Editors' Suggestion

Turning black holes and D-branes inside out of their photon spheres

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Similarly to extremal Reissner-Nordström black holes (BHs), D3-branes and their intersecting bound states in lower dimensions enjoy a peculiar symmetry under conformal inversions that exchanges the horizon with infinity and keeps the photon sphere fixed. We explore the implications of this symmetry for the dynamics of massless and massive Bogomol'nyi-Prasad-Sommerfield (BPS) particles. In particular we find a remarkable identity between the scattering angle of a probe impinging from infinity and the in-spiralling angle of a probe with the very same energy and angular momentum falling into the horizon from inside the photon sphere. We argue for the identity of the radial actions and Shapiro time delays of the two processes, when some cutoff regulator is adopted. We spell out the detailed conditions for the inversion symmetry to hold in the case of large BPS BHs in four dimensions. We address conformal inversions for other BHs and Dp-branes with photon spheres in various dimensions. We briefly discuss the fate of the symmetry at the quantum level as well as for nonspherically symmetric BHs and branes and sketch potential implications for the holographic correspondence.

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I. INTRODUCTION

Direct detection of gravitational waves (GWs) emitted by binary mergers of black holes (BHs) and neutron stars has triggered increasing attention onto the detailed features of the signals that can unveil the inner characteristics of very compact gravitating objects [1–4].

In particular, astonishing progress has been achieved in the determination of the corrections to the GW signal emitted in the "in-spiral" phase, relying on the connection with scattering amplitudes [5–8], as well as in the "ring-down" phase, relying on the peculiar connection between quasinormal modes (QNMs) and quantum Seiberg-Witten curves for $N = 2$ supersymmetric Yang-Mills theories [9–12].

Other important features, such as the presence of echoes [1,13] for exotic compact objects, such as horizonless microstates in the fuzzball proposal [14–19], and memory effects [20–25], based on soft theorems [26–30] have been investigated and precise characteristics have been identified that should allow to distinguish BHs from stringy fuzzballs [31–37].

A common and crucial feature of BHs, D-branes and other compact gravitating objects is the "photon sphere" or light ring (light halo for rotating objects) formed by the unstable bound orbits of massless particles such as photons "surfing" the wall separating the asymptotically flat region from the horizon (or the inner region for fuzzballs) [38–42].

The aim of the present investigation is to point out a remarkable symmetry between these two regions for special classes of BHs and D-branes. Indeed we will prove that

$$\Delta\phi_{\rm fall}(J,E) = \Delta\phi_{\rm scatt}(J,E) \tag{1.1}$$

for impact parameters $b = J/E$ above the critical value b_c. While $\Delta\phi_{\rm scatt}(J,E)$ denotes the deflection angle of a massless probe with "energy" E and orbital angular momentum J outside the photon sphere, $\Delta\phi_{\rm fall}(J,E)$ denotes the angle described by a massless object with the very same energy E and angular momentum J falling into the horizon from inside the photon sphere.

Our "inversion" formula bears some resemblance with the equally remarkable "boundary to bound" (B2B) correspondence in binary processes between periastron advance $\Delta\phi(J,E)$ and the scattering angle $\chi(J,E)$ [7,8]

$$\Delta\phi(J,E) = \chi(J,E) + \chi(-J,E) \quad E < 0. \tag{1.2}$$

While the latter requires analytic continuation in the binding energy E and angular momentum J but seems valid to all orders in the post-Minkowskian (PM) expansion for (nonrotating) compact objects in the "conservative" sector [7,8], the former requires no analytic continuation but seems valid only for massless or massive

Bogomol'nyi-Prasad-Sommerfield (BPS) probes in a restricted number of cases, including extremal Reissner-Nordström (RN) BHs, D3-branes, 2-charge small BHs in $d=5$ and 4-charge "large" BHs in $d=4$.

The geometric peculiarity of these special cases is not only extremality but also their symmetry under conformal inversions of the Couch-Torrence (CT) kind [43], which have been recently revived and generalized [44–46] also in connection with Freudenthal duality [47,48] and BH (in) stability [49–53].

The plan of the paper is as follows. After briefly reviewing the concepts of critical impact parameters and photon spheres for null geodesics in Sec. II, we show that the inversion formula holds true for extremal charged RN BHs in $d=4$ in Sec. III.

In Sec. IV we demonstrate the validity of the inversion formula for D3-branes thanks to both a homological argument based on contour deformation and a geometric argument based on a generalized Couch-Torrence inversion that keeps the photon sphere fixed. We then pass to analyze intersecting D3-D3' "small" BHs in Sec. V and for 4-charge large BPS BHs in Sec. VI. While the former follow very easily from the previous case any numbers of intersecting D3-D3', in the latter case we find restrictions on the four charges for the inversion symmetry to hold.

We consider massive BPS probes in Sec. VII and find similar subtle restrictions on the charges and masses for 4-charge large BPS BHs but no such issues for 2-charge systems.

In Sec. VIII we find that extremal BPS 3-charge large nonrotating BHs in $d=5$ do not seem to enjoy the inversion formula. We also find similar difficulties for other BHs and Dp-branes with photon spheres in various dimensions with flat or AdS asymptotics. We briefly address the issue for rotating black holes, which will be discussed more thoroughly in [54].

In Sec. IX we discuss the fate of the inversion formula at the quantum level, possibly including the regularization of the "radial" action by introducing a cutoff or putting the system in AdS. We also sketch potential implications for the AdS/CFT.

Section X contains a summary of our results, our conclusions and directions for further investigation in the future.

II. CRITICAL NULL GEODESICS AND PHOTON SPHERES

Scattering of (massless neutral) probes impinging from (asymptotically flat) infinity off compact rotationally invariant (nonspinning) gravitating objects,[1,2] such as BHs or D-branes, typically exposes three different regimes, depending on the value of the impact parameter $b=J/E$ [19,38]:

(i) $b>b_c$: above a critical value, the probe scatters off with a deflection angle $\Delta\phi\sim G_N M/b^{d-3}$ in d dimensions (with G_N the Newton constant and M the mass);

(ii) $b=b_c$: at the critical value, the probe is (asymptotically) trapped in an unstable "circular" orbit $r=r_c\sim b_c\sim (G_N M)^{1/d-3}$;

(iii) $b<b_c$: below the critical value, the probe falls into the horizon.

In Fig. 1, we plot the geodesics for D3-branes since these will be studied in detail. The critical impact parameter b_c and the critical radius r_c are determined by the conditions

$$V_{\mathrm{eff}}(r_c)=E^2 \quad \text{and} \quad V'_{\mathrm{eff}}(r_c)=0, \tag{2.1}$$

where V_{eff} is the "effective" potential.

For a spherically symmetric compact object, described by an asymptotically flat (or AdS) metric of the form

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2 ds^2_{S^{D-2}}, \tag{2.2}$$

the null geodesic equation $ds^2=0$ can be separated and put in Hamiltonian form:

$$\mathcal{H}=0=-\frac{E^2}{f(r)}+f(r)P_r^2+\frac{J^2}{r^2}, \tag{2.3}$$

where

$$E=-P_t=f(r)\dot{t}, \quad P_r=\frac{\dot{r}}{f(r)}, \quad J=P_\phi=r^2\dot{\phi}. \tag{2.4}$$

Thanks to spherical symmetry, only the total orbital angular momentum J and its conjugate angular variable, denoted by ϕ, play a role.

Extracting P_r and computing the radial action

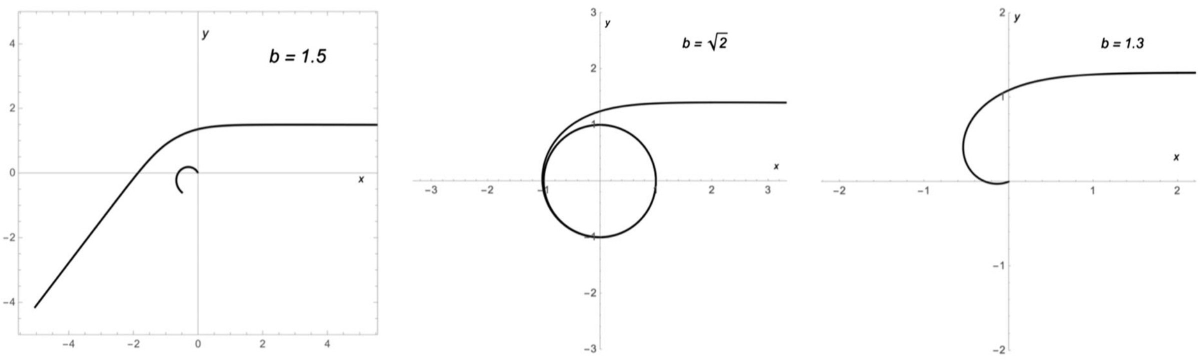
$$\begin{aligned} S_r(J,E;r_i,r_f) &= \int_{r_i}^{r_f} P_r(J,E)dr \\ &= \pm\int_{r_i}^{r_f}\sqrt{E^2-f(r)\frac{J^2}{r^2}}\frac{dr}{f(r)} \end{aligned} \tag{2.5}$$

in terms of the conserved energy E and angular momentum J, as well as of the initial r_i and final r_f radii, one can easily compute the deflection angle[3]

[1]The discussion can be easily generalized to (A)dS asymptotics with very little effort, as we will see, and with some effort to rotating objects.

[2]In the rotating case the critical radius varies in an interval $r_H=r_{\min}<r_c<r_{\max}$ [38–41].

[3]We are assuming that r_i, r_f are either independent of J or are turning points of the radial motion so that $P_r(r_{i/f})=0$.



(a) (b) (c)

FIG. 1. Geodesics in D3-brane geometry ($L = 1$) for (a) $b > b_c$, (b) $b = b_c$ and (c) $b < b_c$.

$$\Delta\phi = \frac{\partial S_r(J,E;r_i,r_f)}{\partial J} = -\int_{r_i}^{r_f} \frac{J dr}{r^2 f(r) P_r(J,E)} = \int_{r_i}^{r_f} \frac{J dr}{E\sqrt{r^4 - b^2 r^2 f(r)}} \tag{2.6}$$

and Shapiro time delay[4]

$$\Delta t = -\frac{\partial S_r(J,E;r_1,r_2)}{\partial E} = -\int_{r_1}^{r_2} \frac{E dr}{f(r)^2 P_r(J,E)}. \tag{2.7}$$

For Dp-branes, their bound states and other extremal/BPS objects it turns out to be convenient to use a different radial variable such that $u = 0$ at the horizon $r = r_H$. The metric in isotropic form reads

$$ds^2 = -\frac{dt^2 - d\mathbf{x}^2}{h(u)} + h(u)[du^2 + u^2 ds^2_{S^{8-p}}]. \tag{2.8}$$

Setting to zero the p "longitudinal" momenta $P_x = 0$, as well as the Kaluza-Klein (KK) ones, if present, one has

$$\mathcal{H} = 0 = -h(u)E^2 + \frac{1}{h(u)}\left[P_u^2 + \frac{J^2}{u^2}\right] \tag{2.9}$$

so that the radial action becomes

$$S_u(J,E;u_1,u_2) = \pm\int_{u_1}^{u_2} \sqrt{h(u)^2 E^2 - \frac{J^2}{u^2}} du \tag{2.10}$$

and the deflection angle and time delay turn out to be given by

$$\Delta\phi = -\int_{u_1}^{u_2} \frac{J du}{u^2 P_u(J,E)}, \quad \Delta t = -\int_{u_1}^{u_2} \frac{E h(u)^2 du}{P_u(J,E)}. \tag{2.11}$$

III. INVERSION SYMMETRY FOR EXTREMAL RN BHs

In Einstein-Maxwell theories, the most general spherically symmetric BH in $d = 4$ is the (nonextremal) RN BHs. Setting $G_N = 1$, the metric is given by (2.2) with

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{3.1}$$

For zero charge $Q = 0$ one gets a Schwarzschild BH. For $|Q| = M$ one gets an extremal RN BH. The two horizons are located at

$$r_H^\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.2}$$

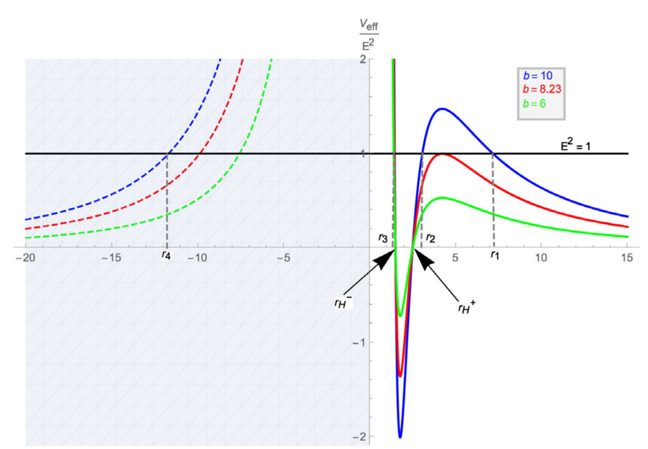


FIG. 2. Nonextremal RN effective potential for $M = 2$, $Q^2 = 3.75$.

[4]This is valid when r_i, r_f are either independent of E or turning points of the radial motion so that $P_r(r_{i/f}) = 0$.

The effective potential is given by

$$V_{\text{eff}}(r) = E^2 - \dot{r}^2 = \frac{J^2}{r^2}\left[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right]. \tag{3.3}$$

See Fig. 2 for non extremal RN BHs and Fig. 3 for extremal RN BHs. Imposing $V_{\text{eff}} = E^2$ and $V'_{\text{eff}} = 0$ one finds the critical radius r_c and the critical impact parameter b_c [40,41]:

$$r_c = \frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2}) \tag{3.4}$$

$$b_c = \sqrt{\frac{2r_c^3}{r_c - M}} = \sqrt{\frac{27M^4 - 36M^2Q^2 + 8Q^4 + M(9M^2 - 8Q^2)^{\frac{3}{2}}}{2(M^2 - Q^2)}}. \tag{3.5}$$

For $b > b_c$ there are four solutions ("turning points") of $V_{\text{eff}} = E^2$ with $r_1 > r_2 > r_H^+ > r_H^- > r_3 > 0 > r_4$.

The scattering angle[5] is given by (2.6) integrated from ∞ to the turning point r_1 (the largest root):

$$\Delta\phi_{\text{scatt}}(E, J) = -b\int_{\infty}^{r_1} \frac{dr}{\sqrt{r^4 - b^2 r^2 f(r)}} = \frac{2b}{\sqrt{r_{13}r_{24}}}\mathcal{K}\left[\arcsin\sqrt{\frac{r_{24}}{r_{14}}}; \sqrt{\frac{r_{14}r_{23}}{r_{13}r_{24}}}\right], \tag{3.6}$$

where $r_{ij} = r_i - r_j$ and $r_{i+} = r_i - r_H^+$ with $i, j = 1, 2, 3, 4$. If $b = b_c$ it is easy to see that $r_1 = r_2$ and (3.6) diverge, so the massless probe gets asymptotically trapped in a circular unstable orbit. The union of such orbits for photons (or other neutral massless probes) impinging from different directions generates the "photon sphere."

In principle a massless probe with the very same energy E and angular momentum J, such that $b > b_c$, can be "emitted" from inside the photon sphere. Due to the strong gravitational attraction it cannot escape to infinity but rather falls into the horizon describing a "spiral." We can compute the in-spiralling angle which is given by integrating (2.6) from r_2 to the horizon r_H^+:

$$\Delta\phi_{\text{fall}}(E,J) = -b\int_{r_2}^{r_H^+} \frac{dr}{\sqrt{r^4 - b^2 r^2 f(r)}} = \frac{2b}{\sqrt{r_{13}r_{24}}}\mathcal{K}\left[\arcsin\sqrt{\frac{r_{13}r_{2+}}{r_{23}r_{1+}}}; \sqrt{\frac{r_{23}r_{14}}{r_{13}r_{24}}}\right]. \tag{3.7}$$

In general (3.6) and (3.7) are different for $Q \neq M$. In the extremal case ($Q = M$) they coincide. In terms of the coordinate $u = r - Q$, the metric reads

[5] Actually the "standard" scattering angle is $\Delta\theta = \pi - 2\Delta\phi_{\text{scatt}}$.

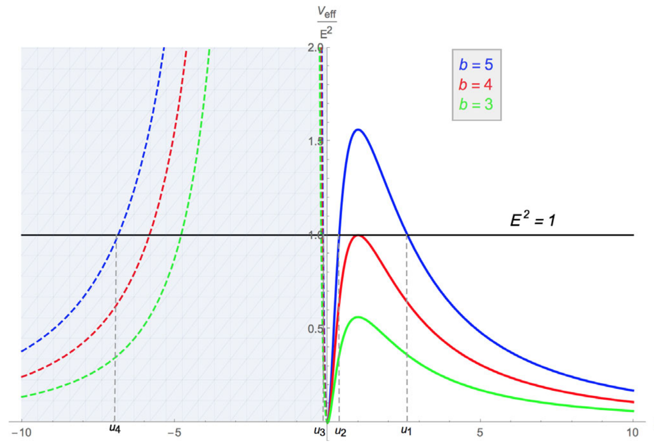


FIG. 3. Extremal RN effective potential $Q = M = 1$. The horizon is a double zero of the metric.

$$ds^2 = -\frac{dt^2}{h(u)} + h(u)(du^2 + u^2 d\theta^2 + u^2 \sin^2\theta d\phi^2)$$
$$h(u) = \left(1 + \frac{Q}{u}\right)^2. \tag{3.8}$$

The zeros of P_u are encoded in the algebraic equation:

$$u^2 + M(2 \mp \beta)u + M^2 = 0, \tag{3.9}$$

where we introduced the (adimensional) impact parameter $\beta = \frac{b}{M} = \frac{J}{ME}$. The four solutions are

$$u_{\pm}^{[\pm]} = M\left[\pm\frac{\beta}{2} - 1[\pm]\sqrt{\frac{\beta^2}{4} \mp \beta}\right], \tag{3.10}$$

where $[\pm]$ indicates the uncorrelated signs. All the solutions are real for $\beta \geq 4 = \beta_c$ that corresponds to the critical impact parameter. For $\beta > 4$ the ordering of the roots is the following:

$$u_1 = u_+^+ > u_2 = u_+^- > 0 > u_3 = u_-^+ > u_4 = u_-^-. \tag{3.11}$$

Since u_1 is the largest root of P_u, the scattering angle is given by

$$\Delta\phi_{\text{scatt}}(E,J) = -b\int_{\infty}^{u_1} \frac{du}{\sqrt{h(u)^2 u^4 - b^2 u^2}} = \frac{2\beta}{\sqrt{v_{13}v_{24}}}\mathcal{K}\left[\arcsin\sqrt{\frac{v_{24}}{v_{14}}}; \sqrt{\frac{v_{23}v_{14}}{v_{13}v_{24}}}\right], \tag{3.12}$$

where we set $u_i = Mv_i$ for $i = 1, 2, 3, 4$.

Following the same procedure, we can derive the in-spiralling angle:

$$\Delta\phi_{\rm fall}(E,J) = -b\int_{u_2}^{0}\frac{du}{\sqrt{h(u)^2u^4 - b^2u^2}} = \frac{2\beta}{\sqrt{v_{13}v_{24}}}\mathcal{K}\left[\arcsin\sqrt{\frac{v_{13}v_2}{v_{23}v_1}};\sqrt{\frac{v_{23}v_{14}}{v_{13}v_{24}}}\right] \tag{3.13}$$

and observe that the expressions (3.12) and (3.13) are equal if and only if

$$\frac{v_{24}}{v_{14}} = \frac{v_{13}v_2}{v_{23}v_1} \tag{3.14}$$

that indeed holds true for $Q = M$, so much so that the angle described by the massless probe from infinity to the turning point and the in-spiralling angle are equal for extremal RN BHs:

$$\Delta\phi^{{\rm extr}RN}_{\rm fall}(E,J) = \Delta\phi^{{\rm extr}RN}_{\rm scatt}(E,J). \tag{3.15}$$

We will have more to say about this remarkable relation after discussing D3-branes and their bound states. Suffice it to say here that the extremal RN metric admits a conformal inversion $u \to Q^2/u$ that preserves the light-cone ($ds^2 = 0 \to ds^2 = 0$) up an overall Weyl factor $W = u^2/Q^2$ and keeps the photon sphere $u_c = Q$ ($r_c = 2Q > r_H = Q$) fixed. These kinds of transformations were introduced by Couch and Torrence [43] and recently revived by [44,45,47,48].

Before concluding this section, let us briefly consider the fate of the inversion symmetry for spherically symmetric charged BHs in AdS_4,[6] whereby

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}. \tag{3.16}$$

Computing the radial momentum P_r one finds

$$r^4f^2(r)P_r^2E^{-2} = r^4 - b^2r^2f(r) = r^4\left(1-\frac{b^2}{\ell^2}\right) - b^2r^2 + 2Mb^2r - Q^2b^2. \tag{3.17}$$

Positivity of the leading term at infinity requires $|b| < \ell$. In the asymptotically AdS case, for $M = Q$ one finds

$$\Delta\phi^{\rm AdS}(b = J/E) = \int_{r_i}^{r_f}\frac{bdr}{\sqrt{r^4 - b^2r^2f(r)}} = \Delta\phi^{\rm flat}\left(\tilde{b} = b/\sqrt{1-\frac{b^2}{\ell^2}}\right). \tag{3.18}$$

[6]In principle dS corresponds to $\ell^2 \to -\ell^2$

As a result $\Delta\phi^{\rm AdS}_{\rm scatt}(J,E) = \Delta\phi^{\rm AdS}_{\rm fall}(J,E)$ for charged BHs in AdS with $M = Q$ as in flat space-time.[7] Even though the radial action is not invariant under $r - r_0 \to (r_c - r_0)^2/(r - r_0)$, with $r_c = 2r_0 = 2M = 2|Q|$, one expects Couch-Torrence conformal inversions [43] to admit a generalization in gauged (super)gravity [44,45,47,48].

Let us now pass to consider Dp-branes with metric (2.8) where

$$h(u) = H(u)^{\frac{1}{2}}, \qquad H(u) = 1 + \frac{L^{7-p}}{u^{7-p}}. \tag{3.19}$$

In particular we will mostly focus on D3-branes and their bound states.

IV. D3-BRANES

In the case of D3-branes, the turning points, i.e. the zeros of $h(u)$ satisfy

$$u^4 - b^2u^2 + L^4 = 0 \tag{4.1}$$

and are thus given by

$$u^2_\pm = \frac{b^2}{2}\left(1 \pm \sqrt{1 - \frac{4L^4}{b^4}}\right) \tag{4.2}$$

so that the critical "radius" and impact parameter are $u_c = L$ and $b_c = \sqrt{2}L$ [19,39–41]. Setting

$$\gamma = \frac{\sqrt{2}L}{b} \tag{4.3}$$

we can see from (4.2) that for $0 < \gamma < 1$ we have two real positive solutions such that $u_+ > u_-$ and two real negative solutions $-u_+ < -u_-$. The scattering angle is obtained from (2.11) integrating from $u = \infty$ up to u_+:

$$\Delta\phi_{\rm scatt} = -b\int_\infty^{u_+}\frac{du}{\sqrt{u^4 - b^2u^2 + L^4}} = -b\int_\infty^{u_+}\frac{du}{\sqrt{(u^2-u_+^2)(u^2-u_-^2)}}. \tag{4.4}$$

Setting $v = \frac{u_*}{u}$ one can express the integral in terms of a complete elliptic function of the first kind:

[7]Strictly speaking the solution with $M = Q$ is a naked singularity without a proper horizon [55]. Regular charged BHs in AdS have nonzero angular momentum [56]. Yet a photon sphere is present at $r_c = 2r_0 = 2M = 2|Q|$, independent of ℓ, in the nonrotating case.

$$\Delta\phi_{\rm scatt} = \frac{b}{u_+}\int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-\frac{u_-^2}{u_+^2}v^2)}} = \frac{b}{u_+}\mathcal{K}\left[\frac{u_-^2}{u_+^2}\right] \tag{4.5}$$

that admits a representation in terms of a Gaussian hypergeometric function

$$\Delta\phi_{\rm scatt} = \sqrt{\frac{2}{1+\sqrt{1-\gamma^4}}}\mathcal{K}\left[\frac{\gamma^4}{(1+\sqrt{1-\gamma^4})^2}\right] = \frac{\pi}{2}\sqrt{\frac{2}{1+\sqrt{1-\gamma^4}}}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1\middle|\frac{\gamma^4}{(1+\sqrt{1-\gamma^4})^2}\right). \tag{4.6}$$

For $0<\gamma<1$ the argument of the hypergeometric function is smaller than 1 and this ensures convergence of the series. For $\gamma=1$ ($b=b_c$) the series (4.6) diverges and the massless probe gets trapped in the photon sphere. For $\gamma>1$ ($b<b_c$) it falls into the horizon hovering the photon sphere.

The in-spiralling angle is (2.11) integrated from the internal turning point u_2, which exists for $\gamma<1$, to the horizon $u_H=0$:

$$\Delta\phi_{\rm fall} = -b\int_{u_-}^0 \frac{dr}{\sqrt{u^4-b^2u^2+L^4}} = b\int_0^{u_-}\frac{du}{\sqrt{(u^2-u_+^2)(u^2-u_-^2)}}. \tag{4.7}$$

If we choose the new variable $u=\frac{v}{u_-}$, we can express (4.7) in terms of a complete elliptic integral of the first kind obtaining exactly the same expression as in (4.5). In other words in a D3-brane background, for impact parameters larger the critical one ($b>\sqrt{2}L$), the angle described by the massless particle coming from radial infinity and approaching the turning point is exactly equal to the in-spiralling angle.

We would like to provide two arguments to explain this peculiar and far-reaching property. The first one is a homological argument based on contour deformation for elliptic integrals. The second one is a geometric argument based on the hitherto un-noticed symmetry of the D3-brane metric under conformal inversions, generalizing the ones in [43].

The homological argument runs as follows. The integral $\Delta\phi_{\rm fall}=\mathcal{I}(0,u_-)$ can be extended to negative values of u so that $2\Delta\phi_{\rm fall}=\mathcal{I}(-u_-,u_-)$, closing the contour in the ${\rm Im}(u)<0$ half-plane one has $4\Delta\phi_{\rm fall}=\oint_a\omega$; i.e. it can be viewed as the a-period of the elliptic curve (torus). Similarly extending beyond infinity $2\Delta\phi_{\rm scatt}=\mathcal{I}(+u_+,-u_+)$, closing the contour in the ${\rm Im}(u)<0$ half-plane one has $4\Delta\phi_{\rm scatt}=\oint_{a'}\omega$; i.e. it can be viewed as the a'-period of the elliptic curve (torus). But the cycles a and a' are homologous so that $\Delta\phi_{\rm fall}=\Delta\phi_{\rm scatt}$. See Fig. 4.

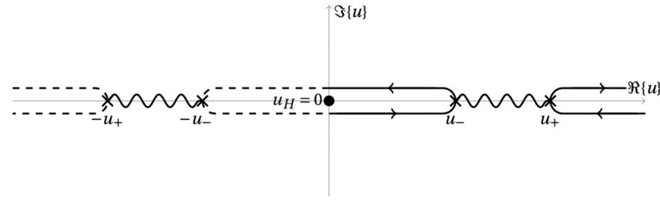


FIG. 4. The position of the branch cut in the complex u plane.

The geometric arguments runs as follows. Performing the transformations $u\to L^2/v$ the (relevant part of the) D3-brane metric suffers a Weyl rescaling:

$$ds^2 = -\left(1+\frac{L^4}{u^4}\right)^{-\frac{1}{2}}dt^2 + \left(1+\frac{L^4}{u^4}\right)^{-\frac{1}{2}}(du^2+u^2d\phi^2) \to \frac{L^2}{v^2}\left[-\left(1+\frac{L^4}{v^4}\right)^{-\frac{1}{2}}dt^2+\left(1+\frac{L^4}{v^4}\right)^{-\frac{1}{2}}(dv^2+v^2d\phi^2)\right]. \tag{4.8}$$

Since massless geodesics are Weyl invariant and $u_-=L^2/u_+$ the integrals corresponding to the two angles get exchanged under conformal inversion $u\to L^2/u$, whose fixed locus is the photon sphere $u_c=L$. This is a remarkable symmetry of D3-branes that—if not spoiled by quantum corrections—can reveal new insights into the holographic AdS/CFT correspondence. We will see that the same property is enjoyed by intersecting D3-brane systems with two and four charges. In the latter case this is related to generalized Freudenthal duality [44,45,47,48]. After imposing suitable restrictions on the charges, it amounts to a generalized Couch-Torrence inversion.

V. D3-D3' SMALL BHs

Let us consider a massless probe in an intersecting D3-D3' background[8] compactified on $T^4\times S^1$. We denote by y the coordinate compactified on a (large) circle S^1, by $\vec{x}$ the coordinates along the four (1–4) noncompact spatial with Dirichlet-Dirichlet (DD) boundary conditions directions and by $\vec{z}$ the coordinates along the four with Neuman-Dirichlet (DD) boundary conditions directions (6–9) compactified on a (small) T^4. After smearing each D3' along the transverse T^2 directions, the metric is given by

$$ds^2 = -(H_3H_{3'})^{-\frac{1}{2}}(dt^2-dy^2)+(H_3H_{3'})^{\frac{1}{2}}dx^2 + \left(\frac{H_3}{H_{3'}}\right)^{\frac{1}{2}}ds^2_{T^2_{67}} + \left(\frac{H_{3'}}{H_3}\right)^{1/2}ds^2_{T^2_{89}} \tag{5.1}$$

with

[8] This system is T-dual to the D1-D5 system. We work in the D3-D3' U-duality frame to keep a uniform notation.

$$H_3(u) = 1 + \frac{L_3^2}{u^2}, \qquad H_{3'}(u) = 1 + \frac{L_{3'}^2}{u^2}. \tag{5.2}$$

Setting

$$x_1 + ix_2 = u\cos\theta e^{i\psi}, \qquad x_3 + ix_4 = u\sin\theta e^{i\phi} \tag{5.3}$$

the four-dimensional metric in the DD directions reads

$$dx^2 = du^2 + u^2[d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2]. \tag{5.4}$$

The zero mass shell condition Hamiltonian formalism reads

$$\mathcal{H} = \sqrt{H}(P_y^2 - P_t^2) + \frac{1}{\sqrt{H}}\left[P_u^2 + \frac{P_\theta^2}{u^2} + \frac{P_\phi^2}{u^2\sin^2\theta} + \frac{P_\psi^2}{u^2\cos^2\theta}\right] + |\vec{P}_z|^2\sqrt{\frac{H_3}{H_{3'}}} + |\vec{P}'_z|^2\sqrt{\frac{H_{3'}}{H_3}} = 0, \tag{5.5}$$

where $H = H_3 H_{3'}$.

Thanks to spherical symmetry, without loss of generality, one can consider motion in the equatorial plane ($\theta = \pi/2$). As a result $J_\psi = 0$ and $J_\phi = J$ is the total angular momentum. For simplicity, for the time being, we also take vanishing KK momenta $\vec{P}_z = \vec{P}_z{}' = 0$ along the compact directions. We will consider massive BPS probes later on.

The only difference between D3 and intersecting D3-D3’ lies in the definition of the (harmonic) function $H(u)$. Indeed we expect D3-D3’ systems to enjoy the same property that characterizes D3-branes. In the present case the turning points satisfy

$$\left(1 + \frac{L_3^2}{u^2}\right)\left(1 + \frac{L_{3'}^2}{u^2}\right)\frac{u^4}{b^2} - u^2 = 0 \tag{5.6}$$

whose solutions are

$$u_\pm^2 = \frac{1}{2}\left\{b^2 - L_3^2 - L_{3'}^2 \pm \sqrt{(b^2 - L_3^2 - L_{3'}^2)^2 - 4L_3^2 L_{3'}^2}\right\}. \tag{5.7}$$

Analogously to the D3 case we can compute the scattering angle, described by the probe from infinity to the “external” turning point $u = u_+$, as well as the in-spiralling angle, from the “internal” turning point $u = u_-$ to the horizon $u_H = 0$. Taking into account the difference between roots (4.1) and (5.6) the integrations can be performed in an identical way and the formulas obtained for the scattering and the spiralling angle are the same:

$$\Delta\phi_{\rm scatt}(\infty, u_+) = \Delta\phi_{\rm fall}(u_-, 0) = \frac{b}{u_+}\mathcal{K}\left[\frac{u_-^2}{u_+^2}\right], \tag{5.8}$$

where $u_\pm$ in (5.8) are given by (5.6).

Once again the conformal inversion symmetry

$$u \to \frac{L_3 L_{3'}}{u} \tag{5.9}$$

under which $u_+ = L_3 L_{3'}/u_-$ and the photon sphere $u_c = \sqrt{L_3 L_{3'}}$ is fixed, is crucial to explain the geometric origin of the result.

We do not repeat here the homological argument based on contour deformation, because it runs exactly the same way as for D3-branes.

VI. INTERSECTING D3-BRANES AS LARGE BPS BHs

Intersecting four stacks of D3-branes, such that any pair has four common N-D (internal) directions, one gets a large BPS BH solution in STU supergravity (STU supergravity with 3 complex scalars usually denoted by the symbols S, T and U) with four charges Q_i. Neglecting the internal T^6, to which we will turn our attention later, the (4-d) metric is given by

$$ds^2 = -\prod_{i=1}^4\left(1 + \frac{Q_i}{u}\right)^{-\frac{1}{2}}dt^2 + \prod_{i=1}^4\left(1 + \frac{Q_i}{u}\right)^{\frac{1}{2}}[du^2 + u^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{6.1}$$

Our aim is to prove the property that the scattering angle and the in-spiral angle holds true for all backgrounds of the form (6.1). The case in which all the charges are equal coincides with an extremal RN background that we already dealt with in Sec. III. We first identify the most general set of charges that lead to solutions admitting a conformal inversion symmetry *à la* Couch-Torrence. Then we will analyze the case $Q_1 = Q_2 > Q_3 = Q_4$ that still admits an analytical treatment and briefly discuss the unequal charge case at the end.

A. Couch-Torrence conformal inversion

In the case of 4-charge BHs obtained from intersecting D3-branes the relevant inversion is

$$u \to \frac{\sqrt{Q_1 Q_2 Q_3 Q_4}}{u}. \tag{6.2}$$

This is a conformal isometry of the metric if and only if[9]

$$\sum_i Q_i = \sqrt{Q_1 Q_2 Q_3 Q_4}\sum_i \frac{1}{Q_i}. \tag{6.3}$$

This happens to be the case when all Q’s are equal or when they are equal in pairs. Setting

[9] Otherwise one has to perform a (symplectic) transformation on the charges as well [44,45,47,48].

$$Q_1 = xQ_4, \qquad Q_2 = yQ_4, \qquad Q_3 = zQ_4, \tag{6.4}$$

the condition boils down to

$$1 + x + y + z = \sqrt{xyz}\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right). \tag{6.5}$$

In general setting $z = \lambda^2 xy$ one finds that the only three solutions are

$$\lambda^2 = 1, \qquad \lambda^2 = \frac{x^2}{y^2}, \qquad \lambda^2 = \frac{y^2}{x^2} \tag{6.6}$$

that in turn mean (assuming all charges to be positive)

$$z = xy, \qquad x = yz, \qquad y = zx \tag{6.7}$$

or even more simply

$$Q_1 Q_2 = Q_3 Q_4 \tag{6.8}$$

or permutations thereof. It is amusing to see that the simplest nontrivial integer solution is (a permutation of) $Q_1 = 1$, $Q_2 = 2$, $Q_3 = 3$, $Q_4 = 6$. In general we have a three-parameter family of solutions admitting conformal inversion as a symmetry. In all these cases the photon sphere, located at

$$u_c = \sqrt[4]{Q_1 Q_2 Q_3 Q_4}, \tag{6.9}$$

is fixed under inversion and $u_1 u_2 = u_c^2 = \sqrt{Q_1 Q_2 Q_3 Q_4}$. As a consequence the identity $\Delta\phi_{\rm scatt} = \Delta\phi_{\rm fall}$ has a deep geometric origin that allows to turn these 4-charge BHs inside out their photon spheres.

B. Pairwise equal charges

For simplicity we set $Q_1 = Q_2 = Q$ and $Q_3 = Q_4 = \tilde{Q}$ and we choose $Q > \tilde{Q}$. The turning points satisfy

$$\left(1 + \frac{Q}{u}\right)^2 \left(1 + \frac{\tilde{Q}}{u}\right)^2 - b^2 u^2 = 0. \tag{6.10}$$

Setting

$$z = \frac{u}{Q} \qquad \beta = \frac{b}{Q} \qquad q = \frac{\tilde{Q}}{Q} < 1, \tag{6.11}$$

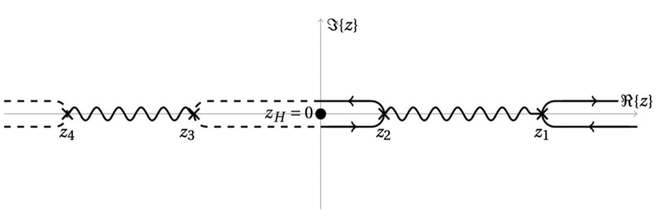


FIG. 5. The position of the branch cuts in the complex z plane.

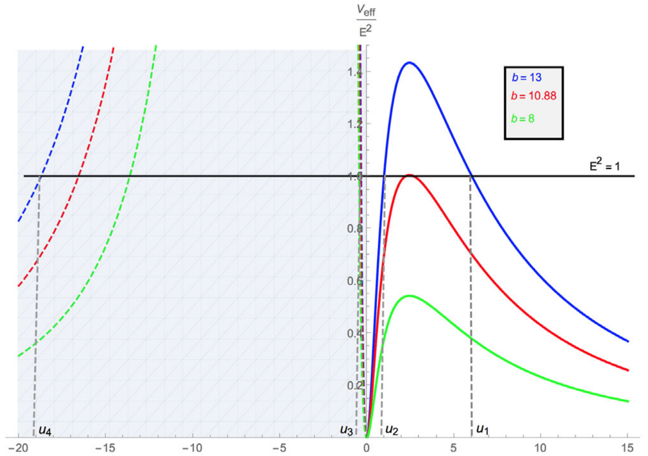


FIG. 6. D3D3D3D3-brane effective potential for $Q_1 = 1$, $Q_2 = 2$, $Q_3 = 3$ and $Q_4 = 4$.

the square root of (6.10) reads

$$(1 + z)(q + z) = \pm\beta z \tag{6.12}$$

which admits the following solutions:

$$z_{\pm}^{[\pm]} = \frac{-(q + 1 \mp \beta)[\pm]\sqrt{(q-1)^2 + \beta^2 \mp 2\beta(q+1)}}{2}, \tag{6.13}$$

where the square parentheses mean the uncorrelated signs. Let us notice that in the limit $q \to 1$ we recover the roots (3.10) that allows us to order the roots in (6.13) as follows:

$$z_1 = z_+^+ > z_2 = z_+^- > 0 > z_3 = z_-^+ > z_-^- = z_4. \tag{6.14}$$

The position of the branch points in the z plane is depicted in Fig. 5. The structure of the scattering and inspiralling angle is the same as in (3.12) and (3.13) and, as for extremal RN, the roots (6.13) satisfy a relation analogous to (3.14).

C. All unequal charges

In general, for different charges the effective potential for massless particles is given by

$$V_{\rm eff}(u) = \frac{J^2}{u^2 \prod_{i=1}^4 (1 + \frac{Q_i}{u})}. \tag{6.15}$$

The effective potential for a special choice of different charges is plotted in Fig. 6. Setting

$$z = \frac{u}{Q_4} \qquad \beta = \frac{b}{Q_4} \qquad q_i = \frac{Q_i}{Q_4} \qquad i = 1, 2, 3 \tag{6.16}$$

without loss of generality one can take $0 < q_i < q_j < 1$ with $i < j = 1, 2, 3$. The effective potential has a minimum in $z = 0$ and a maximum in $z = z_c > 0$. The critical impact parameter is identified by the following relations:

$$V_{\text{eff}}(z = z_c) = E^2, \qquad V'_{\text{eff}}(z = z_c) = 0$$
$$\beta_c = \frac{\sqrt{(1 + z_c)(q_2 + z_c)(q_3 + z_c)(q_4 + z_c)}}{z_c}. \tag{6.17}$$

We are interested in the regime in which $\beta > \beta_c$. The zeros of P_u are encoded in the following algebraic equation:

$$(1 + z)(q_1 + z)(q_2 + z)(q_3 + z) = \beta^2 z^2. \tag{6.18}$$

The expressions for the scattering and in-spiralling angles are the same as in (3.12) and (3.13), where now the z_i are the solutions of (6.18).

Although (6.18) can be solved by quadrature, the expressions for the z_i are quite cumbersome to manipulate. In order to check that even in this case

$$\Delta\phi^{4Q}_{\text{scatt}} = \Delta\phi^{4Q}_{\text{fall}} \tag{6.19}$$

holds true it turns out convenient to use numerical methods that confirm indeed the validity of our inversion formula, provided $Q_1 Q_2 = Q_3 Q_4$ or permutations thereof.

One may try and extend the analysis to 4-charge STU BHs in AdS with metric [55]

$$ds^2 = -f(u)\prod_{i=1}^{4} H_i^{-\frac{1}{2}} dt^2 + \prod_{i=1}^{4} H_i^{\frac{1}{2}}\left[\frac{du^2}{f(u)} + u^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.20}$$

with $H_i = 1 + \frac{Q_i}{u}$ and $f = 1 + \frac{r^2}{\ell^2}\prod_{i=1}^4 H_i$. It is easy to check that

$$\Delta\phi^{4Q}_{\text{AdS}}(b = J/E) = \Delta\phi^{4Q}_{\text{fall}}\left(\tilde{b} = \frac{b}{\sqrt{1 - \frac{b^2}{\ell^2}}}\right) \tag{6.21}$$

for generic choice of the charges Q_i so much so that

$$\Delta\phi^{4Q,\text{scatt}}_{\text{AdS}}(J, E) = \Delta\phi^{4Q,\text{fall}}_{\text{AdS}}(J, E) \tag{6.22}$$

if $Q_1 Q_2 = Q_3 Q_4$ or permutations thereof. Yet, as already mentioned in the case of singly charged BHs with $M = Q$ in AdS, strictly speaking (6.20) has a naked singularity and no proper horizon [55], even though a photon sphere at the same $u = u_c$ as in flat space-time is present that is fixed under CT transformations $u \to u_c^2/u$, exchanging infinity and the "putative" horizon at $u = 0$. Regular BHs with arbitrary charge in AdS require nonzero angular momentum [56].

VII. MASSIVE BPS PROBES

So far we have only considered massless probes. Let us try and consider some massive probe. For simplicity we will focus on massive BPS particles that owe their mass to their (generalized) KK momentum along internal directions. Since in the case of D3-branes there are none, we will consider intersecting D3-branes. For non-BPS particles with arbitrary masses and couplings to the geometry we do not expect the identity to hold.

A. D3-D3 with massive BPS probes

In the intersecting D3-brane configuration (5.1) with $L_3 = L_{3'} = L$, one can consider the hyperplane $\theta = \pi/2$ without loss of generality. The relevant part of the metric reads

$$ds^2 = H^{-\frac{1}{2}}(-dt^2 + dy^2) + H^{1/2}(du^2 + u^2 d\theta^2) + d\mathbf{z}^2, \tag{7.1}$$

where $H = H_3 H_{3'} = h^2$ with $h = 1 + \frac{L^2}{u^2}$.

The mass shell condition $\mathcal{H} = g^{MN} p_M p_N = 0$ holds exactly in ten dimensions, but we allow nonzero momenta associated to the internal $\mathbf{z}$ coordinates. In the Hamiltonian formalism, imposing the mass-shell condition yields

$$P_u^2 = h^2(u)\mathcal{E}^2 - m^2 h(u) - \frac{J^2}{u^2}, \quad \mathcal{E}^2 = E^2 - p_y^2, \quad m^2 = |\mathbf{p}_\mathbf{z}|^2. \tag{7.2}$$

The effective potential, the massive particle is subject to, is given by

$$\frac{V_{\text{eff}}(u)}{\mathcal{E}^2} = \frac{u^2}{u^2 + L^2}\left(\mu^2 + \frac{b^2}{u^2 + L^2}\right), \qquad \mu = \frac{m}{\mathcal{E}}. \tag{7.3}$$

The radial turning points are

$$u_\pm^2 = \frac{\lambda^2}{\nu}\left(1 \pm \sqrt{1 - \nu^2}\right) \tag{7.4}$$

with

$$\nu = \frac{2L^2\sqrt{1 - \mu^2}}{b^2 + \mu^2 L^2 - 2L^2}, \qquad \lambda = \frac{L}{\sqrt[4]{1 - \mu^2}}. \tag{7.5}$$

Obviously $\mu^2 < 1$ and we are interested in the regime in which $\nu < 1$. The effective potential (7.3) is symmetric under exchange $u \to -u$ and this fact is reflected in the position of the roots on real axis:

$$u_+ > u_- > 0 > -u_- > -u_+. \tag{7.6}$$

The expressions for scattering and in-spiralling angles are, respectively,

$$\Delta\phi_{\rm scatt} = -\beta \int_{\infty}^{u_+} \frac{du}{\sqrt{(u^2-u_+^2)(u^2-u_-^2)}} \tag{7.7}$$

$$\Delta\phi_{\rm fall} = -\beta \int_{u_-}^{0} \frac{du}{\sqrt{(u^2-u_+^2)(u^2-u_-^2)}}, \tag{7.8}$$

where $\beta = b/\sqrt{1-\mu^2}$. It is crucial to note that the roots of (7.4) obey $u_1 u_2 = \lambda^2$, so starting from the scattering angle and performing the coordinate transformation $v = \frac{u_1 u_2}{u} = \frac{\lambda^2}{u}$, which leaves the photon sphere at $u_c(L,\mu) = \lambda$ fixed, it is very easy to demonstrate that $\Delta\phi(\infty, u_+) = \Delta\phi(u_-, 0)$.

The generalization to $L_3 \neq L_{3'}$ is straightforward. We only write down the expression for the radial momentum

$$P_u^2 = H_3 H_{3'}\mathcal{E}^2 - m^2 H_3 - m'^2 H_{3'} - \frac{J^2}{u^2}$$
$$= \mathcal{E}^2(1-\mu^2-\mu'^2)\left[1 + \frac{L^2(1-\mu^2)+L'^2(1-\mu'^2)-b^2}{(1-\mu^2-\mu'^2)u^2} + \frac{L^2L'^2}{(1-\mu^2-\mu'^2)u^4}\right] \tag{7.9}$$

with $m^2 = |\mathbf{p}|^2 = \mu^2\mathcal{E}^2$ and $m'^2 = |\mathbf{p}'|^2 = \mu'^2\mathcal{E}^2$. The angular deflection is given by

$$\Delta\phi = \tilde{b} \int_{u_i}^{u_f} \frac{du}{\sqrt{u^4 + (\tilde{\alpha}^2 - \tilde{b}^2)u^2 + \lambda^4}} \tag{7.10}$$

with

$$\tilde{b} = b/\sqrt{1-\mu^2-\mu'^2}, \qquad \tilde{\alpha}^2 = \frac{L^2(1-\mu^2)+L'^2(1-\mu'^2)}{1-\mu^2-\mu'^2}$$
$$\lambda^2 = LL'/\sqrt{1-\mu^2-\mu'^2}. \tag{7.11}$$

The turning points are

$$u_\pm^2 = \frac{1}{2}[\tilde{b}^2 - \tilde{\alpha}^2 \pm \sqrt{(\tilde{b}^2-\tilde{\alpha}^2)^2 - 4\lambda^4}] \tag{7.12}$$

so that the critical impact parameter is given by

$$\tilde{b} = \sqrt{\tilde{\alpha}^2 + 2\lambda^2} \tag{7.13}$$

while the photon sphere is located at

$$\tilde{u}_c = \lambda = \frac{\sqrt{LL'}}{\sqrt[4]{1-\mu^2-\mu'^2}}. \tag{7.14}$$

Using the homological argument or the conformal inversion $u \to \lambda^2/u$ one easily proves the identity

$$\Delta\phi_{\rm scatt}^{mKK} = \Delta\phi_{\rm fall}^{mKK} \tag{7.15}$$

for generic KK masses and D3 and D3' charges.

B. D3-D3-D3-D3 with massive BPS probes

The generalization to 4-d BHs with four charges associated to four stacks of intersecting D3-branes is subtler. For six generic KK momenta $\mathbf{p}_{ij} = \mathbf{p}_{ji}$ with $i \neq j$, $i, j = 1, \ldots 4$, such that $m_{ij}^2 = |\mathbf{p}_{ij}|^2 = \mu_{ij}^2 E^2$, satisfy the 10-d mass-shell condition, the radial momentum is given by

$$P_u^2 = \mathcal{E}^2 \prod_{i=1}^{4} H_i(u) - \sum_{i<j}^{6} m_{ij}^2 H_i H_j - \frac{J^2}{u^2}$$
$$= \mathcal{E}^2(1-\mu^2)\left[1 + \frac{\sigma_1}{u} + \frac{\sigma_2 - \tilde{b}^2}{u^2} + \frac{\sigma_3}{u^3} + \frac{\sigma_4}{u^4}\right], \tag{7.16}$$

where $\mu^2 = \sum_{i<j}\mu_{ij}^2$, $\tilde{b} = b/\sqrt{1-\mu^2}$ and

$$\sigma_1 = \frac{\sum_i Q_i(1-\sum_{j\neq i}\mu_{ij}^2)}{1-\mu^2}, \qquad \sigma_2 = \frac{\sum_{i<j} Q_iQ_j(1-\mu_{ij}^2)}{1-\mu^2} \tag{7.17}$$

$$\sigma_3 = \frac{\sum_{i<j<k} Q_iQ_jQ_k}{1-\mu^2}, \qquad \sigma_4 = \frac{Q_1Q_2Q_3Q_4}{1-\mu^2}. \tag{7.18}$$

Conformal inversions of the Couch-Torrence kind correspond to

$$u \to \frac{\sqrt{\sigma_4}}{u}. \tag{7.19}$$

This is a symmetry of the metric if and only if

$$\sigma_1\sqrt{\sigma_4} = \sigma_3 \tag{7.20}$$

which is a nontrivial constraint on Q_i and m_{ij}, whose solution, up to permutations, is

$$Q_1 = xQ_4 \qquad Q_2 = yQ_4 \qquad Q_3 = \lambda^2 xyQ_4 \tag{7.21}$$

with

$$\mu_{12}^2 = \mu^2 \quad \lambda = \sqrt{1-\mu^2} \quad \mu_{ij}^2 = 0 \quad \text{for } (i,j) \neq (1,2) \tag{7.22}$$

or

$$\mu_{34}^2=\mu^2 \quad \lambda=\frac{1}{\sqrt{1-\mu^2}} \quad \mu_{ij}^2=0 \ \text{ for } (i,j)\neq(3,4). \tag{7.23}$$

In these cases (constrained charges and KK momentum) the photon sphere is located at $u_c = \sqrt[4]{\sigma_4}$ and it is easy to check that

$$\Delta\phi_{\text{scatt}}^{KK} = \Delta\phi_{\text{fall}}^{KK} \tag{7.24}$$

either by algebraic or numerical means.

However, for non-BPS particles with arbitrary masses and couplings to the geometry we do not expect the identity to hold.

VIII. HIGHER-DIMENSIONAL BHs AND BRANES

After the success obtained for D3-branes and intersecting D3-brane systems, it seems quite natural to inquire whether BHs and branes in higher dimensions that expose a photon sphere admit a similar inversion symmetry. We anticipate that the answer is negative. The basic reason is the very different behavior of the geometry at infinity from the geometry at the horizon.

Nevertheless we briefly analyze the five-dimensional nonrotating case and even more briefly sketch the generalization to higher dimensions and AdS asymptotics.

A. Nonrotating (BPS) BHs in five dimensions

In type IIB compactifications on T^5, five-dimensional nonrotating BHs with nonzero horizon area can be constructed by superposing Q_5 D5-branes, Q_1 D1-branes and Kaluza-Klein momentum Q_p. The Q_5 D5-branes are wrapped on T^5. The Q_1 D-strings are wrapped along one of the directions of the torus and the KK momentum $P = N/R$ along the string. The solution is given in terms of three harmonic functions H_1, H_5 and H_P:

$$H_1=1+\frac{Q_1}{u^2}, \quad H_5=1+\frac{Q_5}{u^2}, \quad H_p=1+\frac{Q_p}{u^2} \tag{8.1}$$

with $u^2 = x_1^2 + \cdots + x_4^2$. The metric reduced to five dimensions in spherical coordinates is

$$ds_5^2 = -\frac{dt^2}{(H_1H_5H_p)^{\frac{2}{3}}} + (H_1H_5H_p)^{\frac{1}{3}}[du^2 + u^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)]. \tag{8.2}$$

In the hyperplane $\theta = \pi/2$, setting $H = H_1H_5H_p$, the Hamiltonian for massless probes can be written as

$$0 = \mathcal{H} = \frac{1}{2}\left[-H^{2/3}E^2 + H^{-1/3}\left(P_u^2 + \frac{J^2}{u^2}\right)\right]. \tag{8.3}$$

If, for simplicity, we consider $Q_1 = Q_5 = Q_p = Q$, the zeros of P_u satisfy

$$u^6 + (3Q - b^2)u^4 + 3Q^2u^2 + Q^3 = 0. \tag{8.4}$$

Setting $\zeta = u^2/Q, \beta^2 = b^2/Q$, (8.4) becomes

$$\zeta^3 + (3 - \beta^2)\zeta^2 + 3\zeta + 1 = 0. \tag{8.5}$$

This third degree equation has three real roots ζ_i, $i = 1, 2, 3$ only for $\beta > 3\sqrt{3}/2$, which are such that

$$\zeta_1 > \zeta_2 > 0 > \zeta_3. \tag{8.6}$$

The angle described by the particle coming from infinity and reaching the turning point $u_1 = \sqrt{Q\zeta_1}$ is given by

$$\begin{aligned}\phi(\infty, r_1) &= -b\int_\infty^{u_1}\frac{rdr}{\sqrt{(u^2-Q\zeta_1)(u^2-Q\zeta_2)(u^2-Q\zeta_3)}}\\ &\overset{u^2=x}{=} -\frac{b}{2}\int_\infty^{u_1^2=Q\zeta_1}\frac{dx}{\sqrt{(x-Q\zeta_1)(x-Q\zeta_2)(x-Q\zeta_3)}}\\ &\overset{Q\zeta_1/x=\xi}{=} \frac{b}{2\sqrt{Q}}\sqrt{\frac{-\zeta_1}{\zeta_2\zeta_3}}\int_0^1\frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-\frac{\zeta_1}{\zeta_2})(\xi-\frac{\zeta_1}{\zeta_3})}}.\end{aligned} \tag{8.7}$$

Since $\zeta_1/\zeta_2 > 1 > 0 > \zeta_1/\zeta_3$, the last integral in (8.7) can be written in terms of the complete elliptic integral of the first kind:

$$\phi(\infty, u_1) = \frac{b}{\sqrt{Q\zeta_{13}}}\mathcal{K}\left[\sqrt{\frac{\zeta_{23}}{\zeta_{13}}}\right]. \tag{8.8}$$

The in-spiralling angle can be computed in a similar way:

$$\begin{aligned}\phi(u_2, 0) &= -b\int_{u_2}^0\frac{udu}{\sqrt{(u^2-Q\zeta_1)(u^2-Q\zeta_2)(u^2-Q\zeta_3)}}\\ &\overset{u^2x}{=} -\frac{b}{2}\int_{u_2^2}^0\frac{dx}{\sqrt{(x-Q\zeta_1)(x-Q\zeta_2)(x-Q\zeta_3)}}\\ &\overset{\xi x/u_2^2}{=} \frac{b}{2u_2}\int_0^1\frac{d\xi}{\sqrt{(\xi-\frac{\zeta_1}{\zeta_2})(\xi-1)(\xi-\frac{\zeta_3}{\zeta_2})}}.\end{aligned} \tag{8.9}$$

Since $\zeta_1/\zeta_2 > 1 > 0 > \zeta_3/\zeta_2$, the last integral in (8.9) can be written in terms of incomplete elliptic integral:

$$\phi(u_2, 0) = \frac{b}{\sqrt{Q\zeta_{13}}}\mathcal{K}\left[\arcsin\sqrt{\frac{\zeta_2\zeta_{13}}{\zeta_1\zeta_{23}}}; \sqrt{\frac{\zeta_{23}}{\zeta_{13}}}\right]. \tag{8.10}$$

Notice that (8.8) and (8.10) are different in general and cannot be rendered equal for any choice of β. The deep reason of the inequality is the lack of symmetry between horizon $u_H = 0$ and infinity $u \to \infty$. While the latter is a branching point the former is a regular point. Moreover as we discuss more extensively momentarily, generalized Freudenthal duality in $D = 5$ exchanges particles with strings.

B. Other p-branes

The relevant formula (under square root in the denominator) viz.

$$u^4 P_u^2 = F(u) = u^4 H_p(u) - u^2 b^2 \tag{8.11}$$

suggests absence of a photon sphere (critical geodesics) for $p \geq 5$ and its presence for $p \leq 4$. We already discussed at length the case $p = 3$. Let us consider the other cases with $p \leq 4$, for which

$$F(u) = u^4 H_p(u) - u^2 b^2 = u^4 + L^{7-p} u^{p-3} - b^2 u^2 \tag{8.12}$$

so that

$$u_c = \sqrt{7-p}\sqrt{\frac{5-p}{2}} L_p, \qquad b_c = \sqrt{\frac{7-p}{5-p}} u_c \tag{8.13}$$

and

$$\Delta\phi = \int \frac{b du}{\sqrt{u^4 + L^{7-p} u^{p-3} - b^2 u^2}}. \tag{8.14}$$

More explicitly one has

$$p = 4\colon\ \Delta\phi = \int \frac{b du}{\sqrt{u^4 + L^3 u - b^2 u^2}} \tag{8.15}$$

$$p = 2\colon\ \Delta\phi = \int \frac{b du \sqrt{u}}{\sqrt{u^5 + L^5 - b^2 u^3}} \tag{8.16}$$

$$p = 1\colon\ \Delta\phi = \int \frac{b u du}{\sqrt{u^6 + L^6 - b^2 u^4}} = \int \frac{b d\xi}{2\sqrt{\xi^3 + L^6 - b^2 \xi^2}} \tag{8.17}$$

$$p = 0\colon\ \Delta\phi = \int \frac{b u^{3/2} du}{\sqrt{u^7 + L^7 - b^2 u^5}}. \tag{8.18}$$

In all of the above integrals the behavior at the horizon $u_H = 0$ is very different from the one at infinity $u \to \infty$ and no obvious inversion symmetry can be envisaged that exchanges the two and keeps the photon sphere fixed. At present we cannot exclude a symmetry under a generalized inversion such as $u \to L^{1+a} u^{-a}$ with a not an integer, that would however be noninvolutive.

One argument that should help explaining this problem is the fact that Freudenthal duality would exchange point particles with strings in $D = 5$ [47,48]. Generalized Freudenthal duality should exchange p-branes with $D - 4 - p$-branes. This is why self-dual objects like particles/BHs in $D = 4$, D3 in $D = 10$ or strings in $D = 6$ enjoy this property.[10]

IX. EIKONAL PHASE AND RADIAL ACTION

Let us now discuss possible implications of our classical geodetic analysis for scattering amplitudes in a putative quantum theory of gravity, such as string theory. In the eikonal limit, valid for large impact parameters $b > b_c$, the scattering amplitude of a (massless) probe off a spherically symmetric target is given by the exponential of the eikonal phase [57–62]

$$\widetilde{\mathcal{S}}(b, E) = 1 + i\widetilde{\mathcal{T}}(b, E) = 1 + i\frac{\widetilde{\mathcal{A}}(b, E)}{2E} \approx e^{2i\delta_{eik}(J=bE,E)}, \tag{9.1}$$

where $\widetilde{\mathcal{A}}(b, E)$ is the scattering amplitude in impact parameter space

$$\widetilde{\mathcal{A}}(b, E) = \int \frac{d^{d-2}q}{(2\pi)^{d-2}} e^{i\vec{q}\cdot\vec{b}} \mathcal{A}(\vec{q}, E) \tag{9.2}$$

with $\vec{q}$ the transferred (spacelike) momentum. In turn δ_{eik} can be written in terms of the radial action

$$\delta_{eik}(J, E) \approx S_r(J, E; r_i, r_f) = \int_{r_i}^{r_f} P_r(J, E) dr = \int_{r_i}^{r_f} \sqrt{E^2 - f(r)\frac{J^2}{r^2}} \frac{dr}{f(r)}. \tag{9.3}$$

Observables such as the deflection angle and the time delay are then given as derivatives of δ_{eik}, viz.

$$\Delta\phi(J, E) = \frac{\partial\delta_{eik}}{\partial J}, \qquad \Delta t(J, E) = -\frac{\partial\delta_{eik}}{\partial E}. \tag{9.4}$$

When r_i or r_f are taken to infinity or to the horizon there might be divergences that can be subtracted or regulated by introducing a boundary/wall such as in AdS.

Indeed in asymptotically flat space-times $P_r \approx \sqrt{E}$ at very large r and $S_r \approx \sqrt{E}R$ is linearly divergent with the cutoff R as expected for a nearly free particle. Yet, even in

[10] The case of D2, which admits a photon sphere as we have already seen, in $D = 8$ is subtler since no solution of the form $AdS_4 \times S^4 \times \mathcal{M}$ with $\mathcal{M}$ some compact manifold (such as T^2 for strings or T^3 for M-theory) seems to be known. The best one can do is $AdS_4 \times CP^3$ in type IIA or $AdS_4 \times S^7/Z_k$ in M-theory.

this case, the deflection angle $\Delta\phi$ remains finite for $b < b_c$ and diverges only for $b = b_c$, i.e. for critical geodesics. This is not the case for Δt that obviously diverges, unless the process takes place in an asymptotically AdS (aAdS) [60–62]. The raising term r^2/ℓ^2 in $f(r)$ regulates the integral for $J < \ell E$, i.e. for $b < \ell$ as well as it derivatives.

Whether in aAdS or not, the (regulated) radial action is the crucial ingredient in the semiclassical dynamics of massless or massive probes in the presence of BHs, D-branes or other gravitating objects. Thanks to the identity $\Delta\phi_{\rm scatt}(J,E) = \Delta\phi_{\rm fall}(J,E)$, which extends to $S_r(J,E;r_1,\infty) = S_r(J,E;r_2,r_H)$ when both are finite or up to subtractions, one can explore near-horizon dynamics by performing experiments in the asymptotically flat region.

Let us stress once again that the identity relies on generalized inversions *à la* Couch-Torrence [43], which is a conformal symmetries of the metric for D3-branes, D3-D3' and for 4-d large BPS BHs with four charges, which satisfy the condition $Q_1Q_2 = Q_3Q_4$ or permutations thereof, such that the photon sphere is the fixed locus of conformal inversion. Moreover it extends *mutatis mutandis* to massive (BPS) probes whose geodetic equations are not conformal invariant.

Another story is the fate of the inversion symmetry at the quantum level. Since it acts by conformal transformations of the metric it is likely to be anomalous. Yet, being an element of the U-duality group in special cases (when e.g. $\sum Q = \sqrt{\prod Q \sum Q^{-1}}$) it may survive in a full quantum theory of gravity as string theory. In this context, as already mentioned, it may help bypassing issues of extrapolating the results for large $b \gg b_c$ to small $b < b_c$ since physics at the horizon may be captured by physics at flat infinity thanks to the remarkable property we found.

X. SUMMARY AND CONCLUSIONS

Let us summarize the results of our investigation and identify some lines for future study.

We have shown that many BPS systems admitting a photon sphere enjoy a peculiar symmetry under conformal inversions that keep the photon sphere fixed and exchange horizon and flat infinity. Since the dynamics of "massless" probes in backgrounds of this form is Weyl invariant, we have found that the scattering angle $\Delta\phi_{\rm scatt}(J,E)$ for a probe impinging from infinity and scattering off the compact gravitating center for $b > b_c$ exactly coincides with the in-spiralling angle $\Delta\phi_{\rm fall}(J,E)$ for a probe emitted from inside the photon sphere and falling into the horizon with the very same energy E and angular momentum J. Despite similarity with the B2B formula [7,8] relating periastron advance to scattering angle, we should stress once again that the latter requires an analytic continuation to negative E, while in our case E is positive and thus measurable.

Playing with numbers one can formally increase the mass m of the probe but the validity of our analysis would be jeopardized. Yet for extreme mass ratio in spirals where $m \ll M$ our analysis is reliable and may shed some light on the highly nonlinear merging phase that can only be tackled by numerical methods at present.

On astrophysical grounds, the obvious limitation is the BPS nature of the systems we have analyzed. This reflects in their charge(s) and the absence of angular momentum. In higher dimensions, i.e. $d \geq 5$, rotating BHs are compatible with BPS conditions and we plan to further investigate this issue despite the lack of a simple inversion symmetry already for (BPS) nonrotating BHs in $d \geq 5$. Yet the presence of a photon sphere or rather a photon halo (with u_c varying in some interval depending on b_c) suggests that one should try and find a way to explore its interior (up to the horizon) by some generalized inversion that goes beyond Couch-Torrence inversions, whereby $u \to u_c^2(b)/u$ depends on E and J along the lines of [44].

Even more intriguing is fate of the identity for horizonless objects or fuzzballs such as 2-charge microstates or the family of solutions found by Jejjala, Marsden, Ross and Teschner in Ref. [63], including their BPS limit (the family of solutions found by Giusto, Mathur and Saxena in Ref. [64]). It is tempting to conjecture that the relevant inversion keeping the photon sphere (or rather photon halo in these cases) fixed should exchange infinity with the "regular" origin or cap. Similar issues are raised by the study of QNMs [9–12] that, needless to say, crucially depend on the photon sphere. We plan to study the inversion properties for waves, quantum particles and strings in the near future.

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