Toward the primordial black hole formation threshold in a time-dependent equation-of-state background

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The study of the primordial black hole (PBH) gravitational collapse process requires the determination of a critical energy density perturbation threshold δ_c, which depends on the equation of state of the universe at the time of PBH formation. Up to now, the majority of analytical and numerical techniques calculate δ_c by assuming a constant equation-of-state (EoS) parameter w at the time of PBH formation. In this work, after generalizing the constant w prescription of [T. Harada *et al.*, Phys. Rev. D **88**, 084051 (2013); **89**, 029903(E) (2014).] for the computation of δ_c and commenting its limitations we give a first estimate for the PBH threshold in the case of a time-dependent w background. In particular, we apply our formalism in the case of the QCD phase transition, where the EoS parameter varies significantly with time and one expects an enhanced PBH production due to the abrupt softening of w. At the end, we compare our results with analytic and numerical approaches for the determination of δ_c assuming a constant EoS parameter.

I. INTRODUCTION

Primordial black holes (PBHs), first proposed in the early 1970s [1–3], form in the very early universe before star formation out of the gravitational collapse of very high overdensities whose energy density is higher than a critical threshold. According to recent arguments, PBHs can naturally account for a part or the totality of dark matter [4,5]. They can potentially explain as well the generation of large-scale structures through Poisson fluctuations [6,7] and seed also the supermassive black holes residing in galactic centers [8,9]. At the same time, PBHs are connected with numerous gravitational-wave (GW) signals like the stochastic GW background associated to black-hole merging events [10–14], like the ones recently detected by VIRGO-LIGO [15], as well the second order GW signal induced from primordial curvature perturbations [16–20] (for a recent review see [21]) or from Poisson PBH energy density fluctuations [22–24]. In particular, through the aforementioned GW portal PBHs can act as well as novel probes constraining modified gravity theories [25,26]. Other hints in favor of PBHs can be found here [27].

In the standard PBH formation scenario, where PBHs form out of the collapse of enhanced energy density perturbations, the PBH formation threshold δ_c depends in general on the shape of the energy density perturbation profile of the collapsing overdensity region [28,29] as well as on the equation-of-state parameter at the time at which the gravitational collapse is taking place [30]. This threshold value is very important since it can affect significantly the abundance of PBHs, a quantity which is constrained by many observational probes [31].

Historically, the first attempt to compute the PBH formation threshold was done by B. Carr and S. Hawking between 1974 and 1975 [30,32] where by using a Newtonian Jeans instability criterion they were led to the conclusion that $\delta_c \sim w$. Afterwards, δ_c was studied through numerical hydrodynamic simulations by the pioneering works of Nadezhin, Novikov, and Polnarev in 1978 [33], Bicknell and Henriksen in 1979 [34] and Novikov and Polnarev in 1980 [35] and later after a pause of 20 years by high sophisticated numerical codes by Niemeyer and Jedmazik [36] and Shibata and Sasaki [37].

Within the last decade, there have been a huge progress regarding the determination of the PBH formation threshold both at the analytic as well as at the numerical level. In particular, at the analytic level, T. Harada, C.-M. Yoo, and K. Kohri (HYK) in 2013 [38] refined the δ_c value obtained by Carr in 1975 by confronting the gravitational force which pushes the fluid matter of the collapsing overdensity inwards and enhances as such the gravitational collapse with the pressure gradient force which in general pushes the fluid outwards, thus disfavoring the collapsing process. At the end, by comparing the time at which the pressure sound wave crosses the overdensity collapsing to a PBH with the onset time of the gravitational collapse they found that the expression for δ_c in the comoving gauge reads as:

$$\delta_c = \frac{3(1+w)}{5+3w} \sin^2\left(\frac{\pi\sqrt{w}}{1+3w}\right). \tag{1}$$

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Some years later, [28,29] studied the effect of the shape of the initial energy density perturbation which collapses to a PBH on δ_c by introducing the shape parameter α in terms of a compaction function $\mathcal{C}$ defined as

$$\alpha \equiv -\frac{r_m^2 \mathcal{C}''(r_m)}{4\mathcal{C}(r_m)}, \qquad (2)$$

where the compaction function $\mathcal{C}$ is defined as $\mathcal{C} \equiv 2\frac{\delta M(r,t)}{R(r,t)}$ with $\delta M(r,t)$ being the mass excess of a local overdense region and $R(r,t) = a(t)r$ being the areal radius of this region with respect to a spatially flat background metric. The parameter r_m is the position where the compaction function is maximized giving in practice the characteristic scale of the collapsing overdensity while primes denote spatial derivatives.[1] This shape parameter quantifies actually the broadness or sharpness of the energy density perturbation around its peak. In particular, large values of $\alpha \gg 1$ correspond to a broad peak of the collapsing energy density perturbation, whereas for smaller values of α the energy density profile is steeper. The work of [29] was then generalized for an arbitrary EoS parameter w and it was found that for $w > 1/3$ one can find an analytic formula for δ_c as a function of α and w. For $w < 1/3$, the determination of an analytic expression for δ_c remains an open issue given that in this regime the full shape of the compaction function is necessary [39].

At this point, it is very important to highlight as well the immense interest raised in the recent years concerning the effect of nonlinearities [40–43] and non-Gaussianities [44–49] for the determination of the PBH formation threshold as well as the dependence of δ_c and the PBH abundance on the details of the initial power spectrum of curvature perturbations which gave rise to PBHs [50–53]. In addition, some first research works were also performed regarding the dependence of the PBH formation threshold on nonsphericities [54,55], on anisotropies [56] as well as on the underlying theory of gravity [57].

All of the above mentioned research works while determining δ_c made the assumption that w is constant in time which is a good approximation for the vast majority of the cosmic epochs [58]. However, according to the current cosmological paradigm, the universe's EoS parameter varies with time and there are cases where someone is met with very abrupt changes in w such the QCD phase transition [59] and the (pre)reheating era [60], which intermediates between inflation and the hot big bang (HBB) era. In all these regimes, a refined calculation of the PBH formation threshold is required accounting for the effect of a dynamical w profile.

Furthermore, one should point out here that the associated to PBHs scalar induced stochastic gravitational wave background (SGWB) is strongly dependent on the underlying cosmological background [61,62] and can serve as such as a probe of the thermal history of the universe [63]. Thus, a better understanding of the dependence of the PBH formation threshold on the EoS parameter will unavoidably entail a better understanding of the PBH formation process and the associated to it scalar induced SGWB signal potentially detected by ET [64], LISA [65–67] and SKA observational probes [68]. Interestingly, as stated recently in the literature, the NANOGrav signal [69] can be interpreted as an induced SGWB from a close to a dustlike stage with $-0.091 < w < 0$, a result which is dependent however on the exact dependence of δ_c on the dynamical profile of w [70].

Therefore, given all the above motivation regarding the effect of a dynamical w profile on δ_c a legitimate question to ask is how δ_c is calculated in a time-dependent w background? A first attempt toward this direction was performed by the pioneering work of [71] where δ_c was computed during the QCD phase transition by making a time average of the EoS parameter between the horizon crossing time and the time of maximum expansion of the collapsing overdensity region. Then, δ_c was interpolated by using tabulated data of δ_c for different values of w obtained from numerical simulations but under the assumption of constant w. In this work, we make a first step toward a semianalytic scheme for the computation of δ_c in time-dependent w backgrounds based on simple physical arguments.

Consequently, following this introduction where we present a historic overview of the literature regarding the determination of the PBH formation threshold, we perform in Sec. II a refined computation of δ_c in a time-dependent w background by generalizing the work of [38] and commenting its limitations. Then, in Sec. III we apply our formalism in the case of the QCD phase transition, during which w varies significantly with time and PBH production is enhanced due to the softening of the EoS parameter. Finally, Sec. IV is devoted to conclusions.

II. MATHEMATICAL FORMULATION

In this section, after revising the HYK prescription [38] for the computation of the PBH formation threshold valid for constant w we then generalize it by determining δ_c accounting for the time dependence of the EoS.

A. The "three-zone" model

We introduce first the spherically symmetric "three-zone" model where the overdense region is a homogeneous core (closed universe) surrounded by a thin underdense spherical shell which compensates the overdensity and separates the overdense region from the expanding background universe. See Fig. 1.
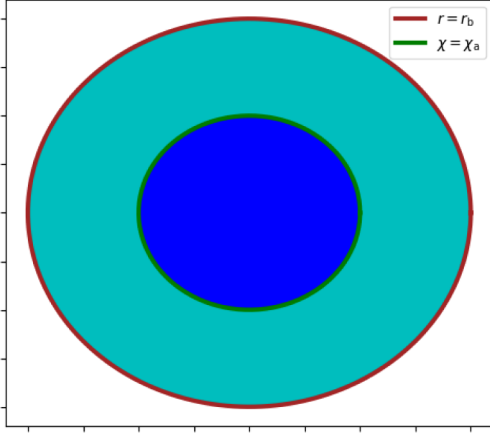
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[1]The compaction function $\mathcal{C}$ is considered here in the super-horizon regime where one can perform a gradient expansion approximation [28] and thus is time-independent.

FIG. 1. The spherical "three-zone" model: The overdensity region is shown in blue and it is surrounded by a spherical underdense layer displayed with cyan. The boundary between the overdensity region and the spherical underdense layer is depicted with the green circumference at $\chi = \chi_a$ whereas the boundary between the underdense layer and the FLRW flat background is shown with the brown circumference at $r = r_b$.

On the one hand, the background metric corresponding to a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe can be recast as

$$ds^2 = -dt^2 + a_b^2(t)(dr^2 + r^2 d\Omega^2), \qquad (3)$$

where $d\Omega^2$ is the line element of a unit two-sphere and $a_b(t)$ is the scale factor of the background universe. The respective Friedmann equation reads as

$$H_b^2 = \left(\frac{\dot{a}_b}{a_b}\right)^2 = \frac{\rho_b}{3M_{Pl}^2}, \qquad (4)$$

where ρ_b and H_b is the energy density and the Hubble parameter of the background.

On the other hand, the overdense region associated with a close ($K = 1$) FLRW universe is described with the following metric

$$ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sin^2\chi d\Omega^2) \qquad (5)$$

and the Friedmann equation reads as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{Pl}^2} - \frac{1}{a^2}, \qquad (6)$$

where ρ is the energy density of the overdense region.

The underdense spherical shell is matched to the closed FLRW universe describing the overdensity at $\chi = \chi_a$ while the flat FLRW background universe is matched to the compensating underdense layer at $r = r_b$. Therefore, the

areal radius at the edge of the overdense region, R_a as well as that at the edge of the surrounding underdense spherical shell read as

$$R_a = a\sin\chi_a, \qquad R_b = a_b r_b. \qquad (7)$$

B. The PBH formation threshold in the uniform Hubble gauge

We proceed then to the computation of δ_c on the uniform Hubble gauge where the Hubble parameters of the overdensity and that of the background are the same, i.e., $H = H_b$. Doing so, we define the energy density parameter Ω as

$$\Omega \equiv \frac{\rho}{3M_{Pl}^2 H^2} = 1 + \frac{1}{a^2 H^2}, \qquad (8)$$

where in the last equality we have used Eq. (6). Then, using the expression for the areal radius at $\chi = \chi_a$, i.e., $R_a = a\sin\chi_a$, as well as the definition of the horizon scale, i.e., $R_H = H^{-1}$, one can find straightforwardly that

$$(\Omega - 1)\left(\frac{R_a}{R_H}\right)^2 = \sin^2\chi_a, \qquad (9)$$

an expression which relates Ω with the scale of the overdensity. In addition, one can relate Ω with the energy density contrast of the overdensity defined as

$$\delta \equiv \frac{\rho - \rho_b}{\rho_b}. \qquad (10)$$

Specifically, by solving Eq. (10) for ρ and substituting ρ in Eq. (8) one gets that

$$\Omega = (1 + \delta)\left(\frac{H_b}{H}\right)^2, \qquad (11)$$

where ρ_b has been expressed in terms of H_b through Eq. (4). Then, one can extract the energy density perturbation at horizon crossing time, δ_{hc}, at the time when $R_a = H_b^{-1}$, by solving for δ Eq. (11) and substituting Ω from Eq. (9). Finally, one obtains that

$$\delta_{hc} = \left(\frac{H}{H_b}\right)^2 - \cos^2\chi_a. \qquad (12)$$

At the end, on the uniform Hubble gauge, where $H = H_b$, Eq. (12) becomes

$$\delta_{hc}^{UH} = \sin^2\chi_a, \qquad (13)$$

We should stress out here that the above expression for δ_{hc}^{UH} was extracted independently on the equation of state of the universe at PBH formation time.

C. Refining the PBH formation threshold

We perform now a refined computation regarding the PBH formation threshold in the case of a time-dependent EoS parameter. In particular, following the same reasoning as in [38], we compute δ_c by establishing a criterion in order not to avoid the gravitational collapse. In particular, we confront the gravitational force which pushes the fluid matter of the collapsing overdensity inwards and enhances in this way the collapse to a black hole with the pressure gradient force which pushes matter outwards, thus delaying the collapsing process. Practically, the criterion adopted is the requirement that the time at which the pressure sound wave crosses the radius of the collapsing overdensity should be larger than the free fall time from the maximum expansion to complete collapse. Thus, the pressure gradient force will not have time to disperse the collapsing fluid matter to the background medium and prevent at the end the gravitational collapse process.

To do so, we put firstly the Friedmann equation for the overdensity, namely Eq. (6), in a Tolman-Bondi form, valid for the case of dust matter, which has an analytic parametric solution. To do so, we redefine appropriately the scale factor a and the cosmic time t as follows:

$$\tilde{a} = a e^{3I(a)} \qquad (14)$$

$$d\tilde{t} = dt e^{3I(a)}[1 + 3w(a)], \qquad (15)$$

where $I(a) \equiv \int_{a_{\text{ini}}}^{a} \frac{w(x)}{x} dx$ and the index ini denotes the initial time. Then, solving the continuity equation $\dot{\rho} + 3H(1+w)\rho = 0$ for a time-dependent EoS parameter and using the coordinate transformation of Eqs. (14) and (15), the Friedmann equation for the collapsing overdensity (6) can be recast in a dust form as

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{A}{\tilde{a}} - 1, \qquad (16)$$

where $A = \frac{\rho_{\text{ini}} a_{\text{ini}}^3}{3M_{\text{Pl}}^2}$ and we have used the fact that $\frac{d\tilde{a}}{d\tilde{t}} = \frac{da}{dt}$. The above equation once integrated gives the following parametric solution:

$$\tilde{a} = \tilde{a}_{\text{max}} \frac{1 - \cos\eta}{2}, \qquad \tilde{t} = \tilde{t}_{\text{max}} \frac{\eta - \sin\eta}{\pi}, \qquad (17)$$

with $\eta \in [0, 2\pi]$. In the above solution, η is the conformal time defined in terms of the redefined scale factor and cosmic time, i.e., $d\tilde{t} \equiv \tilde{a} d\eta$ whereas $\tilde{a}_{\text{max}}$ and $\tilde{t}_{\text{max}}$ are the redefined scale factor and cosmic time at the maximum expansion time respectively which read as:

$$\tilde{a}_{\text{max}} = \frac{\Omega_{\text{ini}}}{\Omega_{\text{ini}} - 1} \tilde{a}_{\text{ini}}, \qquad \tilde{t}_{\text{max}} = \frac{\pi}{2} \tilde{a}_{\text{max}}. \qquad (18)$$

Concerning now the sound wave propagation in a close Friedmann geometry, the latter is dictated by the following equation:

$$a \frac{d\chi}{dt} = c_s(t), \qquad (19)$$

where c_s^2 is the sound speed of a perfect fluid with a time-dependent EoS parameter. In the case of an adiabatic perfect fluid, c_s^2 reads as [See also Appendix]

$$c_s^2(\eta) = w(\eta) - \frac{1}{3[1 + w(\eta)]\mathcal{H}(\eta)} \frac{dw}{d\eta}, \qquad (20)$$

where one sees the presence of the time derivative of w in the expression for the sound speed, which makes $c_s^2(\eta)$ different from its value when w is constant, i.e., $c_s^2(\eta) = w$.

Finally, making use of the conformal time definition η the above equation can be recast as

$$\frac{d\chi}{d\eta} = \frac{c_s(\eta)}{1 + 3w(\eta)}. \qquad (21)$$

One then can establish the PBH formation criterion as described before by demanding that the time at which the sound wave crosses the radius of the overdensity, i.e., $\eta(\chi_a)$ is larger than the time at which the overdensity reaches the maximum expansion, i.e., $\eta_{\text{max}} = \pi$. In this way, the pressure gradient force will not have time to prevent the gravitational collapse whose onset time is considered here as the time of maximum expansion. To do so, in contrast with the treatment of [38] one should solve numerically Eq. (21) and demand that

$$\eta_{\text{num}}(\chi_a) = \pi, \qquad (22)$$

where $\eta_{\text{num}}(\chi)$ is the numerical solution of Eq. (21) and χ_a is the comoving scale at which the sound wave crosses the overdensity at the time of the maximum expansion. Consequently, from Eq. (13) the PBH formation threshold in the uniform Hubble gauge reads as

$$\delta_c^{\text{UH}} = \sin^2\chi_a, \qquad (23)$$

with χ_a being the solution of $\eta_{\text{num}}(\chi_a) = \pi$.

At this point, one should stress that the black hole apparent horizon should form after the onset of the gravitational collapse, i.e., the time of the maximum expansion. Thus, one should demand as well that $\eta_h > \eta_{\text{max}} = \pi$ where η_h is the time of formation of the apparent horizon which is obtained when $\frac{2M}{R} = 1$, where M is the Misner-Sharp mass in spherical symmetric spacetimes

[See in [72,73] for more details]. A rigorous analysis shows that in the case of a closed FLRW universe, the condition $\frac{2M}{R} = 1$ gives that [38]

$$\eta_{\mathrm{h}} = 2\chi_{\mathrm{a}} \quad \text{or} \quad 2\pi - 2\chi_{\mathrm{a}}. \tag{24}$$

Given the fact that the coordinates in Eq. (5) cannot cover entirely the overdense region of perturbations for which $\pi/2 < \chi_{\mathrm{a}} < \pi$ we work here with perturbations for which $0 < \chi_{\mathrm{a}} < \pi/2$ and therefore $\eta_{\mathrm{h}} = 2\pi - 2\chi_{\mathrm{a}}$. Demanding then that $\eta_{\mathrm{h}} > \eta_{\max} = \pi$ one has that $\chi_{\mathrm{a}} < \pi/2$. Here, one should point out that in the case w is constant then $c_{\mathrm{s}}^2 = w$ and Eq. (21) can be solved analytically. In this regime, solving the equation $\eta(\chi_{\mathrm{a}}) = \pi$ with $0 < \chi_{\mathrm{a}} < \pi/2$ leads to the formula for δ_{c} obtained in [38].

One then can use the aforementioned semianalytic scheme and apply it in the case of time-dependent w epochs such the preheating epoch during which PBHs can be abundantly produced [13,74–77] or the QCD phase transition where one also expects enhanced PBH production due to the softening of the EoS [71,78,79].

However, it is important to stress out that the prescription described above for the computation of δ_{c} can be only viewed as an approximate one since it requires the homogeneity of the central overdense core that is not the case when one is met with strong pressure gradients. It is valid then for situations in which $w \ll 1$. As noticed also in [28,29], the "three-zone" model initially introduced by [38] gives δ_{c} for a very sharply peaked homogeneous overdensity profile, where the shape parameter $\alpha \to 0$, but it does not take into account the shape dependence of the energy density profile discussed in Sec. I and the role of pressure gradients. These two effects can potentially disfavor the gravitational collapse and increase the value of δ_{c}. In particular, broader energy density perturbation profiles with $\alpha \gg 1$ have the tendency to bounce back to the background medium and not collapse to a PBH. For this reason, the critical PBH formation threshold needs to be high enough in order for such perturbations to collapse. Consequently, the PBH formation threshold computed within the "three-zone" model can be viewed as a lower bound for δ_{c}.

D. The PBH formation threshold in the comoving gauge

Let us now express the PBH formation threshold in the comoving gauge which is the one mostly used in numerical simulations [80–83]. In the comoving gauge, the energy density perturbation at horizon crossing, $\delta_{\mathrm{hc}}^{\mathrm{com}}$ can be written as [84]

$$\delta_{\mathrm{hc}}^{\mathrm{com}} = Q(\eta) \frac{1}{3r^2} \frac{\mathrm{d}}{\mathrm{d}r} [r^3 K(r)] r_{\mathrm{m}}^2, \tag{25}$$

where r_{m} is the comoving scale of the collapsing overdensity region, defined as the position of the maximum of

the compaction function, i.e., $\mathcal{C}'(r_{\mathrm{m}}) = 0$, $K(r)$ is the curvature profile in the quasihomogeneous solution regime [28] and Q is a function of time which is given by

$$Q(\eta) = 1 - \frac{H(\eta)}{a(\eta)} \int_{a_{\mathrm{ini}}}^{a(\eta)} \frac{\mathrm{d}a'}{H(a')}. \tag{26}$$

In the case of a constant equation of state, $Q = \frac{3(1+w)}{5+3w}$. For the case of the "three-zone" model considered here, $K(r) = 1$ and $r_{\mathrm{m}} = \sin\chi_{\mathrm{a}}$ and as a consequence

$$\frac{1}{3r^2} \frac{\mathrm{d}}{\mathrm{d}r} [r^3 K(r)] r_{\mathrm{m}}^2 = \sin^2\chi_{\mathrm{a}} = \delta_{\mathrm{hc}}^{\mathrm{UH}}. \tag{27}$$

Therefore, the energy density perturbation at horizon crossing time in the comoving and the uniform Hubble gauge are related as follows

$$\delta_{\mathrm{hc}}^{\mathrm{com}} = Q(\eta)\delta_{\mathrm{hc}}^{\mathrm{UH}}. \tag{28}$$

E. Limitations of the HYK prescription and further refinement

In the prescription described above for the computation of δ_{c} in a time-varying w background, $\delta_{\mathrm{hc}}^{\mathrm{UH}}$ in Eq. (28) is computed by extracting the location in comoving coordinates where the sound wave crosses the collapsing overdensity at the time of maximum expansion, denoted here with $\chi_{a,\max}$.

Given the fact that within the "three-zone" model the conformal time range within which the gravitational collapse is completed is $\Delta\eta = 2\pi$ as it can bee seen from Eq. (17), the location χ_{a} in the HYK prescription is uniquely fixed once and for all by the constant value of w independently on the time variation of w given that it is constant in time.

However in the current regime, given that w varies with time, one should subdivide the conformal time range in subranges where $\Delta\eta = 2\pi$ and extract the location $\chi_{a,\max}$ for every time subrange by solving numerically the equation $\eta(\chi_a) = \pi$. This aspect is very important to be taken into account since in the case of a dynamical w profile $\chi_{a,\max}$ is not uniquely fixed by the value of w once and for all but rather depends on the full time evolution of the EoS parameter. At the end, by computing the location $\chi_{a,\max}$ for every conformal time subrange one can make an interpolation between the values $\chi_{a,\max}$ computed within the different time subranges and extract $\chi_{a,\max}$ as a function of time. Consequently, $\delta_{\mathrm{hc}}^{\mathrm{UH}}$ will read as

$$\delta_{\mathrm{hc}}^{\mathrm{UH}} = \sin^2[\chi_{a,\max}(\eta)]. \tag{29}$$

At the end, this refined treatment takes into account the details of the full dynamical profile of the EoS parameter and constitutes a first step toward a more accurate computation of δ_{c} in a time-varying w background. It also shed

light on the limitations of the HYK prescription which can clearly not be used in a time-dependent w regimes.

III. THE PBH FORMATION THRESHOLD DURING THE QCD PHASE TRANSITION

We choose to illustrate our formalism regarding the computation of δ_c with a rather physical regime observed in the universe where the EoS parameter changes with time. One should in principle choose regimes where $w \ll 1$ in which the effect of pressure gradients is small and our formalism is precise. Such a regime is the phase of (pre) reheating where in the context of canonical inflation the universe effectively behaves as dust ($w \simeq 0$). However, given that the details of the physics during reheating remain uncertain we do not have yet a robust answer of what is the dynamical evolution of w during this phase [60].

For this reason, we choose to illustrate our formalism with the case of the QCD phase transition where the EoS parameter and the sound speed are robustly computed through lattice QCD simulations [85]. During this period, one expects to have an enhanced PBH production with mass $M \sim M_\odot$ due to the softening of the equation of state [71,78,79]. However, during the QCD phase transition phase w varies within the range $0.23 < w < 0.33$ where one expects a strong effect of the pressure gradients. For this reason, the results presented in this section regarding the computation of δ_c in a time-varying w background should be viewed as an example case where our formalism is applied rather than a defined result of the paper.

Regarding now the period of the QCD phase transition, the EoS parameter w, defined as the ratio between the pressure and the energy density, i.e., $w \equiv p/\rho$, can be computed with the help of the number of energy and entropy relativistic degrees of freedom, g_ρ and g_s respectively, defined through the expressions for the energy and the entropy density, i.e., $g_\rho(T) = 30\rho/(\pi^2 T^4)$ and $g_s(T) = 45s/(2\pi^2 T^3)$. At the end, using the relationship between ρ, s and p, i.e., $p = sT - \rho$, and the definition of w one can easily find that the EoS parameter and the sound speed square read as:

$$w(T) = \frac{4g_s(T)}{3g_\rho(T)} - 1 \tag{30}$$

$$c_s^2(T) = \frac{4(4g_s(T) + Tg_s'(T))}{3(4g_\rho(T) + Tg_\rho'(T))} - 1. \tag{31}$$

Using now tabulated data for T, g_ρ, and g_ρ/g_s during the QCD phase transition [85] and making a cubic spline interpolation one can extract the EoS parameter w and the sound speed square c_s^2 as a function of the temperature. See in the bottom panel of Fig. 2 their variation with temperature.

Then, applying the formalism presented before we compute δ_c in the comoving gauge during the QCD phase
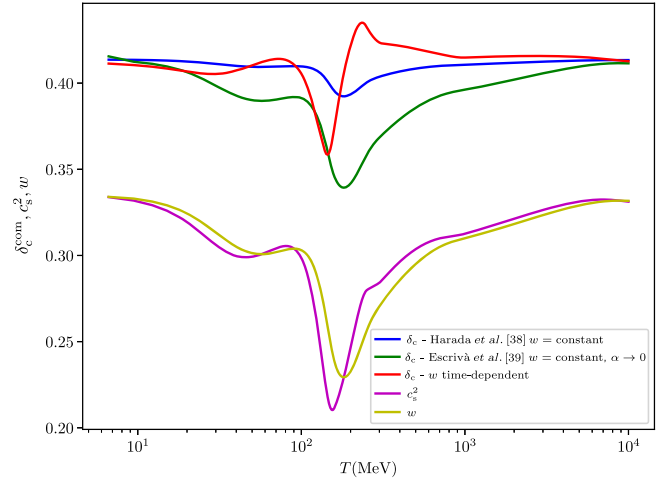

FIG. 2. In the bottom panel we show the EoS parameter w (yellow curve) and the sound speed square, c_s^2 (magenta curve) as a function of the temperature during the QCD phase transition. In the top panel, we plot the PBH formation threshold, δ_c in the comoving gauge during the QCD phase transition. With the blue curve we depict δ_c derived within the HYK prescription, valid for a constant EoS parameter w while the green curve shows δ_c computed within numerical simulations for constant w by Escrivà et al. [39] in the case of a peaked energy density perturbation profile with shape parameter $\alpha \to 0$. The red curve corresponds to δ_c derived within our prescription where w varies with time.

transition (red curve in Fig. 2) by dividing the conformal time range of variation of w in $N_p = 10^5$ subranges with $\Delta\eta = 2\pi$ following the procedure described in Sec. II E. By increasing more the number of subranges, the picture does not change significantly. Then, we compare it with δ_c obtained within the HYK treatment (blue curve in Fig. 2) valid for constant w as well with the value of δ_c obtained from numerical simulations (green curve in Fig. 2) with constant w run by Escrivà and Germani and Sheth (EGS) [39] making at the end a linear interpolation between w and δ_c values in the regime where $w < 1/3$. Regarding, the shape of the collapsing overdensity region we account for the fact that in our setup we have a peaked overdense region matched to the unperturbed background through an underdense layer. Therefore, the shape parameter α introduced in Sec. I should be close to zero, i.e., $\alpha \to 0$.

As one may notice from Fig. 2, there are places in particular around the region of the minimum of w or c_s^2 where one is met with regimes where $c_s^2 \neq w$ and where w varies within 30% from its background radiation value $w_{rad} = 1/3$. The case where $c_s^2 \neq w$ is actually the non static regime where in the context of a perfect adiabatic fluid, the EoS parameter w varies with time as it can be seen from Eq. (20). Clearly, in these regimes the range of validity of constant w analytic or numerical prescriptions for the computation of δ_c is limited.

Interestingly, one can also notice from Fig. 2 that δ_c follows the course of c_s^2 with its minimum being situated at

the same location with the minimum of c_s^2. One should also point out that in the regions where $c_s^2 \neq w$ our prescription differs significantly from the EGS and the HYK prescrtiptions where $c_s^2 = w$. In particular, when $c_s^2 < w$, i.e., when the fluid becomes softer compared to the static case, δ_c is decreased compared to the constant w prescriptions enhancing in this way the PBH gravitational collapse whereas when $c_s^2 > w$, i.e., when the fluid becomes harder compared to the static regime, δ_c is increased disfavoring the gravitational collapse. Expectedly, when $c_s^2 \sim w$, δ_c approaches its value computed within the EGS and the HYK prescrtiptions.

The above mentioned behavior can be explained also mathematically if one studies carefully the sound wave equation (21) where one can see that the conformal time derivative $\frac{d\chi}{d\eta}$ is proportional to the ratio $\frac{c_s(\eta)}{1+3w(\eta)}$. So, in regions where $c_s^2 > w$, the conformal time derivative increases and as a consequence the location at which the sound wave crosses the overdensity region at the time of maximum expansion shifts to higher values of $\chi_{a,\mathrm{max}}$ always in the range $0 < \chi_a < \pi/2$ as mentioned in Sec. II C. Consequently, given the fact that the $\sin^2$ function is a monotonically increasing function in the range $0 < \chi_a < \pi/2$ this leads to an increase of δ_c as it can be clearly seen from Eq. (29). The inverse phenomenology is followed when $c_s^2 < w$ leading to a decrease of δ_c compared to the static case.

One should also highlight here that as noticed in [39] in the case of a constant w the HYK analytic formula gives a lower bound for δ_c in the regime where $w > 1/3$, i.e., when pressure gradients are not negligible. However, in the regime $w < 1/3$ there is a significant discrepancy between the HYK limit and the results from numerical simulations in the case of peaked energy density perturbation profiles, i.e., $\alpha \ll 1$, in contradiction with what it was believed up to recently, namely that the HYK formula gives a lower bound of δ_c. This fact can be confirmed as well here since as one sees in Fig. 2 in the region of the abrupt decrease of w where $w < 1/3$, δ_c with the HYK prescription is higher of the order 10% compared to the EGS results based on numerical simulations for constant w.

However, within our prescription where the time variation of w is taken into account we find indeed a lower bound for δ_c at least in the regimes where the fluid becomes softer compared to the static case, i.e., $c_s^2 < w$. In the region, where the fluid becomes harder, i.e., $c_s^2 > w$ the δ_c value computed within our formalism is found to be higher compared to both the HYK and the EGS prescriptions.

In particular, there is a spike in δ_c around $T \sim 200$ MeV which gives rise to a variation of the threshold of about 0.08, a feature which is absent in previous results on the topic which assume however a constant EoS parameter. This behavior may be explained by the fact that our prescription considers only the time variation of the background values of w and c_s^2 not accounting for the effect of the backreaction of the perturbations to these

thermodynamic quantities which may alter our results. In principle, one should consider the full perturbed w and c_s^2 and extract δ_c from numerical simulations. According to ongoing numerical research work on the topic [86], this spike is not present when one includes the perturbations' backreaction effect and extract δ_c from numerical simulations. Thinking in physical terms, this result can be explained from the fact that the pressure gradients constitute a form of gravitational energy so while initially they delay the gravitational collapse once the collapse is triggered they mostly favor it since as it was found in [39] the formation time of a black-hole apparent horizon is decreasing as w increases. In view of this physical understanding, the effect of the pressure gradient is expected to wash out the enhancement of δ_c in the regions where $c_s^2 > w$ signaling that the respective spike in the δ_c value in the region around $T \sim 110$ MeV should not be interpreted as physical.

IV. CONCLUSIONS

The PBH formation threshold δ_c constitutes a very important quantity in the field of PBH physics since it determines the criterion for PBH formation and affects crucially the computation of the PBH abundance, a quantity which is constrained by many observational probes [31]. The vast majority of the works in the literature has not accounted for the dynamical evolution of w during the PBH formation process considering it as static, i.e., constant in time, an approximation which is good for the vast majority of the cosmic epochs.

However, in general w is a time-dependent parameter and in particular in some periods of the cosmic history such as preheating or the QCD phase transition when one expects an enhanced PBH production, can experience very abrupt changes. Consequently, motivated by the above mentioned phenomenology, in this work we studied the effect of a time-dependent EoS parameter on the computation of the PBH formation threshold δ_c based on physical arguments. In particular, generalizing the mathematical formalism of [38] and commenting its limitations we computed δ_c by comparing the time the pressure sound wave crosses the radius of the collapsing overdensity with the time of the onset of the gravitational collapse.

We should point out however that our refined semi-analytic treatment for the calculation of the PBH formation threshold is an approximate one working well in regimes where pressure gradients are small. Thus, the value of δ_c derived here can be viewed as a lower bound of the true δ_c at least in the regimes where the fluid becomes softer compared to the static case, namely when $c_s^2 < w$. It also does not take into account the dependence of δ_c on the shape of the collapsing energy density perturbations as well as perturbation backreaction effects on the EoS parameter and the sound speed which may alter our conclusions.

We applied our refined prescription for the computation of δ_c in the case of the QCD phase transition, when the EoS

parameter changes significantly with time and one expects an enhanced PBH production due to the abrupt softening of w. Interestingly, we found that in the static regime, where $c_s^2 \sim w$, which is mainly realized when w and c_s^2 varies slowly with time, our formalism gives results comparable with analytic and numerical approaches assuming a constant w.

However, in the nonstatic regime, i.e., regions where $c_s^2 \neq w$, our prescription differs significantly from prescriptions assuming constant w. In particular, by accounting only for the background dynamical evolution of w and c_s^2, when $c_s^2 < w$, i.e., when the fluid becomes softer compared to the static case, δ_c is decreased compared to the constant w prescriptions enhancing in this way the PBH gravitational collapse whereas when $c_s^2 > w$, i.e., when the fluid becomes harder, δ_c is increased disfavoring the gravitational collapse.

These interesting results show the big effect of a dynamical w profile to the determination of the PBH formation threshold and points out to the need of the development of full numerical techniques to study the effect of a time-varying EoS parameter on the value of δ_c.

ACKNOWLEDGMENTS

APPENDIX: THE SOUND SPEED IN A TIME-DEPENDENT w BACKGROUND

Here, we extract the sound speed of a general adiabatic fluid, c_s^2, with a time-dependent equation-of-state

parameter, w. In a general system, the pressure density p is a function of the energy density ρ as well as of the entropy density S, i.e., $p = p(\rho, S)$. Consequently, one can write the following equation:

$$\delta p = c_s^2 \delta \rho + \left(\frac{\partial p}{\partial S}\right)_\rho \delta S, \tag{A1}$$

where the sound speed c_s^2 is defined as $c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S$. If one considers then an adiabatic system then they should require that $\left(\frac{\partial p}{\partial S}\right)_\rho = 0$, i.e., there is no entropy production. Consequently, for such a system c_s^2 becomes

$$c_s^2 = \frac{\delta p}{\delta \rho}, \tag{A2}$$

Given then the fact that the background pressure and energy densities of an adiabatic fluid system, p and ρ depend only on time, one can rewrite Eq. (A2) by introducing the derivation with respect to the conformal time using the chain rule, as

$$c_s^2 = \frac{p'}{\rho'}, \tag{A3}$$

where the prime denotes differentiation with respect to the conformal time, η. Finally, by using the continuity equation $\rho' + 3\mathcal{H}(1 + w)\rho = 0$, written with the conformal time as the time variable as well as time differentiating w defined as $w \equiv p/\rho$, one can straightforwardly obtain that

$$c_s^2(\eta) = w(\eta) - \frac{1}{3[1 + w(\eta)]\mathcal{H}(\eta)} \frac{dw}{d\eta}, \tag{A4}$$

where $\mathcal{H} \equiv \frac{a'}{a}$ is the conformal Hubble parameter.

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Correction: The previously published Fig. 2 had a scaling error in temperature T and has been replaced. Corresponding values in text in the fourth and tenth paragraphs of Sec. III have been fixed.