Group quantization of the black hole minisuperspace

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The emergence of nontrivial symmetries for black holes minisuperspaces has been recently pointed out. These Noether symmetries possess non-null charges and hence map physical solutions to different ones. The symmetry group is isomorphic to the finite-dimensional Poincaré group $\mathrm{ISO}(2,1)$, whose irreducible representations are well known. This structure is used to build a consistent quantum theory of black hole minisuperspace. This has, among other consequences, the striking consequence of implying a continuous spectrum for the mass operator. Following loop quantum cosmology, we obtain a regularization scheme compatible with the symmetry structure. It is possible to study the evolution of coherent states following the classical trajectories in the low curvature regime. We show that this produces an effective metric where the singularity is replaced by a Killing horizon merging two asymptotically flat regions. The quantum correction comes from a fundamental discreteness of spacetime, and the uncertainty on the energy of the system. Remarkably, the effective evolution of semiclassical states is described by an effective Hamiltonian, related to the original one through a canonical transformation.

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I. INTRODUCTION

Black holes are one of the most fascinating predictions of general relativity, and the interest in their properties has been growing since the rise of experimental ability, thanks to gravitational wave detection and very recent black hole imagery. But black holes in classical general relativity are inevitably associated with singularities, signals of the breakdown of the classical theory. It is widely expected that approaching the classical singularity, where the curvature becomes Planckian, quantum effects become important. In a full quantum theory of spacetime, singularities would be replaced by a unitary evolution through a fuzzy geometry.

Unfortunately, extracting information about the fate of black hole singularity from a specific quantum theory of spacetime remains an outstanding challenge. For any known quantum gravity theory, there is indeed no straightforward way to determine the physical quantum states representing black holes, even for the simplest case of spherically symmetric geometries.

It is nonetheless possible to incorporate some features of the full theories into minisuperspace models with a finite number of degrees of freedom. Consistent efforts have been done in this direction, starting from the nonperturbative and background independent theory of Loop Quantum Gravity (LQG) and related approaches [1–4].

Concerning black hole quantization, the majority of the works are built on the isometry between black hole interior and Kantowski-Sachs cosmology [5–25], allowing one to import techniques originally developed in the cosmological setting. In Loop Quantum Cosmology (LQC), the quantum effects are claimed to be captured by a phase space regularization, so-called *polymerization*, that encodes the fundamental discreteness of spacetime at small scales. In the first place, the symmetries of spacetime (e.g., homogeneity or spherical symmetry) are imposed classically; then, quantization is done on a latticelike regularization of the minisuperspace model, where the essential operators are holonomies of connection and areas. See e.g., Ref. [26] for a review on LQC.

In flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, the calculation of expectation values on suitable semiclassical states provides an effective evolution where the big bang singularity is replaced by a bounce [27–29]. The effective evolution can be seen as generated by a regularized classical Hamiltonian, where the canonical momenta (say p) are replaced by the polymerized version $\sin(\lambda p)/\lambda$, where λ is a UV cutoff, typically related to the Planck length, that could be phase space dependent.

As with any canonical quantization, this procedure suffers from ambiguities. On top of the usual ordering issues, polymerization adds another level of ambiguity. To define the size of the lattice, we should indeed specify a phase space (in)dependence of the regularization parameter(s). Different choices of UV regularizations lead to drastically different dynamics [30–32]. In FLRW models, there is a solid consensus in favor of the choice know as $\bar{\mu}$-scheme, even if some ambiguities still remain [33–36].

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Despite the large effort, for the black hole dynamics, no agreement has been found yet [17–25,37–43]. Moreover, the majority of the works about black hole dynamics start directly by the heuristic effective dynamics introducing by hand the polymer correction. In the absence of fully controlled quantum dynamics, the question of the equivalence between the effective evolution and the expectation values of quantum states is rarely addressed.

Nevertheless, it is worth noting that, despite the technical differences between the various approaches, the effective models share common features, like the replacement of the singularity with a transition surface from a black hole to a white hole interior.

An important question that is raised by the effective dynamics approach is whether the resulting spacetime posses covariance. In a more general setting (inhomogeneous), this consistency check is provided by the requirement that the constraints algebra remains anomaly free after regularization [44,45]. In the homogeneous minisuperspace model, this criterion is useless, as we are left only with a sole scalar constraint, trivially commuting with itself.

In a recent work [46], it has been unraveled that the black hole interior homogeneous model actually posses a nontrivial and finite-dimensional symmetry algebra, isomorphic to the $\mathfrak{iso}(2,1)$ Poincaré algebra, that fully encodes the dynamics on the phase space. This is a generalization of what happens in flat FLRW cosmology coupled with a scalar field, in the isotropic case and for the Bianchi I model [47–54], and it has very recently extended to (anti-)de Sitter Schwarzschild solutions [55].

This invariance has been used as a guiding principle to build a polymerization scheme in which the symmetry is protected, replacing the argument of preservation of the constraint algebra in the full theory. The new symmetry also opens the door to a group quantization of the model, in the spirit of what has been developed for the cosmological model [48,49]. The present work aims to exploit the representation theory of the $2+1$ Poincaré group to build a quantum theory of the black hole interior.

The outline of the paper is as follows. I start by reviewing the classical setup for black holes minisuperspace in Sec. II. I recall there the construction of Ref. [46], embedding a massless and zero spin realization of the $\mathfrak{iso}(2,1)$ algebra into the mechanical phase space. This algebra corresponds to the evolving version of the Noether charges of the $\mathrm{ISO}(2,1)$ symmetry [46]. Arguing that such asymmetry could be used as a guiding principle for quantization, we build a quantum theory of black hole minisuperspace in Sec. III. Thanks to the knowledge of the energy eigenstates, we can impose the dynamics (Sec. III B) on the Hilbert space. This is done in two different ways by respectively strongly and weakly fixing the energy level of the system, and the resulting evolution of the coherent states leads to substantially different effective metrics. The group quantization also gives a very strong prediction about the continuity of the mass spectrum (III C). I end this work with a last section, Sec. IV, with an analysis of the possible LQC-like deformation of the phase space, such that the symmetry is preserved.

II. CLASSICAL SYMMETRIES OF BLACK HOLE MINISUPERSPACE

I start by recalling the notations introduced in the precedent work [46]. To study the dynamics of the spherically symmetric homogeneous Kantowski-Sachs cosmology, describing the Schwarzschild interior, we consider the line element

$$\mathrm{d}s^2 = -N(t)^2 \frac{V_1(t)}{2V_2(t)}\mathrm{d}t^2 + \frac{8V_2(t)}{V_1(t)}\mathrm{d}x^2 + L_s^2 V_1(t)\mathrm{d}\Omega^2, \quad (2.1)$$

where x runs over the real line and $\mathrm{d}\Omega^2$ is the metric on the unit sphere at constant x and t, meaning that the spatial slices have the topology $\mathbb{R}\times S^2$. To get dimensionless fields V_i's, we need to introduce a fiducial radius of the two-sphere L_s. The system is homogeneous in the sense that the dynamical fields N, V_i depend only on time.

Considering the case of gravity, without matter, and a vanishing cosmological constant, the dynamics is described by Einstein-Hilbert action, integrating the Ricci scalar of (2.1) over a finite slice at constant t. For this, we must introduce another fiducial scale L_0 in the noncompact x direction. It plays the role of an infrared cutoff that regulates the integration. Because of homogeneity, the finite slice contains information about the whole spacetime, and the action now describes the dynamics of a mechanical system evolving in time,

$$\begin{aligned} \mathcal{S}^{(t)}_{\mathrm{EH}}[N,V_i] &= \frac{1}{16\pi L_P^2}\int \mathrm{d}^4x\sqrt{|g|}\mathcal{R} \\ &= \frac{L_0}{L_P^2}\int \mathrm{d}t\left[N + L_s^2\frac{V_1'(V_2V_1' - 2V_1V_2')}{2NV_1^2}\right. \\ &\quad \left.+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L_s^2}{2NV_1}(V_1V_2)'\right)\right], \end{aligned} \quad (2.2)$$

where L_P is the Planck length and the prime denotes the derivative with respect to coordinate time t. The total derivative at the end is exactly compensate by the contribution of the Gibbons-Hawking term associated to constant time hypersurfaces. We will drop it in the following, leaving only the first order Lagrangian to describe the classical dynamics of the system. For the following, we will also consider the dimensionless quantity

$$\kappa = \frac{L_0 L_s^2}{L_P^3}, \quad (2.3)$$

which encodes the ratio between IR fiducial length and UV scale of the theory, represented by the Planck length. The diffeomorphism invariance of general relativity has been completely fixed, except for the time reparametrization. We face now two possibilities concerning the role of the lapse and the gauge freedom of the time coordinate [46]. On the one hand, we can work in coordinate time t, and the equation of motion obtained varying the action with respect to the lapse N will correspond the so-called Hamiltonian constraint. This is explicitly given by

$$\frac{\delta \mathcal{S}_{\rm EH}^{(t)}}{\delta N}=0 \Leftrightarrow 0 \approx \mathcal{C}_{\mathcal{H}}=-\frac{\kappa L_P}{L_s^2}+\frac{\kappa L_P}{N^2}\left[\frac{V_2 V_1'^2}{2V_1^2}-\frac{V_1' V_2'}{V_1}\right]. \quad (2.4)$$

On the other hand, this is completely equivalent to introducing a *proper time* gauge $\mathrm{d}\tau = N\mathrm{d}t$. With this redefinition, the lapse completely disappears from the action. Moreover, the potential term coming from the intrinsic curvature of the slice becomes a boundary term, and as such can be simply discarded from the action. The latter becomes

$$\mathcal{S}_0[V_i] \equiv \kappa L_P \int \mathrm{d}\tau \frac{\dot{V}_1(V_2\dot{V}_1 - 2V_1\dot{V}_2)}{2V_1^2}, \quad (2.5)$$

where now the dot represents the derivative with respect to τ. The scalar constraint $\mathcal{C}_{\mathcal{H}} \approx 0$ is now translated into a relationship between the on-shell value of the physical Hamiltonian of $\mathcal{S}_0$, the fundamental scale, and the IR cutoff. This relation is crucial when inserting the on-shell fields into the line element to recover the right solution. In order to better see the equivalence between the two approaches, we perform the canonical analysis of the two actions. In both cases, the conjugate momenta are given by

$$\left|\begin{array}{l} P_1 = \kappa L_P \frac{V_2\dot{V}_1 - V_1\dot{V}_2}{V_1^2} = \kappa L_P \frac{V_2 V_1' - V_1 V_2'}{N V_1^2}, \\ P_2 = -\kappa L_P \frac{\dot{V}_1}{V_1} = -\kappa L_P \frac{V_1'}{N V_1}, \end{array}\right. \quad (2.6)$$

and the Legendre transform gives the form of the Hamiltonian:

$$\mathcal{S}_0 = \int \mathrm{d}\tau (P_i \dot{V}_i - H),$$
$$H = -\frac{1}{\kappa L_P}\left(P_1 P_2 V_1 + \frac{P_2^2 V_2}{2}\right). \quad (2.7)$$

The canonical Poisson structure is $\{V_i, P_j\} = \delta_{ij}$. The dynamics in the two descriptions is described respectively by an Hamiltonian constraint $\mathcal{C}_{\mathcal{H}}$ for the t dependent theory and a true Hamiltonian H for the evolution with respect to the gauge fixed proper time. They satisfy the equality

$$\mathcal{C}_{\mathcal{H}} \equiv N\left(H - \frac{\kappa L_P}{L_s^2}\right)$$
$$= -\frac{N}{\kappa L_P}\left(\frac{\kappa^2 L_P^2}{L_s^2} + V_1 P_1 P_2 + \frac{1}{2} V_2 P_2^2\right). \quad (2.8)$$

For an arbitrary phase space observable $\mathcal{O}$, we could equivalently describe the dynamics in terms of t or τ, by computing the Poisson brackets of the observable with the respective Hamiltonian density: $\mathcal{O}' = \{O, \mathcal{C}_{\mathcal{H}}\}$, $\dot{\mathcal{O}} = \{\mathcal{O}, H\}$. The *on-shell* equivalence between the two dynamics is ensured by the vanishing of the scalar constraint, even for lapse choices that are field dependent[1]:

$$\mathcal{O}' = \{\mathcal{O}, \mathcal{C}_{\mathcal{H}}\} = N\left\{\mathcal{O}, H - \frac{\kappa L_P}{L_s^2}\right\} + \cancel{\left(H - \frac{\kappa L_P}{L_s^2}\right)}\{\mathcal{O}, N\}$$
$$\approx N\{\mathcal{O}, H\} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\dot{\mathcal{O}}. \quad (2.9)$$

In the following, we will work in proper time and consider the one-dimensional action $\mathcal{S}_0$, without any lapse or potential term, but we shall remember to relate the latter to the value of the Hamiltonian. Doing so, the equations of motion of the reduced action (2.5) are equivalent to the Einstein equations for the metric (2.1) (with $N = 1$). A straightforward calculation leads to the classical solutions [46]

$$V_1 = \frac{A}{2L_P^2\kappa^2}(\tau - \tau_0)^2, \quad (2.10a)$$

$$P_1 = \frac{2BL_P^2\kappa^2}{A(\tau - \tau_0)^2}, \quad (2.10b)$$

$$V_2 = \frac{B}{L_P\kappa}(\tau - \tau_0) - \frac{1}{2L_s^2}(\tau - \tau_0)^2, \quad (2.10c)$$

$$P_2 = -\frac{2L_P\kappa}{\tau - \tau_0}, \quad (2.10d)$$

where the constraint on the value of the Hamiltonian $H = \kappa L_P/L_s^2$ has already been imposed. The other quantities A, B, τ_0 are integration constants. If we insert these solutions back into the line element (2.1), and perform the change of coordinates

$$\tau - \tau_0 = \sqrt{\frac{2}{A}}\frac{L_P\kappa}{L_s}T, \qquad x = \frac{L_s}{2L_P\kappa}\sqrt{\frac{A}{2}}r, \quad (2.11)$$

we find the standard Schwarzschild black hole interior metric, where the mass is given by

$$M = \frac{B\sqrt{A}L_s^3}{\sqrt{2}L_P^2\kappa^2} = \frac{B\sqrt{A}L_P^4}{\sqrt{2}L_0^2 L_s}. \quad (2.12)$$

Notice that the singularity is located at $\tau = \tau_0$, when both the classical solutions for V_1 and V_2 vanish, while the horizon is at $\tau - \tau_0 = \frac{2BL_s^2}{L_P\kappa}$, where only V_2 is zero. Although the action $\mathcal{S}_0$ has been introduced starting from the black hole interior

[1]For the two theories to be equivalent and self-consistent, the fiducial scales must be constant.

and should be in principle limited to the range where τ is timelike, the classical solutions for the fields V_i are regular on the whole real line, smoothly crossing the singularity at $\tau = \tau_0$.

In Ref. [46], it is shown that all the dynamical information of the metric fields is encapsulated into a phase space structure isomorphic to the $\mathfrak{iso}(2,1)$ Poincaré algebra. This is provided by the evolving generators associated with the ISO$(2,1)$ invariance of the mechanical system described by the action (2.5). The main idea behind the construction in Ref. [46] is that solving the Hamilton equations is the same as exponentiating the flow of the Hamiltonian vector field, by iteratively computing the Poisson's bracket with the Hamiltonian $\{..\{V_i, H\}, ..\}, H\}$. One can check if at some point the iteration closes and forms a Lie algebra. For both the fields, the algorithm stops at the second step

$$C \coloneqq \kappa L_P\{V_2, H\} = -P_1 V_1 - P_2 V_2, \qquad \{\{V_2, H\}, H\} = -\frac{H}{\kappa L_P}, \tag{2.13a}$$

$$D \coloneqq -\kappa L_P\{V_1, H\} = P_2 V_1, \qquad A \coloneqq (\kappa L_P)^2\{\{V_1, H\}, H\} = \frac{V_1 P_2^2}{2}, \tag{2.13b}$$

where A is a first integral of motion, commuting with H. As the Hamiltonian trivially commutes with itself, for both the fields, the third iteration (i.e the third derivative in proper time τ) vanishes. The generators can be rearranged into the usual $\mathfrak{iso}(2,1)$ basis

$$J_z = \frac{V_2}{2\lambda} - \lambda\kappa L_P H, \quad K_x = \lambda\kappa L_P H + \frac{V_2}{2\lambda}, \quad K_y = C, \tag{2.14a}$$

$$\Pi_0 = \frac{V_1}{2\lambda} + \lambda A, \quad \Pi_x = D, \quad \Pi_y = \frac{V_1}{2\lambda} - \lambda A, \tag{2.14b}$$

where λ is a real dimensionless constant. Computing the Poisson brackets between the six generators gives the $\mathfrak{iso}(2,1)$ algebra

$$\{J_z, K_i\} = \epsilon_{ij} K_j, \quad \{K_x, K_y\} = -J_z, \quad \{J_z, \Pi_i\} = \epsilon_{ij}\Pi_j, \quad \{K_i, \Pi_0\} = \Pi_i, \quad \{K_i, \Pi_j\} = \delta_{ij}\Pi_0. \tag{2.15}$$

The two Poincaré Casimirs, representing respectively the *mass* and *spin* of the corresponding irreducible representation, are

$$\mathfrak{C}_1 = -\Pi_0^2 + \Pi_x^2 + \Pi_y^2, \quad \mathfrak{C}_2 = J_z\Pi_0 + K_x\Pi_y - K_y\Pi_x. \tag{2.16}$$

Rewriting the generators in terms of canonical variables, we find that the two Casimirs identically vanish $\mathfrak{C}_i = 0$. This condition is necessary to reduce the six-dimensional Lie algebra back to the original four-dimensional phase space. We have already included one of the first integrals of motion (A) into the algebra. The other one is related to the Casimir operator of the $\mathfrak{sl}(2,\mathbb{R})$ sector as

$$\mathfrak{C}_{\mathfrak{sl}(2,\mathbb{R})} = -J_z^2 + K_x^2 + K_y^2 \coloneqq B^2, \qquad B \coloneqq V_1 P_1, \tag{2.17}$$

that by construction commutes with the Hamiltonian. We shall nonetheless remark that it is not a Casimir operator of the whole algebra and does not select the representation of the algebra, unlike what happens in cosmology [49,50] and conformal mechanics [56].

The intriguing property of this structure is that it can be exponentiated to an actual symmetry of the Lagrangian, generated by the Noether charges corresponding to the initial conditions of the $\mathfrak{iso}(2,1)$ generators (see Ref. [46] for further development and Ref. [50] for the corresponding construction in the isotropic setting).

It is interesting and useful for what follows to rewrite the (squared) mass in terms of Poincaré generators:

$$M^2 = B^2 A \frac{L_P^8}{2L_0^4 L_s^2} \coloneqq L_M^2(-J_z^2 + K_x^2 + K_y^2)\frac{\Pi_0 - \Pi_y}{4\lambda}, \qquad L_M \coloneqq \frac{L_P^4}{L_0^2 L_s}. \tag{2.18}$$

For the sake of simplicity, I introduce L_M, which is a constant length indicating how the UV fundamental length and the fiducial scales couple into the definition of the mass observable. We shall remark that, despite the apparent dependence on the fiducial scale, the mass is unchanged by a rescaling of the IR length, because also A and B change under the rescaling [46].

The quantization of the mechanical system representing the black hole interior will be the subject of the next session, but we can already see from here that the vanishing of the Casimir $\mathfrak{C}_i$ implies that we will deal with massless and spin zero representations of the $2+1$ Poincaré group.

Before moving to the quantum theory, we would like to spend some words on the role of the IR regulator and the presence of boundaries. First of all, we needed to introduce L_0 for the action principle of (2.2) to be well defined. The presence of the second length scale L_s is just a matter of convenience to deal with dimensionless quantities and allows translating from L_0 to κ. These length scales appear in the energy level for the Hamiltonian, but once we put the solution into the line element (2.1), we recover the one-parameter family of black holes with a general mass M, as expected from the no-hair theorem.

We have then an apparent clash between the general relativity (GR) point of view (with a one-parameter family of solutions due to no-hair theorem) and the mechanical point of view with a four-dimensional phase space. Furthermore, the symmetries of the mechanical models presented in Ref. [46] interplay with the fiducial scales.

This is because the symmetry changes the energy value of the solution, and so we must rescale the length in order to restore the constraint (2.4). This is actually a quite common situation in gravity, where the boundary seems to carry some physical information, making some gauge redundancy to become physical. Nonetheless, the precise meaning of the physical relevance of the boundary is yet to be determined. For example, some recent work [57] has pointed out a possible relationship between the conformal properties of the Schwarzschild background and its static perturbation. We will see in the following how the fiducial lengths control the quantum modification of the classical line element.

III. ISO(2, 1) GROUP QUANTIZATION

It is a common expectation that the quantum theory of general relativity should provide an anomaly free representation of the constraint algebra. Going into the study of minisuperspaces, such a requirement is meaningless because we are left with the scalar constraint alone, trivially commuting with itself. The unveiling of the conformal symmetry for cosmology and its extension to black holes suggest that for minisuperspaces we can replace the constraint algebra with the new symmetry algebra. In the case considered in this article, this means that the quantum Hilbert space must contain an irreducible representation of Poincaré algebra. It is then smart to directly start by exploiting the well-known irreducible representation of (the universal cover of) $\mathrm{ISO}(2,1)$. For this, the reader shall refer to an exhaustive discussion in Ref. [58], of which I recall here some key features in the Appendix A. We consider the realization of the algebra (2.15) as self-adjoint operators, acting on wave functions on $\mathbb{R}^2$ in polar coordinates, with the scalar product

$$\langle\psi|\chi\rangle = \int_0^\infty \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \psi^*(\rho,\phi)\chi(\rho,\phi). \tag{3.1}$$

The $\mathfrak{iso}(2,1)$ generators are realized as a one-parameter family, with $s \in \mathbb{R}(\bmod 2)$

$$(\widehat{\Pi_0}^{(s)}\psi)(\rho,\phi) = \rho\psi(\rho,\phi), \tag{3.2a}$$

$$(\widehat{\Pi_x}^{(s)}\psi)(\rho,\phi) = \rho\sin\phi\psi(\rho,\phi), \tag{3.2b}$$

$$(\widehat{\Pi_y}^{(s)}\psi)(\rho,\phi) = \rho\cos\phi\psi(\rho,\phi), \tag{3.2c}$$

$$(\widehat{J_z}^{(s)}\psi)(\rho,\phi) = \left[i\frac{\partial}{\partial\phi} - \frac{s}{2}\right]\psi(\rho,\phi), \tag{3.2d}$$

$$(\widehat{K_x}^{(s)}\psi)(\rho,\phi) = \left[i\rho\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right) - \frac{s}{2}\cos\phi\right] \times \psi(\rho,\phi), \tag{3.2e}$$

$$(\widehat{K_y}^{(s)}\psi)(\rho,\phi) = \left[i\rho\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right) + \frac{s}{2}\sin\phi\right] \times \psi(\rho,\phi). \tag{3.2f}$$

By a straightforward calculation, we can verify that they satisfy the quantum version of the algebra (2.15), replacing the Poisson bracket by the commutator between operators. This realization satisfies the condition that both the Casimirs are zero,

$$\widehat{\mathfrak{C}}_1 \coloneqq -\widehat{\Pi_0}^2 + \widehat{\Pi_x}^2 + \widehat{\Pi_y}^2, \tag{3.3a}$$

$$\widehat{\mathfrak{C}}_2 \coloneqq \frac{1}{2}(\widehat{J_z}\,\widehat{\Pi_0} + \widehat{\Pi_0}\,\widehat{J_z} + \widehat{K_x}\,\widehat{\Pi_y} + \widehat{\Pi_y}\,\widehat{K_x} - \widehat{K_y}\,\widehat{\Pi_x} - \widehat{\Pi_x}\,\widehat{K_y}),$$
$$\widehat{\mathfrak{C}}_2\psi(\rho,\phi) = 0 = \widehat{\mathfrak{C}}_1\psi(\rho,\phi). \tag{3.3b}$$

We could explicitly calculate the action of the $\mathfrak{sl}(2,\mathbb{R})$ Casimir and see that it does not depend on s,

$$(\hat{\mathfrak{C}}^{(s)}_{\mathfrak{sl}(2,\mathbb{R})}\psi)(\rho,\phi) = (-\widehat{J_z}^2 + \widehat{K_x}^2 + \widehat{K_y}^2)\psi(\rho,\phi) = -\rho\left(2\frac{\partial}{\partial\rho} + \rho\frac{\partial^2}{\partial\rho^2}\right)\psi(\rho,\phi). \tag{3.4}$$

Let us finally remark that we can also define the *square root* of this operator, which corresponds to the classical integral B:

$$(\hat{B}\psi)(\rho,\phi) \coloneqq i\left(\rho\frac{\partial}{\partial\rho} + \frac{1}{2}\right)\psi(\rho,\phi),$$
$$\hat{\mathfrak{C}}_{\mathfrak{sl}(2,\mathbb{R})} = \hat{B}^2 + \frac{1}{4}. \tag{3.5}$$

The factor $1/4$ represents a quantum correction to the Casimir with respect to its value in terms of the classical integration constant B.

An important remark should be done at this point. The Hilbert space presented here is unitarily equivalent to the so-called Wheeler-DeWitt (WdW) quantization of our system, where we simply promote the fields V_i to multiplicative operators and $\hat{P}_i = -i\partial_{V_i}$ (see Appendix B for the proof of this statement). Whenever we would like to introduce a quantum phase space inequivalent to Wheeler-DeWitt (via e.g., a polymerization), we expect to brake this unitary equivalence, and at first glance, it could seem impossible to preserve the realization of $\mathfrak{iso}(2,1)$ on the polymerized space. We will see in Sec. IV that the solution is provided by mapping the regularized metric coefficients to operators on the *same* Hilbert space, that are not unitary equivalent to the Wheeler-DeWitt operators. For the time being, we stick to the classical mapping between gravitational reduced phase space and Poincaré algebra (2.14) and further study the Hilbert space and how to impose the dynamics at the quantum level.

A. Hilbert space and energy eigenstates

First of all, we search for eigenstate of the rotation generator J_z and the Casimir B. These provide a complete basis of the Hilbert space and are labeled by a real number B and an integer m,

$$\hat{J}_z|B,m\rangle = m|B,m\rangle, \qquad m\in\mathbb{Z}, \tag{3.6a}$$

$$\hat{B}|B,m\rangle = B|B,m\rangle, \tag{3.6b}$$

$$\widehat{K_\pm}|B,m\rangle = \left(m\pm\left(\frac{1}{2}+iB\right)\right)\Big|B,m\pm 1\rangle, \tag{3.6c}$$

$$\widehat{\Pi_0}|B,m\rangle = |B+i,m\rangle, \tag{3.6d}$$

$$\widehat{\Pi_\pm}|B,m\rangle = |B+i,m\pm 1\rangle, \tag{3.6e}$$

where $K_\pm = K_x \pm iK_y$, and $\Pi_\pm = \Pi_y \mp i\Pi_x$.[2]

The formulas above can be verified by direct computation of the realization (3.2) on the normalized wave functions

$$\langle\rho,\phi|B,m\rangle = \frac{1}{2\pi}\frac{1}{\sqrt{\rho}}e^{-iB\log(\rho)}e^{-\frac{1}{2}i(2m+s)\phi},$$
$$\langle B',m'|B,m\rangle = \delta_{m,m'}\delta(B-B'). \tag{3.7}$$

We shall remark that the firsts lines in (3.6) provide a representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra at fixed Casimir, but the presence of the Abelian sector, represented by translations of the Poincaré group, allows us to move between different values of B.

An interesting role is played by the parameter s; it is very similar to the one played by the superselection parameter in LQC [26]. With respect to the scalar product (3.1), two states with different s are always orthogonal and belong to different irreducible representations of the Poincaré group, exactly as two superselected lattices in LQC with respect to the polymer representation of Weyl algebra [59,60]. Hereafter, without loss of generality, I will set $s=0$.

If we want to impose some dynamics on the Hilbert space, we must recall that the classical evolution is generated by the Hamiltonian H. At the quantum level, it means that we need to search for eigenstates of $K_x - J_z$,

$$(\hat{H}\psi)(\rho,\phi) = \frac{1}{2\lambda\kappa L_P}(\widehat{K_x}-\widehat{J_z})\psi(\rho,\phi) = E\psi(\rho,\phi). \tag{3.8}$$

Hopefully, the associated differential equation has an analytical solution. Diagonalizing with respect to the Casimir $\mathfrak{C}_{\mathfrak{sl}(2,\mathbb{R})}$, we obtain again a complete basis for the Hilbert space,

$$\hat{H}|B,E\rangle = E|B,E\rangle, \tag{3.9a}$$

$$\hat{B}|B,E\rangle = B|B,E\rangle, \tag{3.9b}$$

$$\langle\rho,\phi|B,E\rangle = \frac{\sqrt{\lambda\kappa L_P}}{2\pi}\frac{1}{\sqrt{\rho}\sin(\phi/2)}e^{-iB\log(\rho\sin^2(\phi/2))} \times e^{-i(2\lambda\kappa L_P E\cot(\frac{\phi}{2}))}, \tag{3.9c}$$

$$\langle B',E'|B,E\rangle = \delta(E-E')\delta(B-B'). \tag{3.9d}$$

We shall remark that the spectrum of the Hamiltonian is continuous and unbounded from below, exactly as it happens for cosmology and conformal mechanics [56]. This should lead to a catastrophic instability when we consider (multiple) interacting systems, but our formalism is valid only for a single static black hole, without any matter content, and thus the question of stability cannot be addressed here. For instance, it does not make any sense to couple with a thermal bath and to look at the partition function $e^{-\beta H}$.

In order to verify the orthogonality, as well as the completeness, of the basis, it is convenient to perform a change from the (ρ,ϕ) polarization to a new set of variables, with respect to which the eigenfunctions look like plane waves (see Appendix A for the realization of the Poincaré algebra on the new variables):

$$z \coloneqq 2\lambda\kappa L_P\cot\left(\frac{\phi}{2}\right), \qquad a\coloneqq\log\left(\rho\sin^2\frac{\phi}{2}\right), \qquad (z,a)\in\mathbb{R}^2, \tag{3.10}$$

$$\langle z,a|B,E\rangle = \frac{1}{2\pi}e^{-iBa}e^{-iEz}. \tag{3.11}$$

I choose to call the second variable a because it is actually related to the A operator:

$$(\hat{A}\psi)(\rho,\phi) = \frac{1}{2\lambda}(\Pi_0-\Pi_y)\psi(\rho,\phi) = \frac{1}{\lambda}\left(\rho\sin^2\frac{\phi}{2}\right)\psi(\rho,\phi) \coloneqq \frac{1}{\lambda}(\widehat{e^a}\psi)(\rho,\phi). \tag{3.12}$$

On the other hand, z, being conjugated to the energy, is expected to be related to time. This is indeed what happens once we consider physical states satisfying the quantum

[2] In the last two lines, the complex shift of the Casimir must be read as a formal replacement of B into the wave function, for example $\langle\rho,\phi|\hat{\Pi}_0|B,m\rangle = \frac{1}{2\pi}\frac{1}{\sqrt{\rho}}e^{-i(B+i)\log(\rho)}e^{-\frac{1}{2}i(2m+s)\phi} = \frac{1}{2\pi}\sqrt{\rho}e^{-iB\log(\rho)}e^{-\frac{1}{2}i(2m+s)\phi}$. Actually, the momenta operator Π_i acting on a single eigenstate of the Casimir labeled by B maps it to a combination of eingenstates, exactly like the momentum operator acting on a position eigenstate in standard quantum mechanics.

dynamics. The imposition of the dynamics on the Hilbert space will be the subject of the next subsection.

B. Physical solution and semiclassical states

In the previous section, we found the eigenvectors that diagonalize both the Hamiltonian ($K_x - J_z$) and one of the classical first integral represented by the $\mathfrak{sl}(2,\mathbb{R})$ Casimir (B). We can exploit this basis to impose the dynamics of the system. We face two different possibilities to do so: we recall that at the classical level we can both impose the constraint $\mathcal{C}_{\mathcal{H}} = H - \kappa L_P/L_s^2 = 0$ or equivalently see H as a true Hamiltonian generating the time evolution with respect to τ. On the quantum level, the two interpretations (hereafter denoted respectively by *strong* and *weak* constraints) will lead to drastically different semiclassical evolutions. In both cases, we will reconstruct the metric as an emergent quantity, based on the expectation values of the fundamental operators.

Strong constraint.— The most natural way of implementing the strong constraint is to require that the physical states are the ones that satisfy $\mathcal{C}_{\mathcal{H}}|\psi\rangle_{\text{phys}} = 0$, or equivalently

$$|\psi\rangle_{\text{phys}} \coloneqq \int \mathrm{d}E \int \mathrm{d}B\,\delta\left(E - \frac{\kappa L_P}{L_s^2}\right)\psi(B)|B,E\rangle = \int_{\mathbb{R}} \mathrm{d}B\,\psi(B)\left|B, \frac{\kappa L_P}{L_s^2}\right\rangle, \tag{3.13}$$

where the energy scales and their ratio κ are given *a priori* and they act as multiples of the identity operator. These physical states are of course not normalized within the original Hilbert space, and we need to introduce a new inner product on the physical space. To this purpose, we make use of *group averaging* (or refined algebraic quantization) [1,61,62] and define the projector

$$\delta(\mathcal{C}_{\mathcal{H}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x \exp(ix\mathcal{C}_{\mathcal{H}}), \qquad \delta(\mathcal{C}_{\mathcal{H}}) : \mathcal{H}_{\text{kin}} \to \mathcal{H}_{\text{phys}}, \tag{3.14}$$

that induces the inner product

$$\langle\chi|\psi\rangle_{\text{phys}} = \langle\chi|\delta(\mathcal{C}_{\mathcal{H}})|\psi\rangle_{\text{kin}} = \int \mathrm{d}B\,\chi(B)^*\psi(B). \tag{3.15}$$

Notice that on H_{phys}, only the quantum operators $\hat{\mathcal{O}}$ that commute with the constraint are well defined $[\hat{\mathcal{O}}, \widehat{\mathcal{C}_{\mathcal{H}}}] = 0$; otherwise, their action will map out of the physical subspace. We need then to deparametrize the dynamics with respect to a preferred clock (here τ). More explicitly, this means that the observables we can measure are one-parameter families, e.g.,

$$\widehat{V_1}(\tau) = \frac{\tau^2}{2L_P^2\kappa^2}\hat{A}, \qquad \widehat{V_2}(\tau) = \hat{B}\frac{\tau}{L_P\kappa} - \frac{\tau^2}{2L_P\kappa}\hat{H}. \tag{3.16}$$

From the expression (3.11), we can infer that B and a are conjugated variables, while z is conjugated to the energy E, and it is thus traced out in the group averaging (3.14). The most convenient way of representing the physical space is in terms of functions of the variable a [i.e., Fourier transform of $\psi(B)$], upon which the physical observables in (3.16) act as

$$\hat{A}\psi(a) = \frac{1}{\lambda}e^a\psi(a), \qquad \hat{B}\psi(a) = i\partial_a\psi(a), \qquad \hat{H}\psi(a) = \frac{\kappa L_P}{L_s^2}\psi(a). \tag{3.17}$$

The semiclassical states can be obtained by picking as $\psi(a)$ a Gaussian distribution peaked around some classical values (a_*, B_*),

$$\psi_*(a) = \frac{1}{(2\pi\sigma^2)^{1/4}}e^{-\frac{(a-a_*)^2}{4\sigma^2}}e^{-iB_*a},$$
$$\left|\begin{array}{l}\langle\widehat{V_1}(\tau)\rangle = \frac{e^{a_*+\frac{\sigma^2}{2}}}{2\lambda}\frac{\tau^2}{2L_P^2\kappa^2} \coloneqq \frac{A_*}{2}\frac{\tau^2}{2L_P^2\kappa^2},\\ \langle\widehat{V_2}(\tau)\rangle = \frac{B_*\tau}{L_P\kappa} - \frac{\tau^2}{2L_s^2}.\end{array}\right. \tag{3.18}$$

This means that the expectation values follow the classical trajectories, up to a constant rescaling of the first integral A, due to quantum indetermination. This comes without much surprise, as the classical evolution has already been imposed in the deparametrization of the dynamics, in the definition of the one-parameter family of Dirac observables (3.16).

Weak constraint.— The other possibility to impose the dynamics consists in asking that the constraint is satisfied in a weaker sense: $\langle\psi|H|\psi\rangle = \frac{\kappa L_P}{L_s^2}$. One could imagine that somehow this would account for some deep fuzziness of the geometry, contributing as an effective stress-energy tensor, that allows some fluctuations around the classical constraint. In the following, we will see how this statement should be correctly interpreted, the uncertainty on the energy level coming explicitly into the game.

The imposition of the weak constraint is easily achieved by Gaussian wave packets, peaked on some semiclassical values for the pairs of conjugated variables (B_*, a_*) and $(\kappa L_P/L_s^2, z_*)$:

$$|\psi_*\rangle \coloneqq \int \mathrm{d}E \int \mathrm{d}B \frac{1}{(2\pi\sigma_B\sigma_E)^{1/2}}e^{-\frac{(B-B_*)^2}{4\sigma_B^2}}e^{-\frac{(E-\kappa L_P/L_s^2)^2}{4\sigma_E^2}} \times e^{iBa_*}e^{iEz_*}|B,E\rangle, \tag{3.19a}$$

$$\langle a,z|\psi_*\rangle = \sqrt{\frac{2\sigma_B\sigma_E}{\pi}}e^{-(a-a_*)^2\sigma_B^2}e^{-(z-z_*)^2\sigma_E^2}e^{iB_*(a_*-a)} \times e^{i(z_*-z)\kappa L_P/L_s^2}. \tag{3.19b}$$

Now, all the operators corresponding to the $\mathfrak{iso}(2,1)$ generators are well defined. If we want to reconstruct the evolution of the black hole, we simply need to map these generators back to the gravitational phase space:

$$\widehat{V_1}\psi(\rho,\phi) = \lambda(\widehat{\Pi_0} + \widehat{\Pi_y})\psi(\rho,\phi),$$
$$\widehat{V_2}\psi(\rho,\phi) = \lambda(\widehat{J_z} + \widehat{K_x})\psi(\rho,\phi). \tag{3.20}$$

A straightforward calculation, using the properties of Gaussian integrals, gives the expectation values

$$\langle\widehat{V_1}\rangle = \frac{1}{2\lambda L_P^2\kappa^2} e^{a_* + \frac{1}{8\sigma_B^2}} \left(z_*^2 + \frac{1}{4\sigma_E^2}\right) \coloneqq \frac{A_*}{2L_P^2\kappa^2}\left(z_*^2 + \frac{1}{4\sigma_E^2}\right), \tag{3.21a}$$

$$\langle\widehat{V_2}\rangle = \frac{B_*}{L_P\kappa} z_* - \frac{1}{2L_s^2}\left(z_*^2 + \frac{1}{4\sigma_E^2}\right). \tag{3.21b}$$

Comparing with the classical solutions, we identify $\tau = z_*$, and we see that the quantum evolution closely follows a classical trajectory, for large time $\tau \gg 1/4\sigma_E^2$, up to a shift in the relationship between the classical constant of motion A_* and its quantum realization e^{a_*}, due to the quantum indetermination. On the other hand, for small τ, the quantum correction comes into play, and it actually prevents V_1 from being zero, avoiding the singularity.

We must also notice that the imposition of the weak constraint coincides with considering H as a true Hamiltonian so that the states must satisfy the Schrödinger equation

$$i\partial_\tau|\psi(\tau)\rangle = H|\psi(\tau)\rangle, \tag{3.22}$$

whose solution is indeed given by (3.19) replacing $z - z_* \mapsto \tau - \tau_0$. In the following Fig. 1, there is a comparison between the classical solution and the expectation values of a Gaussian semiclassical state.

Inserting the expectation values into the line element (2.1),[3] with a suitable change of coordinates similar to (2.11),[4] we can recast the metric into the form

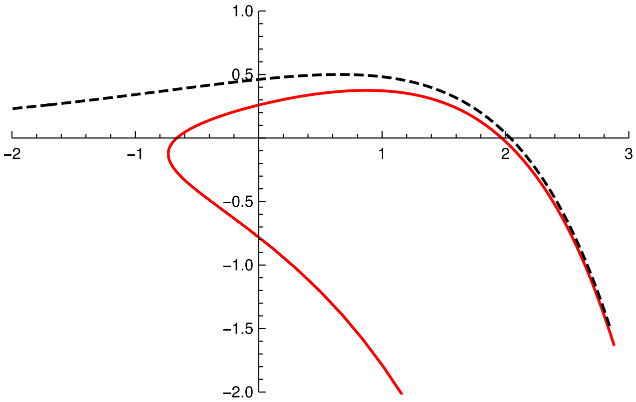

FIG. 1. Plot of the expectation values of the Gaussian state (3.21), with $a_* = 1, \lambda = 1/\sqrt{2}$, $L_0 = 1$, $L_s = 1$, $\sigma_E = 1$, $\sigma_B = 100$, and $B_* = 1$, compared with the respective classical trajectory labeled by the first integrals $B = B_*, A = A_*$.

$$\mathrm{d}s_{\mathrm{eff}}^2 = -f(T)^{-1}\mathrm{d}T^2 + f(T)\mathrm{d}r^2 + (T^2 + \Delta)\mathrm{d}\Omega^2,$$
$$f(T) = \frac{2MT - T^2 - \Delta}{T^2 + \Delta}, \tag{3.23}$$

where the quantum corrections are encoded in the parameter

$$\Delta = \frac{L_s^2}{8\lambda\sigma_E^2 L_P^2\kappa^2} e^{a_* + \frac{1}{8\sigma_B^2}} = \frac{A_* L_P^4}{8\sigma_E^2 L_0^2 L_s^2}. \tag{3.24}$$

It depends on the scales of the system but also on the quantum states through the uncertainty on the energy σ_E and the classical first integral A_*. For small quantum correction ($\Delta/M^2 \ll 1$), the region where τ is timelike is bounded by two horizons for the Killing vector ∂_r, in correspondence of the zero of V_2. The outer one is close to $T \approx 2M$ and represents the event horizon for the outside of the black hole. The inner one is close to $T \approx 0$. The interior structure resembles closely the Reissner-Nordström solution of general relativity, bounded by two null horizons. Extending the solution outside the horizons [17,55], we actually merge two asymptotically flat regions at $T \to \pm\infty$, without any singularity.

Looking at the exterior region for positive T, this effective solution will give a new class of stationary modified black holes, and it would be interesting to further study the correction to standard black hole physics (e.g., Hawking radiation or quasinormal modes) on such an effective spacetime [63,64]. Concerning the region behind the inner horizon, it represents a white hole *outside* region, where ∂_r is timelike, and the *effective* mass is negative $-M$.

Figure 2 represents a schematic diagram for the light cones structure in the three regions.

We shall nevertheless remark that the locations (and even their existence) of the two horizons depend on the quantum states. Moreover, in the extremal limit $\Delta \to M^2$, the two horizons coincide, and the quantum correction becomes

[3] We shall remark here that we have actually calculated the expectation values of the fundamental fields V_i, and not of the metric coefficients; the two can differ from some σ correction. We have considered the effective metric to be $\mathrm{d}s_{\mathrm{eff}}^2 = -\frac{\langle V_1\rangle}{2\langle V_2\rangle}\mathrm{d}\tau^2 + \frac{8\langle V_2\rangle}{\langle V_1\rangle}\mathrm{d}x^2 + \langle V_1\rangle L_s^2\mathrm{d}\Omega^2$. Doing so, we have a well-defined operator V_i; otherwise, we should introduce some regularization to deal with the inverse operator.

[4] The change of coordinate here is $\tau = z_* = \sqrt{\frac{2}{A_*}}\frac{L_P\kappa}{L_s}T$, $x = \frac{L_s}{2L_P\kappa}\sqrt{\frac{A_*}{2}}r$.

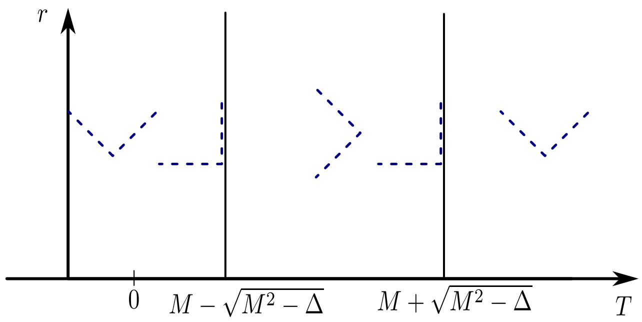

FIG. 2. Diagram of the future directed light cones in the effective solution with two horizons. The T coordinates spans the whole real line, and the vertical lines correspond to the locations of the horizons.

relevant at a macroscopic scale, meaning that there are large deviations from the classical solution in the low curvature regime, near the horizon. In order to see this, we could also look at the value of the Kretschmann scalar at the transition surface ($T=0$), where the radius of the two-sphere (V_1) is minimal. It is given by

$$\mathcal{K}_{T=0}=\frac{12}{\Delta^2}. \tag{3.25}$$

All the corrections to the standard Schwarzschild solution are encoded in the parameter Δ, which in turn depends on the uncertainty on the energy σ_E and the classical first integral A. The importance of the quantum correction is directly proportional to Δ, so inversely proportional to σ_E. By calculating the expectation values of the squared operators, it is possible to show that the uncertainties on the metric coefficients ($\delta V_i \coloneqq \langle V_i^2\rangle - \langle V_i\rangle^2$) near the minimal radius ($z_*=0$) also grow inverse proportionally to the dispersion σ_E,

$$\delta V_i|_{z_*=0} \underset{\sigma_E\to 0}{\longrightarrow} \mathcal{O}\left(\frac{1}{\sigma_E^4}\right), \qquad \begin{matrix} \delta V_1|_{z_*=0} \underset{\sigma_E\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{\sigma_E^4}\right) \\ \delta V_2|_{z_*=0} \underset{\sigma_E\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{\sigma_E^2}\right) \end{matrix}. \tag{3.26}$$

It is then logical to expect that for a heavily fluctuating metric (small σ_E), the quantum correction becomes important, and this is indeed what happens. The problem of this model is that for any given dispersion σ_E, playing with A, coupled with the other integral B, it is possible to make Δ as big as desired, without changing M. Even for small metric fluctuation (big σ_E), the deviation from the Schwarzschild solution could be appreciable as close as desired to the horizon, or even cancel the horizon itself (if $\Delta > M^2$). This means that we need to add by hand a first-class constraint on the integral A, that fixes its value. We also would like to eliminate the dependence on the fiducial scale of the quantum correction. This uniquely fixes $A_* = L_0^2 L_s^2/L_P^4$ and implies that all the corrections come from the uncertainty on the energy. However, at this point, this constraint might seem a little bit *ad hoc*, and it seems hard to believe that we can infer this kind of constraint from the full theory or its quantization.

We shall remark that the behavior of the black hole minisuperspace presents a huge difference with respect to cosmology, where the appearance of quantum correction for the Wheeler-DeWitt quantization has not been observed [26]. Nonetheless, this dependence on the energy uncertainty disappears for nonsqueezed states, where the dispersion on conjugate variables is minimized, by e.g., fixing $\sigma_{E,B}=1/2$.

We will see in the last section (Sec. IV) how the results are modified if we introduce a regularization. Before moving to the study of possible regularizations, we could exploit again the Poincaré structure to discuss the mass spectrum; this is the subject of the next subsection.

C. Mass operator

We recall that at the classical level we have a degeneracy on the definition of the mass; we have indeed two first integrals A and B, that combine into (2.12) to give the only physical quantity that is relevant in the GR framework, the mass, that labels diffeoinequivalent solutions. But if we look at the quantum theory, we expect that both B and A acquire some fluctuation contributing to the mass. Moreover, the two observables do not commute, and we had to build coherent states to represent semiclassical solutions with a finite spread on both of them.

This in turn inevitably forces us to work with semiclassical states that are not eigenvectors of the mass operator. Nevertheless, the group quantization provides interesting information about the mass spectrum. For this purpose, we need to further investigate the properties of the mass operator.

In the first section, we found how we can map the classical observable measuring the squared mass to a combination of Poincaré generators (2.18). With the definition of the *square root* of the $\mathfrak{sl}(2,\mathbb{R})$ Casimir, provided at the beginning of this section, we can easily build the self-adjoint mass operator

$$\begin{aligned}\widehat{M^2}\psi(a,z) &\coloneqq \frac{L_M^2}{2}\hat{B}\hat{A}\hat{B}\psi(a,z)\\ &= -\frac{L_M^2}{2\lambda}\partial_a(e^a\partial_a\psi(a,z)).\end{aligned} \tag{3.27}$$

We shall remark that this operator sees only the a dependence of the wave function. Unsurprisingly, this means that the mass is a Dirac observable commuting with the Hamiltonian, which in turn implies that it can be measured without any problem on both the weakly and strongly constrained states. For the sake of simplicity, in the following, we will consider states on the strongly constrained physical space, tracing out the time-energy dependence of the wave function. The conclusions about the spectrum will not be affected by this simplification.

We can explicitly calculate the wave functions that diagonalize the mass operator and provide a complete basis for the physical wave functions. These are given by the set

$$\langle a|M\rangle \coloneqq \psi_M(a) = 2\frac{\sqrt{\lambda M}}{L_M}e^{-a/2}\mathrm{J}_1\left(\sqrt{8\lambda}\frac{M}{L_M}e^{-a/2}\right),$$
$$\widehat{M^2}|M\rangle = M^2|M\rangle, \tag{3.28}$$

where M is a real positive continuous parameter and J_1 is the first order Bessel function of the first kind. By virtue of the integral properties of the Bessel functions, we can prove the orthogonality and completeness of the basis, and with $y = \sqrt{8\lambda}e^{-a/2}/L_M$, we have indeed

$$\langle M'|M\rangle = \int_0^\infty \mathrm{d}y \mathrm{J}_1(My)\mathrm{J}_1(M'y)\sqrt{MM'}y = \delta(M-M')$$
$$\int_0^\infty |M\rangle\langle M| = \mathbb{I}. \tag{3.29}$$

The existence of the Poincaré structure forces the mass to have a continuous spectrum, as has been pointed out in Ref. [55]. This property is in contrast with several other investigations of black hole spectra [65,66] where a discrete spectrum is postulated or obtained [67–69]. In particular, it means that the black hole could emit particles with any given mass and not only the ones corresponding to the gap between eigenstates.

IV. SINGULARITY AND REGULARIZATION

In this section, I will discuss how it is possible to define a "polymer" quantization that preserves the $\mathrm{ISO}(2,1)$ symmetry. For this regularization, the coherent states evolution reproduces the effective metric (3.23) for both the strong and weak constraints.

The main ingredient of Loop Quantum Cosmology is a realization of the Weyl algebra on a nonseparable Hilbert space, inequivalent to the standard Schrödinger representation. For a given configuration variable (say, $q \in \mathbb{R}$), the space is spanned by orthogonal vectors $|q\rangle$, and it contains functions that are nonvanishing only on a countable subset of $\mathbb{R}$. The lack of weak continuity implies that the momentum operator (say $p = -i\partial_q$) is not defined, but only its finite exponential $e^{i\lambda p}$. This leads to the necessity to introduce a regularized Hamiltonian, where the momenta are replaced by (combination of) their exponentiated version. This is usually done by the substitution $p \mapsto \sin(\lambda p)/\lambda$, but other regularizations are possible as well, and the exact form of the effective Hamiltonian has been heavily debated, especially in the context of black hole interior [17,18,21–25]. In any case, the regulator λ is claimed to encode the fundamental discreteness of spacetime, relating its value to the Planck length. In the limit where it becomes negligible $\lambda \to 0$, we shall recover the classical evolution. For a given parameter λ, the Hilbert space is divided into the so-called superselected sector, according to the position eigenstates, the latter taking discrete real values $\epsilon + n\lambda$, with a fixed offset ϵ. The operator $e^{i\lambda p}$ creates a finite shift of step λ and lets us move within a given superselected sector.

The problem with introducing a regularization scheme for the Hamiltonian is that, in general, it spoils the classical Poincaré symmetry, unless we extend the regularization to the other observables. A systematic way to ensure that any Poisson structure on a phase space is preserved is to look at the regularization as a canonical transformation [46,49], where we then replace the new variables v_i at the place of the corresponding *classical* V_i into the line element. This last step is crucial to make the polymerization describe different physics.

We shall remark that a canonical transformation could be implemented by a nonunitary transformation at the quantum level. Nevertheless, we should be capable of rewriting the *polymer* variables in terms of $\mathfrak{iso}(2,1)$ generators and calculating their quantum expectation values on both the weakly and strongly constrained wave functions. In other words, the difference between the Wheeler-DeWitt quantization and the polymer one is not seen as the result of Hilbert spaces that are unitary inequivalent but as the consequence of considering inequivalent operators (not related by a unitary transformation) on the same Hilbert space, that in addition carries an irreducible representation of $\mathrm{ISO}(2,1)$.

Nevertheless, we cannot freely choose any transformation, but we want it to satisfy a set of properties:

(i) The transformation must be such that the effective metrics is asymptotically equivalent to the Schwarzschild solution.

(ii) We want that the phase space functions representing the polymer coefficients to have a quantum realization with discrete spectra.

We already have at our disposal an operator whose eigenvalues are discrete, and with a superselected sector, it is the rotation generator J_z; the idea is then to take the regularized metric coefficient to be [70]

$$v_2 = 2\lambda J_z = V_2 - 2\kappa L_P \lambda^2 H. \tag{4.1}$$

The λ parameter must be the same as in the mapping from the original phase space to the $\mathfrak{iso}(2,1)$ generator to ensure the right limit $\lambda \to 0$, that maps back to the original phase space. For the superselected sector chosen in the previous sections ($s = 0$), the eigenvalues of v_2 are discrete real values $2n\lambda, n \in \mathbb{Z}$.

Concerning V_1, the polymerization is less straightforward. Assuming the transformation (4.1) for V_2, we find that a compatible canonical transformation is of the form

$$\left| \begin{aligned} & v_1 = v_1(\Pi_0, B) = v_1(V_1 + \lambda V_1 P_2^2, V_1 P_1), \\ & p_1 = p_1(\Pi_0, B) = p_1(V_1 + \lambda V_1 P_2^2, V_1 P_1), \\ & v_2 = 2\lambda J_z = V_2 - 2\lambda^2 \kappa L_P H, \\ & p_2 = \tfrac{1}{\lambda}\arctan(\lambda P_2), \end{aligned} \right. \tag{4.2}$$

with two functions satisfying $\{v_1, p_1\} = 1$. We thus need to find an operator, composed of Π_0 and B that has a discrete spectrum. Unfortunately, this is not achievable through a linear combination, but we need at least a quadratic operator. The simplest one is given by

$$\widehat{v_1^2} \coloneqq 4\lambda^2 \hat{\Pi}_0^2 - \mu^2 \hat{B}^2, \tag{4.3}$$

with a real parameter μ. Its discrete eigenvalues (see Appendix C for the technical details) are $4\mu^2 n^2, n \in \mathbb{Z}$. In this case, we will not have access to the quantum operator measuring v_1 but only its square value. From the point of v_1, this is similar to what happens in cosmology, where the fundamental discreteness is imposed on the volume, the third power of the scale factor. From the point of view of the scale factor in FLRW cosmology and v_1 here, it looks like a so-called $\bar{\mu}$-scheme. Nevertheless, on the semiclassical level, it is possible to take the square root and implement the canonical transformation:

$$\left| \begin{aligned} & v_1 = \sqrt{4\lambda^2 \Pi_0^2 - \mu^2 B^2} = V_1 \sqrt{(1 + P_2 \lambda^2)^2 - \mu^2 P_1^2}, \\ & p_1 = \tfrac{1}{\mu}\arctan\left(\tfrac{\mu P_1}{\sqrt{(1+P_2\lambda^2)^2 - \mu^2 P_1^2}}\right), \\ & v_2 = V_2 + \lambda^2 P_2 (2 P_1 V_1 + P_2 V_2), \\ & p_2 = \tfrac{1}{\lambda}\arctan(\lambda P_2). \end{aligned} \right. \tag{4.4}$$

For the previous construction to make sense, we need to add a constraint on μ. The minimal value of Π_0 during the classical evolution is provided by $2\lambda A$. If we want a definite positive square v_1^2, we need

$$\mu \leq \frac{2\lambda^2 A}{B}, \tag{4.5}$$

when the strict inequality holds; we have that v_1 is never zero and there is no singularity in the effective metric. On the other hand, if $\mu B = 2\lambda^2 A$, there is a singularity at $T = 0$. If the inequality is not satisfied, the canonical transformation is not defined on the whole trajectory, and the singularity occurs before the classical one. The only way to have a canonical transformation that is well defined on the whole phase space is to take the limit $\mu \to 0$, which corresponds to not polymerise V_1. This is somehow analogous to what has been found for the deformation of the constraint algebra for spherically symmetric spacetimes [71], where only one of the momenta gets polymerized. When μ goes to zero, we gain back an operator measuring directly the metric coefficient v_1, instead of its square value, but now it has continuous eigenvalues:

$$\mu \to 0 \Rightarrow \left| \begin{aligned} & v_1 = 2\lambda \Pi_0 = V_1(1 + P_2 \lambda^2), \\ & p_1 = \tfrac{P_1}{1 + P_2 \lambda^2}, \\ & v_2 = 2\lambda J_z, \\ & p_2 = \tfrac{1}{\lambda}\arctan(\lambda P_2). \end{aligned} \right. \tag{4.6}$$

Inverting the canonical transformation, we can express H in terms of the polymerised variables, and it gives

$$\begin{aligned} \kappa L_P H &= -v_1 \frac{\tan \mu p_1}{\mu} \frac{\sin(2\lambda p_2)}{2\lambda} - v_2 \frac{\sin^2(\lambda p_2)}{2\lambda^2} \\ &\underset{\mu \to 0}{\longrightarrow} - v_1 p_1 \frac{\sin(2\lambda p_2)}{2\lambda} - v_2 \frac{\sin^2(\lambda p_2)}{2\lambda^2}. \end{aligned} \tag{4.7}$$

The evolution of v_1 and v_2, generated by the effective Hamiltonian, can be easily solved by inverting the canonical transformation above.

The main advantage of looking at the regularization as a canonical transformation is that the description of both classical and quantum mechanics in terms of Poincaré generators is not modified. In other words, the evolution is always generated by $K_x - J_z$, and the dynamical quantum states are the same as in the previous section, according to which kind of constraint we want to impose. We simply need to change the mapping between Poincaré operators and metric coefficients. We will study here the case where $\mu \to 0$, and the effective metric corresponds to

$$\mathrm{d}s_{\mathrm{eff}}^2 = -\frac{\langle \Pi_0 \rangle}{2\langle J_z \rangle} \mathrm{d}\tau^2 + \frac{8\langle J_z \rangle}{\langle \Pi_0 \rangle} \mathrm{d}x^2 + 2\lambda \langle \Pi_0 \rangle L_s^2 \mathrm{d}\Omega^2. \tag{4.8}$$

Without much surprise, deparametrizing the dynamics with respect to the time τ, the evolution of J_z and Π_0 on the physical Hilbert space satisfying the strong constraint follows the respective classical trajectories

$$\begin{aligned} \langle \lambda \widehat{\Pi_0}(\tau) \rangle &= \frac{A_*}{2\kappa^2 L_P^2} (\tau^2 + 4\kappa^2 L_P^2 \lambda^2), \\ \langle \lambda \widehat{J_z}(\tau) \rangle &= \frac{B_*}{\kappa L_P} \tau - \frac{1}{2L_s^2} (\tau^2 + 4\kappa^2 L_P^2 \lambda^2). \end{aligned} \tag{4.9}$$

The expectation values are on the Gaussian physical state, as in (3.18), and the deparametrization has been done by using the classical solution for Π_0 and J_z given directly by (2.10), and then we replace the first integrals A, B by the corresponding quantum operator. This leads to the same effective metric as in (3.23), where now $\Delta = 2\lambda^2 A_* L_s^2$.

The interesting thing is that the σ correction appearing in the evolution on the weakly constrained states goes in the same direction. More precisely, on the weakly constrained Gaussian wave packets (3.19), we have

$$\langle\widehat{\lambda\Pi_0}\rangle = \frac{1}{2\lambda L_P^2\kappa^2} e^{a_* + \frac{1}{8\sigma_B^2}}\left(z_*^2 + 4\lambda^2\kappa^2 L_P^2 + \frac{1}{4\sigma_E^2}\right)$$
$$\coloneqq \frac{A_*}{2\kappa^2 L_P^2}\left(z_*^2 + 4\lambda^2\kappa^2 L_P^2 + \frac{1}{4\sigma_E^2}\right), \tag{4.10a}$$

$$\langle\widehat{\lambda J_z}\rangle = \frac{B_*}{L_P\kappa} z_* - \frac{1}{2L_s^2}\left(z_*^2 + 4L_P^2\kappa^2\lambda^2 + \frac{1}{4\sigma_E^2}\right). \tag{4.10b}$$

The effective structure is again given by (3.23), but now the quantum correction is encoded in

$$\Delta = \left(2\lambda^2 L_s^2 + \frac{L_P^4}{8\sigma_E^2 L_0^2 L_s^2}\right)A_*. \tag{4.11}$$

We shall remark that this does not solve the apparent paradox of quantum correction at a macroscopic scale. If we leave A free, even for metrics with small fluctuation (big σ_E), the inner horizon can come as close as desired to the external horizon. We still need to add a condition on A.

Taking a closer look at the parameter Δ, we see that it is exactly the sum of the one obtained for the weakly constrained WdW states and the strongly constrained polymer states. It is natural to interpret the two contributions as taking into account respectively the quantum uncertainty on the metric coefficients and the deep discreteness of the spacetime. For wave functions that are well localized, the first one is negligible compared to the second one, i.e., $\frac{\kappa^2}{\sigma_E^2 L_P^2} \ll \lambda^2$. In this case, we expect that the quantum corrections are of Planck size, meaning that the first-class constraint would impose

$$2\lambda^2 L_s^2 A \approx L_P^2 \Rightarrow \Delta \approx \left(L_P^2 + \frac{\kappa^2}{16\sigma_E^2\lambda^2}\right). \tag{4.12}$$

This in turn implies that it is impossible to fully get rid of the fiducial scale, entering the game through the central charge κ. It would be worth studying the role of the cutoff scales, looking at them as running renormalization parameters. The imposition of a first-class constraint relating the polymerization parameter to one of the first integrals is analogous to the construction in Ref. [17]. Despite (4.12) being more reasonable than the one imposed in the WdW setup, the question of if and how we can infer this kind of constraint from the full LQG theory is still unanswered. However, we can have a hint about its origin by remarking the presence of A in the coordinate redefinition (2.11). The relationship (4.12) is expected to be related somehow to the introduction of a Planck length ruler on spacetime. The impossibility of completely eliminating the dependence of the effective metric on the fiducial scales points again toward a physical role of the boundary. We would like to stress that similar behavior has been observed in cosmology [72].

In the previous section, we saw that the evolution of the operators measuring the metric coefficients produces drastically different metrics, depending on whether we allow some energy fluctuation or not. Here, the main features of the effective metric are the same in both cases, and moreover they agree with the classical line element corresponding to (3.23). The evolution of quantum coherent states on the polymer Hilbert space follows the effective evolution described by the corresponding polymer Hamiltonian, and this is stable for nonzero energy fluctuation. This feature has been used as a consistency check in favor of the robustness of FLRW polymerization and can here be extended to the black hole interior.

V. DISCUSSION

The existence of a hidden symmetry, leaving on top of the time reparametrization, has been revealed for some simple and yet physically relevant minisuperspaces. This is for instance the case of cosmology [47,49–53], or the black hole interior [46,55]. In the quantization of the theory we should deal with the presence of this structure, because of its ability to encode the classical dynamics. Taking here the conservative approach of preserving the classical structure has provided a criterion to constrain the quantization. Concretely, this means that any Hilbert space we would like to choose, be it the standard Schrödinger picture or a regularized polymer space, must contain an irreducible representation of the symmetry group we want to preserve. In this work, I focus attention on the black hole interior, but the construction can be generalized to any minisuperspace model that exhibits similar symmetries. The question of the existence of such structures for a general minisuperspace is currently under investigation. We can as well beg the question of whether the Poincaré group for black holes is the maximal symmetry group or is part of a larger structure.

In the article, the well-known irreducible representations of $\mathrm{ISO}(2,1)$ are used to build a consistent quantum theory, providing an explicit example of observables and their spectra. The most striking consequence of this construction is that we obtain a mass operator with a continuous spectrum. This has important consequences on the emission spectra of black holes and is in contrast to what has been postulated in various works on black hole quantum physics. The existence of this *hidden* symmetry could also explicitly play a role in perturbation theory, providing an interesting interpretation in terms of conserved quantities associated to test fields propagating on the black hole background.

On a more concrete playground, I have started with a quantization equivalent to the standard Schrödinger representation of Wheeler-DeWitt gravity, calculating the expectation values of the metric coefficients on some semiclassical states. Classically, only a particular combination of first integrals (namely the mass) is physically relevant. On the other hand, if we allow some fluctuation on the energy, we have seen that the effective metric, emerging as a result of the quantum evolution, strongly depends on both the first integrals and the amplitude of the fluctuations.

In the last section, I propose a *half-polymerized* regularization, reminiscent of the modification allowed in the context of deformed constraint algebra for spherically symmetric spacetime [71]. The apparent puzzle of introducing a discretization on the configuration space, keeping the invariance under Poincaré group, is solved here by looking at the regularized variables as a set of operators that satisfy the polymer-Weyl algebra on the same Hilbert space as the usual Schrödinger operator, but the two sets are not related by a unitary transformation.

We find out that the quantum corrections come from two terms going in the same direction, summing up into the parameter Δ that modifies the classical spacetime structure as in (3.23). The singularity is replaced by a Killing horizon, leading to a white hole region. The two contributions have been interpreted as the effect of a quantum uncertainty on the metric coefficients and a constant piece proportional to the Planck length, encoding the fundamental discreteness of spacetime. This implies that the light cone structure is the same for both the weakly and strongly constrained states, where the effective structure is achieved by evolving the metric coefficients with respect to the polymerized Hamiltonian (4.7).

Despite the common feature of replacing the singularity with a black-to-white hole transition, the metric presented in this article is different than the one usually considered for the study of properties of regular black holes (see Refs. [63,64] and references therein), and it would be interesting to see how this affects the phenomenology.

The existence of the hidden Poincaré structure has been recently extended to the case with a nonvanishing cosmological constant [55], both for de Sitter and anti-de Sitter cases, meaning that the results of this article are easily generalizable in the presence of a cosmological constant.

Finally, a puzzling role is played by the boundaries. On the one hand, the boundary seems not to play any role in the physics of the system, because of homogeneity the boundary of the spacelike slice trivially carries the same information as the bulk. On the other hand, in order to reduce the action to a mechanical model, we need to introduce such a boundary to regulate the divergent integration of the action. And it is precisely the IR regulator that interplays with the symmetries, being modified when we act on a physical trajectory [46,55]. It also plays a crucial role in the quantum theory, by labeling the solution states. How this could be related to renormalization properties, while we consider the IR regulator as a running parameter, is still an open question. Finally, on an effective level, the fiducial scales (and so the boundary) appear explicitly in the modified line element, suggesting a physical role of the boundary for the quantum effects and pointing toward some holographic properties of gravity.

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APPENDIX A: IRREDUCIBLE REPRESENTATIONS OF ISO(2,1)

I report here some key insight on the unitary representation of (the universal cover of) the three-dimensional Poincaré group. I followed the notations of Ref. [58], where the reader can find a more exhaustive discussion on the properties of such representations. They are distinguished into three categories according to the sign of the first Casimir $\mathfrak{C}_1$. The latter represents the mass of the particle when we look at ISO$(2,1)$ as the symmetry group of the three-dimensional Minkowski space. On top of them, there is the trivial representation, where all the elements of the algebra act as zero on the states. As we are interested here in the massless representations ($\mathfrak{C}_1 = 0$), I recall here their main features.

Massless representations are labeled by three parameters $\eta = \pm, s \in \mathbb{R}(\mathrm{mod}\, 2), t \in \mathbb{R}$. The Hilbert space is usually presented in terms of function on $\mathbb{R}^2$, where the two coordinates represent the spatial components of the momenta, and the scalar product is given by

$$\langle \psi | \chi \rangle = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}y}{\rho} \psi^*(x,y)\chi(x,y), \qquad \rho = \sqrt{x^2+y^2}. \tag{A1}$$

And the generators of the algebra act like

$$(\Pi_\mu{}^{(\eta,s,t)}\psi)(x,y) = p^\mu \psi(x,y) \qquad p^\mu = (\rho, x, y), \tag{A2a}$$

$$(J_z{}^{(\eta,s,t)}\psi)(x,y) = \left[-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{s}{2}\right]\psi(x,y), \tag{A2b}$$

$$(k_i{}^{(\eta,s,t)}\psi)(x,y) = \left[i\rho\frac{\partial}{\partial p^i} - s\epsilon_{ij}\frac{p^j}{2\rho} - \frac{\eta t}{2}\epsilon_{ij}\frac{p^j}{\rho^2}\right]\psi(x,y). \tag{A2c}$$

By a straightforward calculation, we can verify that they satisfy the quantum version of the algebra (2.15). As expected, the massless representation gives a zero $\mathfrak{C}_1$, while

$$\hat{\mathfrak{C}}_2 \coloneqq \frac{1}{2}(J_z\Pi_0 + \Pi_0 J_z + K_x\Pi_y + \Pi_y K_x - K_y\Pi_x - \Pi_x K_y), \tag{A3}$$

$$\hat{\mathfrak{C}}_2\psi(x,y) = \frac{1}{2}\eta t\psi(x,y). \tag{A4}$$

If now we want to realize the phase space presented in the main text of the article, we shall take $t = 0$ to have also a zero $\mathfrak{C}_2$.

In the paper, we have actually considered another realization of the Hilbert space, unitarily equivalent to the one just presented here. It simply consists in expressing the wave function in polar coordinates instead of the Cartesian ones $(x, y) = (\rho \sin\phi, \rho\cos\phi)$; this makes the $1/\rho$ factor disappear from the measure. The relation between the two wave function expressions is simply given by a change of variable in the argument,

$$\psi(x,y) = \psi(\rho\sin\phi, \rho\cos\phi) \coloneqq \psi_{\mathrm{pol}}(\rho,\phi),$$
$$\langle\psi|\chi\rangle = \int_0^\infty \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \psi^*_{\mathrm{pol}}(\rho,\phi)\chi_{\mathrm{pol}}(\rho,\phi). \tag{A5}$$

By a bit of abuse of notation, we will drop the subscript *pol*, and we denote with the same symbol the two realizations. The action of the Poincaré generators on these functions is given in the main text at (3.2).

The last realization of the Hilbert space that is used in the main text is the $a - z$ polarization, or its dual Fourier transform. This turns out to be practical to deal with eigenstates of the Hamiltonian $K_x - J_z$. In this case, we still do a change of variables, but now we absorb the Jacobian determinant appearing in the measure into the wave function

$$z = 2\lambda\cot\left(\frac{\phi}{2}\right), \quad a = \log\left(\rho\sin^2\frac{\phi}{2}\right), \quad (z,a)\in\mathbb{R}^2$$
$$\psi_{\mathrm{az}}(a,z) \coloneqq \frac{1}{\sqrt{\lambda}} e^{a/2}\psi(\rho(a,z),\phi(z,a))$$
$$\langle\psi|\chi\rangle = \int_{\mathbb{R}^2} \mathrm{d}a\mathrm{d}z \psi^*(a,z)\chi(a,z). \tag{A6}$$

We will drop again the subscript, because we always represent the same state, even if the measure for the two realizations is not the same. The argument of ψ will implicitly denote the measure we need to pick up. We conclude the discussion by rewriting the action of the generators in this realization,

$$(\Pi_0{}^{(s)}\psi)(a,z) = e^a\left(1 + \frac{z^2}{4\lambda^2}\right)\psi(a,z), \tag{A7a}$$

$$(\Pi_x{}^{(s)}\psi)(a,z) = e^a\frac{z}{\lambda}\psi(a,z), \tag{A7b}$$

$$(\Pi_y{}^{(s)}\psi)(a,z) = e^a\left(\frac{z^2}{4\lambda^2} - 1\right)\psi(a,z), \tag{A7c}$$

$$(J_z{}^{(s)}\psi)(a,z) = \left[\frac{i}{2\lambda}\left(z\partial_a + \frac{4\lambda^2 + z^2}{2}\partial_z\right) - \frac{s}{2} - \frac{i}{4\lambda}z\right] \times \psi(a,z), \tag{A7d}$$

$$(K_x{}^{(s)}\psi)(a,z) = \left[\frac{i}{2\lambda}\left(z\partial_a + \frac{4\lambda^2 - z^2}{2}\partial_z\right) - \frac{s}{2}\frac{z^2 - 4\lambda^2}{z^2 + 4\lambda^2} - \frac{i}{4\lambda}z\right]\psi(a,z), \tag{A7e}$$

$$(K_y{}^{(s)}\psi)(a,z) = \left[i(-\partial_a + z\partial_z) + s\frac{4\lambda z}{z^2 + 4\lambda^2} + \frac{i}{2}\right]\psi(a,z). \tag{A7f}$$

And we see that B is simply conjugated to a, while H has en extra term depending on s and λ, that vanishes for the *superselected* sector $s = 0$,

$$\hat{B}\psi(a,z) = i\partial_a\psi(a,z).$$
$$\hat{H}\psi(a,z) = i\partial_z\psi(a,z) + \frac{2s\lambda}{4\lambda^2 + z^2}\psi(a,z). \tag{A8}$$

APPENDIX B: WHEELER-DEWITT QUANTIZATION

In this Appendix, we will see how the Poincaré irreducible representations are equivalent to the so-called Wheeler-DeWitt quantization; by the latter, I mean the natural quantization scheme that consists in promoting the configuration variables V_i to multiplicative operators and the momenta to derivatives $\hat{P}_i = -i\partial_{V_i}$. The Hilbert space is given by square normalizable wave functions $L^2(\mathbb{R}^2, \mathrm{d}V_1\mathrm{d}V_2)$.

This corresponds to the *position* polarization of the wave functions, that is known to be equivalent to the *momenta* polarization up to a Fourier transform. The key idea is to perform the Fourier transform only on the variable V_2, meaning that the multiplicative operators are now V_1 and P_2, and

$$\widehat{V_1}\psi(V_1,P_2) = V_1\psi(V_1,P_2),$$
$$\widehat{P_1}\psi(V_1,P_2) = -i\partial_{V_1}\psi(V_1,P_2),$$
$$\widehat{P_2}\psi(V_1,P_2) = P_2\psi(V_1,P_2),$$
$$\widehat{V_2}\psi(V_1,P_2) = i\partial_{P_2}\psi(V_1,P_2). \tag{B1}$$

With these fundamental operators at hand, we can build the observables corresponding to the $\mathfrak{iso}(2,1)$ generators on the classical phase space. We simply need to consider the corresponding combination of fundamental operators and make it self-adjoint; for example, if we take $K_y = -P_iV_i$, the corresponding hermitian operator is given by

$$(\widehat{K_y}\psi)(V_1,P_2) = -\frac{1}{2}(\widehat{V_1}\,\widehat{P_1} + \widehat{P_1}\,\widehat{V_1} + \widehat{V_2}\,\widehat{P_2} + \widehat{P_2}\,\widehat{V_2}) \times \psi(V_1,P_2). \tag{B2}$$

The whole algebra is then

$$(\widehat{\Pi_0}\psi)(V_1,P_2)=\frac{V_1(1+P_2^2\lambda^2)}{2\lambda}\psi(V_1,P_2), \tag{B3a}$$

$$(\widehat{\Pi_x}\psi)(V_1,P_2)=V_1P_2\psi(V_1,P_2), \tag{B3b}$$

$$(\widehat{\Pi_y}\psi)(V_1,P_2)=\frac{V_1(1-P_2^2\lambda^2)}{2\lambda}\psi(V_1,P_2), \tag{B3c}$$

$$(\widehat{J_z}\psi)(V_1,P_2)=i\left[\frac{1+P_2^2\lambda^2}{2\lambda}\partial_{P_2}-P_2V_1\right]\psi(V_1,P_2), \tag{B3d}$$

$$(\widehat{K_x}\psi)(V_1,P_2)=i\left[\frac{1-P_2^2\lambda^2}{2\lambda}\partial_{P_2}+P_2V_1\right]\psi(V_1,P_2), \tag{B3e}$$

$$(\widehat{K_y}\psi)(V_1,P_2)=-i[P_2\partial_{P_2}-V_1\partial_{V_1}]\psi(V_1,P_2). \tag{B3f}$$

This already has the form of the operator in the previous Appendix, where the *translation* operator is multiplicative, and the *rotation* and *boosts* contain some derivatives. The unitary equivalence is proven by considering the following change of coordinate in the wave function:

$$V_1=\lambda y+\lambda\sqrt{x^2+y^2}=\lambda\rho(1+\cos\phi),\qquad P_2=-\frac{y-\sqrt{x^2+y^2}}{x\lambda}=\frac{1}{\lambda}\tan\frac{\phi}{2}. \tag{B4}$$

Absorbing the Jacobian of the transformation into the measure of the new realization, we recover the realization presented in the main text and in Appendix A, meaning that the two are unitarily equivalent.

APPENDIX C: $\hat{v}_1$ EIGENSTATES

In Sec. IV, I claim that the simplest combination of operators B and Π_0 that have a discrete spectra is the quadratic functional (4.3). In order to see this explicitly, we shall in the first place look at the linear combination

$$\hat{v}_1 \coloneqq 2\lambda\widehat{\Pi_0}+\mu\hat{B}. \tag{C1}$$

By acting on a wave function in the polar representation, we can analytically solve the eigenvalue problem and find the eigenvectors

$$\langle\rho,\phi|v_1,m\rangle=\frac{1}{2\pi\sqrt{\mu\rho}}e^{-i(2\frac{\rho\lambda}{\mu}-\frac{v_1}{\delta}\log\rho)}e^{-im\phi},\qquad \widehat{v_1}|v_1,m\rangle=v_1|v_1,m\rangle, \tag{C2}$$

that provide an orthonormal basis for a continuous spectrum $v_1\in\mathbb{R}$:

$$\langle v_1',n|v_1,m\rangle=\delta_{n,m}\delta(v_1-v_1'). \tag{C3}$$

On the other hand, the quadratic combination (4.3) considered in the main text has eigenvectors

$$\langle\rho,\phi|v_1^2,m\rangle=\sqrt{\frac{v_1}{2\lambda\rho}}J_{\frac{v_1}{\mu}}\left(\frac{2\lambda\rho}{\mu}\right)\frac{1}{\sqrt{2\pi}}e^{-im\phi},\qquad \widehat{v_1^2}|v_1^2,m\rangle=v_1^2|v_1^2,m\rangle, \tag{C4}$$

where J_n are the first kind Bessel functions. The eigenstates are normalized for a discrete spectrum $\frac{v_1}{2\mu}\in Z$. This is easily shown by using the integral property of the Bessel functions

$$\int_0^\infty\frac{\mathrm{d}y}{y}J_{2n}(y)J_{2m}(y)=\frac{1}{2\pi}\frac{\sin(\pi(n-m))}{n^2-m^2}=\frac{1}{2n}\delta_{n,m}\Leftrightarrow n,m\in\mathbb{Z}. \tag{C5}$$

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