \qquad \delta\psi^\mu = -\frac{1}{2}\epsilon\psi^\mu,$$
$$\delta h_{\alpha\beta} = 2\epsilon h_{\alpha\beta}, \qquad \delta\chi_a = \frac{1}{2}\epsilon\chi_\alpha \quad (6)$$

and the superconformal transformation

$$\delta X^\mu = 0, \quad \delta\psi^\mu = 0, \quad \delta h_{\alpha\beta} = 0, \quad \delta\chi^\alpha = \gamma^\alpha \epsilon. \quad (7)$$

At the critical dimensions, we set the conformal factor to $e^\phi = 1$ and fix the superconformal invariance, setting

$$\gamma^\alpha \chi_\alpha = 0. \quad (8)$$

To construct the Hamiltonian describing the dynamics of the string, an extended one-dimensional object, we express the metric $h_{\alpha\beta}$ in terms of the lapse and shift functions

$$h^{\alpha\beta} = \frac{1}{N_1}\begin{pmatrix} -1 & N_2 \\ N_2 & (N_1)^2 - (N_2)^2 \end{pmatrix}. \quad (9)$$

Accordingly, zweibein $e_\alpha{}^a$ is in terms of the lapse, and the shift functions are expressed as

$$(e_\alpha{}^a) = \frac{1}{\sqrt{N_1}}\begin{pmatrix} 1 & -N_2 \\ 0 & N_1 \end{pmatrix}. \quad (10)$$

Two orthogonal vectors on the world-sheet are given as

$$\boldsymbol{e}_{\hat{0}} = \frac{1}{\sqrt{N_1}}(\boldsymbol{\partial}_\tau - N_2 \boldsymbol{\partial}_\sigma)$$
$$\boldsymbol{e}_{\hat{1}} = \sqrt{N_1}\boldsymbol{\partial}_\sigma \quad (11)$$

The Hamiltonian can be obtained by taking the Legendre transformation of the Lagrangian and defining the canonical conjugates $(P^\mu, \Pi^\alpha, \zeta)$ to $(X_\mu, N_\alpha, \lambda = -\sqrt{2}N_1^{\frac{5}{4}}\bar{\chi}_1)$:

$$P^\mu = \frac{\partial L}{\partial \dot{X}_\mu}, \qquad \Pi^\alpha = \frac{\partial L}{\partial \dot{N}_a}, \qquad \zeta = \frac{\partial L}{\partial \dot{\lambda}}. \quad (12)$$

From the defining equations of the momenta, we obtain

$$P^\mu = \frac{1}{N_1}(\dot{X}^\mu - N_2 X'^\mu) + \bar{\chi}^0 \psi^\mu, \quad (13)$$

and the first class primary constraints

$$\Pi_\alpha = 0, \qquad \zeta = 0. \quad (14)$$

With some algebra, we determine

$$\bar{H} = P^\mu \dot{X}_\mu - L$$
$$= \frac{N_1}{2}(P^2 + X'^2) + N_2 P^\mu X'_\mu + \frac{\lambda}{\sqrt{2}}(\gamma^5 P^\mu + X'^\mu)N_1^{-\frac{1}{4}}\psi_\mu + \frac{1}{2}\bar{\psi}^\mu \gamma^\alpha \frac{\partial}{\partial \xi^\alpha}\psi_\mu. \quad (15)$$

Here, we adopt a simple two- dimensional γ matrix algebra $\gamma^\alpha \gamma^\beta \gamma_\alpha = 0$, the Fierz rearrangement

$$\bar{\psi}_a \psi_2 \psi_3 = -\frac{1}{2}\sum_i \bar{\psi}_1 \Gamma_i \psi_3 \Gamma^i \psi_2, \qquad \Gamma_i = 1, \gamma_a, \gamma_5. \quad (16)$$

We need to scale $\psi \to N_1^{-\frac{1}{4}}\psi^\mu$ to express the fermion in canonical form:

$$H = \frac{N_1}{2}(P^2 + X'^2 + i\psi^\mu\gamma^5\psi'_\mu) + N_2\left(P^\mu X'_\mu + \frac{i}{2}\psi^\mu\psi'_\mu\right) + \frac{\lambda}{\sqrt{2}}(\gamma^5 P^\mu + X'^\mu)\psi_\mu - v_\alpha\Pi^\alpha - u\zeta \tag{17}$$

where Lagrangian multipliers v_α and u are introduced to enforce the primary constraints. The primary constraints $\Pi_\alpha = 0$ and $\zeta = 0$ lead to the secondary constraints given as follows:

$$\begin{aligned} P^2 + X'^2 + i\psi^\mu\gamma^5\psi'_\mu &= 0, \\ P^\mu X'_\mu + \frac{i}{2}\psi^\mu\psi'_\mu &= 0, \\ \frac{1}{\sqrt{2}}(\gamma^5 P^\mu + X'^\mu)\psi_\mu &= 0. \end{aligned} \tag{18}$$

III. NSR OPEN SUPERSTRING

We may extend the domain of σ, initially defined as $[0,\pi]$ for the open superstring, to $[-\pi,\pi]$, such that the field variables for the open string and those for the closed superstring are defined in the same domain. Furthermore, for the open superstring, we impose the following folding conditions on the dynamical variables:

$$\begin{aligned} X^\mu(\sigma) &= X^\mu(-\sigma), & P^\mu(\sigma) &= p^\mu(-\sigma), \\ N_1(\sigma) &= N_1(-\sigma), & N_2(\sigma) &= -N_2(-\sigma), \\ \Pi_1(\sigma) &= \Pi_1(-\sigma), & \Pi_2(\sigma) &= -\Pi_2(-\sigma), \\ \psi^{\mu+}(\sigma) &= \psi^{\mu-}(-\sigma), & \lambda^{\mu+}(\sigma) &= -\lambda^{\mu-}(-\sigma) \end{aligned} \tag{19}$$

The conditions are also read in terms of normal modes as follows

$$X^\mu_n = X^\mu_{-n}, \quad P^\mu_n = P^\mu_{-n}, \quad N^1_n = N^1_{-n}, \quad N^2_n = -N^2_{-n} \tag{20}$$

and

$$\begin{cases} \psi^{\mu+}_n = \psi^{\mu-}_n, \quad \lambda^+_n = -\lambda^-_{-n} & \text{for the Ramond sector,} \\ \psi^{\mu+}_{n+\frac{1}{2}} = \psi^{\mu-}_{n+\frac{1}{2}}, \quad \lambda^+_{n+\frac{1}{2}} = -\lambda^-_{-n-\frac{1}{2}} & \text{for the Neuveu-Schwarz sector.} \end{cases} \tag{21}$$

With these conditions, we can write action S in the Ramond sector as follows:

$$S = \int_{\tau_i}^{\tau_f} d\tau \sum_n \left\{ P^\mu_n \dot{X}_{\mu n} + \frac{i}{2}\psi^\mu_{-n}\dot{\psi}_{\mu n} + \bar{\Pi}_n \dot{\bar{N}}_n + \bar{\zeta}_n \dot{\bar{\lambda}}_n - \bar{N}_n L^R_n - \bar{\lambda}_n F_n + v_n \bar{\Pi}_n + u_n \bar{\zeta}_n \right\}, \tag{22}$$

where in the Ramond sector

$$\begin{aligned} &\bar{N}_n = \frac{1}{2}(N_{1n} + N_{2n}), \quad \bar{N}_0 = N_1 0, \quad \bar{\Pi}_n = \frac{1}{2}(\Pi_{1n} + \Pi_{2n}), \\ &\bar{\Pi}_0 = \Pi_{10}, \quad n \neq 0, \\ &\bar{\lambda}_n = \lambda^+_n + \lambda^-_n, \quad \bar{\zeta}_n = \zeta^+_n + \zeta^-_n, \end{aligned} \tag{23}$$

and in the Neveu-Schwarz sector

$$S = \int_{\tau_i}^{\tau_f} d\tau \sum_n \left\{ P^\mu_n \dot{X}_{\mu n} + \frac{i}{2}\psi^\mu_{-n-\frac{1}{2}}\dot{\psi}_{\mu n+\frac{1}{2}} + \bar{\Pi}_n \dot{\bar{N}}_n + \bar{\zeta}_{n+\frac{1}{2}} \dot{\bar{\lambda}} 1_{n+\frac{1}{2}} - \bar{N}_n L^R_n - \bar{\lambda}_{n+\frac{1}{2}} G_{n+\frac{1}{2}} + v_n \bar{\Pi}_n + u_{n+\frac{1}{2}} \bar{\zeta}_{n+\frac{1}{2}} \right\}, \tag{24}$$

with

$$\begin{aligned} &\bar{N}_n = \frac{1}{2}(N_{1n} + N_{2n}), \qquad \bar{N}_0 = N_1 0, \\ &\bar{\Pi}_n = \frac{1}{2}(\Pi_{1n} + \Pi_{2n}), \qquad \bar{\Pi}_0 = \Pi_{10}, \qquad n \neq 0, \\ &\bar{\lambda}_{n+\frac{1}{2}} = \lambda^+_{n+\frac{1}{2}} + \lambda^-_{n+\frac{1}{2}}, \qquad \bar{\zeta}_{n+\frac{1}{2}} = \zeta^+_{n+\frac{1}{2}} + \zeta^-_{n+\frac{1}{2}}, \end{aligned} \tag{25}$$

IV. COVARIANT PROPER-TIME GAUGE CONDITION FOR THE NSR OPEN SUPERSTRING

A. Neveu-Schwarz Sector

In the Neveu-Schwarz sector, the secondary constraints L^R_n, $G_{n+\frac{1}{2}}$ form the super-Virasoro algebra with the central charge

$$\begin{aligned} &[L^{\rm NS}_n, L^{\rm NS}_m] = (n-m)L^{\rm NS}_{n+m} + \frac{d}{8}n(n^2-1)\delta(n+m), \\ &[L^{\rm LS}_n, G_{m+\frac{1}{2}}] = \left(\frac{n}{2} - n - \frac{1}{2}\right)G_{n+m+\frac{1}{2}}, \\ &[G_{n+\frac{1}{2}}, G_{m-\frac{1}{2}}] = 2L^{\rm NS}_{n+m} + \frac{d}{2}n(n+1)\delta(n+m). \end{aligned} \tag{26}$$

The canonical generator that generates the reparametrization and local supersymmetric transformation is constructed to be in the NS sector.

$$\Omega_{\rm NS}(\epsilon,\omega) = \dot{\epsilon}_n \bar{P} i_n + \epsilon_n \Big\{ L_n^{\rm NS} - i(2n-j)\bar{N}_{j-n}\bar{\Pi}_j + i\left(\frac{3}{2}n - j - \frac{1}{2}\right)\bar{\lambda}_{j-n+\frac{1}{2}}\bar{\zeta}_{j+\frac{1}{2}} \Big\} + \dot{\omega}_{n+\frac{1}{2}}\bar{\zeta}_{n+\frac{1}{2}} - \omega_{n+\frac{1}{2}}\Big\{ G_{n+\frac{1}{2}} + 2i\bar{\lambda}_{j-n-\frac{1}{2}}\bar{\Pi}_j - \frac{i}{2}(3n-j+1)\bar{N}_{j-n}\bar{\zeta}_{j+\frac{1}{2}} \Big\} \tag{27}$$

To secure the Lorentz covariance, it is desirable to impose gauge fixing conditions on the Lagrangian multipliers $\bar{N}_n$ and $\bar{\lambda}_{n+\frac{1}{2}}$. The Lagrangian multipliers transform the gauge transformations by

$$\begin{aligned} \delta\bar{N}_n &= \dot{\epsilon}_n + i(n-2m)\epsilon_m \bar{N}_{n-m} - 2i\omega_{m+\frac{1}{2}}\bar{\lambda}_{n-m-\frac{1}{2}}, \\ \delta\bar{\lambda}_{n+\frac{1}{2}} &= -\dot{\omega}_{n+\frac{1}{2}} + i\left(n - \frac{3}{2}n + \frac{1}{2}\right)\epsilon_m \bar{\lambda}_{n-m+\frac{1}{2}} - \frac{i}{2}(n-3m-1)\omega_{m+\frac{1}{2}}\bar{N}_{n-m}. \end{aligned} \tag{28}$$

We may choose the covariant gauge condition for the NSR open superstring in the Neveu-Schwarz sector by

$$\bar{N}_{n(\neq 0)} = 0, \qquad \dot{\bar{N}}_0 = 0, \qquad \bar{\lambda}_{n+\frac{1}{2}} = 0. \tag{29}$$

We can confirm that this proper-time gauge fixes the gauge degrees of freedom associated with the reparametrization and local supersymmetry completely and consistently. Near the gauge-fixing hypersurface defined by Eq. (29), the infinitesimal gauge transformation that restores the gauge-fixing condition along the gauge orbits is determined by

$$\begin{aligned} &\dot{\epsilon}_n - in\epsilon_n \bar{n} + \bar{N}_n = 0, \qquad n \neq 0, \qquad \ddot{\epsilon}_0 + \bar{N}_0 = 0, \\ &\dot{\omega}_{n+\frac{1}{2}} - \left(n + \frac{1}{2}\right)\omega_{n+\frac{1}{2}}\bar{n} - \bar{\lambda}_{n+\frac{1}{2}} = 0 \end{aligned} \tag{30}$$

where $\bar{n}$ denotes constant $\bar{N}_0$ in the covariant gauge. This linear differential equation for ϵ_n and $\omega_{n+\frac{1}{2}}$ has a unique solution:

$$\begin{aligned} \epsilon_n(\tau) &= e^{in\bar{n}\tau}\left\{ -\int_{\tau_i}^{\tau} e^{-in\bar{n}\tau'}\bar{N}_n dt' + c_n \right\}, \qquad n \neq 0, \\ \epsilon_0(\tau) &= \left\{ \frac{\tau-\tau_i)}{T}\int_{\tau}^{\tau_f} + \frac{\tau-\tau_f)}{T}\int_{\tau_i}^{\tau} \right\}\bar{N}_0 d\tau', \qquad T = \tau_f - \tau_i \\ \omega_{n+\frac{1}{2}} &= e^{(n+\frac{1}{2})\bar{n}\tau}\left\{ \int_{\tau_i}^{\tau} e^{-i(n+\frac{1}{2})\bar{n}\tau'}\bar{\lambda}_{n+\frac{1}{2}} d\tau' + d_{n+\frac{1}{2}} \right\} \end{aligned} \tag{31}$$

where

$$\begin{aligned} c_n &= (1 - e^{-2in\bar{n}T})^{-1}\left\{ \int_{\tau_i}^{\tau_f} e^{-in\bar{n}\tau'}\bar{N}_n d\tau' + e^{-2in\bar{n}\tau_f}\int_{\tau_i}^{\tau_f} e^{in\bar{n}\tau'}\bar{N}_{-n} d\tau' \right\} \\ d_{n+\frac{1}{2}} &= (1 - e^{-2i(n+\frac{1}{2})\bar{n}T})^{-1} \times \left\{ e^{-2i(n+\frac{1}{2})\bar{n}\tau_f}\int_{\tau_i}^{\tau_f}, e^{i(n+\frac{1}{2})\bar{n}\tau'}\bar{\lambda}_{-n-\frac{1}{2}} d\tau' - \int_{\tau_i}^{\tau_f} e^{-i(n+\frac{1}{2})\bar{n}\tau'}\bar{\lambda}_{n+\frac{1}{2}} d\tau' \right\}. \end{aligned} \tag{32}$$

B. Ramond sector

In the Ramond sector, the constraint operators form a super-Virasoro algebra given by

$$\begin{aligned} [L_n^R, L_m^R] &= (n-m)L_{n+m}^R + \frac{d}{8}n^3\delta(n+m), \\ [L_n^R, F_m] &= \left(\frac{n}{2} - m\right)F_{n+m}, \\ [F_n, F_m] &= 2L_{n+m}^R + \frac{d}{2}n^2\delta(n+m). \end{aligned} \tag{33}$$

The canonical generator of symmetric transformation is obtained in the Ramond sector as

$$\Omega_R(\epsilon,\omega) = \dot{\epsilon}_n \bar{\Pi}_n + \epsilon_n \Big\{ L_n^R - i(2n-j)\bar{N}_{j-n}\bar{\Pi}_j + i\left(\frac{3n}{2} - i\right)\bar{\lambda}_{j-n}\bar{\zeta}_j \Big\} + \dot{\omega}_n \bar{\zeta}_n - \omega_n \Big\{ F_n + 2i\bar{\lambda}_{j-n}\bar{\Pi}_j + i\left(\frac{3n}{2} - \frac{j}{2}\right)\bar{N}_{j-n}\bar{\zeta}_j \Big\}. \tag{34}$$

We observe that the Lagrangian multipliers, $\bar{N}_n$ and $\bar{\lambda}_n$, transform under the representation and local supersymmetric transformation by

$$\begin{aligned} \delta\bar{N}_n &= \dot{\epsilon}_n + i(n-2m)\epsilon_m \bar{N}_{n-m} - 2i\omega_m \bar{\lambda}_{n-m} \\ \delta\bar{\lambda}_n &= -\dot{\omega}_n + i\left(n - \frac{3m}{2}\right)\epsilon_m \bar{\lambda}_{n-m} - \frac{i}{2}(n-3m)\omega_m \bar{N}_{n-m}. \end{aligned} \tag{35}$$

We choose the covariant gauge condition in the Ramond sector by

$$\bar{N}_n = 0, \qquad \dot{\bar{N}}_0 = 0, \qquad \bar{\lambda}_n = 0, \qquad \dot{\bar{\lambda}}_0 = 0, \qquad n \neq 0. \tag{36}$$

The covariant gauge condition in Eq. (36) properly fixes gauge symmetry, we examine the equation for infinitesimal gauge parameters that restore the gauge-fixing condition along the gauge orbits near gauge fixing hypersurface by

$$\begin{aligned} \dot{\epsilon}_n - in\epsilon_n\bar{n} - 2i\omega_n\nu + \bar{N}_n &= 0, \\ \dot{\omega}_n + \frac{n}{2}i\epsilon_n\nu - in\omega_n\bar{n} - \bar{\lambda}_n &= 0, \\ \ddot{\epsilon}_0 - 2i\dot{\omega}_0\nu + \dot{\bar{N}}_0 &= 0, \\ \ddot{\omega}_0 - \dot{\bar{\lambda}}_0 &= 0, \end{aligned}$$

where $n \neq 0$,$\bar{n}$, and ν denote $\bar{N}_0$ and $\bar{\lambda}_0$, respectively. The existence of a unique solution to Eqs. (37) ensures that the chosen covariant gauge condition is complete and consistent. With some algebra, we explicitly determine a unique solution:

$$\begin{aligned} \epsilon_n(\tau) &= e^{in\bar{n}\tau}\left\{-\int_{\tau_i}^{\tau} e^{-in\bar{n}\tau'}(\bar{N}_n - 2i\omega_n\nu)d\tau' + c_n\right\}, \\ \epsilon_0(\tau) &= \left\{\frac{(\tau-\tau_i)}{T}\int_{\tau}^{\tau_f} + \frac{(\tau-\tau_f)}{T}\int_{\tau_f}^{\tau}\right\}(\bar{N}_0 - 2i\omega_0\nu)d\tau', \\ \omega_n(\tau) &= e^{in\bar{n}\tau}\left\{\int_{\tau_i}^{\tau} e^{-in\bar{n}\tau'}\left(\bar{\lambda}_n - \frac{n}{2}i\epsilon_n\nu\right)d\tau' + d_n\right\} \\ \omega_0(\tau) &= \left\{\frac{(\tau_i-\tau)}{T}\int_{\tau}^{\tau_f} + \frac{(\tau_f-\tau)}{T}\int_{\tau_i}^{\tau}\right\}\bar{\lambda}_0 d\tau', \quad n \neq 0. \end{aligned} \tag{37}$$

where

$$\begin{aligned} c_n &= (1-e^{-2in\bar{n}T})^{-1}\left\{\int_{\tau_i}^{\tau_f} e^{-in\bar{n}\tau'}(\bar{N}_n - 2i\omega_n\nu)d\tau' \right. \\ &\quad \left. + e^{-2in\bar{n}\tau_f}\int_{\tau_i}^{\tau_f} e^{in\bar{n}\tau'}(\bar{N}_n - 2i\omega_{-n}\nu)d\tau'\right\} \\ d_n &= (1-e^{-2in\bar{n}T})^{-1}\left\{e^{-2in\bar{n}\tau_f}\int_{\tau_i}^{\tau_f} e^{in\bar{n}\tau'}\left(\bar{\lambda}_{-n} + \frac{n}{2}i\epsilon_{-n}\nu\right)d\tau' \right. \\ &\quad \left. - \int_{\tau_i}^{\tau_f} e^{-in\bar{n}\tau'}\left(\bar{\lambda}_n - \frac{n}{2}i\omega_n\nu\right)d\tau'\right\} \end{aligned} \tag{38}$$

In contrast to the Neveu-Schwarz sector, we cannot gauge away the fermionic zero mode of the Lagrangian multipliers $\bar{\lambda}_0$ completely. This leads to the main difference between the two sectors is that in the Ramond sector, modular and supermodular parameters exist, while only modular parameters exist in the Neveu-Schwarz sector.

V. THE OFF-SHELL PROPAGATOR FOR THE NSR OPEN SUPERSTRING

In this section, we apply BRST quantization to the NSR open superstring. The path integral adopted to represent the off-shell propagator is evaluated explicitly.

A. Neveu-Schwarz sector

First, we construct the BRST generator Q_{NS} with the given structure constants in Eq. (26), introducing the fermionic ghost variables η_n, $\bar{\eta}_n$, ξ_n, and the bosonic ghost variables $\beta_{n+\frac{1}{2}}, \bar{\beta}_{n+\frac{1}{2}}, \gamma_{n+\frac{1}{2}}$

$$\begin{aligned} Q_{\rm NS} &= \eta_n L_n^{\rm NS} + \bar{\eta}_n\bar{\Pi}_n + \beta_{n+\frac{1}{2}}G_{n+\frac{1}{2}} + \bar{\beta}_{n+\frac{1}{2}}\bar{\zeta}_{n+\frac{1}{2}} \\ &\quad - \frac{1}{2}(n-m)\eta_n\eta_m\xi_{n+m}, \\ &\quad + \left(\frac{n}{2} - m - \frac{1}{2}\right)\eta_n\beta_{m+\frac{1}{2}}\gamma_{n+m+\frac{1}{2}} - \beta_{n+\frac{1}{2}}\beta_{m-\frac{1}{2}}\xi_{n+m}, \end{aligned} \tag{39}$$

(the BRST ghost variables η_n and ξ_n may be identified in terms of the usual bc ghost variables as $\eta_n = c_{-n}$, and $\xi_n = b_n$.)

Second, the BRST invariant effective action is constructed as

$$\begin{aligned} S = \int_{\tau_i}^{\tau_f} d\tau\Big\{ & P^\mu\dot{X}_{\mu n} + \frac{i}{2}\psi^\mu_{-n-\frac{1}{2}}\dot{\psi}_{\mu(n+\frac{1}{2})} + \bar{\Pi}_n\dot{\bar{N}}_n + \bar{\zeta}_{n+\frac{1}{2}}\dot{\bar{\lambda}}_{n+\frac{1}{2}} \\ & + i\xi_n\dot{\eta}_n + i\bar{\xi}_n\dot{\bar{\eta}}_n, + i\gamma_{n+\frac{1}{2}}\dot{\beta}_{n+\frac{1}{2}} + i\bar{\gamma}_{n+\frac{1}{2}}\dot{\bar{\beta}}_{n+\frac{1}{2}} \\ & - i[Q_{NS}, \Delta]\Big\}, \end{aligned} \tag{40}$$

Here we choose Δ by

$$\Delta = \xi_n\bar{N}_n + \bar{\eta}_n\chi_n - \gamma_{n+\frac{1}{2}}\bar{\lambda}_{n+\frac{1}{2}} - \bar{\gamma}_{n+\frac{1}{2}}f_{n+\frac{1}{2}}, \tag{41}$$

with

$$\chi_n = \frac{1}{\alpha}\bar{N}_n, \quad (n \neq 0), \quad \chi_0 = 0, \quad f_{n+\frac{1}{2}} = \frac{1}{\alpha}\bar{\lambda}_{n+\frac{1}{2}} \tag{42}$$

to produce covariant gauge conditions at limit $\alpha \to 0$. Scaling of dynamic variables

$$\begin{aligned} (\bar{\Pi}_n, \bar{\eta}_n) &\to \alpha(\bar{\Pi}_n, \bar{\eta}_n), n \neq 0, \\ \left(\bar{\zeta}_{n+\frac{1}{2}}, \bar{\beta}_{n+\frac{1}{2}}\right) &\to \alpha\left(\bar{\zeta}_{n+\frac{1}{2}}, \bar{\beta}_{n+\frac{1}{2}}\right), \end{aligned} \tag{43}$$

we find, $\alpha \to 0$ in the limit,

$$S = \int_{\tau_i}^{\tau_f} d\tau \Bigg\{ \sum_n \left(P_n^\mu \dot{X}_{\mu n} + \frac{i}{2}\psi^\mu_{-n-\frac{1}{2}}\dot{\psi}_{\mu n+\frac{1}{2}} + i\xi_n\dot{\eta}_n + i\gamma_{n+\frac{1}{2}}\dot{\beta}_{n+\frac{1}{2}} \right) + \sum_n \left(\bar{N}_n L_n^{NS} - \dot{\bar{\lambda}}_{n+\frac{1}{2}} G_{n+\frac{1}{2}} \right)$$
$$- \sum_{n,m} \left((n-m)\eta_n \bar{N}_m \xi_{n+m}, + \left(\frac{n}{2} - m - \frac{1}{2}\right)\bar{N}_n \beta_{m+\frac{1}{2}}\gamma_{n+m+\frac{1}{2}} + \left(\frac{n}{2} - m - \frac{1}{2}\right)\eta_n \bar{\lambda}_{m+\frac{1}{2}}\gamma_{n+m+\frac{1}{2}} + 2\beta_{n+\frac{1}{2}}\bar{\lambda}_{m-\frac{1}{2}}\xi_{n+m} \right)$$
$$- \sum_n \left(\bar{\zeta}_{n+\frac{1}{2}}\bar{\lambda}_{n+\frac{1}{2}} + i\bar{\beta}_{n+\frac{1}{2}}\bar{\gamma}_{n+\frac{1}{2}} \right), - {\sum_n}' (\bar{\Pi}_n N_n - i\bar{\eta}_n \xi_n) + \bar{\Pi}_0 \dot{\bar{N}}_0 + i\bar{\xi}\dot{\bar{\eta}}_0 - i\xi_0\bar{\eta}_0 \Bigg\} \quad (44)$$

We integrate the conjugates of Lagrangian multipliers $\bar{\Pi}_n$ and $\bar{\zeta}_{n+\frac{1}{2}}$, and obtain the action parametrized by modular parameter $\bar{n}$. Ghost variables $(\bar{\eta}_n, \bar{\xi}_n)$, $n \neq 0$, and $(\bar{\beta}_{n+\frac{1}{2}}, \bar{\gamma}_{n+\frac{1}{2}})$ can be trivially integrated. By further integrating over the ghost zero modes (η_0, ξ_0) and $(\bar{\eta}_0, \bar{\xi}_0)$, we obtain the factor

$$Det[\partial_\tau^2] = T. \quad (45)$$

Finally, we obtain the resultant action as

$$S = \int_{\tau_i}^{\tau_f} d\tau \Bigg\{ \sum_n \left(P_n^\mu \dot{X}_{\mu n} + \frac{i}{2}\psi^\mu_{-n-\frac{1}{2}}\dot{\psi}_{\mu n+\frac{1}{2}} + i\gamma_{n+\frac{1}{2}}\dot{\beta}_{n+\frac{1}{2}} \right)$$
$$+ i{\sum_n}' \xi_n \dot{\eta}_n$$
$$- \bar{n}\left(L_0^{NS} + \sum_n \left(n\eta_n\xi_n - \left(n + \frac{1}{2}\right)\beta_{n+\frac{1}{2}}\gamma_{n+\frac{1}{2}} \right) \right) \Bigg\}, \quad (46)$$

That is the off-shell propagator for the Neveu-Schwarz sector

$$G_{\rm NS} = T \int d\bar{n} \int D[X, P] D[\psi] D[\eta, \xi] D[\beta, \gamma] e^{iS}, \quad (47)$$

We observe that the path integral in Eq. (47), representing the off-shell propagator, is a typical path integral representation for the transition matrix element with Hamiltonian H:

$$H = L_0^{\rm NS} + L_{\rm gh}^{\rm NS},$$
$$L_0^{\rm NS} = \frac{1}{2}\sum_n (P_n^2 + n^2 X_n^2) + \sum_{n \geq 0} \left(n + \frac{1}{2}\right)\left(\psi^\mu_{-n-\frac{1}{2}}\psi_{\mu n+\frac{1}{2}}\right),$$
$$L_{\rm gh}^{\rm NS} = \sum_n \left(n\eta_n\xi_n - \left(n + \frac{1}{2}\right)\beta_{n+\frac{1}{2}}\gamma_{n+\frac{1}{2}} \right). \quad (48)$$

Defining proper time $s = T\bar{n}$, we obtain the off-shell propagator in the NS sector as follows:

$$G_{\rm NS} = \int_0^\infty ds \langle X_f, \psi_f, \eta_f, \beta_f | \times \exp\{, -s(L_0^{\rm NS} + L_{\rm gh}^{\rm NS})\} | X_i, \psi_i, \eta_i, \beta_i \rangle$$
$$= \langle X_f, \psi_f, \eta_f, \beta_f | \frac{1}{L_0^{\rm NS} + L_{\rm gh}^{\rm NS} - i\epsilon}, | X_i, \psi_i, \eta_i, \beta_i \rangle \quad (49)$$

B. Ramond sector

We can construct the BRST generator Q_R for the Ramond sector using the structural constants given in Eq. (33). Similarly,

$$Q_R = \eta_n L_n^R + \bar{\eta}_n \bar{\Pi}_n + \beta_n F_n + \bar{\beta}_n \bar{\zeta}_n - \frac{1}{2}(n-m)\eta_n\eta_m\xi_{n+m}$$
$$+ \left(\frac{n}{2} - m\right)\eta_n\beta_m\gamma_{n+m} - \beta_n\beta_m\xi_{n+m} \quad (50)$$

The BRST invariant action may be written as

$$S = \int_{\tau_i}^{\tau_f} d\tau \Bigg\{ \sum_n \left(P_n^\mu \dot{X}_{\mu n} + \frac{i}{2}\psi^\mu_{-n}\dot{\psi}_{\mu n} + i\xi_n\dot{\eta}_n + i\gamma_n\dot{\beta}_n \right) + \bar{\Pi}_0 \dot{\bar{N}}_0 + \bar{\zeta}_0 \dot{\bar{\lambda}}_0$$
$$+ i\bar{\xi}_0\dot{\bar{\eta}}_0 + i\bar{\gamma}_0\dot{\bar{\beta}}_0 - \sum_n (\bar{N}_n L_n^R + \bar{\lambda}_n F_n) + \sum_{n,m} \Bigg((n-m)\bar{N}_n\eta_m\xi_{n+m}$$
$$- \left(\frac{n}{2} - m\right)\bar{N}_n\beta_m\gamma_{n+m} + \left(n - \frac{m}{2}\right)\bar{\lambda}_n\eta_m\gamma_{n+m} + 2\bar{\lambda}_n\beta_m\xi_{n+m} \Bigg)$$
$$- {\sum_n}' (\bar{\Pi}_n\bar{N}_n - \bar{\zeta}_n\bar{\lambda}_n - i\bar{\eta}_n\bar{\xi}_n - i\beta_n\bar{\gamma}_n) \Bigg\}, \quad (51)$$

We integrate $\bar{\pi}_n$ and $\bar{\zeta}_n$, and as a result, obtain the action parametrized by the modular and supermodular parameters in the covariant gauge

$$\int d\bar{n} d\nu e^{iS}. \tag{52}$$

Integrations over the ghost variables $(\bar{\eta}_n, \bar{\xi}_n)$ $n \neq 0$ and $(\bar{\beta}_n, \bar{\gamma}_n)$, $n \neq 0$ are trivial. We perform further integrations over the ghost zero modes (η_0, ξ_0), (β_0, γ_0), $(\bar{\eta}_0, \bar{\xi}_0)$, and $(\bar{\beta}_0, \bar{\gamma}_0)$ which do not induce a nontrivial factor in the measure, however, in contrast to the Neveu-Schwarz sector.

Furthermore, we reach the point where the off-shell propagator for the Ramond sector is explicitly evaluated.

$$\begin{aligned} S = \int_{\tau_i}^{\tau_f} d\tau \Bigg\{ & \sum_n \left(P_n^\mu \dot{X}_{\mu n} + \frac{i}{2} \psi_{-n}^\mu \dot{\psi}_{\mu n} \right) \\ & + i {\sum_n}' (\xi_n \dot{\eta}_n + \gamma_n \dot{\beta}_n) \\ & - \bar{n} \left(L_0^R + {\sum_n}' n (\eta_n \xi_n - \beta_n \gamma_n) \right) \\ & - \nu \left(F_0 - {\sum_n}' \left(\frac{n}{2} \eta_n \gamma_n + 2 \beta_n \xi_n \right) \right) \Bigg\}, \end{aligned} \tag{53}$$

after ghost zero modes were integrated. The precise definition of the path integral representation of the off-shell propagator in the Ramond sector is as follows:

$$\begin{aligned} & G_R(X_f, \psi_f, \eta_f, \beta_f; X_i, \psi_i, \eta_i, \beta_i) \\ & \quad = \int d\bar{n} d\nu \int D[X, P] D[\psi] D[\eta, \xi] D[\beta, \gamma] e^{i\bar{S}}. \end{aligned} \tag{54}$$

Defining the proper time $s = T\bar{n}$ and its supersymmetric counterpart $\varrho = T\nu$, we rewrite the path integral as

$$\begin{aligned} & G_R(X_f, \psi_f, \eta_f, \beta_f; X_i, \psi_i, \eta_i, \beta_i) \\ & \quad = \int_0^\infty ds \int d\varrho \langle X_f, \psi_f, \eta_f, \beta_f | \exp\{-is(L_0^R + L_{gh}^R) \\ & \qquad - i\varrho (F_0 + F_{gh})\} | X_i, \psi_i, \eta_i, \beta_i \rangle, \end{aligned} \tag{55}$$

We define

$$\begin{aligned} L_0^R &= \frac{1}{2} \sum_n (P_n^2 + n^2 X_n^2) + \frac{1}{2} \sum_{n>0} n (\psi_{-n}^m \psi_{\mu n} - \psi_n^\mu \psi_{\mu(-n)}), \\ L_{\mathrm{gh}}^R &= {\sum_n}' n (\eta_n \xi_n - \beta_n \gamma_n), \\ F_0 &= \sum_{n \geq 0} \frac{1}{\sqrt{2}} (P_n^\mu + inX_n^\mu) \psi_{\mu n} + \sum_{n>0} \frac{1}{\sqrt{2}} (P_n^\mu - inX_n^\mu) \psi_{\mu(-n)}, \\ F_{\mathrm{gh}} &= {\sum_n}' \left(\frac{n}{2} \eta_n \gamma_n - 2 \beta_n \xi_n \right). \end{aligned} \tag{56}$$

Observing that

$$[L_0^R + L_{\mathrm{gh}}^R, F_0 + F_{\mathrm{gh}}] = 0, \tag{57}$$

and

$$(F_0 + F_{\mathrm{gh}})^2 = L_0^R + L_{\mathrm{gh}}^R, \tag{58}$$

we can simply perform integration over the modular and supermodular parameters

$$\begin{aligned} & G_R(X_f, \psi_f, \eta_f, \beta_f; X_i, \psi_i, \eta_i, \beta_i) \\ & \quad = \langle X_f, \psi_f, \eta_f, \beta_f | \frac{F_0 + F_{gh}}{L_0^R + L_{gh}^R - i\epsilon} | X_i, \psi_i, \eta_i, \beta_i \rangle \\ & \quad = \langle X_f, \psi_f, \eta_f, \beta_f | \frac{1}{F_0 + F_{gh}} | X_i, \psi_i, \eta_i, \beta_i \rangle. \end{aligned} \tag{59}$$

VI. FREE FIELD ACTION FOR THE NSR OPEN SUPERSTRING

The expressions of the propagators obtained in the previous sections reveal the structure of the free-field actions for the NSR open superstring. The free-field action is constructed such that the off-shell propagators can be deduced in terms of the action through the quantum field theoretical expression. The free-string field action is given by:

$$\begin{aligned} S = & \int D[X, \psi, \eta, \beta]_{\mathrm{NS}} \Phi (L_0^{\mathrm{NS}} + L_{\mathrm{gh}}^{\mathrm{NS}} - i\epsilon) \Phi +, \\ & \int D[X, \psi, \eta, \beta]_R \Psi (F_0 + F_{\mathrm{gh}}) \Psi. \end{aligned} \tag{60}$$

We discuss the structure of action S in detail. We may define the "creation" and "annihilation" operators

$$\begin{aligned} \psi_j^{\mu\dagger} &= \psi_{-j}^\mu, \qquad r_j^\dagger = \beta_j, \qquad r_j = -\gamma_j, \\ s_j^\dagger &= \gamma_{-j}, \qquad s_j = \beta_{-j} \end{aligned} \tag{61}$$

where j can appropriately be a positive integer or half-integer. They satisfy the commutation relations of harmonic oscillators or anticommutation relations of Grassman harmonic oscillators:

$$[\psi_i^\mu, \psi_j^{\dagger\nu}]_+ = \eta^{\mu\nu} \delta_{ij}, \quad [r_i, r_j^\dagger] = \delta_{ij}, \quad [s_i, s_j^\dagger] = \delta_{ij} \tag{62}$$

The structure of field action becomes transparent in terms of creation and annihilation operators. In the Neveu-Schwarz sector, the kinetic operator is given as

$$L_0^{\rm NS} = p^2 + N^{\rm NS} - \frac{d}{16}$$
$$= p^2 + \sum_{n>0} n a_n^{\dagger\mu} a_{\mu n} + \sum_{n\geq 0}\left(n+\frac{1}{2}\right)\psi^{\dagger\mu}_{n+\frac{1}{2}}\psi_{\mu n+\frac{1}{2}} - \frac{d}{16}.$$
$$L_{\rm gh}^{\rm NS} = N_{\rm gh}^{\rm NS} + \frac{1}{8}$$
$$= \sum_{n>0}\sum_{i=1}^{2} n a^{\dagger i}_{{\rm gh}^n} a^i_{{\rm gh}^n}$$
$$+ \sum_{n\geq 0}\left(n+\frac{1}{2}\right)\left(r^\dagger_{n+\frac{1}{2}} r_{n+\frac{1}{2}} + s^\dagger_{n+\frac{1}{2}} s_{n+\frac{1}{2}}\right), \quad (63)$$

where $a^1_{{\rm gh}^n}, a^2_{{\rm gh}^n}$ may be written in terms of the usual bc ghost variables as

$$b_{zz}(\sigma) = \frac{b_0}{2} + \frac{1}{2}\sum_{n=1}(a^{\rm gh}_{1n} e^{-in\sigma} - I a^{\dagger{\rm gh}}_{2n} e^{in\sigma}), \quad (64a)$$

$$b_{\bar z\bar z}(\sigma) = \frac{b_0}{2} + \frac{1}{2}\sum_{n=1}(a^{\rm gh}_{1n} e^{in\sigma} - i a^{\dagger{\rm gh}}_{2n} e^{-in\sigma}), \quad (64b)$$

$$c^z(\sigma) = \frac{c_0}{2} + \frac{1}{2}\sum_{n=1}(a^{\dagger{\rm gh}}_{1n} e^{in\sigma} + i a^{\rm gh}_{2n} e^{-in\sigma}), \quad (64c)$$

$$c^{\bar z}(\sigma) = \frac{c_0}{2} + \frac{1}{2}\sum_{n=1}(a^{\dagger{\rm gh}}_{1n} e^{-in\sigma} + i a^{\rm gh}_{2n} e^{in\sigma}). \quad (64d)$$

They satisfy

$$[a^{\dagger i}_{{\rm gh}^n}, a^j_{{\rm gh}^m}]_+ = \delta_{nm}\delta^{ij}. \quad (65)$$

In the Ramond sector,

$$F_0 = p_\mu \psi^\mu_0 + \sum_{n>0}\sqrt{n}(a_n^{\dagger\mu}\psi_{\mu n} + a^\mu_n \psi^\dagger_{\mu n}),$$
$$F_{\rm gh} = -\sum_{n>0}\left(\frac{n}{2}(a^{1\dagger}_{{\rm gh}^n} + a^2_{{\rm gh}^n} s^\dagger_n) + 2(r^\dagger_n a^1_{{\rm gh}^n} + s_n a^{2\dagger}_{{\rm gh}^n})\right). \quad (66)$$

Because ψ^μ_0 satisfies the Clifford algebra

$$[\psi^\mu_0, \psi^\nu_0]_+ = \eta^{\mu\nu}, \quad (67)$$

we may represent them by ten dimensional G matrices

$$\psi^\mu_0 = \Gamma^\mu. \quad (68)$$

We can define a mass operator M to make the structure of the action in the Ramond sector more apparent.

$$M = \sum_{n>0}\sqrt{n}\,(a_n^{\dagger\mu}\psi_{\mu n} + a^\mu_n\psi^\dagger_{\mu n}) \qquad M_{\rm gh} = F_{\rm gh},$$
$$M^2 = \sum_{n>0} n(a_n^{\dagger\mu}A_{\mu n} + \psi_n^{\dagger\mu}\psi_{\mu n}) = N^R$$
$$M^2_{\rm gh} = \sum_{n>0} n(a^{\dagger 1}_{{\rm gh}^n} a^1_{{\rm gh}^n} + a^{\dagger 2}_{{\rm gh}^n} a^2_{{\rm gh}^n} + r^\dagger_n r_n + s^\dagger_n s_n) = N^R_{\rm gh} \quad (69)$$

Thus, the excited state of the superstring in the Neveu-Schwarz sector describes a particle with a mass given by the eigenvalues of mass operators $m^2 = N^{\rm NS} + N^{\rm NS}_{\rm gh} - 1/2$ at the critical dimensions $d = 10$. The ground state of the superstring in the Neveu-Schwarz sector is tachyonic. This tachyonic ground state is removed by the GSO (Gliozzi-Scherk-Olive) projection operator [24]. However, a Dirac-type particle that has a mass m, $m^2 = N^R + N^R_{\rm gh}$, can realize an excited state of the superstring in the Ramond sector. The ground state in the Ramond sector has ten-dimensional spinor index with $E_{\rm ground} = 0$.

The GSO-projected free-field action is defined in terms of the total fermion number, F, which is BRST invariant.

$$F = \begin{cases} \sum_{n=1}(\psi_n^{\dagger\mu}\psi_{\mu n} + r^\dagger_n r_n + s^\dagger_n s_n), & \text{Ramond sector} \\ ,\sum_{n=0}(\psi^{\dagger\mu}_{n+\frac{1}{2}}\psi_{\mu n+\frac{1}{2}} + r^\dagger_{n+\frac{1}{2}} r_{n+\frac{1}{2}} + s^\dagger_{n+\frac{1}{2}} s_{n+\frac{1}{2}}), & \text{Neveu-Schwarz sector} \end{cases}$$

by

$$S = \frac{1}{2}\int [X,\psi,\eta,\beta]_{NS}\Phi(1-(-1)^F)\left(-\partial^\mu\partial_\mu + N^{NS} + N^{NS}_{gh} - \frac{1}{2}\right)\Phi,$$
$$+\frac{1}{2}\int [X,\psi,\eta,\beta]_R \Psi(1+\Gamma^{11}(-1)^F)(\partial_\mu\Gamma^\mu + M + M_{gh})\Psi. \quad (70)$$

VII. CONCLUSIONS AND DISCUSSIONS

We studied the Polyakov string path integral of an NSR open superstring on a strip. The evaluation of the path integral was performed in a manifestly covariant manner. This was achieved by choosing the covariant gauge conditions that were imposed on the metric and its supersymmetric counterpart (gravitino) on the strip. The NSR open superstring had two sectors, depending on the periodicity of the fermion field: the NS sector with a periodic condition and the Ramond sector with an antiperiodic sector. In the proper time gauge, the NS sector had modular (proper-time) parameter, and the R-sector had both modular and supermodular parameters. Integration of modular and supermodular parameters yielded string propagators in both sectors. We demonstrated that the path integrals represent field-theoretical off-shell string propagators, which may be obtained from string free-field actions [25–28]. Hence, this study revealed the connection between geometric and algebraic approaches.

This study can be extended in various directions. The immediate extension may be to evaluate the Polyakov string-path integrals of the NSR super string over the Riemann surface, describing three and four interacting strings. Accordingly, we will obtain full string vertex operators, just as in the case of bosonic string theory. The (super)string theory in the proper-time gauge is deformable to Witten's (super)string. However, in the proper-time gauge, it is easy to deal with string interactions of an arbitrary number of strings because the Riemann surfaces in the proper time gauges are free of conical singularity, which is the main obstacle in evaluating the scattering amplitudes of higher string interactions. In the presence of conical singularity, it is difficult to prove the local (non-Abelian) gauge invariance of scattering amplitudes on multiple Dp-branes. It is also unclear whether the extended BRST impedance is violated by higher-order string interactions. As in the case of bosonic string theory, we expect that the scattering amplitudes of the NSR superstring with three and four strings can be calculated in the proper-time gauge without difficulty.

It is important to derive conventional QFT from corresponding superstring accurately. Otherwise, it may be difficult to systematically calculate the stringy corrections. However, the fermionic action is introduced by hand to compatible with the spectrum of the free superstring. In the case of the interacting theory, things get even worse if we employ the (super)conformal field theory. Unlike the theory in the proper-time gauge, the conventional NSR superstring, based on the conformal field theory possesses zero modes. Because the conformal field theory for the NSR superstring is defined on a sphere in contrast to the theory in the proper-time gauge, which is defined on a cylindrical surface. To deal with these zero modes, we need to insert some (picture changing) operators at the midpoint of the string. Yet this midpoint insertion is difficult to be defined properly. In contrast to the conventional approach, in the proper-time gauge approach the interacting NSR superstring, which is free of this problem, may be derived from the Polyakov string path integral consistently. This work is important as a preliminary work along this direction.

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