Hidden Kac-Moody structures in the fermionic sector
of five-dimensional supergravity

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We study the supersymmetric quantum dynamics of the cosmological models obtained by reducing $D = 5$ supergravity to one timelike dimension. This consistent truncation has fourteen bosonic degrees of freedom, while the quantization of the homogeneous gravitino field leads to a 2^{16}–dimensional fermionic Hilbert space. We construct a consistent quantization of the model in which the wave function of the Universe is a 2^{16}–component spinor depending on fourteen continuous coordinates, which satisfies eight Dirac-like wave equations (supersymmetry constraints) and one Klein-Gordon-like equation (Hamiltonian constraint). The fermionic part of the quantum Hamiltonian is built from operators that generate a 2^{16}-dimensional representation of the (infinite-dimensional) maximally compact subalgebra $K(G_2^{++})$ of the rank-4 hyperbolic Kac-Moody algebra G_2^{++}. The (quartic-in-fermions) squared-mass term $\hat{\mu}^2$ entering the Klein-Gordon-like equation has several remarkable properties: (i) it commutes with the generators of $K(G_2^{++})$; and (ii) it is a quadratic polynomial in the fermion number $N_F \sim \bar{\Psi}\Psi$, and a symplectic fermion bilinear $C_F \sim \Psi C \Psi$. Some aspects of the structure of the solutions of our model are discussed, and notably the Kac-Moody meaning of the operators describing the reflection of the wave function on the fermion-dependent potential walls ("quantum fermionic Kac-Moody billiard").

I. INTRODUCTION

The discovery of a hidden E_7 symmetry of $N = 8$ supergravity in $D = 4$ [1] has initiated the search of hidden symmetries in supergravity, and superstring theories. The hidden symmetry algebra was more generally conjectured to be E_{11-D} for maximal supergravity reduced to D dimensions [2], which implied reaching the affine Kac-Moody algebra E_9 in $D = 2$ [3], and, possibly, the hyperbolic Kac-Moody algebra E_{10} when reducing to one timelike direction [4]. (See Ref. [5] for the definition and basic structure of infinite-dimensional Kac-Moody algebras.) The possible existence of a hidden, mother E_{11} structure has been suggested in [6].

A new angle on the possible relevance of E_{10} came from studies of the chaotic behavior, *à la* Belinskii-Khalatnikov-Lifshitz (BKL) [7–9], of generic solutions of maximal supergravity near a (spacelike) cosmological singularity [10–12]. These studies highlighted the role of the gravitino in the implementation of hidden hyperbolic Kac-Moody structures [13–15]. The gravitino enters the game as a representation of the algebra $K(E_{10})$, defined as the (formal) maximally-compact subalgebra of E_{10}, namely the subalgebra fixed under the Chevalley involution; we use the maximally split real forms of the considered hyperbolic Kac-Moody algebras, and the corresponding real Chevalley involution. The existence of *finite-dimensional* spinorial representations of (infinite-dimensional) involutory subalgebras of hyperbolic Kac-Moody algebras discovered through such supergravity-based works [13–16] was extended in several directions [17–19], and notably from a mathematical point of view [20–23].

Most studies, however, only considered the gravitino dynamics at lowest order, where the gravitino can be treated as a classical, Grassmanian field, undergoing a fermionic analog [16] of the bosonic billiard dynamics. The compatibility of Kac-Moody structures with the fully nonlinear gravitino dynamics (involving up to quartic-in-fermions terms in the Hamiltonian) has only been explored so far within the simpler setting of the reduction of $N = 1$, $D = 4$

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supergravity to homogeneous cosmological models of the Bianchi IX type [18,24,25]. In this setting, the relevant hyperbolic Kac-Moody structures are not E_{10} and $K(E_{10})$, but a rank-3 hyperbolic subalgebra of E_{10} called AE_3, and its maximally compact subalgebra $K(AE_3)$. In these studies the gravitino is treated as a fully nonlinear quantum field (depending only on time). The results of Refs. [18,24,25] have deepened the significance of hidden Kac-Moody structures by showing, in particular, that: (i) the quartic-in-fermion contribution to the quantum Hamiltonian is invariant under the three generators $\hat{J}_{\alpha_1}, \hat{J}_{\alpha_2}, \hat{J}_{\alpha_3}$ of $K(AE_3)$ (which are associated with the three simple roots $\alpha_1, \alpha_2, \alpha_3$ of AE_3); and (ii) the quantum dynamics of the gravitino near the singularity can be described as a sequence of free motions interrupted by reflections on three Toda-like potential walls corresponding to the three simple roots of AE_3. Each such reflection is described (in the short-wavelength limit) by the corresponding quantum reflection operator (with $i = 1, 2, 3$)

$$\hat{\mathcal{R}}_{\alpha_i} = e^{i\frac{\pi}{2}\hat{J}_{\alpha_i}}. \tag{1.1}$$

In addition, the latter reflection operators satisfy a generalized version of the Coxeter relations satisfied by usual hyperplane reflection operators.

The aim of the present work is to extend the work of Refs. [18,24,25] to the case of pure supergravity in $D = 5$, as a step towards understanding the nonlinear aspects of fermions in $D = 11$ supergravity. We recall that pure $D = 5$ supergravity (with eight supercharges) exhibits some similarity with $D = 11$ supergravity [26]. It is therefore interesting to study the compatibility of Kac-Moody structures with the fully nonlinear gravitino dynamics within the simpler setting of $D = 5$ supergravity. Previous works have indicated that, in this case, the relevant hyperbolic Kac-Moody algebra behind the bosonic dynamics was the rank-4 hyperbolic extension of G_2, which we will denote as G_2^{++} [27,28]. [Contrary to E_{10} (but similarly to AE_3) the hyperbolic Kac-Moody G_2^{++} is nonsimply laced.] We therefore expect that the gravitino will enter as a representation of the subalgebra $K(G_2^{++}) \subset G_2^{++}$, fixed under the Chevalley involution. We will indeed find that the $D = 5$ supergravity fermion couplings define a consistent finite-dimensional vector-spinor representation of $K(G_2^{++})$ (of the type defined in Ref. [19]), and we shall prove that analogs of the results found for the $K(AE_3)$ structure of $D = 4$ supergravity cosmological models hold for the simplest homogeneous cosmological models of $D = 5$ supergravity (where all fields are taken to depend only on time). In particular, the quartic-in-fermion contribution to the quantum Hamiltonian will be shown to be invariant under the four generators $\hat{J}_{\alpha_1}, \hat{J}_{\alpha_2}, \hat{J}_{\alpha_3}, \hat{J}_{\alpha_4}$, of $K(G_2^{++})$, associated with the four simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of G_2^{++}.

II. CLASSICAL LAGRANGIAN FORMULATION

We take as starting point the second-order action of the pure supergravity theory in $D = 5$, as given in (the corrected version of) Ref. [29]. In this formulation the gravitino is described by a (complex) Dirac vector-spinor ψ_μ. (This is equivalent to the alternative formulation using a doublet of symplectic Majorana vector-spinors [30].) We follow the normalization and notation of Ref. [29], notably for the Levi-Civita connection $\mathring{\omega}_{\mu\hat{\alpha}\hat{\beta}} = -\mathring{\omega}_{\mu\hat{\beta}\hat{\alpha}}$. Here, μ is a five-dimensional coordinate index, while hatted indices are frame indices with respect to a local Lorentz frame $e_{\hat{\alpha}}^\mu$, with associated coframe $\theta_\mu^{\hat{\alpha}}$ ($e_{\hat{\alpha}}^\mu \theta_\mu^{\hat{\beta}} = \delta_{\hat{\alpha}}^{\hat{\beta}}$). The Levi-Civita connection (with one coordinate index, μ, and two frame indices) is defined as

$$\mathring{\omega}_{\mu\hat{\alpha}\hat{\beta}} \equiv \eta_{\hat{\alpha}\hat{\gamma}} \mathring{\omega}_\mu{}^{\hat{\gamma}}{}_{\hat{\beta}} = -\mathring{\omega}_{\mu\hat{\beta}\hat{\alpha}}, \tag{2.1}$$

with

$$\mathring{\omega}_\mu{}^{\hat{\alpha}}{}_{\hat{\beta}} \equiv +\theta_\nu^{\hat{\alpha}}(\partial_\mu e_{\hat{\beta}}^\nu + \Gamma_{\mu\sigma}^\nu e_{\hat{\beta}}^\sigma), \tag{2.2}$$

where $\Gamma_{\mu\sigma}^\nu$ denote the usual Christoffel symbols of $g_{\mu\nu}$.

The covariant derivatives of the frame components of a vector, and of a vector-spinor, are respectively given by (when using frame indices)

$$\nabla_\mu V^{\hat{\alpha}} = \partial_\mu V^{\hat{\alpha}} + \mathring{\omega}_\mu{}^{\hat{\alpha}}{}_{\hat{\beta}} V^{\hat{\beta}}, \tag{2.3}$$

$$\mathcal{D}_\lambda[\mathring{\omega}]\psi_{\hat{\mu}} = \partial_\lambda \psi_{\hat{\mu}} + \mathring{\omega}_{\lambda\hat{\mu}}{}^{\hat{\nu}}\psi_{\hat{\nu}} + \frac{1}{4}\mathring{\omega}_\lambda{}^{\hat{\rho}\hat{\sigma}}\gamma_{\hat{\rho}\hat{\sigma}}\psi_{\hat{\mu}}. \tag{2.4}$$

As we use here a mostly positive signature, we had to adapt the results of Ref. [29] (which used a mostly negative signature). For instance, we replaced their gamma matrices as follows: $\Gamma_{CN}^{\hat{\mu}} \mapsto -i\gamma^{\hat{\mu}}$, $\Gamma_{\hat{\mu}}^{CN} \mapsto +i\gamma_{\hat{\mu}}$. Our gamma matrices satisfy $\gamma_{\hat{\mu}}\gamma_{\hat{\nu}} + \gamma_{\hat{\nu}}\gamma_{\hat{\mu}} = 2\eta_{\hat{\mu}\hat{\nu}}$ with $\eta_{\hat{\mu}\hat{\nu}} = \text{diag}(-1, +1, +1, +1, +1)$.

Our sign convention for the covariant components of the antisymmetric Levi-Civita tensor is $\eta_{\alpha\beta\gamma\delta\epsilon} = \sqrt{|g|}\varepsilon_{\alpha\beta\gamma\delta\epsilon}$ with $\varepsilon_{01234} = +1$. The antisymmetrized product of five gamma matrices is proportional to the identity matrix and we use (following [29]) a representation where $\gamma_{\mu\nu\rho\sigma\tau} = -i\eta_{\mu\nu\rho\sigma\tau}$, i.e., $\gamma_{01234} = -i\sqrt{|g|}$, or $\gamma_{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}} = -i$, so that $\gamma^{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}} = +i$. We define the Dirac conjugate as

$$\bar{\Psi} \equiv \Psi^\dagger \beta, \tag{2.5}$$

with the β matrix defined such that $\beta\gamma_{\hat{\mu}}\beta^{-1} = -\gamma_{\hat{\mu}}^\dagger$. We take a representation of the (positive-signature) gamma matrices where $\gamma^{\hat{0}}$ is anti-Hermitian, while the $\gamma^{\hat{i}}$'s are Hermitian, and choose

$$\beta \equiv +i\gamma_{\hat{0}} = -i\gamma^{\hat{0}}. \qquad (2.6)$$

Note that β is Hermitian and unipotent,

$$\beta^{\dagger} = \beta; \qquad \beta^2 = 1. \qquad (2.7)$$

The action reads $S = \int d^5x\mathcal{L}$, with Lagrangian density $\mathcal{L} = eL$ (with $e = \det e^{\hat{a}}_{\mu} = \sqrt{|g|}$), and a second-order Lagrangian L given (in units where $4\pi G_5 = 1$) by

$$L = \frac{1}{4}R(\mathring{\omega}) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{6\sqrt{3}}\eta^{\mu\nu\lambda\rho\sigma}A_{\mu}F_{\nu\lambda}F_{\rho\sigma} + \frac{1}{2}(\bar{\psi}_{\mu}\gamma^{\mu\nu\rho}\mathcal{D}_{\nu}(\mathring{\omega})\psi_{\rho} - \overline{\mathcal{D}_{\nu}(\mathring{\omega})\psi_{\mu}}\gamma^{\mu\nu\rho}\psi_{\rho})$$

$$- i\frac{\sqrt{3}}{4}(\bar{\psi}_{\mu}\gamma^{\mu\nu\rho\sigma}\psi_{\nu} + \bar{\psi}^{\rho}\psi^{\sigma} - \bar{\psi}^{\sigma}\psi^{\rho})F_{\rho\sigma} + \bar{\psi}_{[\mu}\gamma^{\mu}\psi_{\alpha]}\bar{\psi}^{[\nu}\gamma_{\nu}\psi^{\alpha]} - \frac{1}{2}\bar{\psi}_{[\mu}\gamma_{|\nu|}\psi_{\rho]}\bar{\psi}^{[\mu}\gamma^{|\rho|}\psi^{\nu]}$$

$$- \frac{1}{4}\bar{\psi}_{[\mu}\gamma^{\nu}\psi_{\rho]}\bar{\psi}^{[\mu}\gamma_{\nu}\psi^{\rho]} + \frac{1}{4}\bar{\psi}_{\mu}\psi_{\nu}\bar{\psi}_{\rho}\gamma^{\mu\nu\rho\sigma}\psi_{\sigma} + \frac{3}{8}(\bar{\psi}_{\mu}\psi_{\nu} - \bar{\psi}_{\nu}\psi_{\mu})\bar{\psi}^{\mu}\psi^{\nu}. \qquad (2.8)$$

A consistent truncation of this theory consists in considering a (Bianchi-I) five-dimensional "minisuperspace" cosmological model where all the fields ($g_{\mu\nu}$, A_{μ}, ψ_{μ}) depend only on time, without any spatial dependence. More precisely, we consider a model where the four-dimensional space is toroidally compactified (with $0 \leq x^i \leq 1$, $i = 1, 2, 3, 4$), so that supergravity reduces to a kind of supersymmetric quantum mechanical model for the zero modes $g_{\mu\nu}(t)$, $A_{\mu}(t)$, and $\psi_{\mu}(t)$. The metric is written as

$$ds^2 = -N(t)^2 dt^2 + h_{ij}(t)(dx^i + N^i(t)dt)(dx^j + N^j(t)dt). \qquad (2.9)$$

The time component A_0 of the A_{μ} field drops out of the dynamics (the associated Gauss constraint being identically zero). Similarly, the shift vector $N^i(t)$ drops out of the dynamics (its associated momentum constraint vanishing identically). We henceforth set both A_0 and N^i to zero. The only constraints that will remain in our cosmological dynamics are (i) the Hamiltonian constraint associated with the lapse function $N(t)$, and (ii) the supersymmetry constraint associated with $\psi_0(t)$.

As in our previous work dealing with a supersymmetric Bianchi IX model in $D = 4$ [25], we shall avoid the presence of constraints linked to local Lorentz rotations by using a local frame that is algebraically defined in terms of the metric components $g_{\mu\nu}$. We use ($i, j, k = 1, 2, 3, 4$)

$$\theta^{\hat{0}} = N dt, \qquad \theta^{\hat{a}} = \theta^{\hat{a}}_i dx^i,$$

$$e_{\hat{0}} = \frac{1}{N}\partial_t, \qquad e_{\hat{a}} = e^i_{\hat{a}}\partial_i, \qquad (2.10)$$

where $\theta^{\hat{a}}_j e^i_{\hat{a}} = \delta^i_j$. Previous work on the approach to cosmological singularities [12] has emphasized the usefulness of parametrizing the gravitational degrees of freedom by means of an Iwasawa decomposition of the spatial coframe $\theta^{\hat{a}}_i$. This means encoding the ten independent components of the spatial metric h_{ij} by means of four diagonal logarithmic scale factors $\exp(-\beta^{\hat{a}})$ and six off-diagonal variable $n^{\hat{a}}_i$ (with $\hat{a} < i$), defined so that

$$\theta^{\hat{a}}_i = e^{-\beta^{\hat{a}}}(\mathcal{N})^{\hat{a}}_i, \qquad e^i_{\hat{a}} = e^{\beta^{\hat{a}}}(\mathcal{N}^{-1})^i_{\hat{a}}, \qquad (2.11)$$

where $\mathcal{N}$ is an upper-triangular, unipotent matrix; namely,

$$\mathcal{N} = (\mathcal{N}^{\hat{a}}_i) = (\delta^{\hat{a}}_i + n^{\hat{a}}_i) = \begin{pmatrix} 1 & n^1_2 & n^1_3 & n^1_4 \\ 0 & 1 & n^2_3 & n^2_4 \\ 0 & 0 & 1 & n^3_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the inverse matrix $(\mathcal{N}^{-1})^i_{\hat{a}}$ is also an unipotent upper-triangular matrix.

As a consequence the spatial metric h_{ij} reads

$$h_{ij} = \sum_{\hat{a}} e^{-2\beta^{\hat{a}}}\mathcal{N}^{\hat{a}}_i\mathcal{N}^{\hat{a}}_j, \text{i.e.,}$$

$$(h_{ij}) = \mathcal{N}^T\mathcal{A}^2\mathcal{N}; \quad \text{with} \quad \mathcal{A} \equiv \text{diag}\{e^{-\beta^{\hat{a}}}\}. \qquad (2.12)$$

It is convenient to use as basic variables in the Lagrangian formulation the quantities

$$\beta^{\hat{a}}; \quad n^{\hat{a}}_i \text{ (with } \hat{a} < i\text{)}; \quad B_{\hat{a}}; \quad \Psi_{\hat{a}}; \quad \tilde{N}, \qquad (2.13)$$

where we defined (to replace A_i, ψ_i, and N)

$$B_{\hat{a}} \equiv A_i(\mathcal{N}^{-1})^i_{\hat{a}}; \qquad \Psi_{\hat{a}} \equiv e^{-\frac{1}{2}\sigma_{\beta}}\psi_{\hat{a}}; \qquad \tilde{N} \equiv Ne^{+\sigma_{\beta}}, \qquad (2.14)$$

with $\sigma_{\beta} \equiv \sum_{\hat{a}=1}^4 \beta^{\hat{a}}$.

The Lagrangian density $\mathcal{L} = eL$ then decomposes into

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{F^2} + \mathcal{L}_{RS} + \mathcal{L}_{F\Psi^2} + \mathcal{L}_{\Psi^4}, \qquad (2.15)$$

where $\mathcal{L}_R = \frac{e}{4} R(\overset{\circ}{\omega})$ corresponds to the first (Einstein-Hilbert) term in Eq. (2.8), $\mathcal{L}_{F^2} = -\frac{e}{4} F_{\mu\nu} F^{\mu\nu}$ to the second (Maxwell) term, $\mathcal{L}_{RS}$ to the Rarita-Schwinger term on the first line, $\mathcal{L}_{F\Psi^2}$ to the $\bar{\psi}\psi F$ coupling on the second line, and where $\mathcal{L}_{\Psi^4}$ corresponds to all the remaining terms, which are quartic in ψ—the Chern-Simons term $A \wedge F \wedge F$ on the first line vanishes, as well as its variation. In our units (where $4\pi G_5 = 1$ and $\int d^4 x = 1$), we can consider $\mathcal{L}$ as the total Lagrangian of a supersymmetric quantum mechanical model, with corresponding action $S = \int dt \mathcal{L}$.

The explicit expressions of the various terms in $\mathcal{L}$, Eq. (2.15), are as follows. The Einstein term reads (henceforth, we cease to systematically put hats on the frame indices $a = \hat{a}$)

$$\mathcal{L}_R = \frac{1}{4\tilde{N}} \left(G_{ab} \dot{\beta}^a \dot{\beta}^b + \frac{1}{2} \sum_{a<b} e^{+2(\beta^b - \beta^a)} (W_{\hat{a}\hat{b}})^2 \right), \quad (2.16)$$

where the quadratic form G_{ab} defining the kinetic terms of the logarithmic scale factors β^a is defined as

$$G_{ab} \dot{\beta}^a \dot{\beta}^b \equiv \sum_a (\dot{\beta}^a)^2 - \left(\sum_a \dot{\beta}^a \right)^2, \quad (2.17)$$

and where we defined (for $\hat{a} < \hat{b}$)

$$W_{\hat{a}\hat{b}} \equiv W^{\hat{a}}{}_{\hat{b}} \equiv \sum_{\hat{a} < i \leq \hat{b}} \dot{n}^{\hat{a}}{}_i (\mathcal{N}^{-1})^i{}_{\hat{b}}. \quad (2.18)$$

The Maxwell kinetic term reads

$$\mathcal{L}_{F^2} = \frac{1}{2\tilde{N}} \sum_a e^{2\beta^a} E_a^2, \quad (2.19)$$

where E_a denotes the electric-field variable

$$E_a \equiv \sum_i (\mathcal{N}^{-1})^i_{\hat{a}} F_{ti}$$

$$= \dot{B}_{\hat{a}} + \sum_{i=2}^{\hat{a}} \sum_{\hat{b}=1}^{i-1} B_{\hat{b}} \dot{n}_i^{\hat{b}} (\mathcal{N}^{-1})^i_{\hat{a}}. \quad (2.20)$$

The Rarita-Schwinger term reads

$$\mathcal{L}_{RS} = \frac{i}{2} G_{ab} (\Phi^{\dagger a} \dot{\Phi}^b - \dot{\Phi}^{\dagger a} \Phi^b) + \frac{N}{2} \tilde{Q}^{\hat{\nu}\hat{\alpha}\hat{\beta}} \overset{\circ}{\omega}_{\hat{\nu}\hat{\alpha}\hat{\beta}}. \quad (2.21)$$

Here we replaced the rescaled gravitino Ψ^a by the useful vector-spinor variable [16]

$$\Phi^a \equiv \gamma^{\hat{a}} \Psi^{\hat{a}}; \qquad (\text{no sum on } \hat{a}), \quad (2.22)$$

while the second term involves the contraction between the Fermion bilinear

$$\tilde{Q}^{\hat{\nu}\hat{\alpha}\hat{\beta}} = \frac{1}{2} (\bar{\Psi}_{\hat{\mu}} \gamma^{\hat{\mu}\hat{\nu}\hat{\alpha}} \Psi^\beta - \bar{\Psi}_{\hat{\mu}} \gamma^{\hat{\mu}\hat{\nu}\hat{\beta}} \Psi^\alpha - \bar{\Psi}^{\hat{\beta}} \gamma^{\hat{\alpha}\hat{\nu}\hat{\mu}} \Psi_{\hat{\mu}} + \bar{\Psi}^{\hat{\alpha}} \gamma^{\hat{\beta}\hat{\nu}\hat{\mu}} \Psi_{\hat{\mu}}) + \frac{1}{2} \bar{\Psi}_{\hat{\mu}} \gamma^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\alpha}\hat{\beta}} \Psi_{\hat{\rho}}$$

$$+ \frac{1}{2} (\bar{\Psi}_{\hat{\mu}} (\gamma^{\hat{\mu}} (\eta^{\hat{\rho}\hat{\alpha}} \eta^{\hat{\nu}\hat{\beta}} - \eta^{\hat{\nu}\hat{\alpha}} \eta^{\hat{\rho}\hat{\beta}}) + \gamma^{\hat{\nu}} (\eta^{\hat{\mu}\hat{\alpha}} \eta^{\hat{\rho}\hat{\beta}} - \eta^{\hat{\rho}\hat{\alpha}} \eta^{\hat{\mu}\hat{\beta}}) + \gamma^{\hat{\rho}} (\eta^{\hat{\nu}\hat{\alpha}} \eta^{\hat{\mu}\hat{\beta}} - \eta^{\hat{\mu}\hat{\alpha}} \eta^{\hat{\nu}\hat{\beta}})) \Psi_{\hat{\rho}}), \quad (2.23)$$

and the Levi-Civita spin-connection, whose only non-vanishing (frame) components are

$$\overset{\circ}{\omega}_{\hat{0}\hat{a}\hat{b}} = -\frac{1}{2N} (e^{-(\beta^a - \beta^b)} W_{\hat{a}\hat{b}} - e^{-(\beta^b - \beta^a)} W_{\hat{b}\hat{a}}),$$

$$\overset{\circ}{\omega}_{\hat{b}\hat{0}\hat{a}} = \frac{1}{N} \left(\dot{\beta}^a \delta_{\hat{a}\hat{b}} - \frac{1}{2} (e^{-(\beta^a - \beta^b)} W_{\hat{a}\hat{b}} + e^{-(\beta^b - \beta^a)} W_{\hat{b}\hat{a}}) \right)$$

$$= \overset{\circ}{\omega}_{\hat{a}\hat{0}\hat{b}}. \quad (2.24)$$

Here the quantities $W_{\hat{a}\hat{b}}$ (which are essentially the time derivatives of $n^{\hat{a}}{}_i$) were defined in Eq. (2.18) above. Note that $W_{\hat{a}\hat{b}}$ vanishes if $\hat{b} \leq \hat{a}$, so that the nonvanishing contributions to $\overset{\circ}{\omega}$ are all multiplied by a factor of the type $e^{+(\beta^b - \beta^a)}$ with $b > a$.

The $\bar{\psi}\psi F$ coupling term, $\mathcal{L}_{F\Psi^2}$, reads

$$\mathcal{L}_{F\Psi^2} = -i \frac{\sqrt{3}}{2} \sum_{\hat{a}} e^{\beta^{\hat{a}}} X^{\hat{0}\hat{a}} E_{\hat{a}}, \quad (2.25)$$

where

$$X^{\hat{0}\hat{a}} \equiv i\eta^{bcd\hat{0}a} \bar{\Psi}_b \gamma_c \Psi_d + \bar{\Psi}^{\hat{0}} \Psi^a - \bar{\Psi}^a \Psi^{\hat{0}}. \quad (2.26)$$

At this stage, we see that the Lagrangian is the sum of four types of terms: (i) the kinetic terms for the bosonic variables $\beta^{\hat{a}}$; $n^{\hat{a}}{}_i$; $B_{\hat{a}}$; namely,

$$\mathcal{L}_{\text{kin b}} = \frac{1}{\tilde{N}} \left(\frac{1}{4} G_{ab} \dot{\beta}^a \dot{\beta}^b + \frac{1}{8} \sum_{a<b} e^{+2(\beta^b - \beta^a)} (W_{\hat{a}\hat{b}})^2 \right.$$

$$\left. + \frac{1}{2} \sum_a e^{2\beta^a} E_a^2 \right); \quad (2.27)$$

(ii) the kinetic terms for the fermionic variables $\Psi_{\hat{a}}$; namely,

$$\mathcal{L}_{\mathrm{kin\,f}} = \frac{i}{2} G_{ab}(\Phi^{\dagger a}\dot{\Phi}^b - \dot{\Phi}^{\dagger a}\Phi^b); \qquad (2.28)$$

(iii) the couplings between the bosonic velocity variables $\dot{\beta}^a$, W_{ab}, E_a and corresponding fermion bilinears, $Q_{\hat{\beta}}^a(\bar{\Psi}, \Psi)$, $Q_W^{ab}(\bar{\Psi}, \Psi)$, $Q_E^a(\bar{\Psi}, \Psi)$, of the form

$$\mathcal{L}_{\dot{q}\Psi^2} = \sum_a \dot{\beta}^a Q_{\hat{\beta}}^a(\bar{\Psi}, \Psi) + \sum_{a<b} e^{\beta^b - \beta^a} W_{ab} Q_W^{ab}(\bar{\Psi}, \Psi)$$

$$+ \sum_a e^{\beta^a} E_a Q_E^a(\bar{\Psi}, \Psi), \qquad (2.29)$$

with, for instance

$$\begin{aligned}
Q_{\hat{\beta}}^a(\bar{\Psi}, \Psi) &= \tilde{Q}^{\hat{a}\hat{0}\hat{a}} \\
&= \frac{i}{2} \sum_{\hat{m}\neq\hat{a}}(\Psi^{\dagger\hat{m}}\gamma^{\hat{m}\hat{a}}\Psi^{\hat{a}} - \Psi^{\dagger\hat{a}}\gamma^{\hat{a}\hat{m}}\Psi^{\hat{m}} + \Psi^{\dagger\hat{0}}\gamma^{\hat{0}\hat{m}}\Psi^{\hat{m}} + \Psi^{\dagger\hat{m}}\gamma^{\hat{m}\hat{0}}\Psi^{\hat{0}}) \\
&= \frac{i}{2} \sum_{\hat{m}\neq\hat{a}}(\Phi^{\dagger\hat{m}}\Phi^{\hat{a}} - \Phi^{\dagger\hat{a}}\Phi^{\hat{m}} + \Psi^{\dagger\hat{0}}\gamma^{\hat{0}}\Phi^{\hat{m}} + \Phi^{\dagger\hat{m}}\gamma^{\hat{0}}\Psi^{\hat{0}}); \qquad (2.30)
\end{aligned}$$

and, finally, (iv) the terms quartic in the fermions that entered the original Lagrangian, Eq. (2.8); namely,

$$\mathcal{L}_{\Psi^4} = \tilde{N} L_{\Psi^4}, \qquad (2.31)$$

with

$$\begin{aligned}
L_{\Psi^4} &= \bar{\Psi}_{[\mu}\gamma^\mu\Psi_{\alpha]}\bar{\Psi}^{[\nu}\gamma_\nu\Psi^{\alpha]} - \frac{1}{2}\bar{\Psi}_{[\mu}\gamma_{|\nu|}\Psi_{\rho]}\bar{\Psi}^{[\mu}\gamma^{|\rho|}\Psi^{\nu]} \\
&\quad - \frac{1}{4}\bar{\Psi}_{[\mu}\gamma^\nu\Psi_{\rho]}\bar{\Psi}^{[\mu}\gamma_\nu\Psi^{\rho]} + \frac{1}{4}\bar{\Psi}_\mu\Psi_\nu\bar{\Psi}_\rho\gamma^{\mu\nu\rho\sigma}\Psi_\sigma \\
&\quad + \frac{3}{8}(\bar{\Psi}_\mu\Psi_\nu - \bar{\Psi}_\nu\Psi_\mu)\bar{\Psi}^\mu\Psi^\nu. \qquad (2.32)
\end{aligned}$$

III. CLASSICAL HAMILTONIAN FORMULATION

We have seen in the previous section that the Lagrangian had a structure of the type

$$\begin{aligned}
\mathcal{L} &= \frac{i}{2}G_{ab}(\Phi^{\dagger a}\dot{\Phi}^b - \dot{\Phi}^{\dagger a}\Phi^b) + \frac{1}{2\tilde{N}}\dot{q}^k g_{kl}\dot{q}^l \\
&\quad + \mathcal{Q}_k(\Psi)\dot{q}^k + \tilde{N} L_{\Psi^4}, \qquad (3.1)
\end{aligned}$$

where q^k denote the bosonic variables, $\beta^{\hat{a}}$; $n^{\hat{a}}_i$; $B_{\hat{a}}$, where the $\mathcal{Q}_k$'s are bilinear in the fermions (and depend on the bosonic variables, notably through various exponential factors $e^{\beta^b - \beta^a}$, e^{β^a}), and where the term quartic in the fermions, L_{Ψ^4}, is given by Eq. (2.32). We recall that Φ^a denote the redefined version (2.22) of the gravitino variables Ψ^a.

Passing to the corresponding Hamiltonian formulation, in terms of the bosonic momenta,

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = \frac{1}{\tilde{N}} g_{kl}\dot{q}^l + \mathcal{Q}_k(\Psi), \qquad (3.2)$$

leads to a first-order action of the form

$$S = \int dt \left(p_k \dot{q}^k + \frac{i}{2}G_{ab}(\Phi^{\dagger a}\dot{\Phi}^b - \dot{\Phi}^{\dagger a}\Phi^b) - \tilde{N} H^{\mathrm{tot}} \right), \qquad (3.3)$$

with

$$\begin{aligned}
H^{\mathrm{tot}} &= \frac{1}{2}g^{kl}(p_k - \mathcal{Q}_k)(p_l - \mathcal{Q}_l) - L_{\Psi^4} \\
&= \frac{1}{2}g^{kl}p_k p_l - g^{kl}p_k\mathcal{Q}_l + \frac{1}{2}g^{kl}\mathcal{Q}_k\mathcal{Q}_l - L_{\Psi^4}, \qquad (3.4)
\end{aligned}$$

where g^{kl} denotes the inverse of the symmetric quadratic form g_{kl} defining the bosonic kinetic terms. As the $\mathcal{Q}_k$'s are bilinear in the fermions, the term $\frac{1}{2}g^{kl}\mathcal{Q}_k\mathcal{Q}_l$ adds to the original quartic-in-fermions term $-L_{\Psi^4}$.

The structure of the Hamiltonian action (3.3) shows that $\tilde{N}$ is a Lagrange multiplier, associated with the Hamiltonian constraint

$$H^{\mathrm{tot}} = 0. \qquad (3.5)$$

In addition, the explicit computation of H^{tot} shows (as guaranteed by the local supersymmetry of the original, unreduced supergravity action) that the time component $\Psi_{\hat{0}}$ of the gravitino (and its Dirac conjugate $\bar{\Psi}_{\hat{0}}$) appear only linearly in H^{tot}. They are therefore two fermionic Lagrange multipliers, associated with two supersymmetry constraints, say

$$\mathcal{S} = 0, \qquad \bar{\mathcal{S}} = 0, \qquad (3.6)$$

whose expressions will be given below.

The computation of H^{tot} leads to an expression of the form

$$H^{\text{tot}} = H^{(0)} + H^{(2)} + H^{(4)} + \bar{\Psi}'_{\hat{0}}\mathcal{S} + \bar{\mathcal{S}}\Psi'_{\hat{0}}, \quad (3.7)$$

where the superscripts indicate the polynomial order in the *spatial* components, $\Psi_{\hat{a}}$, or $\bar{\Psi}_{\hat{a}}$, of the gravitino, and where we introduced the following shifted time component of the gravitino,

$$\Psi'_{\hat{0}} \equiv \Psi_{\hat{0}} - \gamma_{\hat{0}}\sum_a \gamma^{\hat{a}}\Psi^{\hat{a}}. \quad (3.8)$$

The terms in Eq. (3.7) read as follows:

The purely bosonic part of the Hamiltonian reads

$$H^{(0)} = G^{ab}\pi_a\pi_b + 2\sum_{a<b}e^{-2(\beta^b-\beta^a)}(P_{ab})^2 + \frac{1}{2}\sum_a e^{-2\beta^a}(P^a)^2, \quad (3.9)$$

where π_a is the conjugate momentum to β^a, P^a is the momentum conjugate to B_a, and where P_{ab} (with $a < b$) is the following combination of the conjugate momentum $p^i{}_a$ to $n^a{}_i$ and of P^a,

$$P_{ab} \equiv \sum_{a<b\leq i}p^i{}_a\mathcal{N}^b{}_i - B_a P^b. \quad (3.10)$$

The part of the Hamiltonian that is quadratic in fermions reads

$$H^{(2)} = +2\sum_{a<b}e^{-(\beta^b-\beta^a)}P_{ab}J_{ab}(\Psi) - \frac{1}{\sqrt{3}}\sum_a e^{-\beta^a}P^a J_a(\Psi), \quad (3.11)$$

where $J_{ab}(\Psi)$ and $J_a(\Psi)$ are fermion bilinears [defined in Eqs. (3.14) below].

The part of the Hamiltonian that is quartic in fermions is given by the following sum

$$H^{(4)} = \frac{1}{2}\sum_{a<b}(J_{ab}(\Psi))^2 + \frac{1}{6}\sum_a(J_a(\Psi))^2 - L^{\text{cg}}_{\Psi^4}, \quad (3.12)$$

where the superscript cg means that one should replace everywhere in L_{Ψ^4} $\Psi_{\hat{0}}$ by its "coset gauge value", $\Psi^{\text{cg}}_{\hat{0}}$, obtained by setting $\Psi'_{\hat{0}}$ to zero, i.e., in view of Eq. (3.8), by

$$\Psi^{\text{cg}}_{\hat{0}} = \gamma_{\hat{0}}\sum_a \gamma^{\hat{a}}\Psi^{\hat{a}} = \gamma_{\hat{0}}\sum_a \Phi^{\hat{a}}. \quad (3.13)$$

It was found in previous works that this coset gauge has the property of revealing hidden Kac-Moody structures in the fermionic dynamics.

The fermion bilinears $J_{ab}(\Psi)$ (with $a < b$) and $J_a(\Psi)$ entering both $H^{(2)}$ and $H^{(4)}$ have the factorized vector-spinor structure found in Ref. [16] (and generalized in Refs. [17,19]), namely

$$J_{ab}(\Psi) = (G_{cd} - 2\alpha^{(ab)}_c\alpha^{(ab)}_d)\Phi^{\dagger c}\left(\frac{i\gamma^{ab}}{2}\right)\Phi^d,$$

$$J_a(\Psi) = (G_{cd} - 2\alpha^{(a)}_c\alpha^{(a)}_d)\Phi^{\dagger c}\left(\frac{3\gamma^a}{2}\right)\Phi^d, \quad (3.14)$$

where $\alpha^{(ab)}_c$ and $\alpha^{(a)}_c$ denote the (covariant) components of the linear forms in the β's that appear as exponents in several pieces of the Hamiltonian, namely

$$\alpha^{(ab)}(\beta) \equiv \alpha^{(ab)}_c\beta^c \equiv \beta^b - \beta^a,$$

$$\alpha^{(a)}(\beta) \equiv \alpha^{(a)}_c\beta^c \equiv \beta^a. \quad (3.15)$$

For instance, $\alpha^{(a)}_c = \delta^a_c$. The (Kac-Moody) meaning of the linear forms $\alpha^{(ab)}(\beta)$, $\alpha^{(a)}(\beta)$ will be explained in the next section. Note that in the definitions (3.14), a and b are numerical labels (which are not summed over), while c and d are vectorial indices in β space that are summed over as per the Einstein convention.

Finally, the supersymmetry constraint $\mathcal{S}$ has the form

$$\mathcal{S} = \mathcal{S}^{(1)} + \mathcal{S}^{(3)}, \quad (3.16)$$

where the linear in fermion part is

$$\mathcal{S}^{(1)} = \sum_a \pi_a\Phi^a - \sum_{a<b}e^{-(\beta^b-\beta^a)}P_{ab}\gamma^{ab}(\Phi^b - \Phi^a) - i\frac{\sqrt{3}}{2}\sum_a e^{-\beta^a}P^a\gamma^a\Phi^a, \quad (3.17)$$

while the cubic in fermion part reads

$$\mathcal{S}^{(3)} = -\frac{1}{2}\sum_{\hat{p},\hat{q}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{q}}\Psi_{\hat{q}} - \bar{\Psi}_{\hat{q}}\gamma_{\hat{q}}\Psi_{\hat{p}})\gamma_{\hat{0}}\Psi_{\hat{p}} - \frac{1}{2}\sum_{\hat{q},\hat{p}>\hat{q}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{0}}\Psi_{\hat{q}} - \bar{\Psi}_{\hat{q}}\gamma_{\hat{0}}\Psi_{\hat{q}})\gamma_{\hat{p}}\Psi_{\hat{q}} - \frac{i}{2}\sum_{\hat{p},\hat{q},\hat{r},\hat{s}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{q}}\Psi_{\hat{r}})\Psi_{\hat{s}}$$

$$+\frac{i}{2}\sum_{\hat{p},\hat{q},\hat{s},\hat{r}>\hat{s}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{p}}\Psi_{\hat{q}})\gamma_{\hat{s}}\Psi_{\hat{r}} + \frac{i}{4}\sum_{\hat{q},\hat{r},\hat{s},\hat{k},\hat{p}>\hat{k}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{k}}\gamma_{\hat{q}\hat{r}}\Psi_{\hat{s}} + \bar{\Psi}_{\hat{s}}\gamma_{\hat{q}\hat{r}}\Psi_{\hat{k}})(\gamma_{\hat{k}}\Psi_{\hat{p}} + \gamma_{\hat{p}}\Psi_{\hat{k}})$$

$$+\frac{i}{4}\sum_{\hat{q},\hat{r},\hat{s},\hat{p}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{q}\hat{r}}\Psi_{\hat{s}} + \bar{\Psi}_{\hat{s}}\gamma_{\hat{q}\hat{r}}\Psi_{\hat{p}})\gamma_{\hat{p}}\Psi_{\hat{p}}. \quad (3.18)$$

IV. INTERMEZZO ON THE HYPERBOLIC KAC-MOODY ALGEBRA G_2^{++} AND ITS MAXIMALLY COMPACT SUBALGEBRA $K(G_2^{++})$

The bosonic part of the Hamiltonian,

$$H^{(0)} = G^{ab}\pi_a\pi_b + 2\sum_{a<b}e^{-2(\beta^b-\beta^a)}(P_{ab})^2 + \frac{1}{2}\sum_a e^{-2\beta^a}(P^a)^2,$$

$$(4.1)$$

can be viewed (when remembering the constraint $H^{(0)} = 0$) as describing the dynamics of a *massless* particle (submitted to the constraint $g_{kl}\dot{q}^k\dot{q}^l = 0$), with coordinates $q^k = (\beta^{\hat{a}}; n^{\hat{a}}_i; B_{\hat{a}})$ [or, equivalently, $(h_{ij}; A_i)$] moving in a 14-dimensional curved (Lorentzian-signature) spacetime, with metric $ds^2 = g_{kl}dq^k dq^l$ defined by

$$ds^2 = G_{ab}d\beta^a d\beta^b + \frac{1}{2}\sum_{a<b}e^{+2(\beta^b-\beta^a)}(dn^a_i(\mathcal{N}^{-1})^i_b)^2$$

$$+ \frac{1}{2}\sum_a e^{2\beta^a}(dB_a + B_b dn^b_i(\mathcal{N}^{-1})^i_a)^2.$$

$$(4.2)$$

In terms of the coordinates h_{ij}, A_i, this metric reads

$$ds^2 = \frac{1}{4}(h^{ik}h^{jl} - h^{ij}h^{kl})dh_{ij}dh_{kl} + \frac{1}{2}h^{ij}dA_i dA_j.$$

$$(4.3)$$

Though the latter spacetime metric admits as a 20-dimensional symmetry group the semidirect product of $GL(4)$ transformations (Λ^i_j) with R^4 translations $(A_i \mapsto A_i + c_i)$, its dynamics is chaotic, and describes the BKL-type chaos of general solutions of the Einstein-Maxwell theory near a cosmological singularity [31].

The finite-dimensional model defined by Eq. (3.9) is a truncation of an infinite-dimensional model describing the dynamics of a massless particle on the coset space(time) $G_2^{++}/K(G_2^{++})$, where G_2^{++} is the hyperbolic Kac-Moody group defined by the (untwisted) hyperbolic extension of the exceptional Lie group G_2, and where $K(G_2^{++})$ denotes the maximally compact subgroup of G_2^{++}, defined as the fixed point of the Chevalley involution (see below). The original motivation for considering such an hyperbolic Kac-Moody coset is the fact that the four linear forms

$$\alpha^{(1)}(\beta) = \beta^1,$$

$$\alpha^{(12)}(\beta) = \beta^2 - \beta^1,$$

$$\alpha^{(23)}(\beta) = \beta^3 - \beta^2,$$

$$\alpha^{(34)}(\beta) = \beta^4 - \beta^3,$$

$$(4.4)$$

entering the four dominant potential walls [among the Toda-like potentials $e^{-2(\beta^b-\beta^a)}$, $e^{-2\beta^a}$ of the bosonic Hamiltonian Eq. (3.9)] that determine its chaotic behavior,

can be identified with the four simple roots of G_2^{++}. Indeed, the four linear forms $\alpha_i(\beta) = \alpha_{ia}\beta^a$, $i = 1, 2, 3, 4$, with $\alpha_1 \equiv \alpha^{(1)}$, $\alpha_2 \equiv \alpha^{(12)}$, $\alpha_3 \equiv \alpha^{(23)}$, $\alpha_4 \equiv \alpha^{(34)}$, viewed as forms in β space, with metric G_{ab} (so that we have the scalar product $\langle \alpha_i, \alpha_j \rangle \equiv \alpha_{ia}G^{ab}\alpha_{jb}$) have squared lengths equal to

$$\langle \alpha_1, \alpha_1 \rangle = \frac{2}{3}, \qquad \langle \alpha_i, \alpha_i \rangle = 2 \quad \text{for } i = 2, 3, 4. \quad (4.5)$$

The associated Cartan matrix (which define G_2^{++}),

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}, \quad (4.6)$$

is given by

$$(A_{ij}) = \begin{pmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \quad (4.7)$$

The corresponding Dynkin diagram is represented in Eq. (4.8),

$$\underset{\alpha^{(1)}}{\bullet}\!\!\Lleftarrow\!\!\underset{\alpha^{(12)}}{\bullet}\!\!-\!\!\underset{\alpha^{(23)}}{\bullet}\!\!-\!\!\underset{\alpha^{(34)}}{\bullet} \qquad (4.8)$$

The Chevalley-Serre-Kac presentation is then defined by the four $sl(2)$ triplets (e_i, h_i, f_i), $i = 1, 2, 3, 4$ (associated with the four simple roots α_i), satisfying the standard defining relations of a Kac-Moody algebra associated with A_{ij},

$$[h_i, h_j] = 0; \qquad [e_i, f_j] = \delta_{ij}h_j;$$

$$[h_i, e_j] = A_{ij}e_j; \qquad [h_i, f_j] = -A_{ij}f_j \qquad (4.9)$$

together with the crucial Serre relations

$$\text{ad}(e_i)^{1-A_{ij}}(e_j) = 0; \qquad \text{ad}(f_i)^{1-A_{ij}}(f_j) = 0. \quad (4.10)$$

Summarizing the present section so far, the linear forms $\alpha^{(ab)}(\beta)$, $\alpha^{(a)}(\beta)$ entering the (Bianchi-I-reduced) bosonic Hamiltonian (3.9) suffice to characterize the hyperbolic Kac-Moody algebra G_2^{++}. Similarly to the decomposition of E_{10} associated with eleven-dimensional supergravity [11,12], one can decompose the Lie algebra of G_2^{++} with respect to the $gl(4)$ subalgebra defined by the gravity-related roots $\alpha^{(ab)}(\beta)$ (together with the Cartan element h_1) [28]. One adds to this level-0 subalgebra (K^a_b) the level-1 generators E^a and F_a associated with the electric-related roots $+\alpha^{(a)}(\beta)$ and $-\alpha^{(a)}(\beta)$. The rest of the algebra is then defined by taking commutators, starting with the level-2 defined by $E^{[ab]} \equiv [E^a, E^b]$, the level 3 defined by

$E^{[abc]} \equiv [E^a, E^{[bc]}]$, etc. It it then checked [28] that the bosonic dynamics defined by Eq. (3.9) is equal to the reduction of the infinite-dimensional coset dynamics on $G_2^{++}/K(G_2^{++})$ obtained by setting to zero the momenta corresponding to all the positive roots of levels $\ell \geq 2$ (similarly to the truncation of $E_{10}/K(E_{10})$ beyond level 1 [11,12]).

The general conjecture made in Refs. [11,12] is that there is a gravity-coset correspondence under which the dynamics of any supergravity theory would be equivalent to a corresponding hyperbolic Kac-Moody coset dynamics, having the same asymptotic cosmological billiard. The purpose of the present work is to obtain new evidence for such a correspondence by focussing on the fermionic sector of $5D$ supergravity, and particularly on the terms quartic in fermions, which have been neglected in most of the previous investigations of the gravity-coset conjecture. In this respect, we need to consider in detail the coset analog of the R-symmetry, i.e., the symmetry group under which the coset fermions are conjectured to rotate. This group is supposed (in each coset model) to be the maximally compact subgroup of the considered Kac-Moody group. In the case of $5D$ supergravity, this is $K(G_2^{++})$, whose Lie algebra is defined as the fixed point of the Lie algebra of G_2^{++} under the Chevalley involution. The Chevalley involution θ is defined by its action on the Chevalley-Serre-Kac basis,

$$\theta(h_i) = -h_i, \qquad \theta(e_i) = -f_i, \qquad \theta(f_i) = -e_i. \quad (4.11)$$

The θ-fixed subalgebra $K(G_2^{++})$ of G_2^{++} is then generated by the four Lie-algebra elements

$$x_i \equiv e_i - f_i. \quad (4.12)$$

Previous work on supergravity in $D = 11$ has shown that the gravitino field belonged to a finite-dimensional representation of the (infinite-dimensional) Lie algebra $K(E_{10})$ [13–15]. Analog results were found for other supergravity theories [27]. In our present context, we therefore expect that the $5D$ gravitino Ψ_a will belong to a finite-dimensional representation of $K(G_2^{++})$. The main results of the present work will indeed be to show not only that this is true, but to further show that the $O(\Psi^4)$ term in the (quantum) Hamiltonian is invariant under the $K(G_2^{++})$ rotations defining the representation of the quantized gravitino. In order to investigate technically this issue we will need to characterize the conditions defining a representation of $K(G_2^{++})$.

A linear representation of $K(G_2^{++})$ is characterized by a vector space on which acts four linear operators J_i satisfying the same defining relations as the four abstract Lie-algebra elements $x_i = e_i - f_i$ defined above. As each operator J_i (corresponding to x_i) is associated to the specific simple root α_i, it will be convenient to label the

linear operators J_i by the same label as the associated simple root of G_2^{++}, as listed in Eq. (4.4). Therefore, we will denote them simply as J_1, J_{12}, J_{23}, and J_{34}, respectively associated with $\alpha^{(1)}(\beta)$, $\alpha^{(12)}(\beta)$, $\alpha^{(23)}(\beta)$, and $\alpha^{(34)}(\beta)$.

The set of defining (Serre-Berman) relations that the four operators J_1, J_{12}, J_{23}, J_{34} must satisfy is [20,32]

$$\text{ad}^4(J_1)J_{12} - 10 \, \text{ad}^2(J_1)J_{12} + 9J_{12} = 0, \quad (4.13)$$

$$\text{ad}^2(J_{12})J_1 - J_1 = 0, \quad (4.14)$$

$$\text{ad}^2(J_{12})J_{23} - J_{23} = 0, \quad (4.15)$$

$$\text{ad}^2(J_{23})J_{12} - J_{12} = 0, \quad (4.16)$$

$$\text{ad}^2(J_{23})J_{34} - J_{34} = 0, \quad (4.17)$$

$$\text{ad}^2(J_{34})J_{23} - J_{23} = 0, \quad (4.18)$$

$$[J_1, J_{23}] = [J_1, J_{34}] = [J_{12}, J_{34}] = 0. \quad (4.19)$$

Note that in the present work we will be dealing with Hermitian-like rotation operators J_i, instead of the anti-Hermitian ones x_i used in mathematical contexts. In other words, a J_i rotation will be of the type $\exp(\sqrt{-1}\theta_i J_i)$, instead of $\exp(\theta_i \text{ad}x_i)$.

Several different representations of $K(G_2^{++})$ play a role in our present $5D$ supergravity context. First, there are representations associated with classical (i.e., Grassmannian-valued) fermions, of spin $\frac{1}{2}$ and $\frac{3}{2}$. Second, there are representations of $K(G_2^{++})$ associated with quantum fermions. Let us describe now the representations of classical spinors, of spin $\frac{1}{2}$ and $\frac{3}{2}$.

Note first that the relations involving only the J_{ab}'s among Eq. (4.13) express the fact that the J_{ab}'s are usual $SO(4)$ rotation operators. The four (complex) components of a spin $\frac{1}{2}$ Dirac field Ψ define a representation space for the J_{ab}'s if we define their action in the usual Spin(4) way, namely

$$J_{ab}^s = i\frac{\gamma^{ab}}{2}; \qquad 1 \leq a < b \leq 4. \quad (4.20)$$

We defined here not only the three simple-root generators J_{12}, J_{23}, J_{34} (associated with the symmetry-wall simple roots), but also the three others needed to describe the rotations in all the two-planes ab of R^4; the factor i is needed because we are working with Hermitian-like operators. It is then easy to check that if we tentatively define the generators J_a associated with the electric roots $\alpha^{(a)}(\beta) = \beta^a$ as

$$J_a^s = \frac{C^s \gamma^a}{2}; \qquad 1 \le a \le 4, \qquad (4.21)$$

the defining relations Eq. (4.13) will be satisfied if the factor C^s is equal to

$$C^s = \pm 1; \quad \text{or} \quad C^s = \pm 3 \quad \text{(for a spinor representation)}. \qquad (4.22)$$

Indeed, the first defining relation can actually be factorized as

$$(\mathrm{ad}^2(J_1) - 3^2)(\mathrm{ad}^2(J_1) - 1^2)J_{12} = 0. \qquad (4.23)$$

We can then define two types of four-dimensional spinor representations of $K(G_2^{++})$ (with $C^s = \pm 1$, or $C^s = \pm 3$).

Let us now consider the possible vector-spinor representations of $K(G_2^{++})$, i.e., matrices J_{ab}, J_a, acting on the sixteen components of a gravitinolike object $\Phi^a = \gamma^a \Psi^a$. Both the vector index $a = 1, 2, 3, 4$, and the hidden Dirac-spinor index, of Φ^a take four values. Here we consider the actions on Φ^a rather than on Ψ^a because it was found in Ref. [16] that this reveals a hidden factorized structure for the vector-spinor representations associated with $K(E_{10})$ and $K(AE_3)$. We found that such a factorized structure also holds for $K(G_2^{++})$, in spite of the fact that G_2^{++} is not simply laced (remember that the simple root α_1 has length-squared $\frac{2}{3}$). More precisely, we define, for any one of the simple roots (and more generally for any of the basic gravitational or electric roots entering the levels 0 and 1), the action of J_α on a vector spinor Φ^{aA} as

$$(J_\alpha^{vs} \cdot \Phi)^{aA} \equiv (\delta_b^a - 2\alpha^a \alpha_b)(J_\alpha^s)_B^A \Phi^{bB}, \qquad (4.24)$$

where $\alpha^a \equiv G^{ac}\alpha_c$, and where J_α^s is the above-defined action of J_α in the (four-dimensional) spinor representation. Here, for clarity, we have explicitly indicated the (usually implicit) spinor indices A, B. We then found (in agreement with Ref. [19]) that the vector-spinor matrices $(J_\alpha^{vs})^{aA}{}_{bB}$ satisfy the defining relations (4.13) if, and only if, the factor C^s entering the electric operator (4.21) is taken to be

$$C^s = \pm 3 \quad \text{(for a vector-spinor representation)}. \qquad (4.25)$$

The value $C^s = \pm 1$ fails to define a vector-spinor representation of $K(G_2^{++})$ when inserted in Eq. (4.24). Note that this is precisely the value $C^s = 3$ that appeared in the supergravity-derived bilinear J_a, Eq. (3.14). We shall explain below, after quantizing the supergravity dynamics, the meaning of the quantum avatars of the bilinears in Eq. (3.14) as generators of a 2^{16}-dimensional representation of $K(G_2^{++})$, in which lives the quantum state of our cosmological model.

V. QUANTIZATION

The classical Hamiltonian action of our supersymmetric cosmological model has the form (with $\mathcal{N}^a{}_i \equiv \delta_i^a + n^a{}_i$)

$$S = \int dt \left(\pi_a \dot{\beta}^a + p^i{}_a \dot{n}^a{}_i + P^a \dot{B}_a + \frac{i}{2} G_{ab} (\Phi^{\dagger a} \dot{\Phi}^b - \dot{\Phi}^{\dagger a} \Phi^b) \right.$$
$$\left. - \tilde{N}(H^{\mathrm{cg}} + \overline{\Psi}'_{\hat{0}} \mathcal{S} + \bar{\mathcal{S}} \Psi'_{\hat{0}}) \right). \qquad (5.1)$$

Here we use Einstein's summation convention. The indices a, i of the (strictly) upper triangular matrix $n^a{}_i$, and therefore those of its canonical conjugate $p^i{}_a$ are restricted to the range $a < i$. The contribution H^{cg} to the total Hamiltonian has the structure

$$H^{\mathrm{cg}} = H^{(0)} + H^{(2)} + H^{(4)}, \qquad (5.2)$$

where the explicit values of the terms $H^{(0)}$, $H^{(2)}$, $H^{(4)}$ were given above. The action (5.1) features three independent Lagrange multipliers: $\tilde{N}$, $\tilde{N}\overline{\Psi}'_{\hat{0}}$, and $\tilde{N}\Psi'_{\hat{0}}$, where $\Psi'_{\hat{0}}$ is the shifted value of $\Psi_{\hat{0}}$, defined in Eq. (3.8). These Lagrange multipliers reflect the presence of three local-in-time gauge symmetries: (i) invariance under reparametrization of the time variable; and (ii) the two local-in-time supersymmetries $\epsilon_0(t)$, $\bar{\epsilon}_0(t)$. These gauge symmetries allow one to choose at will the values of the Lagrange multipliers $\tilde{N}$ and $\tilde{N}\Psi'_{\hat{0}}$. It is convenient to choose the coset gauge where $\tilde{N} = 1$ and $\Psi'_{\hat{0}} = 0$.

The action (5.1) defines a constrained dynamics, with first-class constraints

$$H^{\mathrm{cg}} \approx 0; \qquad \mathcal{S}_A \approx 0; \qquad \bar{\mathcal{S}}_A \approx 0, \qquad (5.3)$$

where we explicitly indicated the spinor index A, which takes four values.

The classical (i.e., Grassmannian) consistency of supergravity implies that the constraints (5.3) close under the Poisson(-Dirac) brackets defined by the kinetic terms

$$\{\beta^a, \pi_b\}_P = \delta_b^a; \qquad \{n^a{}_i, p^j{}_b\}_P = \delta_b^a \delta_i^j;$$
$$\{B_a, P^b\}_P = \delta_b^a; \qquad \{\Phi^{aA}, \Phi^{\dagger bB}\}_P = \frac{1}{i} G^{ab} \delta^{AB}. \qquad (5.4)$$

A crucial classical identity (which we checked to hold) is the fact that the Poisson brackets of the supersymmetry constraints close as follows:

$$\{\mathcal{S}_A, \mathcal{S}_B\}_P = 0;$$
$$\{\mathcal{S}_A^\dagger, \mathcal{S}_B^\dagger\}_P = 0;$$
$$\{\mathcal{S}_A, \mathcal{S}_B^\dagger\}_P = L(\Phi)^{\dagger C}_{AB} \mathcal{S}_C - L(\Phi)^C_{AB} \mathcal{S}_C^\dagger + \frac{1}{i} \delta_{AB} H^{\mathrm{cg}}, \quad (5.5)$$

where $L(\Phi)^C_{AB}$ is linear in Φ and does not contain $\Phi^\dagger$, nor any of the bosonic dynamical variables.

We quantize the constrained dynamics defined by the action (5.1) *à la* Dirac, i.e., by (i) replacing Poisson-Dirac brackets by appropriate (anti)commutators, (ii) verifying that this allows one to construct operators providing a deformed version of the classical algebra of constraints, and (iii) imposing the quantum constraints $\hat{\mathcal{C}} = (\hat{H}, \hat{\mathcal{S}}_A, \hat{\mathcal{S}}^\dagger_A)$ as conditions restricting physical states $|\boldsymbol{\Psi}\rangle$: $\hat{\mathcal{C}}|\boldsymbol{\Psi}\rangle = 0$.

For the bosonic degrees of freedom we adopt a Schrödinger picture. The wave function of the Universe is seen as a function of the fourteen configuration-space variables β^a, $n^a{}_i$, and B_a. The corresponding basic conjugate quantum momenta operators are represented as

$$\hat{\pi}_a = \frac{\hbar}{i}\partial_{\beta^a},$$

$$\hat{p}^i{}_a = \frac{\hbar}{i}\partial_{n^a{}_i},$$

$$\hat{P}^a = \frac{\hbar}{i}\partial_{B_a}. \tag{5.6}$$

In the following, we shall often set $\hbar = 1$. The momentum-like combination P_{ab}, Eq. (3.10), associated with the symmetry walls $\alpha^{(ab)}$ are then defined as

$$\hat{P}_{ab} \equiv \sum_{a<i\leq b}\hat{p}^i_a\mathcal{N}^b{}_i - B_a\hat{P}^b$$

$$= \sum_{a<i\leq b}\mathcal{N}^b{}_i\hat{p}^i_a - \hat{P}^b B_a. \tag{5.7}$$

As indicated, there are no ordering ambiguities in defining $\hat{P}_{ab}$ because they are defined only for $a < b$. These operators satisfy an algebra which coincides (modulo a factor $i\hbar$) with the classical one. For instance, we have

$$\{P_{12}, P_{23}\}_P = P_{13}; \quad \text{and} \quad [\hat{P}_{12}, \hat{P}_{23}] = i\hbar\hat{P}_{13}. \tag{5.8}$$

Similarly

$$\{P_{12}, P^1\}_P = -P^2; \quad \text{and} \quad [\hat{P}_{12}, \hat{P}^1] = -i\hbar\hat{P}^2$$

$$\{P_{12}, P^2\}_P = 0; \quad \text{and} \quad [\hat{P}_{12}, \hat{P}^2] = 0. \tag{5.9}$$

The fermionic operators have to obey anticommutations relations dictated by their kinetic term. These anticommutation relations take an especially simple form when using the objects Φ^a and $\Phi^{\dagger a}$ (rather than Ψ^a and $\bar{\Psi}^a$); namely,

$$\{\hat{\Phi}^{aA}, \hat{\Phi}^{bB}\} = 0, \qquad \{\hat{\Phi}^{\dagger aA}, \hat{\Phi}^{\dagger bB}\} = 0,$$

$$\{\hat{\Phi}^{aA}, \hat{\Phi}^{\dagger bB}\} = \hbar G^{ab}\delta^{AB}, \tag{5.10}$$

where, now, the curly brackets (without a P subscript) denote an anticommutator.

When decomposing the Hermitian-conjugated quantum fermionic operators $\hat{\Phi}^{Aa}$, $\hat{\Phi}^{\dagger aA}$ into their (formally) Hermitian parts, $\hat{\phi}^{Aa}_1 \equiv \frac{1}{2}(\hat{\Phi}^{Aa} + \hat{\Phi}^{\dagger aA})$, $\hat{\phi}^{Aa}_2 \equiv \frac{1}{2i}(\hat{\Phi}^{Aa} - \hat{\Phi}^{\dagger aA})$, the 32 fermionic operators, $\hat{\phi}^{Aa}_i$, $i = 1, 2$, are found to satisfy a Clifford algebra in a *real* 32-dimensional space endowed with the quadratic form $\frac{1}{2}\hbar\delta_{ij}G^{ab}\delta^{AB}$, which has signature $24^+, 8^-$. Thus the gravitino operators can be represented by $2^{16} \times 2^{16} = 65536 \times 65536$ Dirac matrices[1] and the wave function of the Universe can be viewed as a 65536-dimensional spinor of Spin(24,8), depending on the fourteen configuration-space variables β^a, $n^a{}_i$ and B_a: $\boldsymbol{\Psi} = \Psi_\sigma(\beta^a, \varphi^b)$, with $\sigma = 1, ..., 65536$.

VI. QUANTUM CONSTRAINTS AND THEIR CONSISTENCY

A crucial issue in the quantization of our system is to promote the classical constraints (5.3) into corresponding quantum operators, say $\hat{\mathcal{S}}_A$, $\hat{\bar{\mathcal{S}}}_A$, and $\hat{H}$, so as to impose them, *à la* Dirac, on the state $|\boldsymbol{\Psi}\rangle$,

$$\hat{\mathcal{S}}_A|\boldsymbol{\Psi}\rangle = 0, \qquad \hat{\bar{\mathcal{S}}}_A|\boldsymbol{\Psi}\rangle = 0, \qquad \hat{H}|\boldsymbol{\Psi}\rangle = 0. \tag{6.1}$$

However, such a quantization scheme will be consistent only if we can define an ordering such that the quantum constraints operators $\hat{\mathcal{S}}_A$, $\hat{\bar{\mathcal{S}}}_A$, and $\hat{H}$ do close on themselves by satisfying a quantum version of the classical identities (5.5). Let us indicate how we succeeded in defining such an ordering and then in proving its quantum consistency.

The structure of the classical supersymmetry constraint is, sketchily,

$$\mathcal{S} \sim \pi\Phi + \sum_\alpha e^{-\alpha(\beta)}P_\alpha\Phi + \Phi^\dagger\Phi\Phi. \tag{6.2}$$

There are no ordering ambiguities in the dependence of $\mathcal{S}$ on bosonic variables because (i) the bosonic variables commute with the fermionic ones, (ii) the wall forms $\alpha(\beta)$ commute with the momentalike variables P_α, and (iii) we have seen that the P_α's have no internal ordering ambiguities. Finally, the only ordering ambiguity in the definition of $\hat{\mathcal{S}}$ is contained in the last, cubic-in-fermions term $\Phi^\dagger\Phi\Phi$. The ordering of the latter term is, however, uniquely fixed by the natural requirement of respecting the symmetry between the Φ's and the $\Phi^\dagger$'s that is present in the basic quantization conditions (5.10).

Starting from the classical (Grassmannian) expression of the cubic contribution,

$$\mathcal{S}^{(3)}_A = +\sigma^{\bar{\mathcal{C}}[\mathcal{AB}]}_A\bar{\Psi}_{\bar{\mathcal{C}}}\Psi_\mathcal{A}\Psi_\mathcal{B}, \tag{6.3}$$

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[1]In view of the signature $24^+, 8^-$, these matrices can be chosen to be real.

we define its quantum version by

$$\hat{\mathcal{S}}_A^{(3)} \equiv -\sigma_A^{\bar{\mathcal{C}}[\mathcal{AB}]} \hat{\Psi}_{\mathcal{A}} \hat{\bar{\Psi}}_{\bar{\mathcal{C}}} \hat{\Psi}_{\mathcal{B}}. \qquad (6.4)$$

Here, the calligraphic indices $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{C}}$ denote combined vector-spinor indices aA, bB, cC (with an additional bar on the indices pertaining to a $\bar{\Psi}$), and we use the Einstein summation convention on these indices. The coefficients $\sigma_A^{\bar{\mathcal{C}}[\mathcal{AB}]}$ are numerical factors (involving products of gamma matrices) that are defined so as to be antisymmetric in $\mathcal{AB}$. When reexpressing $\hat{\mathcal{S}}_A^{(3)}$ in terms of the $\Phi^{\mathcal{A}} \equiv \Phi^{aA}$'s and $\Phi^{\dagger \mathcal{A}} \equiv \Phi^{\dagger aA}$'s we have

$$\hat{\mathcal{S}}_A^{(3)} = c_{\mathcal{ABC}}^A \hat{\Phi}^{\mathcal{A}} \hat{\Phi}^{\dagger \mathcal{B}} \hat{\Phi}^{\mathcal{C}}, \qquad (6.5)$$

with corresponding numerical coefficients $c_{\mathcal{ABC}}^A = -c_{\mathcal{CBA}}^A$. We henceforth use such an ordering.[2]

We have checked that this ordering enforces a symmetry under which Φ^a and $\tilde{\Phi}^a \equiv \Phi^{\dagger a}$ are swapped: $\Phi \leftrightarrow \Phi^\dagger$. This is most easily seen by using a representation where γ^0, γ^1, γ^2, γ^3 are real while $\gamma^4 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ is purely imaginary. In such a representation the numerical coefficients $c_{\mathcal{ABC}}^A$ entering Eq. (6.5) are found to be purely imaginary. This ensures that

$$(c_{\mathcal{ABC}}^A \hat{\Phi}^{\mathcal{A}} \hat{\Phi}^{\dagger \mathcal{B}} \hat{\Phi}^{\mathcal{C}})^\dagger = (c_{\mathcal{ABC}}^A)^* \hat{\Phi}^{\dagger \mathcal{C}} \hat{\Phi}^{\mathcal{B}} \hat{\Phi}^{\dagger \mathcal{A}}$$
$$= -c_{\mathcal{ABC}}^A \hat{\Phi}^{\dagger \mathcal{C}} \hat{\Phi}^{\mathcal{B}} \hat{\Phi}^{\dagger \mathcal{A}} = +c_{\mathcal{ABC}}^A \hat{\Phi}^{\dagger \mathcal{A}} \hat{\Phi}^{\mathcal{B}} \hat{\Phi}^{\dagger \mathcal{C}}, \quad (6.6)$$

where we used $(c_{\mathcal{ABC}}^A)^* = -c_{\mathcal{ABC}}^A$ together with the antisymmetry $c_{\mathcal{ABC}}^A = -c_{\mathcal{CBA}}^A$ and a relabeling of indices, $\mathcal{C} \leftrightarrow \mathcal{A}$ (which are summed over).

Defining $\mathcal{S}_A$ in the way just explained, we have shown that the following quantum versions of the classical identities (5.5) hold. First,

$$\{\hat{\mathcal{S}}_A, \hat{\mathcal{S}}_B\} = 0,$$
$$\{\hat{\mathcal{S}}_A^\dagger, \hat{\mathcal{S}}_B^\dagger\} = 0, \qquad (6.7)$$

and, second,

$$\{\hat{\mathcal{S}}_A, \hat{\mathcal{S}}_B^\dagger\} = \frac{i\hbar}{2}[L(\hat{\Phi})_{AB}^{\dagger \mathcal{C}}, \hat{\mathcal{S}}_C] - \frac{i\hbar}{2}[L(\hat{\Phi})_{AB}^{\mathcal{C}}, \hat{\mathcal{S}}_C^\dagger]$$
$$+ \hbar \delta_{AB} \hat{H}_0, \qquad (6.8)$$

where $L(\hat{\Phi})_{AB}^{\mathcal{C}} = L_{AB\mathcal{A}}^{\mathcal{C}} \hat{\Phi}^{\mathcal{A}}$ is the *same* linear form in $\hat{\Phi}$ that entered the classical identity (5.5). (The $L_{AB\mathcal{A}}^{\mathcal{C}}$ being purely numerical coefficients made of gamma matrices.) Note the

presence of quantum anticommutators ($\{,\}$) on the left-hand side, and the presence of quantum commutators ($[,]$) on the right-hand side. The quantum operator $\hat{H}_0$ appearing on the last right-hand side is a (formally) Hermitian operator ($\hat{H}_0 = \hat{H}_0^\dagger$), which is a quantum version of the classical Hamiltonian H^{cg}. It has the structure

$$\hat{H}_0 = \hat{H}^{(0)} + \hat{H}^{(2)} + \hat{H}_0^{(4)}, \qquad (6.9)$$

where the bosonic part reads

$$\hat{H}^{(0)} = G^{ab} \hat{\pi}_a \hat{\pi}_b + 2 \sum_{a<b} e^{-2(\beta^b - \beta^a)} (\hat{P}_{ab})^2 + \frac{1}{2} \sum_a e^{-2\beta^a} (\hat{P}^a)^2, \qquad (6.10)$$

the part quadratic in fermions reads

$$\hat{H}^{(2)} = +2 \sum_{a<b} e^{-(\beta^b - \beta^a)} \hat{P}_{ab} \hat{J}_{ab}(\Psi) - \frac{1}{\sqrt{3}} \sum_a e^{-\beta^a} \hat{P}^a \hat{J}_a(\Psi), \qquad (6.11)$$

where the quantum bilinears $\hat{J}_\alpha$ are the quantum avatars of Eqs. (3.14); namely,

$$\hat{J}_{ab}(\Psi) = (G_{cd} - 2\alpha_c^{(ab)} \alpha_d^{(ab)}) \hat{\Phi}^{\dagger c} \left(\frac{i\gamma^{ab}}{2}\right) \hat{\Phi}^d,$$

$$\hat{J}_a(\Psi) = (G_{cd} - 2\alpha_c^{(a)} \alpha_d^{(a)}) \hat{\Phi}^{\dagger c} \left(\frac{3\gamma^a}{2}\right) \hat{\Phi}^d, \qquad (6.12)$$

and where the quartic-in-fermions part $\hat{H}_0^{(4)}$ is a uniquely-defined (Hermitian) ordered version of the classical expression (3.12). There are no ordering ambiguities in the definition (6.12) of the $\hat{J}_\alpha$'s (because the matrices $\frac{i\gamma^{ab}}{2}$ and $\frac{3\gamma^a}{2}$ are traceless). Actually, one can also check that the only ordering ambiguity in a Hermitian-ordered version of Eq. (3.12) lies in a double Wick contraction, corresponding to an additive c-number ambiguity. Anyway, what is important at this stage is that the existence of the last identity, Eq. (6.8), uniquely defines $\hat{H}_0$ and, in particular, $\hat{H}_0^{(4)}$. If we define an empty state $|0\rangle_-$ as being annihilated by all the $\hat{\Phi}$'s,

$$\Phi^{aA} |0\rangle_- = 0, \qquad (6.13)$$

we found that

$$\hat{H}_0^{(4)} |0\rangle_- = c_0 \hbar^2 |0\rangle_-, \qquad (6.14)$$

where

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[2]Actually, any other ordering will lead to the same final physical results because we have shown that any ordering of the cubic terms can be absorbed in a linear shift of the π's of the type $\pi_a \mapsto \pi_a' = \pi_a + i\delta\pi_a$, where $\delta\pi_a$ are some real numbers.

$$c_0 = -\frac{743}{24}, \tag{6.15}$$

which characterizes the c-number ordering ambiguity in the quantization path leading from the classical $H^{(4)}$ to $\hat{H}_0$.

The identity Eq. (6.8) has the advantage of featuring only manifestly Hermitian building blocks. However, it is, by itself, not of the form needed for proving the consistency of our Dirac quantization scheme. Indeed, the quantum constraints, Eq. (6.1), which are of the form $\hat{\mathcal{C}}_i\Psi = 0$, will close on themselves only if all the (anti)commutators between the quantum constraint operators $\hat{\mathcal{C}}_i$ close on this same set of constraints in the following way

$$[\hat{\mathcal{C}}_i, \hat{\mathcal{C}}_j]_\pm = \sum_k \hat{L}_{ij}^k \hat{\mathcal{C}}_k, \tag{6.16}$$

with the constraint operators $\hat{\mathcal{C}}_k$ appearing *on the right* of the coefficient operators $\hat{L}_{ij}^k$. This is not the case for the identity (6.8), which contains commutators on the right-hand side. However, we have shown that the difference between the *anticommutator* of $L(\hat{\Phi})_{AB}^{\dagger C}$ with $\hat{S}_C$, and the one of $L(\hat{\Phi})_{AB}^C$ with $\hat{S}_C^\dagger$, is such that it leads to an identity of the required form, modulo a redefinition of the quantum Hamiltonian $\hat{H}$ entering the last term. More precisely, we found that Eq. (6.8) implies the identity

$$\{\hat{S}_A, \hat{S}_B^\dagger\} = i\hbar L(\hat{\Phi})_{AB}^{\dagger C}\hat{S}_C - i\hbar L(\hat{\Phi})_{AB}^C\hat{S}_C^\dagger + \hbar\delta_{AB}\hat{H}_1, \tag{6.17}$$

where the new Hamiltonian $\hat{H}_1$ reads

$$\hat{H}_1 = \hat{H}_0 - 2i\varpi^a\hat{\pi}_a. \tag{6.18}$$

Here the real vector ϖ^a (living in β space, or Cartan space) has the following components ($a = 1, 2, 3, 4$)

$$\varpi^a = \frac{1}{4}\{1, 2, 3, 4\}, \tag{6.19}$$

or, in covariant form (i.e., in root space)

$$\varpi_a \equiv G_{ab}\varpi^a = -\frac{1}{4}\{9, 8, 7, 6\}. \tag{6.20}$$

The Hamiltonian $\hat{H}_0$ which appeared in the identity Eq. (6.8) was (formally) Hermitian, while the shifted Hamiltonian $\hat{H}_1$ entering the new identity Eq. (6.17) is formally non-Hermitian. A similar situation arose in our previous work [25]. Like in the latter case, a simple redefinition of the wave function of the universe allows us to work again with a formally hermitian Hamiltonian. Indeed, if we writes the quantum-state wave function $\langle\beta^a, \varphi^b; \sigma|\Psi\rangle = \Psi_\sigma(\beta^a, \varphi^b)$ as

$$\Psi_\sigma(\beta^a, \varphi^b) = e^{-\varpi_a\beta^a}\Psi_\sigma'(\beta^a, \varphi^b), \tag{6.21}$$

the terms involving the differential operator $\hat{\pi}_a = \frac{\hbar}{i}\partial_{\beta^a}$ in

$$\begin{aligned}
\hat{H}_1 &= G^{ab}\hat{\pi}_a\hat{\pi}_b - 2i\varpi^a\hat{\pi}_a + \ldots \\
&= -\hbar^2 G^{ab}\partial_{\beta^a}\partial_{\beta^b} - 2\hbar\varpi^a\partial_{\beta^a} + \ldots \\
&= -\hbar^2 G^{ab}(\partial_{\beta^a} + \varpi_a)(\partial_{\beta^b} + \varpi_b) + \hbar^2\varpi^2 + \ldots, \tag{6.22}
\end{aligned}$$

where we defined

$$\varpi^2 \equiv G^{ab}\varpi_a\varpi_b = -\frac{70}{16}, \tag{6.23}$$

take the following form when reexpressed through their action on $\Psi_\sigma'(\beta^a, \varphi^b)$,

$$\hat{H}_1\Psi_\sigma(\beta^a, \varphi^b) = e^{-\varpi_a\beta^a}\hat{H}_1'\Psi_\sigma'(\beta^a, \varphi^b), \tag{6.24}$$

where

$$\begin{aligned}
\hat{H}_1' &= -\hbar^2 G^{ab}\partial_{\beta^a}\partial_{\beta^b} + \hbar^2\varpi^2 + \ldots \\
&= G^{ab}\hat{\pi}_a'\hat{\pi}_b' + \hbar^2\varpi^2 + \ldots. \tag{6.25}
\end{aligned}$$

In the last expression the notation $\hat{\pi}_a'$ denotes the differential operator $\frac{\hbar}{i}\partial_{\beta^a}$ when acting on the primed wave function.

Finally, $\hat{H}_1'$ can be written as

$$\hat{H}_1' = \hat{H}'^{(0)} + \hat{H}^{(2)} + \hat{H}_1'^{(4)}. \tag{6.26}$$

Here

$$\hat{H}'^{(0)} = G^{ab}\hat{\pi}_a'\hat{\pi}_b' + 2\sum_{a<b}e^{-2(\beta^b-\beta^a)}(\hat{P}_{ab})^2 + \frac{1}{2}\sum_a e^{-2\beta^a}(\hat{P}^a)^2, \tag{6.27}$$

$\hat{H}^{(2)}$ is given by the same expression (6.11) as above, and the last contribution is given by

$$\hat{H}_1'^{(4)} = \hat{H}_0^{(4)} + \hbar^2\varpi^2 = \hat{H}_0^{(4)} - \frac{70}{16}\hbar^2. \tag{6.28}$$

In view of our previous result (6.14), we conclude that the vacuum value of the new Hamiltonian $\hat{H}_1^{(4)}$ is equal to

$$\hat{H}_1'^{(4)}|0\rangle_- = c_1\hbar^2|0\rangle_-, \tag{6.29}$$

where

$$c_1 = c_0 + \varpi^2 = -\frac{743}{24} - \frac{70}{16} = -\frac{106}{3}. \tag{6.30}$$

VII. KAC-MOODY STRUCTURE $(G_2^{++}, K(G_2^{++}))$ OF THE QUANTUM SUPERGRAVITY DYNAMICS

A. Summary of the quantum supergravity dynamics

Summarizing the results obtained so far, the quantum supergravity dynamics of our five-dimensional cosmological model is described by a 2^{16}-dimensional spinorial wave function $\mathbf{\Psi} = \Psi_\sigma(\beta^a, n^a{}_i, B_a)$ (where the spinorial index σ takes $2^{16} = 65536$ values) that must satisfy the 8×2^{16} constraints

$$\hat{\mathcal{S}}^A|\mathbf{\Psi}\rangle = 0, \qquad \hat{\mathcal{S}}^{A\dagger}|\mathbf{\Psi}\rangle = 0. \qquad (7.1)$$

Here, each of the $\hat{\mathcal{S}}^A$'s and $\hat{\mathcal{S}}^{A\dagger}$'s is represented by a $2^{16} \times 2^{16}$ matrix of first-order differential operators in the fourteen bosonic variables β^a, $n^a{}_i$, B_a. More precisely the structure of $\hat{\mathcal{S}}^A$ is

$$\hat{\mathcal{S}}^A = \sum_a \hat{\pi}_a \hat{\Phi}^{aA} - \sum_{a<b} e^{-(\beta^b - \beta^a)} \hat{P}_{ab} (\gamma^{ab})^A{}_B (\hat{\Phi}^{bB} - \hat{\Phi}^{aB})$$
$$- i\frac{\sqrt{3}}{2} \sum_a e^{-\beta^a} \hat{P}^a (\gamma^a)^A{}_B \hat{\Phi}^{aB} + \hat{\mathcal{S}}^A_{(3)}, \qquad (7.2)$$

where $\hat{\pi}_a$, $\hat{P}_{ab}$, $\hat{P}^a$ are the first-order derivative operators defined in Eq. (5.6), while the sixteen $\hat{\Phi}^{aA}$ are $2^{16} \times 2^{16}$ "gamma matrices" satisfying the Clifford algebra (5.10). The last term $\hat{\mathcal{S}}^A_{(3)}$ in Eq. (7.2) (which is analogous to a matrix-valued mass term $\hat{M}$ in a Dirac equation $\gamma^\mu \hat{P}_\mu \Psi + \hat{M}\Psi = 0$) is cubic in the $\hat{\Phi}^{aA}$'s and independent of bosonic degrees of freedom. It is defined by the ordering displayed in Eq. (6.5), with $\hat{\mathcal{S}}^{\dagger A}_{(3)}$ being correspondingly ordered. Note also that the momenta entering $\hat{\mathcal{S}}^{\dagger A}$ contain $\hat{\pi}_a^\dagger$, which is defined as usual as being $\hat{\pi}_a^\dagger \equiv \hat{\pi}_a$.

Similarly to the fact that the first-order Dirac equation $\gamma^\mu \hat{P}_\mu \Psi + m\Psi = 0$ entails the second-order Klein-Gordon equation $\eta^{\mu\nu} \hat{P}_\mu \hat{P}_\nu \Psi + m^2 \Psi = 0$, the first-order (supersymmetry) constraints (7.2) imply a quantum (Hamiltonian) constraint that is second-order in the bosonic quantum momenta $\hat{\pi}_a$, $\hat{P}_{ab}$, $\hat{P}^a$. The ordering of this quantum Hamiltonian constraint is fully determined by the above-defined ordering of the supersymmetry constraints. When acting on the rescaled wave function

$$\Psi_\sigma'(\beta^a, \varphi^b) = e^{+\varpi_a \beta^a} \Psi_\sigma(\beta^a, \varphi^b), \qquad (7.3)$$

the quantum Hamiltonian constraint reads

$$\hat{H}_1' \Psi'(\beta^a, \varphi^b) = 0, \qquad (7.4)$$

where $\hat{H}_1'$ is a Klein-Gordon-like operator of the form

$$\hat{H}_1' = G^{ab} \hat{\pi}_a' \hat{\pi}_b' + 2\sum_{a<b} e^{-2(\beta^b - \beta^a)} (\hat{P}_{ab})^2$$
$$+ \frac{1}{2} \sum_a e^{-2\beta^a} (\hat{P}^a)^2 + \hat{H}^{(2)} + \hat{\mu}^2. \qquad (7.5)$$

Here $\hat{\pi}_a' \equiv \frac{\hbar}{i} \partial_{\beta^a}$ when acting on $\Psi_\sigma'(\beta^a, \varphi^b)$, the bilinear coupling to the fermions $\hat{H}^{(2)}$ is given by

$$\hat{H}^{(2)} = +2\sum_{a<b} e^{-(\beta^b - \beta^a)} \hat{P}_{ab} \hat{J}_{ab}(\Psi) - \frac{1}{\sqrt{3}} \sum_a e^{-\beta^a} \hat{P}^a \hat{J}_a(\Psi), \qquad (7.6)$$

while the "squared mass term" $\hat{\mu}^2$ is quartic in the fermions Φ and $\Phi^\dagger$, and independent of the bosonic degrees of freedom β^a, n^a_i, B_a. When one is far from all the walls (and on their positive sides), i.e., when all the linear forms β^a, and $\beta^b - \beta^a$ (with $a < b$) are much larger than 1, one can neglect all the exponential terms, so that the Hamiltonian constraint reduces to a simple Klein-Gordon-like equation in the four-dimensional β space,

$$(G^{ab} \hat{\pi}_a' \hat{\pi}_b' + \hat{\mu}^2) \Psi'(\beta^a) = 0. \qquad (7.7)$$

However, the squared-mass term $\hat{\mu}^2 \equiv \hat{H}_1'^{(4)}$ in the latter far-wall Klein-Gordon equation is not a c number, but an operator in the quantum fermionic space, i.e., a $2^{16} \times 2^{16}$ matrix acting on the spinor index σ of the wave function Ψ_σ'.

B. Kac-Moody structures in the quantum constraints

Having summarized the quantum dynamics of our five-dimensional supergravity cosmological model, we can now highlight the hyperbolic Kac-Moody structures it contains.

First, both the supersymmetry constraints, and the Hamiltonian one, involve exponential terms of the form $e^{-\alpha_I(\beta)}$ (in $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^\dagger$) or $e^{-2\alpha_I(\beta)}$ (in $\hat{H}_1'$). Here, the $\alpha_I(\beta)$'s are certain linear forms in the logarithmic scale factors β^a parametrizing the diagonal degrees of freedom of the spatial metric h_{ij}. There are ten such linear forms. Six of them, namely

$$\alpha^{(ab)}(\beta) \equiv \beta^b - \beta^a \quad (\text{with } a < b), \qquad (7.8)$$

are called "symmetry walls forms", and are linked to the off-diagonal degrees of freedom of the spatial metric $h_{ij}(t)$, while the remaining four "electric wall forms", namely

$$\alpha^{(a)}(\beta) \equiv \beta^a, \qquad (7.9)$$

are linked to the time-dependent electric potential $A_i(t)$. When endowing the four-dimensional β space with the Lorentzian-signature metric G_{ab} defining the kinetic terms of the $\beta^a(t)$'s, Eq. (2.17), the wall forms $\alpha^{(ab)}(\beta)$ and

$\alpha^{(a)}(\beta)$ can be identified with real roots of the hyperbolic Kac-Moody algebra G_2^{++}. In addition, the four linear forms $\alpha^{(12)}(\beta)$, $\alpha^{(23)}(\beta)$, $\alpha^{(34)}(\beta)$, $\alpha^{(1)}(\beta)$ that can be identified with the four *simple roots* of G_2^{++} are the ones that enter the four dominant potential walls when considering the BKL-type chaos of general solutions of the (Einstein-Maxwell-like) bosonic dynamics of $5D$ supergravity near a cosmological singularity. [Indeed, in the Weyl chamber defined by the positivity of $\alpha^{(12)}(\beta)$, $\alpha^{(23)}(\beta)$, $\alpha^{(34)}(\beta)$, $\alpha^{(1)}(\beta)$, i.e., in the domain $0 < \beta^1 < \beta^2 < \beta^3 < \beta^4$, the other exponential potentials are subdominant; e.g., as $\alpha^{(13)}(\beta) = \alpha^{(12)}(\beta) + \alpha^{(23)}(\beta)$, we have the subdominance property $e^{-\alpha^{(13)}(\beta)} = e^{-\alpha^{(12)}(\beta)} e^{-\alpha^{(23)}(\beta)}$.]

Besides the appearance of some of the roots of G_2^{++}, including the crucial simple roots (which suffice to generate the full root lattice of G_2^{++}), the other Kac-Moody-related features exhibited by our quantum dynamics concern the fermionic sector. There are two such features.

On the one hand, the bilinear coupling to the fermions $\hat{H}^{(2)}$, Eq. (7.6), associates to each one of the wall roots $\alpha_I(\beta) = (\alpha^{(ab)}(\beta), \alpha^{(a)}(\beta))$, a coupling term of the generic form

$$e^{-\alpha_I(\beta)} \hat{P}_{\alpha_I} \hat{J}_{\alpha_I}, \qquad (7.10)$$

where $\hat{P}_{\alpha_I}$ is a quantum momentum associated with the bosonic variable $\alpha_I(\beta)$ (and contributing to the bosonic part of the Hamiltonian a term $\propto e^{-2\alpha_I(\beta)}(\hat{P}_{\alpha_I})^2$), while $\hat{J}_{\alpha_I}$ is a fermion bilinear. The important point here is that, when normalizing[3] the various fermion bilinears $\hat{J}_{\alpha_I}$ as in Eq. (6.12), they do satisfy the Serre-Berman relations Eq. (4.13) as operators acting on the 2^{16}-dimensional Clifford representation space of the quantum fermions $\Phi, \Phi^\dagger$. This follows from the fact that the Fock quantization (for fermions, as is relevant here) has functorial properties in that it maps classical generators J_α acting on some vectors v, members of some n-dimensional vector space $\mathcal{V}$, onto quantum operators $\hat{J}_\alpha$ acting on the Fock space built by piling up the successive antisymmetric powers of $\mathcal{V}$ (up to the maximum power $\mathcal{V}^{\wedge n}$ allowed by antisymmetry); in our case, $n = 16$ and the space $\mathcal{V}$ is that of classical vector-spinors v^{aA}. More precisely, given a linear endomorphism J_α of $\mathcal{V}$ [explicitly given, in some basis e_i of $\mathcal{V}$, by a matrix $(J_\alpha)^i{}_j$ acting on the vector index of $v = v^i e_i$, i.e., $(J_\alpha \cdot v)^i = (J_\alpha)^i{}_j v^j$], the Fock space is $\mathbb{C} \oplus \mathcal{V} \oplus \mathcal{V}^{\wedge 2} \oplus \cdots \mathcal{V}^{\wedge n}$, and the quantized $\hat{J}_\alpha = \Phi_i^\dagger (J_\alpha)^i{}_j \Phi^j$, with $\{\Phi_i^\dagger, \Phi^j\} = \delta_i^j$, decomposes as a

direct sum of operators acting on each (fermionic) level, from $N_F = 0$, up to $N_F = n$. More precisely, at level $N_F = 0$ (Fock vacuum, $|0\rangle_-$), $\hat{J}_\alpha$ acts like 0, at level $N_F = 1$, $\hat{J}_\alpha$ acts on $\mathcal{V}$ like J_α, and at level $N_F = 2$, $\hat{J}_\alpha$ acts on $\mathcal{V}^{\wedge 2}$ like

$$\hat{J}_\alpha|^{N_F=2} = (J_\alpha \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes J_\alpha). \qquad (7.11)$$

Explicitly, the meaning of the latter equation is that $\hat{J}_\alpha|^{N_F=2}$ acts on a (factorized)[4] element $u \wedge v \in \mathcal{V}^{\wedge 2}$ as $(J_\alpha \cdot u) \wedge v + u \wedge (J_\alpha \cdot v)$. At the fermionic level N_F, $\hat{J}_\alpha$ decomposes as a sum of N_F terms of the same type as indicated in Eq. (7.11), e.g.,

$$\hat{J}_\alpha|^{N_F=3} = (J_\alpha \otimes \mathbb{1} \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes J_\alpha \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes \mathbb{1} \otimes J_\alpha). \qquad (7.12)$$

This nice functorial nature of the map transforming a classical operator J_α into a corresponding quantized one $\hat{J}_\alpha$ allows one to transport many properties satisfied by J_α into corresponding properties of $\hat{J}_\alpha$.

For instance, classical commutators $[J_{\alpha_1}, J_{\alpha_2}]$ are mapped onto their corresponding quantum ones, namely

$$[\hat{J}_{\alpha_1}, \hat{J}_{\alpha_2}] = [\widehat{J_{\alpha_1}, J_{\alpha_2}}]. \qquad (7.13)$$

This functorial property ensures, in particular, that, if we have, say, $[J_{\alpha_1}, J_{\alpha_2}] = c J_{\alpha_3}$, the corresponding quantum commutators satisfy $[\hat{J}_{\alpha_1}, \hat{J}_{\alpha_2}] = c \hat{J}_{\alpha_3}$. This guarantees, in particular, that the Serre-Berman relations, Eq. (4.13), are preserved by the quantization. An important consequence is that the root operators $\hat{J}_{\alpha_I}$ entering the quantized Hamiltonian $\hat{H}^{(2)}$ generate a 2^{16}-dimensional representation of $K(G_2^{++})$, the maximally compact subalgebra of G_2^{++} fixed by the Chevalley involution. We will indicate below another important consequence of these functorial properties concerning the reflection operators of quantum fermions in the short-wavelength limit of the cosmological dynamics.

In addition, we have also explicitly proven that the term quartic in fermions in the quantum Hamiltonian constraint, namely $\hat{\mu}^2$ in Eq. (7.5), *commutes* with all the root operators $\hat{J}_{\alpha_I}$,

$$[\hat{J}_{\alpha_I}, \hat{\mu}^2] = 0; \quad \text{for } I = (ab), (a). \qquad (7.14)$$

Quite remarkably, the latter commutation property is rooted in a hidden simple structure of the quartic-in-fermion term. Indeed, we found that $\hat{\mu}^2$ can be expressed in terms of two simple fermion-bilinears $\hat{N}_F$ and $\hat{C}_F$, which separately commute with the root operators $\hat{J}_{\alpha_I}$. Namely,

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[3]As discussed in Appendix B of [16] the appropriate Kac-Moody-related normalization of the momentum P_{α_I} depends on the squared-length $\alpha_I^2 = G^{ab}\alpha_a^I \alpha_b^I$ of the considered root. The normalization induced by the supergravity dynamics happens to be appropriate for a Kac-Moody interpretation.

[4]A generic element of $\mathcal{V}^{\wedge 2}$ is a linear combination of such factorized elements.

$$\hat{\mu}^2 = \frac{14}{3} - \frac{1}{2}(\hat{N}_F - 8)^2 - \frac{1}{4}(\hat{C}_F^\dagger \hat{C}_F + \hat{C}_F \hat{C}_F^\dagger)$$

$$= -\frac{106}{3} + 9\hat{N}_F - \frac{1}{2}\hat{N}_F^2 - \frac{1}{2}\hat{C}_F^\dagger \hat{C}_F, \qquad (7.15)$$

with

$$\hat{N}_F \equiv G_{ab}\hat{\Phi}^{a\dagger}\hat{\Phi}^b \equiv G_{ab}\hat{\Phi}^{aA\dagger}\delta_{AB}\hat{\Phi}^{bB}, \qquad (7.16)$$

and

$$\hat{C}_F \equiv G_{ab}\hat{\Phi}^{aA}C_{AB}\hat{\Phi}^{bB}. \qquad (7.17)$$

Equation (7.16) defines the quantum fermion number, with eigenvalues $N_F = 0, 1, \ldots, 16$. In Eq. (7.17) the 4×4 matrix C_{AB} is the "charge conjugation" matrix of the (spatial) γ_i matrices, defined so that it is Hermitian, $C^\dagger = C$, and satisfies $C\gamma_i C^{-1} = -\gamma_i^T$; C_{AB} is an antisymmetric matrix in all representations of the γ matrices. We then have

$$\hat{C}_F^\dagger \equiv G_{ab}\hat{\Phi}^{aA\dagger}C_{AB}\hat{\Phi}^{bB\dagger}. \qquad (7.18)$$

As already said, both $\hat{N}_F$ and $\hat{C}_F$ (and therefore also $\hat{C}_F^\dagger$) commute with all the $\hat{J}_{ab}$'s and $\hat{J}_a$'s. Note that while $\hat{N}_F$ is a sesquilinear form $\hat{N}_F \sim \Phi^\dagger \Phi$ that is Hermitian, $\hat{C}_F$ is a symplectic bilinear form in the Φ's (which would vanish if the Φ's would commute rather than anticommute). It is also to be noted that

$$\hat{N}_F - 8 = \frac{1}{2}G_{ab}(\hat{\Phi}^{a\dagger}\hat{\Phi}^b - \hat{\Phi}^a\hat{\Phi}^{b\dagger}), \qquad (7.19)$$

is odd under the up-down fermion symmetry where one swaps $\Phi \leftrightarrow \Phi^\dagger$. The first line in Eq. (7.15) then shows that $\hat{\mu}^2$ is also invariant under the swapping $\Phi \leftrightarrow \Phi^\dagger$; the up-down fermion symmetry was used above as part of our definition of the ordering of the supersymmetry constraints.

From the mathematical point of view, as already mentioned above, any four operators $J_1, J_{12}, J_{23}, J_{34}$ (acting as endomorphisms of some vector space) satisfying the Serre-Berman relations, Eq. (4.13), define a representation of the (formally) maximally compact subalgebra $K(G_2^{++})$ of G_2^{++}. We can therefore summarize the results of the present section by saying that the fermions of our quantized supersymmetric cosmological model live in a 2^{16}-dimensional representation of $K(G_2^{++})$, and that all the building blocks entering the dynamics of the fermions, i.e., the various terms defining $\hat{H}^{(2)} \sim \Phi^\dagger\Phi$ and $\hat{H}^{(4)} \sim \Phi^\dagger\Phi\Phi^\dagger\Phi$ have a direct meaning in terms of the simple-root generators $\hat{J}_1, \hat{J}_{12}, \hat{J}_{23}, \hat{J}_{34}$ of $K(G_2^{++})$.

VIII. SOLUTIONS OF THE QUANTUM CONSTRAINTS

In this final section we briefly discuss some aspects of the solutions of our quantized cosmological model, i.e., the solutions of the supersymmetry constraints (7.1). We recall that the latter supersymmetry constraints entail the Hamiltonian constraint, say Eq. (7.4).

Let us first focus on the structure of the solutions far from all the walls, i.e., in a domain of the β^a's where we can neglect all the exponential terms $e^{-(\beta^b - \beta^a)}$ and $e^{-\beta^a}$ in the $\hat{S}_A$'s, and their squares in $\hat{H}$. In this limit the supersymmetry constraints reduce to

$$\left(\hat{\Phi}^{aA}\frac{\hbar}{i}\partial_{\beta^a} + \hat{S}_{(3)}^A\right)|\Psi(\beta)\rangle = 0, \qquad (8.1)$$

$$\left(\hat{\Phi}^{aA\dagger}\frac{\hbar}{i}\partial_{\beta^a} + \hat{S}_{(3)}^{A\dagger}\right)|\Psi(\beta)\rangle = 0, \qquad (8.2)$$

while the Hamiltonian constraint reads

$$\left[G^{ab}\left(\frac{\hbar}{i}\partial_{\beta^a} - i\varpi_a\right)\left(\frac{\hbar}{i}\partial_{\beta^b} - i\varpi_b\right) + \hat{\mu}^2\right]|\Psi(\beta)\rangle = 0, \qquad (8.3)$$

where $\varpi_a \equiv G_{ab}\varpi^a = -\frac{1}{4}\{9, 8, 7, 6\}$. In these equations we have formally considered that the operator $\hat{\pi}_a$ was Hermitian, and we have used the original, nonrescaled wave function $\Psi(\beta)$, rather than the rescaled wave function $\Psi'(\beta)$, Eq. (7.3), used in Eq. (7.4).

A. Spectrum of $\hat{\mu}^2$

To solve the Hamiltonian constraint we can look for solution states $|\Psi\rangle$ that are eigenstates of the $\hat{\mu}^2$ operator. It is therefore interesting to first discuss the eigenvalues and eigenstates (in fermionic space) of $\hat{\mu}^2$. The explicit expression (7.15) of $\hat{\mu}^2$ show that $\hat{\mu}^2$ commutes with $\hat{N}_F$. The latter operator defines the fermion number with respect to the Fock vacuum of the Φ's, i.e., the empty state $|0\rangle_-$ such that

$$\hat{\Phi}^{aA}|0\rangle_- = 0. \qquad (8.4)$$

Starting from this empty state, the $N_F = 1$ states are obtained by acting on $|0\rangle_-$ with any of the sixteen anticommuting fermionic creation operator $\hat{\Phi}^{aA\dagger}$, etc. The number of states at level N_F is then equal to $\binom{16}{N_F} = \binom{16}{16-N_F}$, i.e., 16 for $N_F = 1$ (and $N_F = 15$), 120 for $N_F = 2, 14$, etc., with a maximum value $\binom{16}{8} = 12870$ for $N_F = 8$. The filled state, say $|0\rangle_+$, at level $N_F = 16$ is unique and such that

$$\hat{\Phi}^{aA\dagger}|0\rangle_+ = 0. \tag{8.5}$$

The explicit expression (7.15) of $\hat{\mu}^2$ allows one to prove that $\hat{\mu}^2$ also commutes with the operators $\hat{C}_F$ and $\hat{C}_F^\dagger$,

$$[\hat{\mu}^2, \hat{C}_F] = 0; \qquad [\hat{\mu}^2, \hat{C}_F^\dagger] = 0. \tag{8.6}$$

This is seen by using the easily checked commutation relations

$$[\hat{N}_F, \widehat{C_F}] = -2\widehat{C_F}, \tag{8.7}$$

$$[\hat{N}_F, \widehat{C_F}^\dagger] = +2\widehat{C_F}^\dagger, \tag{8.8}$$

$$[\widehat{C_F}^\dagger, \widehat{C_F}] = +4\hat{N}_F - 32. \tag{8.9}$$

Noting that $\hat{C}_F^\dagger$ increases the value of N_F by 2, while $\hat{C}_F$ decreases N_F by 2, and that they both commute with $\hat{\mu}^2$, we can use $\hat{C}_F^\dagger$ and $\hat{C}_F$ as ladder operators to map some subeigenspaces of $\hat{\mu}^2$ at fermion level N_F onto corresponding eigenspaces of $\hat{\mu}^2$ at fermion levels $n_F \pm 2$, with the same value of μ^2. This yields the following spectrum of $\hat{\mu}^2$ when N_F varies between 0 and 8 (with symmetric results when $N_F' = 16 - N_F$),

$$N_F = 0, 16 \qquad \mu^2 = -\frac{106}{3}\Big|_1$$

$$N_F = 1, 15 \qquad \mu^2 = -\frac{161}{6}\Big|_{16}$$

$$N_F = 2, 14 \qquad \mu^2 = -\frac{106}{3}\Big|_1, \quad -\frac{58}{3}\Big|_{119}$$

$$N_F = 3, 13 \qquad \mu^2 = -\frac{161}{6}\Big|_{16}, \quad -\frac{77}{6}\Big|_{544}$$

$$N_F = 4, 12 \qquad \mu^2 = -\frac{106}{3}\Big|_1, \quad -\frac{58}{3}\Big|_{119}, \quad -\frac{22}{3}\Big|_{1700}$$

$$N_F = 5, 11 \qquad \mu^2 = -\frac{161}{6}\Big|_{16}, \quad -\frac{77}{6}\Big|_{544}, \quad -\frac{17}{6}\Big|_{3808}$$

$$N_F = 6, 10 \qquad \mu^2 = -\frac{106}{3}\Big|_1, \quad -\frac{58}{3}\Big|_{119}, \quad -\frac{22}{3}\Big|_{1700}, \quad +\frac{2}{3}\Big|_{6188}$$

$$N_F = 7, 9 \qquad \mu^2 = -\frac{161}{6}\Big|_{16}, \quad -\frac{77}{6}\Big|_{544}, \quad -\frac{17}{6}\Big|_{3808}, \quad +\frac{19}{6}\Big|_{7072}$$

$$N_F = 8 \qquad \mu^2 = -\frac{106}{3}\Big|_1, \quad -\frac{58}{3}\Big|_{119}, \quad -\frac{22}{3}\Big|_{1700}, \quad +\frac{2}{3}\Big|_{6188}, \quad +\frac{14}{3}\Big|_{4862}$$

Here the numbers indicated after the eigenvalues of $\hat{\mu}^2$ denote the dimensions of the corresponding eigenspaces. For instance, the one-dimensional eigenspace $\mu^2 = -\frac{106}{3}$ at level $N_F = 2$ is obtained by acting on the unique $N_F = 0$ state by $\widehat{C_F}^\dagger$. In other words, if we define the function

$$f(n) \equiv -\frac{106}{3} + 9n - \frac{1}{2}n^2, \tag{8.10}$$

the possible eigenvalues of $\hat{\mu}^2$ at a given level N_F are of the form $f(n)$, with degeneracy $\binom{16}{n} - \binom{16}{n-2}$, where the integer n runs over the values $N_F, N_F - 2, N_F - 4, \ldots$.

B. Far-wall solutions of the quantum constraints at low (and high) fermion levels

The above-determined spectrum of $\hat{\mu}^2$ yields a necessary constraint on possible solution wave functions, but is far from sufficient to determine whether such solutions exist at some given fermion level N_F. The reasoning given below Eq. (11.21) of Ref. [25] shows that one can look for solutions having a given N_F level. We must tackle the supersymmetry constraints, Eqs. (8.2)–(8.2). We succeeded in doing so for the levels $N_F = 0, 1, 2, 3$ and their up-down symmetric partners $N_F = 16, 15, 14, 13$.

The main result at the levels $N_F = 0, 1, 2, 3$ (and $N_F = 16, 15, 13$) is that there exist solutions of the type

$$|\mathbf{\Psi}(\beta)\rangle = \exp(i\pi_a\beta^a)|\mathbf{\Psi}(0)\rangle, \qquad (8.11)$$

only for certain specific, discrete values of the momenta π_a.

At the level $N_F = 0$, $|\mathbf{\Psi}(0)\rangle$ must be proportional to $|0\rangle_-$, while π_a must take the specific value $\pi_a^{N_F=0} = \frac{i}{4}\{19,16,13,10\}$. Note that the corresponding value π_a' parametrizing the rescaled wave function $|\mathbf{\Psi}'(\beta)\rangle$, namely

$$\pi_a' = \pi_a - i\varpi_a, \qquad (8.12)$$

is also purely imaginary and is fixed to the specific value

$$\pi_a'^{N_F=0} = i\{7,6,5,4\}. \qquad (8.13)$$

It is easily checked that $G^{ab}\pi_a'^{N_F=0}\pi_b'^{N_F=0}$ is equal to $-\mu_{N_F=0}^2 = +\frac{106}{3}$, as it should be.

At the level $N_F = 1$, we found that there does not exist any solution of the supersymmetry constraints.

At the level $N_F = 2$, there exist only five possible, discrete values of the momenta π_a, all of them being purely imaginary. The corresponding linear space of solutions is six-dimensional, because one value of π_a (namely $\pi_a^{(1)} = \frac{i}{4}\{11,8,9,6\}$) admits a two-dimensional space of solutions for the spinor factor $|\mathbf{\Psi}(0)\rangle$. The other possible values of π_a at $N_F = 2$ are: $\pi_a^{(2)} = \frac{i}{4}\{19,16,13,10\} = \pi_a^{N_F=0}$ (with spinor part $\widehat{\mathcal{C}}_F^\dagger|0\rangle$), $\pi_a^{(3)} = \frac{i}{4}\{7,12,9,6\}$, $\pi_a^{(4)} = \frac{i}{4}\{11,8,5,10\}$ and $\pi_a^{(5)} = \frac{i}{4}\{23,12,9,6\}$. The values of μ^2 corresponding to the five possible momenta at level $N_F = 2$ are $(\mu^2)^{(1)} = (\mu^2)^{(3)} = (\mu^2)^{(4)} = (\mu^2)^{(5)} = -\frac{58}{3}$ and $(\mu^2)^{(2)} = -\frac{106}{3}$.

At the level $N_F = 3$, there exists only one possible, discrete value of π_a, namely $\pi_a^{N_F=3} = \frac{i}{4}\{5,8,3,6\}$ (with $\mu_{N_F=3}^2 = -\frac{77}{6}$), with a corresponding four-dimensional eigenspace for the spinor part $|\mathbf{\Psi}(0)\rangle$.

There exist corresponding mirror solutions at $N_F = 16$, 14, 13 with correspondingly equal values of π_a. More generally the up-down symmetry in fermion space guarantees that one can map any solution at any level N_F into a corresponding solution at level $16 - N_F$. Indeed, under the transformation where[5] $\Phi \mapsto \tilde{\Phi} \equiv \Phi^\dagger$ (and therefore $\Phi^\dagger \mapsto \tilde{\Phi}^\dagger \equiv \Phi$) our ordering, Eq. (6.5), shows that $\mathcal{S} \mapsto \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}} \sim \pi\tilde{\Phi} + c\tilde{\Phi}\tilde{\Phi}^\dagger\tilde{\Phi}$ is simply equal to $\mathcal{S}^\dagger$. Then, using Eq. (6.6) [and $(\hat{\pi})^\dagger \equiv \hat{\pi}$], one finds that $\mathcal{S}^\dagger \mapsto \tilde{\mathcal{S}}^\dagger$, where $\tilde{\mathcal{S}}^\dagger$ is simply equal to $\mathcal{S}$. Thereby any solution $|\mathbf{\Psi}(\beta)\rangle$ at some level $N_F = n$ constructed by acting on the empty state $|0\rangle_-$ with n creation operators $\Phi^{\dagger\mathcal{A}} = \Phi^{\dagger a A}$, say

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[5]Here, we omit for simplicity the hats on the various quantum operators.

$$|\mathbf{\Psi}(\beta)\rangle = X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta)\Phi^{\dagger\mathcal{A}_1}\Phi^{\dagger\mathcal{A}_1}\cdots\Phi^{\dagger\mathcal{A}_n}|0\rangle_-, \quad (8.14)$$

with coefficients $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta) = X_{[\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n]}(\beta)$, can be automatically mapped into a corresponding mirror solution at level $N_F = 16 - n$ obtained by acting on the filled state $|0\rangle_+$ (which is annihilated by the $\tilde{\Phi}^{\mathcal{A}}$'s) with the operators $\tilde{\Phi}^{\dagger\mathcal{A}} \equiv \Phi^{\mathcal{A}}$, namely

$$|\tilde{\mathbf{\Psi}}(\beta)\rangle = X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta)\Phi^{\mathcal{A}_1}\Phi^{\mathcal{A}_2}\cdots\Phi^{\mathcal{A}_n}|0\rangle_+. \quad (8.15)$$

Note that this mirror solution at level $16 - n$ involves the same coefficients $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta)$. In particular, when considering plane wave solutions, $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta) = e^{i\pi_a\beta^a}X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(0)$, this up-down symmetry maps a momentum π_a at level n into the same momentum π_a at level $16 - n$.

In addition to this up-down symmetry of the space of solutions of the constraints, there is an additional $\mathbb{Z}_2$ symmetry mapping any solution at level N_F into a corresponding solution at the same level. This second symmetry is rooted in the reality structure of the supersymmetry constraints, namely in the fact that the numerical coefficients $c_{\mathcal{ABC}}^{\mathcal{A}}$ entering Eq. (6.5) are purely imaginary (in a suitable quasi-Majorana representation). Indeed, when decomposing the supersymmetry constraints $\mathcal{S}|\mathbf{\Psi}(\beta)\rangle = 0$, $\mathcal{S}^\dagger|\mathbf{\Psi}(\beta)\rangle = 0$, with a state of the form Eq. (8.14), on the Fock states $\Phi^{\dagger\mathcal{A}_1}\Phi^{\dagger\mathcal{A}_1}\cdots\Phi^{\dagger\mathcal{A}_k}|0\rangle_-$ at levels $k = n - 1$ and $k = n + 1$, one gets a system of first-order differential equations for the coefficients $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta)$ of the symbolic form (using $\hbar = 1$)

$$\frac{1}{i}\partial_\beta X(\beta) + G\delta c X(\beta) = 0. \qquad (8.16)$$

Here, the numerical coefficients $\sim G\delta c$ coming from the cubic-in-fermions contributions involve the coefficients $c = c_{\mathcal{ABC}}^{\mathcal{A}}$ entering Eq. (6.5), multiplied by the real coefficients $G^{ab}\delta^{AB}$ coming from the use of the anticommutation relations Eqs. (5.10). The explicit form of the supersymmetry constraints, Eq. (8.16), are given in Appendix B. Using the pure-imaginary nature of the $c_{\mathcal{ABC}}^{\mathcal{A}}$'s, we see that (after multiplying them by i) the supersymmetry constraint equations yield a system of *real* partial differential equations for the wave function $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta)$. Therefore, to any given (generally) complex solution $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}(\beta)$ at level n, one can associate a solution having the complex-conjugated wave function $X_{\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n}^*(\beta)$. For instance, under this map a plane wave solution of momentum π_a at level n is transformed into a corresponding solution at the same level with momentum $-\pi_a^*$. For generic solutions at the intermediate levels $4 \leq n \leq 12$ such an involutory map acts nontrivially on the space of solutions. On the other hand, it acts trivially on the solutions discussed above at levels $n = 0, 2, 3$ and $n = 13$,

14, 16, which are purely real (up to an arbitrary overall complex factor).

C. Short-wavelength continuous far-wall solutions of the quantum constraints for $4 \leq N_F \leq 12$

It was found in the study of the quantum cosmological dynamics of $D = 4$, $N = 1$ supergravity [18,25], that continuous solutions of the supersymmetry constraints (with real π'_a's taking all possible values on its allowed mass shell $G^{ab}\pi'_a\pi'_b = -\mu^2$) exist only in the middle of fermionic space, namely $N_F = 2, 3, 4$. These solutions were also shown to be continuously connected to their short-wavelength analogs, obtained by taking the limit $\pi'_a \gg 1$. In the latter limit, one can neglect the cubic term $\hat{S}^A_{(3)}$ in the supersymmetry constraint, and the corresponding finite value of the quartic term $\mu^2 = O(\hbar^2)$. We shall here assume that such a general feature holds also in our present $D = 5$, $N = 2$ supergravity case.

Under this (plausible) assumption, we can complete our explicit study of the discrete solutions existing at low (and high) values of N_F by delineating the general structure of the continuous-π' solutions existing for the remaining values, namely $4 \leq N_F \leq 12$. There might also exist additional discrete solutions; e.g., related by the ladder operators $\hat{C}_F$, $\hat{C}^\dagger_F$, to the discrete solutions discussed above.

When considering, short-wavelength states, $|\Psi(\beta)\rangle = \exp(i\pi_a\beta^a)|\Psi(0)\rangle$, with $\pi_a \gg 1$, or equivalently, for the rescaled wave function $|\Psi'(\beta)\rangle = \exp(i\pi'_a\beta^a)|\Psi(0)\rangle$ with $\pi'_a = \pi_a - i\varpi_a$, the supersymmetry constraints yield

$$\hat{\Phi}^{aA}\pi'_a|\Psi(0)\rangle = 0, \qquad (8.17)$$

$$\hat{\Phi}^{aA\dagger}\pi'_a|\Psi(0)\rangle = 0, \qquad (8.18)$$

which imply the (Hamiltonian-constraint) consequence

$$G^{ab}\pi'_a\pi'_b = 0. \qquad (8.19)$$

Let us associate to any real (co)vector v_a in (the dual of the) β space the fermionic operators (putting the spinor index A down for convenience)

$$\hat{\Phi}^v_A \equiv v_a\hat{\Phi}^a_A; \qquad \hat{\Phi}^{v\dagger}_A \equiv v_a\hat{\Phi}^{a\dagger}_A. \qquad (8.20)$$

Given two covectors u and v, the so-defined fermionic operators satisfy the (Clifford) relations

$$\{\hat{\Phi}^u_A, \hat{\Phi}^{v\dagger}_B\} = u \cdot v \delta_{AB}, \quad \{\hat{\Phi}^u_A, \hat{\Phi}^v_B\} = 0, \quad \{\hat{\Phi}^{u\dagger}_A, \hat{\Phi}^{v\dagger}_B\} = 0, \qquad (8.21)$$

where $u \cdot v \equiv G^{ab}u_av_b$.

Given some π'_a on the (Hamiltonian-constraint) light cone $\pi'^2 = G^{ab}\pi'_a\pi'_b = 0$, we can complete π'_a into a null frame π'_a, n_a, t^1_a, t^2_a in the (dual) four-dimensional

Lorentzian β space. Here, π'_a, n_a are both null, $0 = \pi'^2 = n^2$ (with the relative normalization $\pi' \cdot n = 1$), while the two complementary vectors t^1_a, t^2_a are *transverse* to the null direction π'_a, i.e., satisfy $0 = \pi' \cdot t^1 = \pi' \cdot t^2$. One can also require that t^1 and t^2 are orthogonal to n, and between themselves, and (being necessarily spacelike) are normalized to unity. From the basic relations (8.21), and the fact that the supersymmetry constraints read $\hat{\Phi}^{\pi'}_A|\Psi(0)\rangle = 0$, $\hat{\Phi}^{\pi'\dagger}_A|\Psi(0)\rangle = 0$, one easily sees that the lowest value of N_F where there can exist a short-wavelength solution is $N_F = 4$, and that, for this value, there is, for any given (null) π' a one-dimensional space of solutions of the type[6]

$$C\exp(i\pi'_a\beta^a)\hat{\Phi}^{\pi'\dagger}_1\hat{\Phi}^{\pi'\dagger}_2\hat{\Phi}^{\pi'\dagger}_3\hat{\Phi}^{\pi'\dagger}_4|0\rangle_-. \qquad (8.22)$$

Then, at the $N_F = 5$ level, there will be (for any given null π') an eight-dimensional space of solutions generated by acting on the state in Eq. (8.22) with any of the eight independent raising operators $\hat{\Phi}^{t^1\dagger}_A$, and $\hat{\Phi}^{t^2\dagger}_A$, involving the two transverse vectors t^1 and t^2 constructed above. At the $N_F = 6$ level, there will be a $\frac{8 \times 7}{2} = 28$-dimensional space of solutions obtained by acting on the state in Eq. (8.22) with a product of two raising operators of the form $\hat{\Phi}^{t^1\dagger}_A$, or $\hat{\Phi}^{t^2\dagger}_B$. One can continue generating such solutions up to the maximum value $N_F = 12$, corresponding to acting on the state in Eq. (8.22) with the eight different operators $\hat{\Phi}^{t^1\dagger}_A$, or $\hat{\Phi}^{t^2\dagger}_B$.

D. Reflection of short-wavelength solutions on potential walls

Let us finally briefly discuss another consequence of our assumption that there exist solutions of the quantum supersymmetry constraints that are continuously connected to the approximate solutions which one obtains by working in the Wentzel-Kramers-Brillouin (WKB), short-wavelength approximation. This approximation being the quasi-classical approximation ($\hbar \to 0$), we further expect that such solutions will also correspond to the approximation where the spin degrees of freedom are described by anticommuting Grassmann variables ($\{\Phi^\dagger, \Phi\} = 0$) rather than (as we did above) by quantum operators satisfying a Clifford-algebra relation $\{\Phi^\dagger, \Phi\} = O(\hbar)$.

In the Grassmann-fermion approximation, it was generally shown (even in the nonsimply-laced case of relevance here) in Ref. [16] that the law of evolution of a fermion field Φ^i (where we use here, for generality, a generic index i to label the representation space in which lives the considered fermion field) under Hamiltonians

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[6]See Eq. (19) of Ref. [33] for the analogs of these short-wavelength solutions in $D = 11$ supergravity.

containing, in addition to the usual Toda-like bosonic dynamics,

$$H^{(0)} = \frac{1}{2} G^{ab} \pi_a \pi_b + \sum_I e^{-2\alpha_I(\beta)} P_{\alpha_I}^2, \qquad (8.23)$$

fermion couplings of the related Toda-type; namely,

$$H^{(2)} \approx \sum_I e^{-\alpha_I(\beta)} P_{\alpha_I} \hat{J}_{\alpha_I}, \qquad (8.24)$$

where

$$\hat{J}_{\alpha_I} = \Phi_i^{\dagger} (J_{\alpha_I})^i{}_j \Phi^j, \qquad (8.25)$$

could be approximately integrated, and led to a "Fermionic Billiard" picture. More precisely, the latter Fermionic-Billiard picture is based on the fact that the approximate integration of the law of evolution of the fermion field near each separate wall[7]; namely,

$$\partial_t \Phi^i \approx i e^{-\alpha_I(\beta)} P_{\alpha_I} (J_{\alpha_I})^i{}_j \Phi^j, \qquad (8.26)$$

leads to a transformation linking the incident value of the Grassmann-valued Φ^i to its reflected value given by a classical, fermionic reflection operator of the form

$$\mathcal{R}_{\alpha_I}^{\text{classical}} = e^{i\frac{\pi}{2}\epsilon_{\alpha_I} J_{\alpha_I}}, \qquad (8.27)$$

where $\epsilon_{\alpha_I} = \pm$ denotes the sign of the momentum P_{α_I}. In Eq. (8.27), J_{α_I} denotes the matrix $(J_{\alpha_I})^i{}_j$ acting on the representation space defined by a classical homogeneous gravitino, and the resulting classical reflection operator $\mathcal{R}_{\alpha_I}^{\text{classical}}$, obtained by exponentiating $(J_{\alpha_I})^i{}_j$, is also a matrix (or endomorphism) in the representation space of the classical (i.e., Grassmannian) fermion field Φ^i.

When working, as we do here, with quantized fermions, i.e., when replacing the Grassmann fermion field Φ^i by a linear operator $\hat{\Phi}^i$ acting in a fermionic Fock space, we can use the functorial character of the Fock-type quantization [illustrated in our case by the definition, Eq. (6.12), of the quantized $\hat{J}_{\alpha_I}$, and the fact that they have the same algebraic properties as their classical analogs, $(J_{\alpha_I})^i{}_j$] to map the classical reflection matrix $\mathcal{R}_{\alpha_I}^{\text{classical}}$ onto a corresponding reflection operator acting in the representation space of the quantized fermion.

In other words, under our assumption that the quasi-classical limit of our quantum supersymmetric cosmological model does continuously connect quantum states

to quasiclassical states, we conclude that, in the short-wavelength limit, the spinor factor, $|\Psi(0)\rangle$ [stripped of the plane-wave factor $\exp(i\pi_a' \beta^a)$], of the quantum plane-wave solution states discussed in the previous subsection,

$$|\Psi'(\beta)\rangle = \exp(i\pi_a' \beta^a)|\Psi(0)\rangle \qquad (8.28)$$

(see, e.g., Eq. (8.22) in the $N_F = 4$ subspace), considered as states in the 2^{16}-dimensional representation space of the quantized gravitino, will be transformed, upon reflection on each (symmetry or electric) potential wall, by the quantum reflection operator

$$\mathcal{R}_{\alpha_I}^{\text{quantum}} = e^{i\frac{\pi}{2}\epsilon_{\alpha_I} \hat{J}_{\alpha_I}}. \qquad (8.29)$$

The latter operator is a linear endomorphism of the 2^{16}-dimensional quantum spinor space. We note in passing that the validity of the assumptions made here [and the validity of the final result Eq. (8.29)] has been explicitly checked in Ref. [18] in the case of the (Bianchi IX) $D = 4$, $N = 1$, supergravity model.

Using again the simple functorial nature of Fock quantization, we can finally write down some of the relations satisfied both by the classical, and the quantum, reflection operators $\mathcal{R}_{\alpha_I}$. Let us recall that, motivated by the structure of the fermionic billiards arising in the near-singularity behavior of supergravity, Ref. [16] introduced, when working within specific finite-dimensional representations of the maximally compact subalgebras of physically relevant hyperbolic Kac-Moody algebras (namely, $K[E_{10}] \subset E_{10}$, and $K[AE_3] \subset AE_3$) the notion of *spin-extended Weyl groups*, generated by fermion reflection operators associated with the simple roots α_i of the considered Kac-Moody algebra, say G. See Ref. [20] for a mathematical definition of spin-extended Weyl groups (for general simply-laced Kac-Moody algebras) as a part of the definition of spin-covers of maximal compact Kac-Moody subgroups.

As here we are in a setting where we constructed finite-dimensional representations (for a nonsimply-laced case) of $K(G_2^{++})$, we can define *spin-extensions* of the Weyl group[8] of G_2^{++} as the group of linear operators generated by (to be explicit)

$$\mathcal{R}_{\alpha_i} = e^{i\frac{\pi}{2} J_{\alpha_i}}, \qquad (8.30)$$

where i labels the simple roots [in our case $i = (1), (12), (23), (34)$], and where the linear operator J_{α_i} is taken in one of the finite-dimensional representations defined above. Specifically, we can take J_{α_i} in the 16-dimensional vector-spinor representation Eq. (3.14) [corresponding to the classical reflection operators (8.27)], or in

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[7]The billiard approximation consists in treating both the bosonic and the fermionic dynamics as a free far-wall evolution interrupted by time-localized interactions with well-separated potential walls.

[8]See Ref. [34] for a study of the (ordinary) Weyl group of G_2^{++}.

the 2^{16}-dimensional quantum vector-spinor representation defined in Eq. (6.12).

The last point we wish to make here is that, in both these representations, the four reflection operators, $r_i = \mathcal{R}_{\alpha_i}$, listed in Eq. (8.30), associated with the four simple roots of G_2^{++}, satisfy the following generalized Coxeter relations

$$r_i^8 = 1, \qquad (8.31)$$

together with the "braid relations" (see Refs. [20,35])

$$r_i r_j r_i \cdots = r_j r_i r_j \cdots \quad \text{with } m_{ij} \text{ factors on each side.} \quad (8.32)$$

Here, i, and j, with $i \neq j$, are labels for the nodes of the Dynkin diagram of the considered Kac-Moody group. The positive integers m_{ij} entering the braid relations (8.32) are defined from the corresponding values of the nondiagonal elements of the Cartan matrix a_{ij} (which are negative integers, while $a_{ii} = 2$); namely (see [35]),

$$m_{ij} = \{2, 3, 4, 6, 0\} \quad \text{if } a_{ij}a_{ji} = \{0, 1, 2, 3, \geq 4\}$$
$$\text{(respectively).} \qquad (8.33)$$

Note that in our case the values $i = (1)$, $j = (12)$ have $a_{ij}a_{ji} = 3$, corresponding to $m_{ij} = 6$. In that case the braid relation, Eq. (8.32), explicitly reads

$$r_{(1)} r_{(12)} r_{(1)} r_{(12)} r_{(1)} r_{(12)} = r_{(12)} r_{(1)} r_{(12)} r_{(1)} r_{(12)} r_{(1)}. \quad (8.34)$$

The validity of Eq. (8.31) for the 16-dimensional vector-spinor classical representation is easily checked to follow from the half-integral nature of the eigenvalues of the basic gamma matrices $\frac{i\gamma^{ab}}{2}$ and $\frac{\gamma^a}{2}$ entering their definitions. Indeed, let us look again at the definition of the classical action of J_α in the 16-dimensional vector-spinor representation

$$(J_\alpha.\Phi)^{aA} \equiv (\delta_b^a - 2\alpha^a \alpha_b)(J_\alpha^s)^A{}_B \Phi^{bB}, \qquad (8.35)$$

where $J_{ab}^s = i\frac{\gamma^{ab}}{2}$ while $J_a^s = \frac{3\gamma^a}{2}$.

The eigenvectors v^{aA} of J_α can be looked for in factorized form, namely $v^{aA} = v^a \xi^A$ where v^a is an eigenvector of the matrix $\delta_b^a - 2\alpha^a \alpha_b$ [say $(\delta_b^a - 2\alpha^a \alpha_b)v^b = \lambda_v v^a$] while ξ^A is an eigenvector of the spin part J_α^s [say $(J_\alpha^s)^A{}_B \xi^B = \lambda_s \xi^B$]. The eigenvalue of J_α corresponding to $v^{aA} = v^a \xi^A$ is equal to the product $\lambda_{vs} = \lambda_v \lambda_s$. The four eigenvectors of $\delta_b^a - 2\alpha^a \alpha_b$ are (i) any vector parallel to α^a with eigenvalue $1 - 2\langle \alpha\alpha \rangle$. and (ii) three vectors orthogonal to α^a with eigenvalue 1. Using the fact that the squares of the matrices $i\gamma^{ab}$ and γ^a are equal to the unit matrix, one finds that the four eigenvalues of $J_{ab}^s = i\frac{\gamma^{ab}}{2}$ are $\{+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}$, while the four eigenvalues of $J_a^s = \frac{3\gamma^a}{2}$ are $\{+\frac{3}{2}, +\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}\}$. Using

the fact that the squared roots $\langle \alpha\alpha \rangle = G^{ab}\alpha_a \alpha_b$ are equal to 2 for the long symmetry roots $\alpha^{(ab)}$, but equal to $\frac{2}{3}$ for the short electric roots, one finds that the corresponding vector eigenvalues $\lambda_v^{(ab)}$'s are $\{-3, 1, 1, 1\}$, while the $\lambda_v^{(a)}$'s are $\{-\frac{1}{3}, 1, 1, 1\}$. As a consequence the sixteen product eigenvalues $\lambda_{vs} = \lambda_v \lambda_s$ have the values $\{\pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \}$ for the symmetry walls, and the values $\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{2}, \pm\frac{3}{2}, \}$ for the electric walls. [Note the cancellation of the $\frac{1}{3}$ coming from the anomalous $\langle \alpha\alpha \rangle = \frac{2}{3}$ by the extra factor $C^s = 3$ in the definition of J_a^s.]

When passing from the 16-dimensional classical-gravitino representation to the 2^{16}-dimensional quantum-gravitino representation, the explicit forms of the action of $\hat{J}_\alpha$ at various fermion levels [see Eqs. (7.11) and (7.12)] show that the eigenvalues at level $N_F = n$ are given by sums $\lambda^{N_F=n} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, corresponding to a factorized eigenvector $v_1 \wedge v_2 \wedge \cdots \wedge v_n$, where each v_p is itself of the factorized form $v_p^{aA} = v_p^a \xi_p^A$ under the condition that these wedge products do not vanish; the full spectrum of the $\hat{J}_\alpha$'s, with their multiplicities is found in Appendix A. This result immediately shows that all the eigenvalues of $\hat{J}_\alpha$ will be half-integral (or integral). This guarantees that the eighth power of $\hat{\mathcal{R}}_\alpha = e^{i\frac{\pi}{2}\hat{J}_\alpha}$ is equal to 1.

We have verified the validity of the braid relations (8.32) for the classical, 16-dimensional, vector-spinor representation of the J_α's by a direct computation. For instance,

$$\mathcal{R}_1 \mathcal{R}_{12} \mathcal{R}_1 \mathcal{R}_{12} \mathcal{R}_1 \mathcal{R}_{12} = \mathcal{R}_{12} \mathcal{R}_1 \mathcal{R}_{12} \mathcal{R}_1 \mathcal{R}_{12} \mathcal{R}_1, \quad (8.36)$$

while

$$\mathcal{R}_{12} \mathcal{R}_{23} \mathcal{R}_{12} = \mathcal{R}_{23} \mathcal{R}_{12} \mathcal{R}_{23}. \qquad (8.37)$$

These results can then be lifted to the full 2^{16}-dimensional quantum-gravitino representation by using the functorial nature of the Fock-representation expressions Eqs. (7.11), (7.12). Indeed, they imply that the exponentiated operators, $\hat{X}_\alpha = e^{x\hat{J}_\alpha}$, act, when considered at any given level,[9] N_F, as a product of corresponding classical exponentiated factors. For example, at level 2, we have

$$e^{x\hat{J}_\alpha}(u \wedge v) = (e^{xJ_\alpha}u) \wedge (e^{xJ_\alpha}v). \qquad (8.38)$$

Such a general product action applies in particular to the reflection operators $\hat{\mathcal{R}}_\alpha = e^{i\frac{\pi}{2}\hat{J}_\alpha}$, and thereby also to the relevant braid operators which are made of products of $\hat{\mathcal{R}}_\alpha$'s. As a consequence, the equality of two braid classical combinations, e.g., Eq. (8.36), entails the equality

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[9]$\hat{J}_\alpha$ commutes with $\hat{N}_F$ and therefore any function of $\hat{J}_\alpha$ acts within any fixed-N_F space.

of the corresponding quantum combination at all levels, so that, e.g.,

$$\hat{\mathcal{R}}_1\hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_1\hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_1\hat{\mathcal{R}}_{12} = \hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_1\hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_1\hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_1, \quad (8.39)$$

holds in the 2^{16}-dimensional quantum-gravitino representation.

IX. CONCLUSIONS

Let us summarize our main results on the supersymmetric quantum dynamics of the cosmological models obtained by reducing $D = 5$ supergravity to one timelike dimension, i.e., by considering the consistent truncation where the spatial metric, h_{ij}, the vector potential, A_i, and the spatial components of the gravitino, ψ^{iA}, depend only on time.

(1) We constructed a consistent quantization of this model, with the fourteen bosonic coordinates quantized *à la* Schrödinger ($p = \frac{\hbar}{i}\frac{\partial}{\partial q}$), while the suitably redefined spatial gravitino field $\Phi^{aA} = (\det h)^{\frac{1}{4}}\gamma^a\theta_i^a\psi^{iA}$ satisfies simple anticommutation relations $\{\hat{\Phi}^{aA}, \hat{\Phi}^{\dagger bB}\} = \hbar G^{ab}\delta^{AB}$. Here, G^{ab} is the inverse of the metric G_{ab} in the Cartan space of G_2^{++}: $G_{ab}\dot{\beta}^a\dot{\beta}^b \equiv \sum_a(\dot{\beta}^a)^2 - (\sum_a\dot{\beta}^a)^2$, where the β^a's are the logarithmic scale factors of the spatial metric h_{ij}, see Eq. (2.12). In other words, the wave function of the Universe is a 2^{16}–component spinor of Spin(24,8) which depends on the fourteen bosonic configuration variables h_{ij}, A_i (with $i = 1$, 2, 3, 4). The latter variables are usefully replaced by the four logarithmic scale factors, β^a, the six off-diagonal Iwasawa variables $\mathcal{N}^a_i$ (with $a < i$), and the four electric variables $B_a \equiv A_i(\mathcal{N}^{-1})^i_a$.

(2) Quantum states $|\Psi\rangle$ are described by wave functions $\Psi_\sigma(\beta^a, \mathcal{N}^a_i, B_a)$ (where the spin index σ takes 2^{16} values) that must satisfy the eight (Dirac-like) supersymmetry constraints $\hat{\mathcal{S}}_A|\Psi\rangle = 0$, $\hat{\mathcal{S}}_A^\dagger|\Psi\rangle = 0$, as well as the Hamiltonian constraint $\hat{H}|\Psi\rangle = 0$. We have checked the consistency of the algebra of constraints [see Eqs. (6.7), (6.8), and (6.17)] when using an ordering ensuring an up-down symmetry in fermion space (i.e., symmetry under swapping $\Phi \leftrightarrow \Phi^\dagger$).

(3) The hyperbolic Kac-Moody algebra G_2^{++} shows up in the bosonic sector in the fact that the bosonic part of the Hamiltonian describes a null geodesic over the symmetric space $G_2^{++}/K(G_2^{++})$ when setting to zero some higher-level Kac-Moody terms formally corresponding to some spatial gradient terms on the

supergravity side [28]. The root structure of G_2^{++} is reflected in the bosonic Hamiltonian through the presence of a Toda-like structure,

$$H^{(0)} = G^{ab}\pi_a\pi_b + 2\sum_{a<b}e^{-2(\beta^b-\beta^a)}(P_{ab})^2$$
$$+ \frac{1}{2}\sum_a e^{-2\beta^a}(P^a)^2, \quad (9.1)$$

where π_a is the conjugate momentum to β^a, P^a is the momentum conjugate to B_a, and where P_{ab} is a momentumlike variable associated with $\mathcal{N}^a_i$. Here $\alpha^{(ab)}(\beta) = \beta^b - \beta^a$, and $\alpha^{(a)}(\beta)$ are linear forms in the β's which correspond to (real) roots of G_2^{++}. In particular, they feature the four simple roots $\alpha_1 = \alpha^{(1)}$, $\alpha_2 = \alpha^{(12)}$, $\alpha_3 = \alpha^{(23)}$, $\alpha_4 = \alpha^{(34)}$ defining the Cartan matrix, Eq. (4.7), of G_2^{++}.

(4) The $K(G_2^{++})$ structure associated with the fermions shows up in the fermionic sector in several ways. The part $\hat{H}^{(2)}$ of the quantum Hamiltonian that is bilinear in the fermions reads

$$\hat{H}^{(2)} = +2\sum_{a<b}e^{-(\beta^b-\beta^a)}\hat{P}_{ab}\hat{J}_{ab}(\Psi)$$
$$- \frac{1}{\sqrt{3}}\sum_a e^{-\beta^a}\hat{P}^a\hat{J}_a(\Psi). \quad (9.2)$$

This fermion-quadratic contribution associates to each one of the wall roots, $\alpha_I(\beta) = (\alpha^{(ab)}(\beta), \alpha^{(a)}(\beta))$, entering the bosonic Hamiltonian, a corresponding fermion bilinear $\hat{J}_{\alpha_I}$. The latter quantum fermion bilinears generate a 2^{16}-dimensional representation of $K(G_2^{++})$. Indeed, the four operators $\hat{J}_{\alpha_i}$, $i = 1$, 2, 3, 4, corresponding to the four simple roots of G_2^{++}, satisfy the Serre-Berman relations, Eq. (4.13).

(5) In the short-wavelength limit, the propagating-wave solutions of the constraints that exist in the middle of the fermionic Fock space ($4 \le N_F \le 12$) transform, upon reflection on each of the (symmetry or electric) potential wall delimiting the boundary of the billiard chamber (identified with the Weyl chamber of G_2^{++}), under the corresponding four quantum reflection operators

$$\hat{\mathcal{R}}_{\alpha_i} = e^{i\frac{\pi}{2}\hat{J}_{\alpha_i}}. \quad (9.3)$$

These quantum reflection operators satisfy the generalized Coxeter relations given in Eqs. (8.31)–(8.32).

These relations define a spinorial extension of the Weyl group of G_2^{++}.

(6) The quartic-in-fermion contribution to the quantum Hamiltonian $\hat{\mu}^2 \equiv \hat{H}_1'^{(4)}$ (as defined in Sec. VI) satisfy two remarkable $K(G_2^{++})$-related properties. First, it is invariant under the generators $\hat{J}_{\alpha_i}$, $i = 1, 2, 3, 4$ of $K(G_2^{++})$, and second, it happens to be expressible [see Eq. (7.15)] in terms of two $K(G_2^{++})$-invariant *fermion bilinears*, the (sesquilinear) fermion number, $\hat{N}_F \equiv G_{ab}\hat{\Phi}^{a\dagger}\hat{\Phi}^b$, and the bilinear $\hat{C}_F \equiv G_{ab}\hat{\Phi}^{aA}C_{AB}\hat{\Phi}^{bB}$, which involves the "charge conjugation" matrix C_{AB} of the (spatial) γ_i matrices ($C\gamma_i C^{-1} = -\gamma_i^T$).

The invariance of $\hat{N}_F$ and $\hat{C}_F$ under the $\hat{J}_{\alpha_i}$'s stems from the invariance of the two corresponding bilinear forms $H(\Phi_1, \Phi_2) = G_{ab}\Phi_1^{aA\dagger}\delta_{AB}\Phi_2^{bB}$ and $J(\Phi_1, \Phi_2) = G_{ab}\Phi_1^{aA}C_{AB}\Phi_2^{bB}$ under the action of the generators of $K(G_2^{++})$ in the 16-dimensional space defined by the (classical) vector-spinor representation; here, we consider this representation from a mathematical point of view, i.e., within the vector space of complex-valued vector-spinors Φ^{aA}. The quantum representation being correspondingly built by Fock quantization, as discussed in Sec. VII. The sesquilinear form H is *Hermitian*, with signature $(12^+, 4^-)$, while the bilinear form J is *symplectic*. The fact that the representatives of the generators of the (infinite-dimensional) algebra $K(G_2^{++})$ within this 16-dimensional representation leave these two forms invariant indicates that the image of $K(G_2^{++})$ within this representation is the intersection of the Lie algebra of the pseudounitary group $U(12, 4)$ and of the symplectic group $Sp(16, \mathbb{C})$. By definition, this intersection is the Lie algebra $sp(6, 2)$ of the quaternionic pseudo-unitary group $Sp(6, 2)$. The image Lie algebra $sp(6, 2)$ of $\text{Lie}[K(G_2^{++})]$ has real dimension 136 (as can be directly checked by looking at the general solution of the invariance conditions, $u^\dagger H + Hu = 0$, $u^T J + Ju = 0$, of H and J under an infinitesimal $GL(16, \mathbb{C})$ transformation $\delta\Phi = u \cdot \Phi$).

Our results open new perspectives that we hope to explore in future work. The most promising one is that our finding that the quartic-in-fermions term commutes with the generators of the involutory subalgebra $K(G_2^{++})$ (which generalizes the similar property found for $\mathcal{N} = 1$, $D = 4$ supergravity [25]), raises the hope that such a property will also hold for the maximal supergravity $\mathcal{N} = 8$ in $D = 4$, or $\mathcal{N} = 1$ in $D = 11$, i.e., that the quartic-in-fermions term in the Hamiltonian is invariant under $K(E_{10})$. Let us note in this respect that, as shown by Eq. (3.12), the quartic-in-fermions term in the Hamiltonian is the sum of the original fermion-quartic part of the Lagrangian and of a sum of the squares of the J_{α_I} bilinears corresponding to the roots explicitly appearing in the Lagrangian (the latter terms being generated by the Legendre transform associated to the velocity dependence of the couplings of $\mathcal{N}^a{}_i$ and A_i to fermions). Though our analysis has truncated away the couplings to spatial derivatives, it has retained all the velocity-dependent couplings; a similar analysis can be implemented for $D = 11$ supergravity. This suggests that the value of μ^2 obtained in such one-time-dimension reductions is relevant to the exact supergravity dynamics. Therefore, finding a value of μ^2 that is invariant under the relevant involutory algebra is a strong signal of a hidden Kac-Moody-related symmetry. In previous Kac-Moody-coset analyses, it was argued that μ^2 is naturally given by a quadratic Casimir, $\frac{1}{2}\sum_\alpha J_\alpha^2$, involving a formal sum over the infinite number of positive roots [25,36]. It will be therefore important to see whether an extension of our analysis to the $D = 11$-supergravity fermion sector leads to a μ^2 that is invariant under $K(E_{10})$.

If this is the case, besides being a clear confirmation of a hidden $K(E_{10})$ symmetry, it will also probably imply that $\hat{\mu}^2 \equiv \hat{H}_1'^{(4)}$ is a c number, rather than a fermionic operator, because we have checked that there are no nontrivial $K(E_{10})$-invariant (symplectic) bilinears of the type, $C_F = G_{ab}\Phi^a C\Phi^b = G_{ab}\Phi^{aA}C_{AB}\Phi^{bB}$, that allowed expressions of the type Eq. (7.15) to exist. In $D = 4$, $\hat{\mu}^2$ was quadratic in C_F with $C = \gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$, while in $D = 5$, C_F involved the spatial charge conjugation matrix. We recall that, like in $D = 4$, the gravitino is a Majorana spinor in $D = 11$.

APPENDIX A: SPECTRUM OF THE QUANTIZED J OPERATORS

1. $\hat{J}_{[ab]}$ spectrum

$$N_F = 0, 16 \quad \lambda = 0|_1,$$

$$N_F = 1, 15: \quad \lambda = \pm\frac{3}{2}\Big|_2, \quad \pm\frac{1}{2}\Big|_6$$

$$N_F = 2, 14: \quad \lambda = \pm3|_1, \quad \pm2|_{12}, \quad \pm1|_{27}, \quad 0|_{40}$$

$$N_F = 3, 13: \quad \lambda = \pm\frac{7}{2}\Big|_6, \quad \pm\frac{5}{2}\Big|_{36}, \quad \pm\frac{3}{2}\Big|_{94}, \quad \pm\frac{1}{2}\Big|_{144}$$

$$N_F = 4, 12: \quad \lambda = \pm4|_{15}, \quad \pm3|_{76}, \quad \pm2|_{222}, \quad \pm1|_{372}, \quad 0|_{450}$$

$$N_F = 5, 11: \quad \lambda = \pm\frac{9}{2}\Big|_{20}, \quad \pm\frac{7}{2}\Big|_{120}, \quad \pm\frac{5}{2}\Big|_{366}, \quad \pm\frac{3}{2}\Big|_{712}, \quad \pm\frac{1}{2}\Big|_{966}$$

$$N_F = 6, 10: \quad \lambda = \pm5|_{15}, \quad \pm4|_{132}, \quad \pm3|_{466}, \quad \pm2|_{966}, \quad \pm1|_{1527}, \quad 0|_{1776}$$

$$N_F = 7, 9: \quad \lambda = \pm\frac{11}{2}\Big|_6, \quad \pm\frac{9}{2}\Big|_{92}, \quad \pm\frac{7}{2}\Big|_{402}, \quad \pm\frac{5}{2}\Big|_{1020}, \quad \pm\frac{3}{2}\Big|_{1812}, \quad \pm\frac{1}{2}\Big|_{2388}$$

$$N_F = 8: \quad \lambda = \pm6|_1, \quad \pm5|_{36}, \quad \pm4|_{249}, \quad \pm3|_{764}, \quad \pm2|_{1599}, \quad \pm1|_{2400}, \quad 0|_{2772} \tag{A1}$$

2. $\hat{J}_a$ spectrum

$$N_F = 0, 16 \quad \lambda = 0|_1,$$

$$N_F = 1, 15: \quad \lambda = \pm\frac{3}{2}\Big|_6, \quad \pm\frac{1}{2}\Big|_2$$

$$N_F = 2, 14: \quad \lambda = \pm3|_{15}, \quad \pm2|_{12}, \quad \pm1|_{13}, \quad 0|_{40}$$

$$N_F = 3, 13: \quad \lambda = \pm\frac{9}{2}\Big|_{20}, \quad \pm\frac{7}{2}\Big|_{30}, \quad \pm\frac{5}{2}\Big|_{36}, \quad \pm\frac{3}{2}\Big|_{144}, \quad \pm\frac{1}{2}\Big|_{80}$$

$$N_F = 4, 12: \quad \lambda = \pm6|_{15}, \quad \pm5|_{40}, \quad \pm4|_{55}, \quad \pm3|_{180} \ \pm2|_{207}, \quad 1|_{228}, \quad 0|_{370}$$

$$N_F = 5, 11: \quad \lambda = \pm\frac{15}{2}\Big|_6, \quad \pm\frac{13}{2}\Big|_{30}, \quad \pm\frac{11}{2}\Big|_{50}, \quad \pm\frac{9}{2}\Big|_{170}, \quad \pm\frac{7}{2}\Big|_{290}, \quad \pm\frac{5}{2}\Big|_{360}, \quad \pm\frac{3}{2}\Big|_{666}, \quad \pm\frac{1}{2}\Big|_{612}$$

$$N_F = 6, 10: \quad \lambda = \pm9|_1, \quad \pm8|_{12}, \quad \pm7|_{27}, \quad \pm6|_{96}, \quad \pm5|_{235}, \quad \pm4|_{340}$$
$$\pm3|_{720}, \quad \pm2|_{900}, \quad \pm1|_{1005}, \quad 0|_{1336}$$

$$N_F = 7, 9: \quad \lambda = \pm\frac{19}{2}\Big|_2, \quad \pm\frac{17}{2}\Big|_8, \quad \pm\frac{15}{2}\Big|_{30}, \quad \pm\frac{13}{2}\Big|_{108}, \quad \pm\frac{11}{2}\Big|_{192}, \quad \pm\frac{9}{2}\Big|_{470}$$
$$\pm\frac{7}{2}\Big|_{780}, \quad \pm\frac{5}{2}\Big|_{990}, \quad \pm\frac{3}{2}\Big|_{1590}, \quad \pm\frac{1}{2}\Big|_{1550}$$

$$N_F = 8: \quad \lambda = \pm10|_1, \quad \pm9|_4, \quad \pm8|_{25}, \quad \pm7|_{60}, \quad \pm6|_{174}, \quad \pm5|_{396}$$
$$\pm4|_{585}, \quad \pm3|_{1140}, \quad \pm2|_{1425}, \quad \pm1|_{1600}, \quad 0|_{2050} \tag{A2}$$

APPENDIX B: EXPLICIT FORM OF THE SUPERSYMMETRY CONSTRAINTS

In this appendix we use a slightly different notation from the one used in the text. Vector indices are denoted $a, b, \cdots$ (as in the text), while spinor indices are denoted $\alpha, \beta, \cdots$. The composite indices combining these two types of indices (denoted $\mathcal{A} = aA$ in the text) are denoted here as $A = a\alpha$. When a spinor, or a composite index, belongs to some $\Phi^\dagger$ we dot it to indicate its origin, e.g., $(\hat{\Phi}^{a\alpha})^\dagger = \hat{\Phi}^{\dagger a\dot\alpha} = (\hat{\Phi}^A)^\dagger = \hat{\Phi}^{\dagger\dot{A}}$. The right-hand side $G^{ab}\delta^{\alpha\beta}$ (with $\hbar = 1$) of the third (nontrivial) anticommutation relations Eq. (5.10) is denoted $\Delta^{A\dot{B}}$, i.e.,

$$\{\hat{\Phi}^A, \hat{\Phi}^{\dagger\dot{B}}\} = \Delta^{A\dot{B}}. \tag{B1}$$

Though, with our normalization $\Delta^{A\dot{B}} = G^{ab}\delta^{\alpha\beta}$ is real and symmetric, it is useful (for keeping track of Hermitian conjugations in contracted indices) to denote its complex conjugate as $(\Delta^{A\dot{B}})^\star = \Delta^{B\dot{A}}$. We also denote the (purely imaginary) numerical coefficients $c^\alpha_{\mathcal{ABC}} = -c^\alpha_{\mathcal{CBA}}$ entering Eq. (6.5) as $\sigma^\alpha_{\dot{B}[AC]} = -\sigma^\alpha_{\dot{B}[CA]}$, so that $\mathcal{S}^\alpha$ reads

$$\mathcal{S}^\alpha = -i\partial_{\beta^a}\hat{\Phi}^{a\alpha} + \sigma^\alpha_{\dot{P}[BC]}\hat{\Phi}^B\hat{\Phi}^{\dagger\dot{P}}\hat{\Phi}^C. \tag{B2}$$

The following contraction of the σ coefficients plays a distinguished role,

$$\sigma^\alpha_{\dot{P}[BC]}\Delta^{B\dot{P}} = -i\nu_c\delta^\alpha_\gamma. \tag{B3}$$

Here $C = c\gamma$ and the four components of the vector ν are $\nu_a = -\frac{1}{4}\{19, 16, 13, 10\}$.

With this notation, the explicit form of the supersymmetry constraint $\mathcal{S}^\alpha|X\rangle = 0$, when acting on a plane-wave state of fermion level k written as

$$|X\rangle = e^{i\pi_a\beta^a}X_{\dot{A}_1\cdots\dot{A}_k}\hat{\Phi}^{\dagger\dot{A}_1}\cdots\hat{\Phi}^{\dagger\dot{A}_k}|0\rangle_-, \tag{B4}$$

reads

$$k((\pi_a - i\nu_a)\Delta^{a\alpha\dot{P}}X_{\dot{P}\dot{A}_1\cdots\dot{A}_{k-1}} - (k-1)\sigma^\alpha_{[\dot{A}_1|[BC]}\Delta^{C\dot{P}}$$
$$\times \Delta^{B\dot{Q}}X_{\dot{P}\dot{Q}|\dot{A}_2\cdots\dot{A}_{k-1}]}) = 0. \tag{B5}$$

The corresponding explicit form of the constraint $\mathcal{S}^{\dagger\dot\alpha}|X\rangle = 0$ reads

$$((\pi_{a_1} + i\nu_{a_1})\delta^{\dot\alpha}_{\dot\alpha_1}X_{\dot{A}_2\cdots\dot{A}_{k+1}}$$
$$+ k\sigma^{*\dot\alpha}_{B\dot{A}_1\dot{A}_2}\Delta^{B\dot{P}}X_{\dot{P}\dot{A}_3\cdots\dot{A}_{k+1}})_{[\dot{A}_1\dot{A}_2\cdots\dot{A}_{k+1}]} = 0, \tag{B6}$$

where the last subscript indicates antisymmetrization with respect to the composite indices $\dot{A}_1\dot{A}_2\cdots\dot{A}_{k+1}$ (with $\dot{A}_1 = a_1\dot\alpha_1$).

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