New first-order formulation of the Einstein equations exploiting analogies with electrodynamics

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The Einstein and Maxwell equations are both systems of hyperbolic equations which need to satisfy a set of elliptic constraints throughout evolution. However, while electrodynamics and magnetohydrodynamics have benefited from a large number of evolution schemes that are able to enforce these constraints and are easily applicable to curvilinear coordinates, unstructured meshes, or N-body simulations, many of these techniques cannot be straightforwardly applied to existing formulations of the Einstein equations. We develop a $3+1$ a formulation of the Einstein equations that shows a striking resemblance to the equations of relativistic magnetohydrodynamics and to electrodynamics in material media. The fundamental variables of this formulation are the frame fields, their exterior derivatives, and the Nester-Witten and Sparling forms. These mirror the roles of the electromagnetic four potential, the electromagnetic field strengths, the field excitations and the electric current. The role of the lapse function and shift vector, corresponds exactly to that of the scalar electric potential. The formulation is manifestly first order and flux-conservative, which makes it suitable for high-resolution shock capturing schemes and finite-element methods. Being derived as a system of equations in exterior derivatives, it is directly applicable to any coordinate system and to unstructured meshes, and leads to a natural discretization potentially suitable for the use of machine-precision constraint propagation techniques such as the Yee algorithm and constrained transport. Due to these properties, we expect this new formulation to be beneficial in simulations of many astrophysical systems, such as binary compact objects and core-collapse supernovae as well as cosmological simulations of the early Universe.

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I. INTRODUCTION

In the last few years the study of relativistic astrophysics and in particular of compact objects has made significant progress. The theoretical understanding of binary black holes, binary neutron stars, and super-massive black holes has been validated by a string of impressive observations, such as the first detection of gravitational waves from binary black holes [1]; the first and joint detection of gravitational waves, a gamma ray burst, and a kilonova from a binary neutron star system [2,3]; and the first direct imaging of a supermassive accreting black hole [4,5].

These are systems exhibiting extreme complexity, and whose modeling requires the interplay of different areas of modern physics, such as relativistic gravitation, fluid dynamics, electrodynamics, nuclear physics, neutrino physics, and many others. Therefore the theoretical study of these and other systems cannot be accomplished with purely analytical tools. Numerical relativity has instead emerged as a powerful modeling tool.

The core approach of numerical relativity (NR) consists in finding approximate solutions to the partial differential equations (PDEs) describing the system at study, namely the Einstein's field equations (EFE), by numerical integration. To this end, the equations of general relativity (GR) have first to be recast as an initial boundary value problem.

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This can be accomplished in various ways. Examples include the generalized-harmonic formalism [6–8], the characteristic-evolution formalism [9], the conformal approach [10,11], and fully constrained formulations [12]. These approaches however are not the subject of this work. Instead we operate in the context of the most commonly employed formalism, the so-called $3 + 1$ formalism [13–15].

In this formalism, the four-dimensional spacetime of GR is foliated in a succession of purely spatial hypersurfaces; the EFE themselves split in 12 hyperbolic evolution equations, governing the evolution of the fields as time advances, and four elliptic constraint equations. The latter define constraints that the solution has to satisfy, and at the analytical level are always satisfied provided the initial data also satisfy them (and as such they must be solved to generate the initial data itself, see, e.g., [16]). In order to obtain a true solution to the EFE, these constraints need to be satisfied. Violations may easily lead to unstable numerical simulations. While the constraints will be always satisfied at the analytical level, numerical truncation errors will easily cause violations that can accumulate and destabilize the evolution. It can even be shown that the Arnowitt-Deser-Misner (ADM) [17,18] formulation of the EFE can be made strongly hyperbolic by assuming, among other conditions, that the momentum constraints are identically satisfied [13]. These considerations have motivated the search for alternative, more robust formulations of Einstein equations. Several approaches have been pursued to ensure stable numerical evolutions. A widely used and strongly hyperbolic formulation, namely Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima (BSSNOK), was introduced in Refs. [19–23]. In this formulation, the constraint violations cannot be dampened and will accumulate and grow over time. Despite this shortcoming, it allows for stable, long-term evolutions, yet in some particularly challenging test cases constraint violations can grow without bounds, typically crashing the evolution code [24,25].

A simple extension of the EFE to include propagating modes for the constraints, is to generalize a Lagrange multiplier approach, similar to the one adopted for electrodynamics [26]. The resulting family of formulations stemming from the Z4 formalism [27], most notably Z4c [28] and CCZ4 [24,29], include damping terms designed with the twofold aim of propagating the constraint violations away from where they occur and also damping them as they propagate [30].

It is important to understand that this approach does not guarantee exact fulfilment of the constraint equations. Techniques to control the growth of constraint violations however are commonly used in numerical electrodynamics. Maxwell's equations include conditions such as the absence of magnetic monopoles, $\nabla \cdot \boldsymbol{B} = 0$, which similarly to GR are elliptic equations that the solution of the corresponding evolution equations should satisfy at all times [31,32]. An example of a technique designed to handle these requirements is Dedner's *et al.* method [26], employed successfully in numerical magnetohydrodynamics (MHD) and particle-in-cell (PIC) simulations.

Constraint damping was successfully applied in the first successful merger simulation [6], and it has been mainly adopted in simulations using the generalized-harmonic formulation of the EFE [8,33]. One important aspect of the generalized-harmonic system is that the equations can trivially be recast in first-order form [8], which is more difficult for BSSNOK-like systems, such as FO-CCZ4 [34] or first-order BSSNOK [35]. First-order formulations are particularly important when solving the EFE using finite elements or pseudospectral methods [36], see Refs. [34,37,38].

As recently pointed out, these first-order extensions are subject to additional curl constraint, which can render the simulations unstable if not enforced. Generalizing the idea of divergence cleaning, Ref. [39] introduced the notion of curl cleaning, which requires to approximately solve four elliptic equations per constraint (using hyperbolic relaxation), and applied it to FO-CCZ4. This results in a system with a total of more than a hundred evolved variables, making the system very expensive to solve and implement efficiently.

Hence it would be beneficial to have a system of first-order equations that could be solved using simpler and cheaper approaches. In fact numerical electrodynamics has benefited also from another class of methods which are able to maintain a discretized version of the constraints satisfied to machine accuracy during the evolution, without adding additional equations to the system. The common feature of these methods is that the electromagnetic variables are not all defined and stored at the same spatial points in the computational domain, but on staggered grids. Belonging to this class of methods are the popular Yee algorithm [40] and constrained transport schemes [41], widely used in numerical electrodynamics and MHD simulations.

A constraint preserving scheme for GR based on staggered grids was proposed by Ref. [42]. This work identifies as crucial the role played by the second Bianchi identities in propagating the constraints and develops a staggered finite-difference discretization that is able to satisfy them to machine precision in Riemann normal coordinates. However when such discretization is applied to general coordinates, the exact fulfilment of the identities is prevented by the noncancellation of terms that are cubic in the Christoffel symbols, which appear as a result of the noncommutativity of covariant derivatives of the Riemann tensor. As a result the scheme's ability to exactly propagate the constraints is bounded by the truncation error.

In the present work, we realize the importance of expressing equations as a system that relates differential forms with the tool of exterior calculus to obtain discretizations that fulfil the constraints to machine precision, and apply this idea to obtain a $3 + 1$ formulation of GR. Being natural integrands over submanifolds, differential forms are very well suited to represent quantities such as total charges inside volumes or fluxes through surfaces. For this reason, integrating such equations yields a natural discretization that reflects the geometric properties of the equations themselves, and represents in a consistent way both the evolution and the constraint equations. Two important schemes derived from this idea are finite volume and constraint transport methods. In MHD, the former is able to achieve machine precision conservation of volume-integrated quantities (e.g., particle number density) by locating fluxes at the volume boundaries (cells faces), and the latter is able to achieve machine precision conservation of surface-integrated magnetic fluxes (which results in machine-precision fulfilment of $\nabla \cdot \boldsymbol{B} = 0$) by locating electric fields at the surface boundaries (cells edges). In our endeavor we build upon the fact that a formulation of GR in the language of exterior calculus already exists (in fact it has already been proposed to exploit it in order to obtain coordinate invariant formulations suitable for numerical implementation [43]).

The formulation we develop mirrors at the formal level the equations of covariant electrodynamics in a moving material medium [32,44]. We argue that this resemblance would allow us to apply the knowledge and the methods developed in those disciplines to the evolution of dynamical spacetimes; in particular it would allow to develop constrained transport schemes for NR, or to apply divergence- or curl-cleaning methods. In fact, it is conceivable that existing MHD solvers, e.g., [45–47], could be adapted with minimal effort to solve the equations derived in this work to evolve dynamical spacetimes instead. This would hold even when adopting unstructured and moving meshes [48].

This formulation, that we refer to as DGREM (for differential forms, general relativity, and electromagnetism) also posses two other desirable features. First, it contains only first order derivatives in both space and time, which can significantly simplify its discretization especially with some numerical schemes such as discontinuous Galerkin methods [49]. Second, it can be written as a system of flux-balance laws, for the discretization of which a lot of expertise has been amassed over decades of work [50]. To the best of the authors' knowledge, no formulation of the Einstein equations available in the literature combines all of these advantages.

This work is organized as follows: after defining our notation (Sec. II), in Sec. III we introduce our exterior calculus-based techniques by applying them to the wave equation; Sec. IV revisits a formulation of GR as a system of equations written in terms of differential forms. Sections V and VI are the central part of this work, in which we derive and present the proposed DGREM formulation. A summary of the results is given in Sec. VII, while several appendices provide details of derivations hinted at in the main text as well as a primer on the theory of exterior calculus.

II. NOTATION AND DEFINITIONS

In this section we summarize the notation that is used in the rest of this work, since due to our reliance on concepts originating from the framework of exterior calculus, it may not be completely familiar to readers used to the NR literature. We direct the reader to Appendix A and references therein for more details on differential forms and exterior calculus. We also collect some definitions used throughout the article, mainly relating to the $3+1$ split of GR.

We work within the usual spacetime of general relativity, i.e., a four-dimensional, Lorentzian, at least twice differentiable manifold $\mathcal{M}$. We differentiate various type of indices on tensors and differential forms. Letters from the first half of the Latin alphabet $(a, b, c, \ldots)$ shall represent, in any basis, indices ranging from 0 to 3. In a coordinate basis, letters from the first half of the greek alphabet $(\alpha, \beta, \gamma, \ldots)$ shall represent indices ranging from 0 to 3, and latin letters from the second half of the alphabet $(i, j, k, \ldots)$ shall represent indices ranging from 1 to 3 (i.e., spatial components). The same convention will apply in a noncoordinate orthonormal basis, but using hatted characters, i.e., $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \ldots$ for indices from 0 to 3, and $\hat{\imath}, \hat{\jmath}, \hat{k}, \ldots$ for indices from 1 to 3.

In what follows many objects contain nontensorial indices. These objects are collections of differential forms, which we also call tensor-valued differential forms. The indices in these objects simply label the components in the collection and do not necessarily imply that the collection as a whole transform a tensor (see Appendix A for further details on tensor-valued differential forms and comments on the terminology). These indices will not be assigned any particular notation, although their nontensorial nature will be indicated in the text.

Without referring to any particular basis, we indicate both tensors and differential forms with boldface characters; however in the abstract index notation that we preferentially employ, we drop the boldface font.

We define the following symbols:

η_{ab}	Minkowski metric
$\delta^a{}_b$	Kronecker delta
ϵ_{abcd}	Levi-Civita symbol
$\varepsilon_{abcd} = \sqrt{-g}\epsilon_{abcd}$	Volume form
ε^{abcd}	Levi-Civita tensor (dual of volume form)
$\boldsymbol{e}_a$	Vector basis
$\boldsymbol{\theta}^a$	Dual basis
∂	Partial derivative
∇	Covariant derivative
$\boldsymbol{d}$	Exterior derivative
$\boldsymbol{D}$	Covariant exterior derivative
$\mathcal{L}$	Lie derivative
$\star$	Hodge dual

where g denotes the determinant of the metric (see below). Note that all definitions above, even when written with coordinate basis indices, are valid in the case of noncoordinate bases too; and that in the definition of basis vectors and forms, the indices are nontensorial, simply labeling objects in a collection.

While the objects we work with are denoted as scalars, vectors, tensors, and differential forms, we actually always mean scalar fields, vector fields, tensor fields, and fields of differential forms, respectively, even when this is not explicitly stated. The same holds for objects that are not tensorial in nature, such as connection coefficients.

The manifold $\mathcal{M}$ is provided with a metric tensor $g_{\mu\nu}$, for which we choose the "mostly plus" signature $(-,+,+,+)$, and whose determinant is denoted by g. We also summarize here the framework of the $3+1$ split of GR, which we employ in order to recast the Einstein equations as an initial value problem (see standard NR textbooks such as [13–15] for more details). We assume that the spacetime can be foliated in a sequence of tridimensional, purely spatial hypersurfaces Σ_t (i.e., the spacetime is assumed to be hyperbolic), each of which is parametrized by a value of a function t. We define the future-directed unit normal $n_\mu = -\alpha\nabla_\mu t$, where the lapse function α equals $\alpha = -1/g^{tt}$. From n_μ we can construct the metric restricted to each hypersurface $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, which is purely spatial. Considering now the vector $t^\mu = g^{\mu\nu}\nabla_\nu t$, we identify it with our basis's temporal vector (i.e., we choose a basis adapted to the foliation) and decompose it in a part parallel to n^μ and one perpendicular to it: $t^\mu = \boldsymbol{e}_t = \partial_t = \alpha n^\mu + \beta^\mu$. The purely spatial vector β^μ is called the shift vector. With these definitions in place we can then state the expressions of n_μ and $g_{\mu\nu}$ (or the line element ds) in a coordinate basis:

$$n_\mu = (-\alpha, 0, 0, 0) \quad \text{and} \quad n^\mu = \frac{1}{\alpha}(1, -\beta^i)^T$$
$$\mathrm{d}s^2 = -(\alpha^2 + \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where we have denoted with x^i the spatial coordinates in any hypersurface Σ_t and the ^{T} superscript indicates matrix transposition. We indicate with γ the determinant of γ_{ij}, $\gamma = \det(\gamma_{ij})$, and note that $\sqrt{-g} = \alpha\sqrt{\gamma}$.

III. PDEs IN THE LANGUAGE OF EXTERIOR CALCULUS

Differential forms are natural integrands on submanifolds, and PDEs that can be written as relations between differential forms with the tools of exterior calculus can be naturally discretized by integration on appropriate volumes. When such a discretization is applied consistently, the resulting evolution scheme correctly reflects the geometric structure of the equations. In turn, this opens up the possibility of developing constraint-preserving evolution schemes.

In order to introduce the reader to our approach as outlined above, we apply it in this section to a well-known PDE. Namely, we explicitly formulate the standard wave equation on a generic spacetime in terms of differential forms. We then give a brief review of electrodynamics in material media, also written in the language of exterior calculus. This helps us set the stage for reformulating GR and the Einstein equations in the same language in the next section.

A. The wave equation

Rather than stating the usual wave equation (in terms of scalar or vector fields and ordinary derivatives) and showing how it can be expressed in terms of differential forms, we choose here to reverse the exposition order, i.e., stating the equation as a relation between differential forms and then recovering the usual formulation. This better reflects the derivation the DGREM formulation of GR in Sec. IV.

Consider a scalar field (or 0-form) ϕ, and its exterior derivative $\boldsymbol{J} = \boldsymbol{d}\phi$, which is of course a 1-form. $\boldsymbol{J}$ satisfies the equation

$$-\star^{-1}\boldsymbol{d}\star\boldsymbol{J} = 0. \tag{1}$$

Employing the components representation of the exterior derivative and of the Hodge dual, we can rewrite Eq. (1) as

$$\varepsilon^{\alpha\beta\gamma\nu}\partial_{[\nu}(\varepsilon_{\alpha\beta\gamma]\mu}J^\mu) = 0. \tag{2}$$

Note that in this section we assume for simplicity a coordinate basis, hence the indices are labeled by greek letters.

Recalling the definition of $\boldsymbol{\varepsilon}$ it is easy to see that the last equation becomes

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}J^\mu) = 0, \tag{3}$$

expressing that the divergence of $\boldsymbol{J}$ must vanish. This was to be expected since operator in (1) (sometimes called the codifferential) is a generalization of the divergence operator [see Eq. (A32)]. Substituting the definition of $\boldsymbol{J}$ as the exterior derivative of ϕ, this equation immediately implies

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) = 0, \tag{4}$$

i.e., the standard homogeneous wave equation for the field ϕ in a generic spacetime.

We now seek too express Eq. (4) via a $3+1$ formulation, i.e., recasting it as an evolution equation for ϕ. To this end let us define the following projections of $\boldsymbol{J}$:

$$\rho = -n^\mu J_\mu,$$
$$j_i = \gamma^\mu{}_i J_\mu. \tag{5}$$

Substituting these definitions in Eq. (3) and recalling the relationship between the unit normal n^μ, the lapse α, and the shift β^i, yields the equations

$$\partial_t(\sqrt{\gamma}\rho) + \partial_i(\sqrt{\gamma}\rho\mathcal{V}^i) = 0,$$
$$\partial_t j_i + \partial_i(\alpha\rho - \beta^k j_k) = 0, \tag{6}$$

where $\mathcal{V}^i = \alpha j^i/\rho - \beta^i$ is the transport velocity of ρ.

These are evolution equations for (quantities related to) the components of $\boldsymbol{J}$. An evolution equation for ϕ itself can easily be recovered from the definition of ρ and recalling that $J_t = \partial_t\phi$, resulting in

$$\partial_t\phi = -\alpha\rho + \beta^k j_k. \tag{7}$$

The wave equation Eq. (4), or the system (6), is subject to a set of differential constraints. Working with differential forms, this can be seen as follows. The nilpotency of the exterior derivative, Eq. (A13), immediately gives

$$\boldsymbol{dJ} = \boldsymbol{dd}\phi = 0. \tag{8}$$

This of course implies that $\star\boldsymbol{dJ} = 0$, and by comparing with Eq. (A33), we can expect this equation to be requiring the curl of $\boldsymbol{J}$ to vanish. Indeed switching to a components representation and using the variables ρ and j_i, Eq. (8) is equivalent to

$$\partial_i j_k - \partial_k j_i = 0. \tag{9}$$

These are three constraint equations for the spatial components of $\boldsymbol{J}$ (a fourth equation, stemming from considering the time components and involving the variable ρ, turns out to be identical to the evolution equation for j_i).

Equation (9) simply asserts the commutativity of second spatial derivatives of ϕ, but as the wave equation itself it can be stated much more compactly and expressively in terms of differential forms.

As mentioned in the Introduction, writing the system in terms of differential forms can be also useful to determine the spatial localization of variables for a constraint preserving discretization. However, the direct integration of Eqs. (1) and (8), would yield a four-dimensional discretization staggered in time. For methods such as finite volume, it is more convenient to derive a semidiscrete evolution equation with all variables located on the hypersurface Σ_t. In order to achieve this, we employ Cartan's "magic" formula [see Eq. (A14) in Appendix A], and compute the Lie derivative of $\boldsymbol{J}$ and $\star\boldsymbol{J}$, with respect to the basis vector $\boldsymbol{e}_t$, which coincides with ∂_t.

$$\mathcal{L}_{\boldsymbol{e}_t}\boldsymbol{J} = \boldsymbol{d}(\boldsymbol{e}_t\cdot\boldsymbol{J}),$$
$$\mathcal{L}_{\boldsymbol{e}_t}\star\boldsymbol{J} = \boldsymbol{d}(\boldsymbol{e}_t\cdot\star\boldsymbol{J}), \tag{10}$$

or

$$\partial_t\boldsymbol{J} = \boldsymbol{d}(-\alpha\rho + \beta^k j_k), \tag{11}$$

$$\partial_t\star\boldsymbol{J} = \boldsymbol{dF}, \tag{12}$$

where the flux form $\boldsymbol{F}$ is defined as

$$\boldsymbol{F} = \varepsilon_{ijk}(\alpha j^i - \rho\beta^i)(\boldsymbol{d}x^j\wedge\boldsymbol{d}x^k).$$

The nontrivial components of (11) and (12) give identical equations to those in (6); however, the advantage of writing them in this way is that the submanifolds on which they should be integrated become explicit. All terms in (11) are 1-forms, and all terms in (12) are 3-forms, which invites to integrate them, respectively, on curves and volumes. For the purpose of a numerical scheme which decomposes a three-dimensional simulation domain in zones, this corresponds to integrate the equations over zone edges and zone volumes. After applying the Stokes theorem (A30), exterior derivatives are replaced by evaluations of the forms on zone boundaries (i.e., respectively, on zone vertices and zone faces).

It is straightforward to see that such discretization conserves globally the volume-integrated "charge" ρ: since faces are shared by two zones, the amount of flux leaving one zone and entering the other will contribute with opposite signs to the time update of each zone's content, and the total charge content in the simulation domain will remain constant to machine precision as long as there is no flux through the simulation boundaries.

The discretization also fulfils a discretized version of Eq. (9) to machine precision. This can be seen by integrating Eq. (8) over a zone face (i.e., a surface, since it is a 2-form). The application of Stoke's theorem once more transforms the exterior derivative into the sum of the forms $\boldsymbol{J}$ integrated on the contour formed by the edges surrounding that face (i.e., the circulation around it). Also in this case, each of the scalars $\alpha\rho + \beta^k j_k$ defined at zone vertices will be shared by two edges and contribute to their time update of $\boldsymbol{J}$ with opposite signs, canceling their contributions to the circulation. The discretization is therefore able to preserve an integrated version of constraint (9) to machine precision when supplied with constraint-fulfilling initial data.

B. Maxwell equations

In the language of differential forms, Maxwell equations can be written as

$$\boldsymbol{dF} = 0, \tag{13}$$

$$\boldsymbol{du} = \star\boldsymbol{J}, \tag{14}$$

where $\boldsymbol{F}$ is the electromagnetic field strength, a 2-form with the components of the Faraday tensor $(\boldsymbol{F})_{\mu\nu} \coloneqq F_{\mu\nu}$, and $\boldsymbol{u}$ is the electromagnetic field excitation.[1] In the particular case

[1]The most common notation for the electromagnetic field excitation is $H_{\mu\nu}$. Here we have used instead the symbols $\boldsymbol{u}$, $u_{\mu\nu}$ to avoid confusion with the those used for the Eulerian magnetic field in this section, and to highlight the similarity of its role within electrodynamics to that of the Nester-Witten form (defined in Sec. IV) in general relativity.

of vacuum, $\boldsymbol{u} = \star \boldsymbol{F}$, but in general it may be related by more complicated local constitutive equations $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{F})$ [51]. The 1-form $\boldsymbol{J}$ has the components of the electric four current. The form $\boldsymbol{F}$ can be obtained as the exterior derivative of a 1-form potential $\boldsymbol{A}$, and we can thus write

$$\boldsymbol{F} = \boldsymbol{d}\boldsymbol{A}, \tag{15}$$

which allows us to recover (13) due to the nilpotency of the exterior derivative. Similarly, by taking the exterior derivative of (14), we obtain

$$\boldsymbol{d}\star \boldsymbol{J} = 0, \tag{16}$$

which, as discussed for Eq. (1), represents a conservation equation, in this case that of electric charge.

In components form, Eqs. (13) and (14) acquire the form

$$\partial_\mu(\sqrt{-g}\star F^{\mu\nu}) = 0, \tag{17}$$

$$\partial_\mu\sqrt{-g}(-\star u^{\mu\nu}) = \sqrt{-g}J^\nu. \tag{18}$$

The expression in components for the definition of the four potential (15) gives

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{19}$$

and that for the conservation of electric charge (16) reads

$$\partial_\mu(\sqrt{-g}J^\mu) = 0. \tag{20}$$

The electric field E^μ, the magnetic induction B^μ, the electric displacement D^μ, and the magnetic field H^μ, as measured by an observer moving at four-velocity n^μ, are obtained by projecting $\boldsymbol{F}$, $\boldsymbol{u}$, and their duals onto n^μ. Setting n^μ as the velocity of Eulerian observers, we can thus define the Eulerian fields as

$$E^\mu \coloneqq F^{\mu\nu}n_\nu, \tag{21}$$

$$B^\mu \coloneqq \star F^{\mu\nu}n_\nu, \tag{22}$$

$$D^\mu \coloneqq -\star u^{\mu\nu}n_\nu, \tag{23}$$

$$H^\mu \coloneqq u^{\mu\nu}n_\nu, \tag{24}$$

where $\star F_{\mu\nu} = (\star \boldsymbol{F})_{\mu\nu}$ and $\star u_{\mu\nu} = (\star \boldsymbol{u})_{\mu\nu}$. These are purely spatial vectors, as can be verified using the antisymmetry of $F^{\mu\nu}$ and $u^{\mu\nu}$ (for more details on the $3+1$ formulation of electrodynamics, see [52]). These projections allow us to write the components of each of the 2-forms as

$$F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu - \varepsilon_{\mu\nu\lambda\sigma}n^\lambda B^\sigma,$$
$$u_{\mu\nu} = n_\mu H_\nu - n_\nu H_\mu + \varepsilon_{\mu\nu\lambda\sigma}n^\lambda D^\sigma. \tag{25}$$

For the electric current 1-form $\boldsymbol{J}$, we can adopt the same projections defined in Eq. (5) and identify ρ with the charge density and j^i with the electric current in direction i as measured by Eulerian observers. Similar projections can be adopted for the four potential, which allows us to define the usual scalar and vector potentials as

$$\phi = -n^\mu A_\mu, \tag{26}$$

$$A_i = \gamma^\mu{}_i A_\mu. \tag{27}$$

Substituting projections (26) and (27) in Eq. (19) gives the familiar $3+1$ expressions for the electric and magnetic fields in terms of the vector potential

$$\partial_t A_i + \partial_i(\alpha\phi - \beta^i A_i) = -\alpha E_i - \epsilon_{ilk}\beta^l B^k, \tag{28}$$

$$\epsilon_{ijk}B^k = \partial_i A_j - \partial_j A_i. \tag{29}$$

Similarly, substituting projections (21) and (22) in the homogeneous Maxwell Eq. (17) gives an evolution equation and a differential constraint, namely Faraday's law and Gauss's law for magnetism,

$$\partial_t\sqrt{\gamma}B^k + \partial_i\sqrt{\gamma}(\alpha\epsilon^{ijk}E_j - \beta^i B^k + \beta^k B^i) = 0, \tag{30}$$

$$\partial_i\sqrt{\gamma}B^i = 0. \tag{31}$$

Projections (23) and (24) substituted in the inhomogeneous Maxwell Eq. (18) also gives an evolution equation and a differential constraint, this time Ampère-Maxwell's law and Gauss's law for electricity,

$$\partial_t\sqrt{\gamma}D^k - \partial_i\sqrt{\gamma}(\alpha\epsilon^{kij}H_j + \beta^i D^k - \beta^k D^i) = -\sqrt{\gamma}j^k, \tag{32}$$

$$\partial_i\sqrt{\gamma}D^i = \sqrt{\gamma}\rho. \tag{33}$$

The equation for charge conservation (20), when expressed in $3+1$ form, becomes identical to the conservation equation (6) derived for the wave equation in Sec. III A. As mentioned earlier, the system is closed by the constitutive relations, which now express the $3+1$ fields E^i and H^i in terms of D^i and B^i, and depend on the medium considered. For the special case of vacuum, $\boldsymbol{u} = \star \boldsymbol{F}$ implies $E^i = D^i$ and $H^i = B^i$.

Following the same procedure as for the wave equation, Cartan's magic formula (A14) can be used in combination with Eqs. (13)–(15) to obtain a set of expressions that, when integrated, will lead to a constraint-preserving semi-discrete scheme. These are

$$\partial_t\mathcal{A} + \boldsymbol{d}(\alpha\phi - \beta^i A_i) = -\mathcal{E}, \tag{34}$$

$$\partial_t\mathcal{B} + \boldsymbol{d}\mathcal{E} = 0, \tag{35}$$

$$\partial_t\mathcal{D} - \boldsymbol{d}\mathcal{H} = -\mathcal{J}, \tag{36}$$

where

$$\mathcal{A} = A_i \boldsymbol{d}x^i, \tag{37}$$

$$\mathcal{E} = (\alpha E_i + \varepsilon_{ilk}\beta^l B^k)\boldsymbol{d}x^i, \tag{38}$$

$$\mathcal{H} = (\alpha H_i - \varepsilon_{ilk}\beta^l D^k)\boldsymbol{d}x^i \tag{39}$$

are 1-forms and

$$\mathcal{B} = \varepsilon_{ijk}B^i \boldsymbol{d}x^j \wedge \boldsymbol{d}x^k, \tag{40}$$

$$\mathcal{D} = \varepsilon_{ijk}D^i \boldsymbol{d}x^j \wedge \boldsymbol{d}x^k, \tag{41}$$

$$\mathcal{J} = \varepsilon_{ijk}(\alpha j^i - \beta^i \rho)\boldsymbol{d}x^j \wedge \boldsymbol{d}x^k \tag{42}$$

are 2-forms. Integration of Eqs. (34)–(36) suggests a constraint-preserving discretization with 1-forms located at cell edges and 2-forms located at cell faces. This is the discretization adopted for constrained-transport methods in ideal and resistive magnetohydrodynamics. A related discretization is that of the Yee algorithm, widely used in PIC simulations. It employs two shifted staggered grids, one for electric and another for magnetic fields, so that cell faces from one grid coincide with cell edges of the other, which is especially useful to avoid interpolations in flat spacetime with Cartesian coordinates.

Finally, when adopting the different symbols used for the exterior derivative in three-dimensional vector calculus, one recovers the familiar expressions for the definition of the electromagnetic potential,

$$\vec{\mathcal{E}} = -\partial_t \vec{\mathcal{A}} - \nabla(\alpha\phi - \boldsymbol{\beta}\cdot\vec{\mathcal{A}}), \tag{43}$$

$$\mathbf{B} = \nabla \times \vec{\mathcal{A}}, \tag{44}$$

Maxwell's equations,

$$\partial_t \sqrt{\gamma}\mathbf{B} + \sqrt{\gamma}\nabla \times \vec{\mathcal{E}} = 0, \tag{45}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{46}$$

$$\partial_t \sqrt{\gamma}\mathbf{D} - \sqrt{\gamma}\nabla \times \vec{\mathcal{H}} = -\sqrt{\gamma}\vec{\mathcal{J}}, \tag{47}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{48}$$

and charge conservation,

$$\partial_t \sqrt{\gamma}\rho + \sqrt{\gamma}\nabla \cdot \vec{\mathcal{J}} = 0, \tag{49}$$

where the arrow denotes the operation of transforming a 1-form to its dual vector, and $\mathbf{D} = D^i \mathbf{e}_i$, $\mathbf{B} = B^i \mathbf{e}_i$, and $\boldsymbol{\beta} = \beta^i \mathbf{e}_i$. The presence of the $\sqrt{\gamma}$ factors depends on the definition adopted for the ∇ operators. For the conventions used here, we refer the reader to Appendix A.

IV. GENERAL RELATIVITY IN THE LANGUAGE OF EXTERIOR CALCULUS

In this section, we first lay the groundwork to derive the DGREM formulation by outlining a reformulation of the Einstein equations in terms of exterior calculus and using objects known as the Nester-Witten and Sparling forms. This results in writing the Sparling equation, which is fully equivalent to the EFE.

We then introduce a change of variables and a particular choice of connection that ultimately allows us to reexpress the Sparling equation, and therefore the EFE, as a system of evolution equations resembling the Maxwell equation of electrodynamics, i.e., the titular DGREM formulation.

Let us define for convenience the "hypersurface forms" as [53]

$$\boldsymbol{\Sigma}_{a_1\ldots a_r} = \frac{1}{(4-r)!}\varepsilon_{a_1\ldots a_r a_{r+1}\ldots a_4}\boldsymbol{\theta}^{a_{r+1}} \wedge \ldots \wedge \boldsymbol{\theta}^{a_4}. \tag{50}$$

Loosely speaking, they can be thought as (the dual forms to) vectors orthogonal to submanifolds spanned by given subsets of the basis $\boldsymbol{\theta}^{a_1} \wedge \ldots \wedge \boldsymbol{\theta}^{a_4}$, e.g., the 3-form $\Sigma_0 = \varepsilon_{0123}\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}^3$ is orthogonal to the tridimensional hypersurface spanned by $\boldsymbol{\theta}^1$, $\boldsymbol{\theta}^2$, and $\boldsymbol{\theta}^3$. They satisfy the identity

$$\boldsymbol{\theta}^b \wedge \boldsymbol{\Sigma}_{a_1\ldots a_r} = (-1)^{r+1} r \delta^b{}_{[a_1}\boldsymbol{\Sigma}_{a_2\ldots a_r]}. \tag{51}$$

For a manifold with curvature and torsion described, respectively, by the 2-forms $\boldsymbol{\Omega}^a_b$ and $\boldsymbol{\Xi}^a$, the connection forms $\boldsymbol{\omega}^a_b$ (see Appendix A for a definition) are completely specified by Cartan's structure equations,

$$\boldsymbol{\Xi}^a = \boldsymbol{d}\boldsymbol{\theta}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{\theta}^b, \tag{52}$$

$$\boldsymbol{\Omega}^a{}_b = \boldsymbol{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b, \tag{53}$$

and by the condition of metric compatibility of the connection,

$$\boldsymbol{d}g_{ab} = \boldsymbol{\omega}_{ab} + \boldsymbol{\omega}_{ba}. \tag{54}$$

Note that in this last equation the individual components of the metric are seen as 0-forms, i.e., the metric itself is a tensor-valued 0-form, hence it is possible to apply the exterior derivative to it.

The curvature and torsion forms are related to the Riemann and the torsion tensors $R^a{}_{bcd}$ and $T^a{}_{bc}$ by

$$\boldsymbol{\Omega}^a{}_b \wedge \boldsymbol{\Sigma}_{cd} = R^a{}_{bcd}\boldsymbol{\Sigma}, \tag{55}$$

$$\boldsymbol{\Xi}^a = T^a{}_{bc}\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c. \tag{56}$$

It can be shown [43,53] that the curvature form is related to the Ricci tensor $R^b{}_c$, the curvature scalar $R = R^b{}_b$, and the Einstein tensor $G^c{}_d = R^c{}_d - Rg^c{}_d$ in the following ways:

$$\mathbf{\Omega}^{ab} \wedge \mathbf{\Sigma}_{ac} = R^b{}_c \mathbf{\Sigma}, \tag{57}$$

$$\mathbf{\Omega}^{ab} \wedge \mathbf{\Sigma}_{ab} = R\mathbf{\Sigma}, \tag{58}$$

$$-\frac{1}{2}\mathbf{\Omega}^{ab} \wedge \mathbf{\Sigma}_{dab} = G^c{}_d \mathbf{\Sigma}_c. \tag{59}$$

By taking the exterior derivative of Cartan's structure equations [Eqs. (52)–(53)], it is possible to obtain the first and second Bianchi identities,

$$\boldsymbol{d}\mathbf{\Xi}^a = \mathbf{\Omega}^a{}_e \wedge \boldsymbol{\theta}^e - \boldsymbol{\omega}^a{}_e \wedge \mathbf{\Xi}^e, \tag{60}$$

$$\boldsymbol{d}\mathbf{\Omega}^a{}_b = \mathbf{\Omega}^a{}_e \wedge \boldsymbol{\omega}^e{}_b - \boldsymbol{\omega}^a{}_e \wedge \mathbf{\Omega}^e{}_b, \tag{61}$$

which for a manifold with no torsion and in a coordinate basis take the usual form

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0, \tag{62}$$

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} = 0. \tag{63}$$

To formulate general relativity as a system with exterior derivatives, we first define a 2-form $\boldsymbol{u}_a$, known as the Nester-Witten form [43,53,54]:

$$\boldsymbol{u}_a := -\frac{1}{2}\boldsymbol{\omega}^{bc} \wedge \mathbf{\Sigma}_{abc}. \tag{64}$$

Taking its exterior derivative and using the two Cartan structure equations, we obtain

$$\begin{aligned}\boldsymbol{d}\boldsymbol{u}_a = &-\frac{1}{2}\mathbf{\Omega}^{bc} \wedge \mathbf{\Sigma}_{abc} + \frac{1}{2}\mathbf{\Xi}^d \wedge \boldsymbol{\omega}^{bc} \wedge \mathbf{\Sigma}_{abcd} \\ &-\frac{1}{2}(\boldsymbol{\omega}^b{}_d \wedge \boldsymbol{\omega}^{dc} \wedge \mathbf{\Sigma}_{abc} + \boldsymbol{\omega}^d{}_a \wedge \boldsymbol{\omega}^{bc} \wedge \mathbf{\Sigma}_{dbc}).\end{aligned} \tag{65}$$

The terms in parentheses can be grouped in a 3-form known as the Sparling form:

$$\boldsymbol{t}_a := -\frac{1}{2}(\boldsymbol{\omega}^b{}_d \wedge \boldsymbol{\omega}^{dc} \wedge \mathbf{\Sigma}_{abc} + \boldsymbol{\omega}^d{}_a \wedge \boldsymbol{\omega}^{bc} \wedge \mathbf{\Sigma}_{dbc}), \tag{66}$$

whose pullbacks in different bases are related to different expressions for the gravitational energy momentum. In particular, in a coordinate basis it is the Einstein pseudotensor [54]. For convenience, let us define $t^b{}_a$ such that

$$\boldsymbol{t}_a = t^b{}_a \mathbf{\Sigma}_b. \tag{67}$$

Assuming no torsion, relation (59) and Eq. (65) can be used to obtain the Sparling equation:

$$\boldsymbol{d}\boldsymbol{u}_a = \boldsymbol{t}_a + \kappa \boldsymbol{T}_a, \tag{68}$$

where the nongravitational energy-momentum 3-form $\boldsymbol{T}_a$ is defined as

$$\boldsymbol{T}_a = T^\mu{}_a \mathbf{\Sigma}_\mu, \tag{69}$$

and where $T^\mu{}_a$ are the components of the energy-momentum tensor.

At this point a few comments are necessary. First of all, Eq. (68) is equivalent to the Einstein equations [43,53,54], and the sum of the Nester-Witten and Sparling forms is related to the Einstein tensor by

$$\boldsymbol{d}\boldsymbol{u}_a - \boldsymbol{t}_a = G^b{}_a \mathbf{\Sigma}_b, \tag{70}$$

or in components form,

$$G^c{}_a = \frac{1}{\sqrt{-g}}\partial_b[\sqrt{-g}(-\star\boldsymbol{u}_a)^{bc}] - t^c{}_a. \tag{71}$$

This equivalence holds despite the fact that the index in the objects $\boldsymbol{u}_a$ and $\boldsymbol{t}_a$ is nontensorial, i.e., the components of the Nester-Witten form $u_{abc} = (\boldsymbol{u}_a)_{bc}$ [2] are not part of a single three-indices tensor but belong to a collection of four 2-forms labeled by the index a, which transforms as $\binom{0}{2}$ tensors with indices b and c (see also Appendix A).

This also means that the objects $\boldsymbol{u}_a$ and $\boldsymbol{t}_a$ are not unique: a different choice of basis 1-forms from which to compute the connection will lead to different collections of objects, although Eq. (68) will still hold, in the same way as the choice of different bases and connections does not alter the validity of the Einstein equations.

Although the nontensorial behavior of these quantities might be startling, this behavior is natural, as it is linked to the local flatness of spacetime. In the language of tensors, various quantities (such as the metric first partial derivatives or energy-momentum pseudotensors) can be made to vanish locally in a free-falling frame. This is possible owing to the nontensorial nature of these objects, as tensors cannot made to vanish by a coordinate (i.e., linear) transformation. By the same token, the Sparling form, which is related to various kinds of energy-momentum pseudotensors [53,54], displays a similar behavior thanks to its own nontensorial nature.

[2]Here and in the following, we often employ a simplified notation, writing e.g., u_{abc} instead of the more verbose $(\boldsymbol{u}_a)_{bc}$, when dealing with the components of various (collections of) differential forms.

V. EXPLOITING THE ANALOGIES WITH MAXWELL'S EQUATIONS

A. Evolution equations and constraints

Equation (68) presents the Einstein equations as a set of four equations with a structure very similar to that of the inhomogeneous Maxwell equations, i.e., with the exterior derivative of a 2-form at the left-hand side and a conserved current at the right-hand side. In fact, taking the exterior derivative of Eq. (68) it can be seen that the four currents $\boldsymbol{J}_a = \star(\boldsymbol{t}_a + \kappa \boldsymbol{T}_a)$ are globally conserved. Each antisymmetric tensor $u_{a\mu\nu}$ in the Nester-Witten form plays the role of the Maxwell 2-form, and, in a coordinate basis, Eq. (68) takes a form completely analogous to that of the inhomogeneous Maxwell equations,

$$\partial_b \sqrt{-g}(-\star u_a{}^{bc}) = \sqrt{-g}(t^c{}_a + \kappa T^c{}_a). \tag{72}$$

Comparing (72) with (71), its equivalence to the Einstein equations becomes clear.

Exploiting further the similarity with electrodynamics, we can define the following projections of the Nester-Witten form and its dual

$$H_a{}^\mu := u_a{}^{\mu\nu} n_\nu \quad \text{and} \quad D_a{}^\mu := -\star u_a{}^{\mu\nu} n_\nu. \tag{73}$$

This allows us to decompose these forms as

$$u_{a\mu\nu} = n_\mu H_{a\nu} - n_\nu H_{a\mu} + \varepsilon_{\mu\nu\alpha\beta} n^\alpha D_a{}^\beta, \tag{74}$$

$$\star u_{a\mu\nu} = -n_\mu D_{a\nu} + n_\nu D_{a\mu} + \varepsilon_{\mu\nu\alpha\beta} n^\alpha H_a{}^\beta. \tag{75}$$

Defining as well the following projections of the components of the Sparling form and the energy-momentum tensor,[3]

$$\begin{aligned}
\rho_a &:= n_\mu t^\mu{}_a, \\
s^i{}_a &:= \gamma^i{}_\mu t^\mu{}_a, \\
P_a &:= n_\mu T^\mu{}_a, \\
S^i{}_a &:= \gamma^i{}_\mu T^\mu{}_a.
\end{aligned} \tag{76}$$

Equation (72) can be separated into four constraint equations

$$\mathcal{C}_a := \partial_i \sqrt{\gamma} D_a{}^i - \sqrt{\gamma}(\rho_a + \kappa P_a) = 0, \tag{77}$$

and 12 evolution equations

$$\begin{aligned}
\mathcal{F}^k_a := {} & \partial_t \sqrt{\gamma} D_a{}^k - \partial_i \sqrt{\gamma}(\alpha \varepsilon^{kij} H_{aj} + \beta^i D_a{}^k - \beta^k D_a{}^i) \\
& + \sqrt{\gamma}(j^k{}_a + \kappa J^k{}_a) = 0,
\end{aligned} \tag{78}$$

where

$$j^k{}_a = \alpha s^k{}_a - \beta^k \rho_a, \tag{79}$$

$$J^k{}_a = \alpha S^k{}_a - \beta^k P_a. \tag{80}$$

The fulfilment of Eq. (77) is equivalent to that of the Einstein constraints. This can be seen by the definition of the usual Hamiltonian and momentum constraints and the $3+1$ evolution equations [55] as

$$\begin{aligned}
\mathcal{H} &:= n^\mu n^\nu (G_{\mu\nu} - \kappa T_{\mu\nu}) = 0, \\
\mathcal{M}_i &:= \gamma^\mu{}_i n^\nu (G_{\mu\nu} - \kappa T_{\mu\nu}) = 0, \\
\mathcal{E}_{ij} &:= \gamma^\mu{}_i \gamma^\nu{}_j (G_{\mu\nu} - \kappa T_{\mu\nu}) = 0,
\end{aligned} \tag{81}$$

from which

$$\begin{aligned}
\mathcal{C}_0 &= -\mathcal{H}, \\
\mathcal{C}_i &= -\mathcal{M}_i/\alpha, \\
\mathcal{F}^i{}_0 &= \alpha \mathcal{M}^i + \beta^i \mathcal{H}, \\
\mathcal{F}^i{}_j &= \mathcal{E}^i{}_j + \beta^i \mathcal{M}_j/\alpha,
\end{aligned} \tag{82}$$

and therefore $\mathcal{C}_a = 0$ is equivalent to $\mathcal{M}_i = 0$ and $\mathcal{H} = 0$. The twice-contracted second Bianchi identities imply that if the Hamiltonian constraint is fulfilled on a spacelike hypersurface, its fulfilment on the "next" hypersurface is guaranteed as long as the momentum constraints are satisfied exactly and the system is evolved using evolution $3+1$ Einstein equations [55]. Similar equations for the propagation of constraints $\mathcal{C}_a$ can be obtained after taking the exterior derivative of the Sparling equation (68). This results in a set of equations equivalent to the twice-contracted second Bianchi identities of the form

$$\partial_t \sqrt{-g}\mathcal{C}_a + \partial_i \sqrt{-g}\mathcal{F}^i{}_a = 0. \tag{83}$$

Therefore, also in this case the evolution equations for $D_k{}^i$ and the exact fulfilment of the momentum constraints $\mathcal{C}_i$ are sufficient to propagate the fulfilment of $\mathcal{C}_0$ between subsequent hypersurfaces.

B. Energy-momentum conservation

The exterior derivative of Eq. (68) can also be used to obtain evolution equations for the "charge densities" ρ_a and P_a, as it expresses the global conservation of the sum of their currents,

$$\boldsymbol{d}(\boldsymbol{t}_a + \kappa \boldsymbol{T}_a) = 0. \tag{84}$$

[3]Note however that these are different from those usually employed in the literature, where the energy momentum tensor is projected *twice* on the normal vector and on the hypersurface.

Together with the local conservation of matter energy-momentum $\boldsymbol{D}\boldsymbol{T}_a = 0$,[4] this gives

$$\boldsymbol{d}\boldsymbol{T}_a = \boldsymbol{\omega}^b{}_a \wedge \boldsymbol{T}_b \quad \text{and} \tag{85}$$

$$\boldsymbol{d}\boldsymbol{t}_a = -\kappa\boldsymbol{\omega}^b{}_a \wedge \boldsymbol{T}_b, \tag{86}$$

or in component form and in a coordinate basis,

$$\partial_\mu \sqrt{-g} T^\mu{}_a = \sqrt{-g}\omega^b{}_{a\mu} T^\mu{}_b \quad \text{and} \tag{87}$$

$$\partial_\mu \sqrt{-g} t^\mu{}_a = -\kappa \sqrt{-g}\omega^b{}_{a\mu} T^\mu{}_b. \tag{88}$$

Substituting the projections defined above [Eq. (76)],

$$\partial_t \sqrt{\gamma}\rho_a + \partial_i \sqrt{\gamma}(\alpha s^i{}_a - \beta^i \rho_a) = -\kappa\sqrt{\gamma} Q_a, \tag{89}$$

$$\partial_t \sqrt{\gamma} P_a + \partial_i \sqrt{\gamma}(\alpha S^i{}_a - \beta^i P_a) = \sqrt{\gamma} Q_a, \tag{90}$$

where

$$Q_a = -(\omega^b{}_{at} + \omega^b{}_{ai}\beta^i) P_b + \alpha\omega^b{}_{ai} S^i{}_b. \tag{91}$$

The physical interpretation of Eqs. (68), (85), and (86) can be that of four vector fields described by the four 2-forms $\boldsymbol{u}_a$ which have as sources two currents $\star\kappa\boldsymbol{T}_a$ and $\star\boldsymbol{t}_a$. The sum of the latter two is globally conserved, but they exchange charge (in this case, energy and momentum) via the "force" term $\kappa\boldsymbol{\omega}^b{}_a \wedge \boldsymbol{T}_b$. These currents are those of gravitational ($\star\boldsymbol{t}_a$) and nongravitational ($\star\kappa\boldsymbol{T}_a$) energy and momentum. Equations (77) and (78) are the analog of the inhomogeneous Maxwell equations in $3+1$ form, and Eqs. (89) and (90) that of the conservation of the two charges.

While Eqs. (89) and (90) convey an interesting physical picture of energy exchange between the purely gravitational and the matter sector, there is another possibility of how to read these equations in practice. Adding up (89) and (90), we obtain

$$\begin{aligned}\partial_t[\sqrt{\gamma}(\rho_a + \kappa P_a)] + \partial_i[\sqrt{\gamma}(\alpha(s^i{}_a + \kappa S^i{}_a) \\ - \beta^i(\rho_a + \kappa P_a))] = 0.\end{aligned} \tag{92}$$

When comparing this equation with the equation of energy-momentum conservation (87), it is striking to see that using the Sparling form all source terms in (92) have disappeared. In this formulation, the geometric source terms of Eq. (87) have been recast into a fully flux conservative form. A similar observation has recently also been made by [56]. While previously such a formulation was known to exist for the time component of Eq. (87) in static spacetimes [57], this is the case here in any dynamical and nondynamical spacetime. While sounding trivial at first, such a formulation opens up the exciting prospects of applying advanced techniques from flux-balance equations to the Einstein-matter system, such first-order flux limiting [58] to ensure positivity of energy- and momentum densities.

This is particularly interesting when combined with the relativistic (magneto)hydrodynamics description of the matter part, for which nontrivial constraints on the physicality of the energy-momentum density P_a exist. A formulation such as this one, clearly separating gravitational and matter contribution, as well as having no explicit sources, might make it possible to transfer advances made on physicality preserving schemes in special relativity over to general spacetimes [59,60].

C. Choosing a connection

In Sec. IV, we showed that the Einstein equations and the conservation of energy and momentum can be expressed as a system of equations with close similarities to the inhomogeneous Maxwell equations and the equation of charge conservation. However, even assuming that we have equations to evolve the matter energy momentum, in order to close the system we need to specify a way of updating the quantities that appear in the equations for which no evolution equation is provided, that is, $\sqrt{\gamma}$, $\omega^b{}_{ac}$, $H^\mu{}_a$, and $s^\mu{}_a$. To find relations between these quantities and the evolved variables, we start by noticing that the Hodge dual of the Nester-Witten form can be written in terms of the connection as

$$(\star\boldsymbol{u}_a)^{bc} = \omega^{[bc]}{}_a + \delta^b{}_a \omega^{[cd]}{}_d - \delta^c{}_a \omega^{[bd]}{}_d. \tag{93}$$

The detailed calculation is provided in Appendix B. The relation (93) can be contracted to obtain

$$(\star\boldsymbol{u}_c)^{bc} = -2\omega^{[bc]}{}_c, \tag{94}$$

from which

$$\omega^{[bc]}{}_a = (\star\boldsymbol{u}_a)^{bc} - \frac{1}{2}\delta^b{}_a(\star\boldsymbol{u}_d)^{cd} + \frac{1}{2}\delta^c{}_a(\star\boldsymbol{u}_d)^{bd}. \tag{95}$$

This shows that the part of the connection that is antisymmetric with respect to its first two indices is completely determined by the Nester-Witten form. Since the full connection appears in other parts of the system, namely inside $\boldsymbol{t}_a$ [Eq. (66)] and Q_a [Eq. (91)], in principle it could be necessary to evolve also the part that is symmetric with respect to these indices. To simplify calculations, it would be useful to exploit the nonuniqueness of the Nester-Witten and the Sparling forms to build them from a connection that is purely antisymmetric with respect to its first two indices. This is the case for the spin connection (cf. Appendix J of [61]), also known as the Ricci rotation coefficients

[4] In this equation $\boldsymbol{D}$ represents the exterior covariant derivative (see Appendix A), and the equation is equivalent to the usual $\nabla_\mu T^{\mu\nu} = 0$.

(cf. Sec. 3.4b of [62]). It will also turn out to be crucial that the spin connection can be interpreted as a combination of exterior derivatives of 1-forms, which will directly establish a connection to the Maxwell equations, see (15).

1. Tetrads and spin connection

For an orthonormal vector basis $\{\boldsymbol{e}_{\hat{\alpha}}\}$ with dual 1-form basis $\{\boldsymbol{\theta}^{\hat{\alpha}}\}$, the spin connection $\boldsymbol{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} = \omega^{\hat{\alpha}}{}_{\hat{\beta}\hat{\mu}}\boldsymbol{\theta}^{\hat{\mu}}$ is defined by

$$\partial_{\hat{\nu}}\boldsymbol{e}_{\hat{\mu}} \coloneqq A^{\nu}{}_{\hat{\nu}}\partial_{\nu}\boldsymbol{e}_{\hat{\mu}} = \omega^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}}\boldsymbol{\theta}^{\hat{\mu}}\boldsymbol{e}_{\hat{\alpha}}, \tag{96}$$

where $A^{\nu}{}_{\hat{\nu}}$ are the coefficients that relate the orthonormal basis to the coordinate basis $\{\boldsymbol{e}_{\alpha}\}$, $\boldsymbol{e}_{\hat{\nu}} = A^{\nu}{}_{\hat{\nu}}\boldsymbol{e}_{\nu}$. The orthonormal 1-form basis $\{\boldsymbol{\theta}^{\hat{\mu}}\}$ and the coordinate basis $\{\boldsymbol{\theta}^{\mu}\}$ are related by the transformations $\boldsymbol{\theta}^{\hat{\mu}} = A^{\hat{\mu}}{}_{\mu}\boldsymbol{\theta}^{\mu}$ and $\boldsymbol{\theta}^{\mu} = A^{\mu}{}_{\hat{\mu}}\boldsymbol{\theta}^{\hat{\mu}}$. The form of the metric when expressed in an orthonormal basis is that of Minkowski metric, and is therefore constant. From metric compatibility (54), it follows that this connection is completely antisymmetric with respect to its first two indices. This can also be seen from the metricity condition, which states that the covariant derivative of the metric must vanish,

$$\nabla_{\hat{\alpha}}\eta_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\alpha}}\eta_{\hat{\mu}\hat{\nu}} - \omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\alpha}}\eta_{\hat{\beta}\hat{\nu}} - \omega^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}}\eta_{\hat{\mu}\hat{\beta}} = 0. \tag{97}$$

In what follows, we still express the equations in a coordinate basis to keep the convenience of directly integrating p-forms over coordinate submanifolds, but construct an orthonormal tetrad field to obtain the connection from which $\boldsymbol{u}_a$ and $\boldsymbol{t}_a$ are defined.

Given the $3+1$ foliation of the spacetime, a natural choice for the tetrad is that of an Eulerian observer moving at velocity n^{μ}, i.e., we take the vector n^{μ} to be part of the basis we are seeking. In order to accomplish this, the components of the tetrad basis 1-forms in the coordinate basis can be written as

$$A^{\hat{0}}{}_{\mu} = (\alpha, 0_i) = -n_{\mu}, \tag{98}$$

$$A^{\hat{i}}{}_{\mu} = (\beta^{\hat{i}}, A^{\hat{i}}{}_{i}), \tag{99}$$

where

$$\eta_{\hat{\mu}\hat{\nu}}A^{\hat{\mu}}{}_{\mu}A^{\hat{\nu}}{}_{\nu} = g_{\mu\nu}, \tag{100}$$

$$\beta^{\hat{i}} = A^{\hat{i}}{}_{i}\beta^{i}, \tag{101}$$

$$\delta_{\hat{i}\hat{j}}A^{\hat{i}}{}_{i}A^{\hat{j}}{}_{j} = \gamma_{ij}. \tag{102}$$

Conversely, the inverse transformation is given by

$$A^{\mu}{}_{0} = (1/\alpha, -\beta^{i}/\alpha) = n^{\mu} \tag{103}$$

$$A^{\mu}{}_{\hat{i}} = \begin{pmatrix} 0_{\hat{i}} \\ A^{i}{}_{\hat{i}} \end{pmatrix}, \tag{104}$$

where also

$$\eta^{\hat{\mu}\hat{\nu}}A^{\mu}{}_{\hat{\mu}}A^{\nu}{}_{\hat{\nu}} = g^{\mu\nu}, \tag{105}$$

$$\delta^{\hat{i}\hat{j}}A^{i}{}_{\hat{i}}A^{j}{}_{\hat{j}} = \gamma^{ij}. \tag{106}$$

The spin connection is calculated from the commutation coefficients of the basis, $c^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}}$, which in turn can be obtained either as the commutators of the basis vectors, or as the exterior derivatives of the basis 1-forms. While the two quantities coincide when expressed in the orthonormal basis, they obey different transformation laws, transforming, respectively, as a vector and as a 2-form. To keep exploiting the analogies with electromagnetism, we decide to calculate the commutation coefficients in the second way, and define the set of 2-forms

$$\boldsymbol{F}^{\hat{\alpha}} = \boldsymbol{d}\boldsymbol{\theta}^{\hat{\alpha}}, \tag{107}$$

which in a coordinate basis takes the form

$$F^{\hat{\alpha}}{}_{\mu\nu} = \partial_{\mu}A^{\hat{\alpha}}{}_{\nu} - \partial_{\nu}A^{\hat{\alpha}}{}_{\mu}. \tag{108}$$

The commutation coefficients are equal to the components of these forms when expressed in the tetrad basis,

$$c^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} = F^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} = A^{\mu}{}_{\hat{\mu}}A^{\nu}{}_{\hat{\nu}}F^{\hat{\alpha}}{}_{\mu\nu}, \tag{109}$$

and the connection can be calculated as

$$\omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} = \frac{1}{2}(c_{\hat{\mu}\hat{\alpha}\hat{\nu}} + c_{\hat{\nu}\hat{\alpha}\hat{\mu}} - c_{\hat{\alpha}\hat{\mu}\hat{\nu}}). \tag{110}$$

The striking similarity of the spin connection to the Levi-Civita connection is by no means a coincidence. The spin connection can be used to generalize the covariant derivative for general tensors $V^{\nu}_{\hat{\alpha}}$,

$$D_{\mu}V^{\nu}_{\hat{\alpha}} = \partial_{\mu}V^{\nu}_{\hat{\alpha}} + \Gamma^{\nu}_{\mu\beta}V^{\beta}_{\hat{\alpha}} - \omega^{\hat{\gamma}}_{\hat{\alpha}\mu}V^{\nu}_{\hat{\gamma}}. \tag{111}$$

It can be shown that this derivative is covariant in the tetrad and the coordinate frame. The specific form of the spin connection (109) now arises because the choice of 2-forms in (107) is equivalent to demanding metric compatibility of the local flat metric in the tetrad under transformations of the generalized covariant derivative (111),

$$D_{\mu}\eta_{\hat{\alpha}\hat{\beta}} = 0. \tag{112}$$

In the same way, that metric compatibility of the spacetime metric uniquely results in the Levi-Civita connection, the choice of (112) imposes the form of the connection coefficients (109). Put differently, we have defined both the global manifold and the local tetrad to be torsion free.

2. Spin connection and electrodynamics

Conversely, by inverting relation (110) it can be found that the forms $\boldsymbol{F}_{\hat{\alpha}}$ collect the antisymmetric part of the spin connection with respect to the last two indices,

$$F_{\hat{\alpha}\hat{\mu}\hat{\nu}} = \omega_{\hat{\alpha}\hat{\nu}\hat{\mu}} - \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}}. \tag{113}$$

We now define the following projections of $\boldsymbol{F}_{\hat{\alpha}}$ and its dual $\star\boldsymbol{F}_{\hat{\alpha}}$ as

$$\begin{aligned} E^{\hat{\alpha}\mu} &:= F^{\hat{\alpha}\mu\nu} n_\nu \\ B^{\hat{\alpha}\mu} &:= \star F^{\hat{\alpha}\mu\nu} n_\nu, \end{aligned} \tag{114}$$

so that we can write their components as

$$\begin{aligned} F^{\hat{\alpha}}{}_{\mu\nu} &= n_\mu E^{\hat{\alpha}}{}_\nu - n_\nu E^{\hat{\alpha}}{}_\mu - \varepsilon_{\mu\nu\lambda\sigma} n^\lambda B^{\hat{\alpha}\sigma}, \\ \star F^{\hat{\alpha}}{}_{\mu\nu} &= n_\mu B^{\hat{\alpha}}{}_\nu - n_\nu B^{\hat{\alpha}}{}_\mu + \varepsilon_{\mu\nu\lambda\sigma} n^\lambda E^{\hat{\alpha}\sigma}. \end{aligned} \tag{115}$$

Substituting Eqs. (115) and (98) in (108), we find the following evolution equations for the transformation coefficients on the slice

$$\begin{aligned} \partial_t A^{\hat{i}}{}_i - \partial_i \beta^{\hat{i}} &= -\alpha E^{\hat{i}}{}_i - \varepsilon_{ilk} \beta^l B^{\hat{i}k}, \\ &= -\alpha E^{\hat{i}}{}_i - \sqrt{\gamma}\epsilon_{ilk} \beta^l B^{\hat{i}k}, \end{aligned} \tag{116}$$

along with the constraints

$$\begin{aligned} E^{\hat{0}}{}_i &= \partial_i \ln\alpha, \\ \sqrt{\gamma} B^{\hat{0}i} &= 0, \\ \sqrt{\gamma} B^{\hat{i}k} &= \epsilon^{ijk} \partial_i A^{\hat{i}}{}_j. \end{aligned} \tag{117}$$

These equations are in close analogy to electromagnetism, with the role of the three-vector potential played by $A^{\hat{i}}{}_i$ and that of the scalar potential played by α and $\beta^{\hat{i}}$. It is interesting to see that Eq. (107) does not provide evolution equations for these scalar potentials, which is in agreement with the gauge freedom of the spacetime foliation.

By taking the exterior derivative of Eq. (107), we obtain

$$\boldsymbol{d}\boldsymbol{F}^{\hat{\alpha}} = 0, \tag{118}$$

which is nothing more than the first Bianchi identity, as can be seen by comparing Eq. (107) with (52) and (60) with (118). Using the projections in definition (114), Eq. (118) splits in four equations with a form analogous to the Gauss law for magnetism, namely

$$\partial_i \sqrt{\gamma} B^{\hat{\alpha}i} = 0, \tag{119}$$

and 12 evolution equations analogous to the Faraday equation,

$$\partial_t \sqrt{\gamma} B^{\hat{\alpha}k} + \partial_i \sqrt{\gamma}(\alpha\epsilon^{ijk} E^{\hat{\alpha}}{}_j - \beta^i B^{\hat{\alpha}k} + \beta^k B^{\hat{\alpha}i}) = 0. \tag{120}$$

For $\hat{\alpha} = \hat{0}$, Eqs. (119) and (120) are trivially fulfilled, since $B^{\hat{0}i} = 0$, and Eq. (120) becomes simply an expression of the commutativity of the partial derivatives of α.

D. Closing the system

We have now obtained all the evolution equations of the system, and can list the elements of the state vector as $\vec{U} = \{A^{\hat{k}}{}_i, D_{\hat{\alpha}}{}^i, \rho_{\hat{\alpha}}, P_{\hat{\alpha}}\}$, where the first 25 quantities determine the state of the gravitational field, while the four momentum densities $P_{\hat{\alpha}}$ depend on the properties of matter. Additionally, we need a set of relations to obtain the remaining quantities that appear in their evolution equations, namely $\vec{Q} = \{\sqrt{\gamma}, E^{\hat{\alpha}}{}_i, B^{\hat{k}i}, H_{\hat{\alpha}}{}^i, s^i{}_{\hat{\alpha}}, S^i{}_{\hat{\alpha}}, Q_{\hat{\alpha}}\}$, where again the momentum fluxes $S^i{}_{\hat{\alpha}}$ depend on the properties of matter. Although $\sqrt{\gamma}$ and $B^{\hat{k}i}$ can in principle be obtained as the determinant and the curl of $A^{\hat{k}}{}_i$, respectively, it may be useful to evolve them with an independent evolution equation. In the case of $\sqrt{\gamma}$, the reason being to evolve it at the side of conformally rescaled quantities or to avoid errors associated to the numerical computation of the determinant. An evolution equation for $\sqrt{\gamma}$, can be obtained by using (50) to define the hypersurface form orthogonal to $-n_\mu$, that is, to $\boldsymbol{\theta}^{\hat{0}}$, and taking its exterior derivative. The resulting expression has the form of a conservation equation for volume,

$$\partial_t \sqrt{\gamma} - \partial_i \sqrt{\gamma}\beta^i = \frac{5}{2}\alpha\sqrt{\gamma} D_{\hat{k}}{}^{\hat{k}}, \tag{121}$$

in which the rate of change in volume of a small region is related to the amount of volume that enters through its boundaries due to the motion of coordinates (represented by β^i) plus the amount of volume generated within the region due to the presence of a field $D_{\hat{k}}{}^i$. A derivation of this equation can be found in Appendix C.

In the case of $B^{\hat{k}i}$, an independent evolution equation (120) may be needed in constraint-damping schemes (as opposed to constrained transport schemes), where the identity of $B^{\hat{k}i}$ as the curl of $A^{\hat{k}}{}_i$, and therefore the fulfillment of the first Bianchi identity, is not guaranteed and needs to be enforced. The gauge functions $\vec{G} = \{\alpha, \beta^i\}$ may belong to either of the sets $\vec{U}$ or $\vec{Q}$, depending on whether we enforce new differential equations for their evolution, or set them as algebraic functions of $\vec{U}$. Finally, the rest of quantities can be obtained from algebraic relations analogous to the constitutive equations in electrodynamics.

These constitutive relations can be obtained from Eqs. (95) and (110), which determine the relations between the connection coefficients in terms of the Nester-Witten form and the form $\boldsymbol{F}^{\hat{\alpha}}$ in the orthonormal frame.

$$D_{\hat{0}}{}^{\hat{i}} = -\epsilon^{\hat{i}\hat{j}\hat{k}} B_{\hat{j}\hat{k}}, \tag{122}$$

$$D_{\hat{k}}{}^{\hat{i}} = -\frac{1}{2}(E_{\hat{k}}{}^{\hat{i}} + E^{\hat{i}}{}_{\hat{k}}) + \delta_{\hat{k}}{}^{\hat{i}} E^{\hat{l}}{}_{\hat{l}}, \tag{123}$$

$$H_{\hat{0}\hat{i}} = \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}} E^{\hat{j}\hat{k}}, \tag{124}$$

$$H_{\hat{k}\hat{i}} = -B_{\hat{i}\hat{k}} + \frac{1}{2}\delta_{\hat{k}\hat{i}} B_{\hat{l}}{}^{\hat{l}} - \epsilon_{\hat{k}\hat{i}\hat{l}} E_{\hat{0}}{}^{\hat{l}}, \tag{125}$$

$$E^{\hat{0}}{}_{\hat{i}} = \frac{3}{2} D_{\hat{0}\hat{i}} - \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}} H^{\hat{j}\hat{k}}, \tag{126}$$

$$E^{\hat{j}}{}_{\hat{k}} = -D_{\hat{k}}{}^{\hat{j}} - \frac{1}{2}\delta^{\hat{j}}{}_{\hat{k}} D_{\hat{l}}{}^{\hat{l}} + \epsilon^{\hat{j}}{}_{\hat{k}\hat{l}} H_{\hat{0}}{}^{\hat{l}}, \tag{127}$$

$$B^{\hat{0}\hat{i}} = -\epsilon^{\hat{i}\hat{j}\hat{k}} D_{\hat{j}\hat{k}}, \tag{128}$$

$$B^{\hat{i}\hat{j}} = \delta^{\hat{i}\hat{j}} H_{\hat{l}}{}^{\hat{l}} - H^{\hat{i}\hat{j}} + \frac{1}{2}(H^{\hat{j}\hat{i}} - H^{\hat{i}\hat{j}} + \epsilon^{\hat{i}\hat{j}\hat{k}} D_{\hat{0}\hat{k}}). \tag{129}$$

We are interested in obtaining the unknown quantities ($E^{\hat{k}}{}_{\hat{i}}$ and $H_{\hat{\alpha}}{}^{\hat{i}}$) needed for evolution from the known evolved variables ($D_{\hat{\alpha}\hat{i}}$ and $B^{\hat{k}\hat{i}}$). We have already expressions for $B^{\hat{0}\hat{i}}$, and $E^{\hat{0}}{}_{\hat{i}}$, since they are determined by the gauge from Eq. (117). Therefore, the required relations are given by Eqs. (127) and (125).

The system (122)–(125) and (126)–(129) also gives constraints on some of the variables determined by evolution. In particular, Eqs. (122) and (123) imply that the $D_{\hat{i}\hat{j}}$ is symmetric, and that $D_{\hat{0}}{}^{\hat{k}}$ is related to the antisymmetric part of $B_{\hat{i}\hat{j}}$. This is a consequence of the symmetry of the Einstein equations, which allows us to express some of the quantities as linear combinations of the others. In principle this could help us reducing the number of necessary evolution equations, as one could evolve just $D_{\hat{i}}{}^{\hat{j}}$ for $\hat{i} \geq \hat{j}$, and obtain their derived quantities when they are needed. However, the variables involved in these constraints have different geometric meanings. For example, $D_{\hat{i}}{}^{\hat{j}}$ is jth component of the three-vector field $\boldsymbol{D}_{\hat{i}}$, while $D_{\hat{j}}{}^{\hat{i}}$ is the ith component of $\boldsymbol{D}_{\hat{j}}$, and they are orthogonal to different surfaces. This will become relevant when designing a staggered scheme that allows us to keep the constraints fulfilled to machine precision, and where $D_{\hat{i}}{}^{\hat{j}}$ and $D_{\hat{j}}{}^{\hat{i}}$ will have different spatial representations, so it may be convenient to evolve them separately. The case of $D_{\hat{0}}{}^{\hat{k}}$ is slightly different, since the propagation of constraint $\mathcal{C}_0$ is ensured by the exact fulfilment of $\mathcal{C}_i$, so it might be possible to drop completely its evolution as well as that of the gravitational energy $\rho_{\hat{0}}$. An approximate value of $D_{\hat{0}}{}^{\hat{k}}$ can then always be obtained from $B_{\hat{i}\hat{j}}$ and an approximate value of $\rho_{\hat{0}}$ from calculating the divergence of $D_{\hat{0}}{}^{\hat{k}}$ and taking the difference with the matter energy $P_{\hat{0}}$ according to Eq. (77). However, their evolution can still be useful to keep track of the transport of gravitational energy and to provide information on the differences between the components of $B_{\hat{i}\hat{j}}$, which might increase the accuracy of interpolations.

Finally, another interesting feature of the constitutive relations (122)–(129) is that they provide no means of calculating $H_{\hat{0}\hat{i}}$ from the evolved variables. Similarly as for the gauge variables α and β^i, this indicates that $H_{\hat{0}\hat{i}}$ represents an additional freedom of the formulation, and in fact, it can be related to the custom choice of rotating the tetrad bases between different hypersurfaces. To see this, let us consider a special case of a spacetime devoid of matter and gravitational energy-momentum, for which $D_{\hat{\alpha}}{}^{i} = 0$ is a solution to constraints (77). Choosing a gauge in which the shift is zero and the lapse is one (geodesic gauge), the evolution equations for the tetrad coefficients [Eq. (116)] read

$$\partial_t A^{\hat{i}}{}_i = -E^{\hat{i}}{}_i = -A^{\hat{j}}{}_i \epsilon^{\hat{i}}{}_{\hat{j}\hat{k}} H_{\hat{0}}{}^{\hat{k}}, \tag{130}$$

so that

$$A^{\hat{i}}{}_i(t+\delta t) \approx (\delta^{\hat{i}}{}_{\hat{j}} - \delta t \epsilon^{\hat{i}}{}_{\hat{j}\hat{k}} H_{\hat{0}}{}^{\hat{k}}) A^{\hat{j}}{}_i, \tag{131}$$

where δt represent an infinitesimal displacement along the time coordinate. This is an infinitesimal rotation of the spatial part of the tetrad basis about the angular velocity vector $\boldsymbol{H}_{\hat{0}}$.

We will now obtain explicit algebraic expressions in terms of the three-vector fields $\boldsymbol{H}_{\hat{\alpha}}$, $\boldsymbol{D}_{\hat{\alpha}}$, $\boldsymbol{E}^{\hat{\alpha}}$, $\boldsymbol{B}^{\hat{\alpha}}$ for the projections of the Sparling form $\rho_{\hat{\alpha}}$ and $s^{j}{}_{\hat{\alpha}}$, of which the latter are needed for evolution. Expressing Eq. (66) in component form in the orthonormal frame and using the definition in Eq. (67), we obtain

$$t^{\hat{\gamma}}{}_{\hat{\alpha}} = \frac{1}{2}(\omega^{\hat{\sigma}}{}_{\hat{\alpha}\hat{\mu}} \omega^{\hat{\rho}\hat{\xi}}{}_{\hat{\nu}} \delta^{\hat{\tau}}{}_{\hat{\sigma}} + \omega^{\hat{\rho}}{}_{\hat{\sigma}\hat{\mu}} \omega^{\hat{\sigma}\hat{\xi}}{}_{\hat{\nu}} \delta^{\hat{\tau}}{}_{\hat{\alpha}}) \delta^{\hat{\nu}\hat{\mu}\hat{\gamma}}_{\hat{\tau}\hat{\rho}\hat{\xi}}, \tag{132}$$

where we have made use of the generalized Kronecker delta to keep the notation compact.[5] Using the relations given by Eq. (110), for a connection that is antisymmetric with respect to its first two indices Eq. (132) can be rewritten as

$$t^{\hat{\gamma}}{}_{\hat{\alpha}} = F^{\hat{\delta}}{}_{\hat{\beta}\hat{\alpha}} \star u_{\hat{\delta}}{}^{\hat{\beta}\hat{\gamma}} - \frac{1}{4}\delta^{\hat{\gamma}}{}_{\hat{\alpha}} F^{\hat{\delta}}{}_{\hat{\beta}\hat{d}} \star u_{\hat{\delta}}{}^{\hat{\beta}\hat{d}}, \tag{133}$$

and taking the projections defined in Eq. (76), we obtain

$$\rho_{\hat{0}} = -\frac{1}{2}(E^{\hat{\alpha}\hat{k}} D_{\hat{\alpha}\hat{k}} + B^{\hat{\alpha}\hat{k}} H_{\hat{\alpha}\hat{k}}), \tag{134}$$

$$\rho_{\hat{i}} = -\epsilon_{\hat{i}\hat{j}\hat{k}} B^{\hat{\alpha}\hat{j}} D_{\hat{\alpha}}{}^{\hat{k}}, \tag{135}$$

$$s^{\hat{i}}{}_{\hat{0}} = -\epsilon^{\hat{i}\hat{j}\hat{k}} E^{\hat{\alpha}}{}_{\hat{j}} H_{\hat{\alpha}\hat{k}}, \tag{136}$$

[5]The generalized Kronecker delta $\delta^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_p}$ is defined so that it equals:

$$\begin{cases} +1 & \text{if } \nu_1\ldots\nu_p \text{are an even permutation of} \mu_1\ldots\mu_p \\ -1 & \text{if } \nu_1\ldots\nu_p \text{are an odd permutation of} \mu_1\ldots\mu_p \\ 0 & \text{otherwise} \end{cases}.$$

$$s^{\hat{i}}{}_{\hat{j}} = E^{\hat{\alpha}}{}_{\hat{j}} D_{\hat{\alpha}}{}^{\hat{i}} + B^{\hat{\alpha}\hat{i}} H_{\hat{\alpha}\hat{j}} - \frac{1}{2}\delta^{\hat{i}}{}_{\hat{j}}(E^{\hat{\alpha}\hat{k}} D_{\hat{\alpha}\hat{k}} + B^{\hat{\alpha}\hat{k}} H_{\hat{\alpha}\hat{k}}). \tag{137}$$

Although Eqs. (134) and (135) express algebraic constraints between variables that are evolved with their own differential equation, if the momentum densities are evolved using a finite volume scheme, these relations between the numerical representation of the variables should not be expected to hold strictly. The reason is that the representation of the momentum densities is that of a volume average, which does not need to coincide with the value of the right-hand side of the equations calculated at a given point (or with values interpolated from a set of given points). However, these expressions may still be useful to obtain additional information on these quantities, e.g., to improve interpolations. In this case the scheme would sacrifice the exact fulfilment of these expressions in favor of machine precision conservation of energy and momentum.

The last quantity for which we need to give an explicit expression is the "force" term given by Eq. (91). After substituting (67) in (91) and decomposing $\boldsymbol{F}_{\hat{\alpha}}$ as in Eq. (115), we obtain

$$Q_{\hat{0}} = E^{\hat{i}\hat{j}} S_{\hat{i}\hat{j}} - E^{\hat{0}\hat{j}} P_{\hat{j}}, \tag{138}$$

$$Q_{\hat{i}} = E^{\hat{k}}{}_{\hat{i}} S_{\hat{k}\hat{0}} + E^{\hat{0}}{}_{\hat{i}} P_{\hat{0}} + E_{\hat{i}\hat{j}\hat{k}} B^{\hat{l}\hat{j}} S^{\hat{k}}{}_{\hat{l}}. \tag{139}$$

In order to close the system completely, it is necessary to specify a set of relations between the nongravitational energy and momentum $P_{\hat{\alpha}}$ and their associated fluxes $S^k{}_{\hat{\alpha}}$, which will depend on the kind of nongravitational fields considered (e.g., ideal fluid, electromagnetic fields, or a scalar field).

VI. THE DGREM FORMULATION

Finally, we can summarize here the equations obtained in the previous section in order to describe the system completely. For each equation we indicate its common name (or that of the equations more closely related to it) and the number that labels it in the part of the text where it is discussed.

A. Evolution equations

First Cartan structure equations:

$$\partial_t A^{\hat{i}}{}_i - \partial_i \beta^{\hat{i}} = -\alpha E^{\hat{i}}{}_i - \epsilon_{ilk} \beta^l B^{\hat{i}k}. \tag{116}$$

First Bianchi identities:

$$\partial_t \sqrt{\gamma} B^{\hat{\alpha}k} + \partial_i \sqrt{\gamma}(\alpha \epsilon^{ijk} E^{\hat{\alpha}}{}_j - \beta^i B^{\hat{\alpha}k} + \beta^k B^{\hat{\alpha}i}) = 0. \tag{120}$$

Einstein evolution equations:

$$\partial_t \sqrt{\gamma} D_{\hat{\alpha}}{}^k - \partial_i \sqrt{\gamma}(\alpha \epsilon^{kij} H_{\hat{\alpha}j} + \beta^i D_{\hat{\alpha}}{}^k - \beta^k D_{\hat{\alpha}}{}^i) = -\sqrt{\gamma}(j^k{}_a + \kappa J^k{}_a). \tag{78}$$

Conservation of gravitational energy-momentum:

$$\partial_t \sqrt{\gamma} \rho_{\hat{\alpha}} + \partial_i \sqrt{\gamma} j^i{}_{\hat{\alpha}} = -\kappa \sqrt{\gamma} Q_{\hat{\alpha}}. \tag{89}$$

Conservation of "matter" energy-momentum:

$$\partial_t \sqrt{\gamma} P_{\hat{\alpha}} + \partial_i \sqrt{\gamma} J^i{}_{\hat{\alpha}} = \sqrt{\gamma} Q_{\hat{\alpha}}. \tag{90}$$

Auxiliary evolution equation for $\sqrt{\gamma}$:

$$\partial_t \sqrt{\gamma} - \partial_i \sqrt{\gamma} \beta^i = \frac{5}{2} \alpha \sqrt{\gamma} D_{\hat{k}}{}^{\hat{k}}. \tag{121}$$

B. Differential constraints

First Cartan structure equations:

$$E^{\hat{0}}{}_i = \partial_i \ln \alpha, \tag{117a}$$

$$B^{\hat{0}i} = 0, \tag{117b}$$

$$B^{\hat{i}k} = \epsilon^{ijk} \partial_i A^{\hat{i}}_j. \tag{117c}$$

First Bianchi identities:

$$\partial_i \sqrt{\gamma} B^{\hat{\alpha}i} = 0. \tag{119}$$

Hamiltonian and momentum constraints:

$$\partial_i \sqrt{\gamma} D_{\hat{\alpha}}{}^i = \sqrt{\gamma}(\rho_{\hat{\alpha}} + \kappa P_{\hat{\alpha}}). \tag{77}$$

C. Constitutive relations

$$H_{\hat{k}\hat{i}} = -B_{\hat{i}\hat{k}} + \frac{1}{2}\delta_{\hat{k}\hat{i}} B_{\hat{l}}{}^{\hat{l}} - \epsilon_{\hat{k}\hat{i}\hat{l}} E_{\hat{0}}{}^{\hat{l}} \tag{125}$$

$$E^{\hat{j}}{}_{\hat{k}} = -D_{\hat{k}}{}^{\hat{j}} - \frac{1}{2}\delta^{\hat{j}}{}_{\hat{k}} D_{\hat{l}}{}^{\hat{l}} + \epsilon^{\hat{j}}{}_{\hat{k}\hat{l}} H_{\hat{0}}{}^{\hat{l}} \tag{127}$$

Gravitational energy-momentum current:

$$j^k{}_{\hat{\alpha}} = \alpha s^k{}_{\hat{\alpha}} - \beta^k \rho_{\hat{\alpha}}, \tag{79}$$

$$s^{\hat{i}}{}_{\hat{0}} = -\epsilon^{\hat{i}\hat{j}\hat{k}} E^{\hat{\alpha}}{}_{\hat{j}} H_{\hat{\alpha}\hat{k}}, \tag{136}$$

$$s^{\hat{i}}{}_{\hat{j}} = E^{\hat{\alpha}}{}_{\hat{j}} D_{\hat{\alpha}}{}^{\hat{i}} + B^{\hat{\alpha}\hat{i}} H_{\hat{\alpha}\hat{j}} - \frac{1}{2}\delta^{\hat{i}}{}_{\hat{j}}(E^{\hat{\alpha}\hat{k}} D_{\hat{\alpha}\hat{k}} + B^{\hat{\alpha}\hat{k}} H_{\hat{\alpha}\hat{k}}). \tag{137}$$

"Matter" energy-momentum current:

$$J^k{}_{\hat{\alpha}} = \alpha S^k{}_{\hat{\alpha}} - \beta^k P_{\hat{\alpha}}. \tag{80}$$

"Gravitational force":

$$Q_{\hat{0}} = E^{\hat{i}\hat{j}} S_{\hat{i}\hat{j}} - E^{\hat{0}\hat{j}} P_{\hat{j}}, \tag{138}$$

$$Q_{\hat{i}} = E^{\hat{k}}{}_{\hat{i}} S_{\hat{k}\hat{0}} + E^{\hat{0}}{}_{\hat{i}} P_{\hat{0}} + \epsilon_{\hat{i}\hat{j}\hat{k}} B^{\hat{l}\hat{j}} S^{\hat{k}}{}_{\hat{l}}. \tag{139}$$

D. Algebraic constraints

$$D_{\hat{0}}{}^{\hat{i}} = -\epsilon^{\hat{i}\hat{j}\hat{k}} B_{\hat{j}\hat{k}}, \tag{123}$$

$$D_{\hat{i}\hat{j}} = D_{\hat{j}\hat{i}}, \tag{128}$$

$$\rho_{\hat{0}} = -\frac{1}{2}(E^{\hat{\alpha}\hat{k}} D_{\hat{\alpha}\hat{k}} + B^{\hat{\alpha}\hat{k}} H_{\hat{\alpha}\hat{k}}), \tag{134}$$

$$\rho_{\hat{i}} = -\epsilon_{\hat{i}\hat{j}\hat{k}} B^{\hat{\alpha}\hat{j}} D_{\hat{\alpha}}{}^{\hat{k}}. \tag{135}$$

E. Free quantities

The fields α, β^i, and $H_{\hat{0}}{}^k$ are not determined by any equation and can be chosen arbitrarily. The matter energy-momentum fluxes $S^k{}_{\hat{\alpha}}$ are not determined by any of the equations here but depend on the specific properties of the matter fields.

F. Properties of the formulation

The final system of equations is in a form that closely resembles those of electromagnetism in the $3+1$ decomposition, with the difference that the gravitational field is represented not by one, but by four "electromagneticlike" fields $(E^{\hat{\alpha}}{}_i, B^{\hat{\alpha}i})$, and that due to the particular choice of the observers frame the field corresponding to $\hat{\alpha} = 0$ is purely "electric."

Being more explicit in this analogy, the gauge variables α and $\beta^{\hat{i}}$, or more specifically the quantities $-\beta^{\hat{i}}$ and $\ln \alpha$, play a role analogous to that of the scalar potential in electromagnetism; while the components of the spatial part of the tetrad play the role of the vector potential, as can be seen from Eqs. (116) and (117).

The first Bianchi identities take a form analogous to that of the Faraday equation (120) and the Gauss law for magnetism (119), while the Einstein equations take that of the Ampère-Maxwell equation (78) and the Gauss law for electricity (77), with the sum of matter and gravitational energy-momentum playing the role of the electric current, which satisfies an exact conservation law [see Eqs. (89), (90), and (92)].

Although not of immediate use for a numerical implementation, it is interesting to notice other similarities of the equations with those of electromagnetism. For instance, the expressions for the gravitational energy-momentum density and fluxes are analogous to those given by Minkowski's energy-momentum tensor for the electromagnetic field in material media [32], and contain an expression related to the transport of gravitational energy (136) that is analogous to the Poynting vector in electrodynamics. The force terms that describe the exchange between matter and the gravitational field in Eqs. (89) and (90) have a form similar to that of the work done by the electric field on a system of charges (138) and to the Lorentz force (139).

However, there are also important differences with respect to Maxwell's equations. The most noticeable one is that the inhomogeneous equations contain source terms quadratic in the fields, which represent the fact that the gravitational energy-momentum current $j^\mu{}_{\hat{\alpha}}$ is itself a source for the gravitational field $D_{\hat{\alpha}}{}^\mu$. Another important difference is that the presence of the square root of the metric determinant $\sqrt{\gamma} = \det(A^{\hat{i}}_k)$ eliminates the gauge freedom that in electrodynamics allows one to replace $A'_\mu \to A_\mu + \partial_\mu \psi$, where ψ is a scalar function and A_μ the vector potential. This prevents us from choosing to solve the "Faraday equation" (120) in place of the evolution equation for the vector potential (116) and forces us to solve the latter in order to know the transformation coefficients from the "laboratory frame" to the tetrad frame where the constitutive relations (125) and (127) are valid.

Although the gauge freedom of electrodynamics does not exist for this system, it posses other gauge freedoms. These come in through the quantities for which neither the Cartan structure equations nor the Einstein equations provide an evolution equation, namely the components of the vector normal to the hypersurface $n^\mu = (1/\alpha, -\beta^i/\alpha)$ and the "magnetic field" $H_{\hat{0}\mu}$. While the freedom in choosing n^μ represents the freedom to foliate the spacetime in different sets of 3D hypersurfaces and to perform spatial translations of the lines of constant spatial coordinates, the freedom to choose $H_{\hat{0}\mu}$ represents the liberty to perform rotations of the spatial part of the tetrad from one slice to the other (see Sec. V D). Although in contrast to electromagnetism these gauge freedoms do not leave unchanged the vector fields $E^{\hat{\alpha}}_i, B^{\hat{\alpha}i}, D^i_{\hat{\alpha}}, H_{\hat{\alpha}i}$, the Einstein tensor at a given point, given by Eq. (71) will be the same object regardless of the foliation and the orientation of the basis vectors. Going beyond GR to include torsion, the system does contain an additional freedom that leaves the fields unchanged.[6] It is conceivable that, similarly to the gauge variables α and β^i, the vector $\boldsymbol{H}_{\hat{0}}$ could play an important

[6]This freedom comes from regarding the field strength $\boldsymbol{F}^{\hat{\alpha}}$ as the sum of the torsion $\boldsymbol{\Xi}^{\hat{\alpha}}$ and the product $\boldsymbol{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} \wedge \boldsymbol{\theta}^{\hat{\beta}}$ [cf. Eqs. (60) and (107)]. For a set of 2-forms $\boldsymbol{S}_{\hat{\alpha}}(\boldsymbol{F}^{\hat{\alpha}})$ which has the same functional dependence on $\boldsymbol{F}^{\hat{\alpha}}$ as that of $\star\boldsymbol{u}_{\hat{\alpha}}(\boldsymbol{F}^{\hat{\alpha}})$ in GR, the Lagrangian

$$L[A^{\hat{\alpha}}_\mu, \partial_\lambda A^{\hat{\alpha}}_\mu] = \frac{\sqrt{-g}}{4\kappa} F^\alpha_{\hat{\mu}\hat{\nu}} S^{\hat{\mu}\hat{\nu}}_\alpha$$

will lead to equations of motion identical to those presented here regardless of the amount of torsion contained in $\boldsymbol{F}^{\hat{\alpha}}$. For $\boldsymbol{\Xi}^{\hat{\alpha}} = 0$, this Lagrangian is equivalent to the Einstein-Hilbert Lagrangian up to a boundary term, and for the extreme case $\boldsymbol{F}^{\hat{\alpha}} = \boldsymbol{\Xi}^{\hat{\alpha}}$ it corresponds to that of the teleparallel equivalent of GR, with $\boldsymbol{S}_{\hat{\alpha}}$ identified as the superpotential (cf. Appendix C of [63]).

role in the numerical stability of the system, and more studies on a proper way to handle this additional freedom are required.

Related to its similarity to the Maxwell equations, the DGREM system also posses the important properties of being first order in spatial and temporal derivatives, and being expressible as a system of flux-balanced laws. As mentioned in the Introduction, such properties make possible the use of the huge amount of technology developed to simulate such systems.

Finally, being formulated as a system of equations in differential forms and exterior derivatives, it is possible to retrieve a natural constraint-preserving discretization, which would also make redundant some of the evolution equations, reducing the number of variables needed for evolution. An example of such discretization with a reduced number of variables will be presented in the next section.

G. A geometric interpretation

One of the advantages of using a constrained transport scheme is that many of the equations in the system described in Sec. VI become redundant when using the proper discretization. The reason is that if a consistent discretization is adopted for all the equations, those that are exterior derivatives of others are automatically fulfilled. In particular, the scheme described here requires only the evolution of Eqs. (116) and (78) to satisfy all equations in the system summarized in Sec. VI. The equations presented in this section are only those related to the evolution of spacetime, while the matter sector is assumed to be evolved with an unspecified scheme that is conservative for energy-momentum.

Similarly as done for the wave equation in Sec. III, we will obtain a constraint-preserving discretization on the hypersurface Σ_t by first applying Cartan's "magic" formula (A14), followed by integrating the differential forms on their respective submanifolds and applying Stoke's theorem (A30).

The first step of the procedure yields the equations

$$\mathcal{L}_{e_t}\boldsymbol{\theta}^{\hat{\alpha}} - \boldsymbol{d}(\boldsymbol{e}_t \cdot \boldsymbol{\theta}^{\hat{\alpha}}) = \boldsymbol{e}_t \cdot \boldsymbol{F}^{\hat{\alpha}}, \tag{140}$$

$$\mathcal{L}_{e_t}\boldsymbol{u}_{\hat{\alpha}} - \boldsymbol{d}(\boldsymbol{e}_t \cdot \boldsymbol{u}_{\hat{\alpha}}) = \boldsymbol{e}_t \cdot (\boldsymbol{t}_{\hat{\alpha}} + \kappa \boldsymbol{T}_{\hat{\alpha}}), \tag{141}$$

which can also be written as

$$\partial_t \mathcal{A}^{\hat{i}} - \boldsymbol{d}\beta^{\hat{i}} = -\mathcal{E}^{\hat{i}}, \tag{142}$$

$$\partial_t \mathcal{D}_{\hat{\alpha}} - \boldsymbol{d}\mathcal{H}_{\hat{\alpha}} = -\mathcal{J}_{\hat{\alpha}}, \tag{143}$$

where

$$\mathcal{A}^{\hat{i}} = A^{\hat{i}}{}_i \boldsymbol{d}x^i, \tag{144}$$

$$\mathcal{E}^{\hat{i}} = (\alpha E^{\hat{i}}{}_i + \varepsilon_{ilk}\beta^l B^{\hat{i}k})\boldsymbol{d}x^i, \tag{145}$$

$$\mathcal{H}_{\hat{\alpha}} = (\alpha H_{\hat{\alpha}i} - \varepsilon_{ilk}\beta^l D_{\hat{\alpha}}{}^k)\boldsymbol{d}x^i, \tag{146}$$

and

$$\mathcal{D}_{\hat{\alpha}} = \varepsilon_{ijk} D_{\hat{\alpha}}{}^i (\boldsymbol{d}x^j \wedge \boldsymbol{d}x^k), \tag{147}$$

$$\mathcal{J}_{\hat{\alpha}} = \varepsilon_{ijk} (j^i{}_{\hat{\alpha}} + \kappa J^i{}_{\hat{\alpha}})(\boldsymbol{d}x^j \wedge \boldsymbol{d}x^k), \tag{148}$$

and where the system is closed by the constitutive relations and by adopting a consistent discretization for the forms

$$\mathcal{B}^{\hat{\alpha}} = \varepsilon_{ijk} B^{\hat{\alpha}i} (\boldsymbol{d}x^j \wedge \boldsymbol{d}x^k), \tag{149}$$

in order to obtain $B^{\hat{\alpha}i}$ from Eq. (117).

Each term in Eq. (142) [(143)] is a 1-form (a 2-form in), and thus an integrand over a 1D (2D) submanifold. We then choose to integrate them over zone edges and zone faces, respectively. After applying Stokes's theorem and replacing exterior derivatives with evaluations of forms at zone vertices and zone edges, the resulting discretization is as shown in Fig. 1, and in principle could be able to preserve to machine accuracy simultaneously the Bianchi identities (119), the Einstein constraints (77), as well as the global conservation of the sum of gravitational plus matter energy-momentum (92), provided that they are satisfied in the initial data, by the mechanism described in Sec. III.

For completeness, we can also express the equations of the system in the language of three-dimensional vector calculus, in a way completely analogous as it was done for electromagnetism in Sec. III B. These are the first Cartan structure equations,

$$\vec{\mathcal{E}}^{\hat{i}} = -\overrightarrow{\partial_t \mathcal{A}}^{\hat{i}} + \nabla(\boldsymbol{\beta} \cdot \vec{\mathcal{A}}^{\hat{i}}), \tag{150}$$

$$\vec{\mathcal{E}}^{\hat{0}} = \nabla\alpha, \tag{151}$$

$$\boldsymbol{B}^{\hat{i}} = \nabla \times \vec{\mathcal{A}}^{\hat{i}}, \tag{152}$$

the first Bianchi identities,

$$\partial_t \sqrt{\gamma}\boldsymbol{B}^{\hat{i}} + \sqrt{\gamma}\nabla \times \vec{\mathcal{E}}^{\hat{i}} = 0, \tag{153}$$

$$\nabla \cdot \boldsymbol{B}^{\hat{i}} = 0, \tag{154}$$

the Sparling equation (i.e., Einstein equations),

$$\partial_t \sqrt{\gamma}\boldsymbol{D}_{\hat{\alpha}} - \sqrt{\gamma}\nabla \times \vec{\mathcal{H}}_{\hat{\alpha}} = -\sqrt{\gamma}\vec{\mathcal{J}}_{\hat{\alpha}}, \tag{155}$$

$$\nabla \cdot \boldsymbol{D}_{\hat{\alpha}} = \rho_{\hat{\alpha}} + \kappa P_{\hat{\alpha}}, \tag{156}$$

and the conservation of energy momentum,

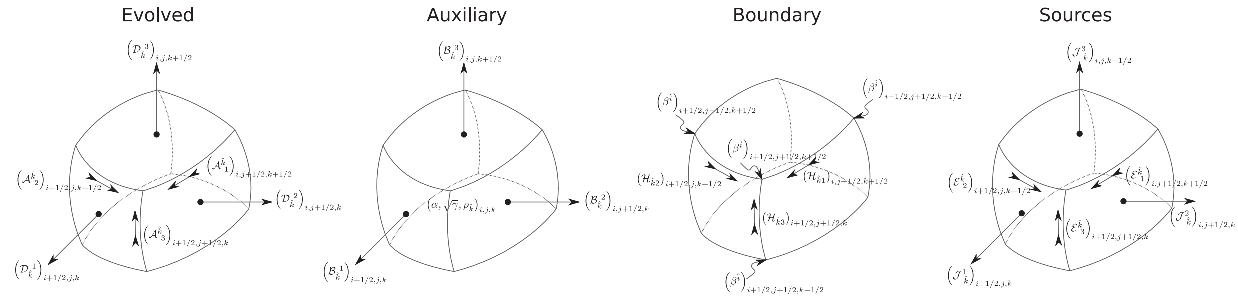

FIG. 1. Collocation of variables for a constraint-preserving discretization. These are classified in four categories: "evolved" variables are those obtained by integrating the evolution equations of the scheme, "boundary" variables are those localized at the boundaries of the regions where the evolved variables are defined, and "source" variables are those sharing the same spatial location as the evolved variables. Finally, "auxiliary" variables are those that can be obtained from the evolved variables, but are neither sharing their spatial location nor that of their boundaries.

$$\partial_t \sqrt{\gamma}(\rho_{\hat{\alpha}} + \kappa P_{\hat{\alpha}}) + \sqrt{\gamma}\nabla \cdot \vec{\mathcal{J}}_{\hat{\alpha}} = 0, \tag{157}$$

where similarly as in Sec. III B, $\mathbf{D}_{\hat{\alpha}} = D_{\hat{\alpha}}{}^i \boldsymbol{e}_i$, $\mathbf{B}^{\hat{j}} = B^{\hat{j}i}\mathbf{e}_i$, and $\boldsymbol{\beta} = \beta^i \mathbf{e}_i$.

H. Gravity and electrodynamics as gauge theories

Although when deriving the equations of the DGREM formulation in Sec. V we have discussed the similarities with electromagnetism from a purely operational point of view, it may be useful to reinterpret these equations in the language of gauge theory to gain more insight into the origin of these similarities.

Yang-Mills theory, which can be considered as a generalization of electromagnetism, is the prototypical example of a gauge theory. Its basic ingredients are a Lie algebra-valued one-form, known as the gauge potential, a covariant derivative, and a field strength. The gauge potential can be written as

$$\boldsymbol{A} \coloneqq A_\alpha \boldsymbol{d}x^\alpha = A^{\hat{\alpha}}{}_\alpha \lambda_{\hat{\alpha}} \boldsymbol{d}x^\alpha, \tag{158}$$

where $\lambda_{\hat{\alpha}}$ are the generators of the Lie algebra (indexed by hatted symbols) and $\boldsymbol{d}x^\alpha$ are the spacetime basis 1-forms (for simplicity, in a coordinate frame). The (gauge) covariant derivative is related to parallel transport in the abstract space of the Lie group, and takes the form

$$D_\alpha = \partial_\alpha + A_\alpha, \tag{159}$$

from which it can be seen that the gauge potential also plays a role analogous to the connection in general relativity, and in fact is often also called a "connection" [64]. The field strength quantifies the failure of covariant derivatives to commute, and takes the form

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta], \tag{160}$$

where the brackets denote the commutator of the Lie algebra. Because of its role analogous to that of the Riemann tensor in general relativity, the field strength is often also known as the "curvature" [64]. A field constructed in this way satisfies a set of Bianchi identities, which state that certain combinations of its derivatives vanish. The final ingredient of the theory is an action quadratic in the field,

$$S_{\mathrm{YM}}[A_\mu, \partial_\mu A_\nu] = \frac{1}{4}\int \mathrm{d}x^4 \mathrm{tr}(F_{\alpha\beta}F^{\alpha\beta}), \tag{161}$$

from which the remaining equations of motion can be derived.

Electromagnetism is the Yang-Mills theory built on the Lie algebra of the $U(1)$ group. Since this group is Abelian, the commutator in (160) vanishes and one can see a direct correspondence between the equations in this section and those in Sec. III B. The Bianchi identities and the equations resulting from the action (161) are, respectively, the homogeneous and inhomogeneous Maxwell equations (with no sources, unless coupling with charge is included in the action). Applying the Noether machinery to this action yields the conservation of electric charge.

However, the analogy between the structure of gravity and that of Yang-Mills theory is less direct. The approach most commonly adopted in textbooks (see, e.g., [64,65]) is to reinterpret the second Cartan structure equation (53) as giving a field strength from a gauge potential,

$$\mathcal{F}^{\hat{\alpha}\hat{\beta}}{}_{\mu\nu} \coloneqq R^{\hat{\alpha}\hat{\beta}}{}_{\mu\nu} = \partial_\mu \omega^{\hat{\alpha}\hat{\beta}}{}_\nu - \partial_\nu \omega^{\hat{\alpha}\hat{\beta}}{}_\mu + [\omega^{\hat{\alpha}\hat{\beta}}{}_\mu, \omega^{\hat{\alpha}\hat{\beta}}{}_\nu], \tag{162}$$

so that the true connection and the Riemann tensor also acquire the roles of "connection" and "curvature" in the Yang-Mills sense. Exploiting this analogy, it is always possible to introduce generalized Yang-Mills electric, $\mathcal{E}$, and magnetic, $\mathcal{B}$, fields via

$$\mathcal{E}^{\hat{\alpha}\hat{\beta}}{}_\mu = n^\nu \mathcal{F}^{\hat{\alpha}\hat{\beta}}{}_{\mu\nu}, \tag{163}$$

$$\mathcal{B}^{\hat{\alpha}\hat{\beta}}{}_\mu = n^{\nu *} \mathcal{F}^{\hat{\alpha}\hat{\beta}}{}_{\mu\nu}. \tag{164}$$

A definition of “electromagneticlike” fields either as projections or components of the Riemann tensor is present in several of the formulations that are reviewed in Sec. VI I, see also Refs. [66,67] for interpretations of these quantities. However, when expressed in these variables, the Einstein equations do not acquire a structure similar to that of the inhomogeneous Maxwell equations. In fact, they do not contain derivatives of the field strength, and to obtain evolution equations for them, one needs to go one order higher in differentiation. This can be traced back to the form of the Einstein-Hilbert action,

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \int \mathrm{d}x^4 \sqrt{-g} R, \tag{165}$$

which is only linear in the components of the Riemann tensor. Strictly speaking, an action closer to that of expression (161) is that of Palatini,

$$S_{\mathrm{P}}[A^{\hat{\mu}}{}_{\mu}, \omega^{\hat{\mu}\hat{\nu}}{}_{\lambda}] = \int \mathrm{d}x^4 A_{\hat{\alpha}}{}^{\mu} A_{\hat{\beta}}{}^{\nu} \mathcal{F}^{\hat{\alpha}\hat{\beta}}{}_{\mu\nu}, \tag{166}$$

which is equivalent to (165) in the tetrad formulation, but makes explicit its dependence on the gauge potential adopted, i.e., the connection [64,68].

However, the connection $\boldsymbol{\omega}^{ab}$ is not the only possible choice for a gauge potential for gravity, and different potentials result in different field strengths and conserved quantities [69]. If the basis 1-forms are adopted as the gauge potential, then the first Cartan structure equation (52) can be reinterpreted to give the torsion as the field strength,

$$T^{\hat{\alpha}}{}_{\mu\nu} = \partial_{\mu} A^{\hat{\alpha}}{}_{\nu} - \partial_{\nu} A^{\hat{\alpha}}{}_{\mu} + c^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}} A^{\hat{\mu}}{}_{\mu} A^{\hat{\nu}}{}_{\nu}, \tag{167}$$

where $c^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}}$ are the commutation coefficients defined in Eq. (109). In general relativity, however, the torsion vanishes. Therefore, with the only purpose of obtaining an equation with the same form as for the electromagnetic case, we identify instead the field strength with the last term of Eq. (167) [cf. Eq. (109)], and obtain (108).

A general torsion-free connection can be written as a linear combination of the gradients of the metric and the commutation coefficients:

$$\omega_{abc} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc} + c_{bac} + c_{cab} - c_{abc}). \tag{168}$$

On the other hand, also for a general connection, the following action is equivalent to the Einstein-Hilbert one up to a boundary term [53]:

$$S = \int \left(\frac{1}{2\kappa} \boldsymbol{\omega}^{a}{}_{e} \wedge \boldsymbol{\omega}^{eb} \wedge \boldsymbol{\Sigma}_{ab} \right). \tag{169}$$

For the special case of the spin connection, the metric derivatives in (168) vanish, so that it becomes a linear combination of the commutation coefficients only and, therefore, of the components of our field strength $\boldsymbol{F}^{\hat{\alpha}}$. As a result, the action is now quadratic in the field strength. The fact that both the field strength and the Nester-Witten form are completely determined by the spin connection allows us to obtain the “constitutive equations” derived in Sec. V D. Using them, for the special case of the spin connection, it is possible to rewrite the action (169) as

$$S[A^{\hat{\alpha}}{}_{\mu}, \partial_{\lambda} A^{\hat{\alpha}}{}_{\mu}] = \int \mathrm{d}x^4 \left(\frac{\sqrt{-g}}{4\kappa} F^{\hat{\alpha}}{}_{\mu\nu} \star u_{\hat{\alpha}}{}^{\mu\nu} \right). \tag{170}$$

This action is closer in form to that of Eq. (161), which results in a form of the Einstein equations closer to that of Maxwell equations when written in these variables. Interestingly, the canonical momentum associated to the one-form basis for this action is precisely the Nester-Witten form. A schematic comparison of the structure of general relativity in this formulation and that of electromagnetism is shown in Fig. 2.

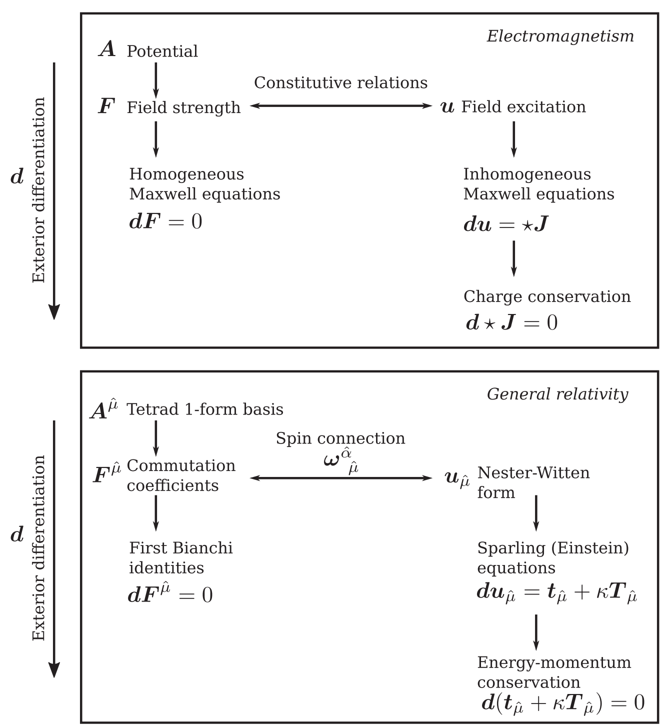

FIG. 2. Comparison of the structure of electromagnetism (top panel) and general relativity (bottom panel) in the formulation employed in this work.

I. Relation to other formulations

Here we briefly describe some of the points in common and differences between the DGREM formulation and those present in the literature which, to the best of our knowledge, are most closely related to it. Before going into more details, we will summarize the most important properties that characterize DGREM. DGREM is a $3+1$ formulation of the Einstein equations specifically targeted towards numerical relativity simulations. To this end, it has been designed to have a structure very close in form to Maxwell equations. Most importantly, the resulting equations are flux conservative and admit both constraint-preserving discretizations (in the vacuum case), as well as constraint damping approaches to enforce the first Bianchi identities and the Einstein constraints. It adopts an orthonormal tetrad formulation equipped with a torsion-free spin connection. More specifically, the fundamental variables are the spatial coefficients of an orthonormal tetrad field and a subset of the projections of the Nester-Witten form. These are evolved, respectively, using the evolution parts of the first Cartan structure equation (52) and the Sparling equation (68), which is equivalent to the Einstein field equations.

Although not a formulation of general relativity by itself and not developed for numerical spacetime evolution, gravitoelectromagnetism (GEM) also provides a way of casting general relativity in a form that resembles Maxwell equations (see [70] for a review). GEM is extremely useful to study effects such as the spin-gravity coupling, in which almost direct analogies exist between gravity and electromagnetism. The main difference between our formulation and GEM is that the latter constitutes an approximation valid in the linear perturbation regime, or in special frames on arbitrary curved spacetimes for which the spatial curvature can be ignored. This results in a single electromagneticlike field built from some of the projections of the Riemann tensor, as opposed to the four electromagneticlike fields that appear in our formulation, each of them with a basis 1-form acting as the four-vector potential.

When coming to $3+1$ formulations designed for numerical relativity, the main differences between DGREM and most of them is that it does not include the spatial metric and the extrinsic curvature among the fundamental variables evolved. The Einstein-Bianchi formulation described in [71] expands the set of evolution equations and constraints with relations derived from the Bianchi identities in order to obtain a hyperbolic system. This introduces additional electromagneticlike evolved variables labeled $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$. These fields are, however, defined as subsets of the components of the Riemann tensor. In contrast, the DGREM fields are projections of two different objects and their duals, namely, $\mathbf{E}^{\hat{a}}$ and $\mathbf{B}^{\hat{a}}$ are obtained from the field strength (the tetrad commutation coefficients), while $\mathbf{D}_{\hat{a}}$ and $\mathbf{H}_{\hat{a}}$ come from the Nester-Witten form. As a result, the electromagneticlike variables in DGREM are "one order lower" in derivatives than those in [71], since they can be obtained as linear combinations of the connections coefficients, while the Riemann tensor already involves derivatives of the connection. This different definition of the electromagneticlike fields results in evolution equations for them that, although reminiscent of Maxwell's equations, do not mirror them directly as it happens for DGREM. An alternative definition of electromagneticlike fields E_{ij} and B_{ij}, in this case as projections of the Weyl tensor, appears in the elegant formulation of [72], which presents hyperbolic reductions of the Einstein equations in the orthonormal frame and in the ADM representation.

The work of [73] presents a formulation closely related to that of [71], and highlights the similarities in the procedure to obtain a flux-conservative, symmetric hyperbolic system for a Yang-Mills field and for general relativity, although it differs slightly from the approach presented here. While in [73] the role of the Yang-Mills gauge potential and field strength is associated, respectively, with that of the metric and the extrinsic curvature, in our formulation these are associated, also respectively, with the basis 1-forms and the commutation coefficients.

Other formulations closely related to DGREM are the symmetric hyperbolic ones by [74,75], written in terms of Ashtekar variables. Similarly to the present work, they do not use directly the spatial metric as an evolution variable, but use instead some form of "square root" of it, being it the soldering forms in [74] or the complex components of the spatial orthonormal basis in [75]. Another similarity is that the self-dual connection used to define the Ashtekar variables (also evolved and used to close the system), is closely related to the spin connection employed in this work. A crucial difference is, however, that the use of complex variables requires imposing reality conditions, which are not needed in our formulation.

However, the formulation that is probably closest to ours is that described in [76]. In that work, a system that evolves the coefficients of the spin connection is shown to be symmetric hyperbolic. The formulation allows us to freely specify the lapse and the shift, and being all quantities real, it does not require imposing reality conditions. A difference with respect to our formulation is that in such a formulation the evolution is carried out completely on the tetrad frame, and there is no need to evolve the tetrad transformation coefficients. Although this gives a very simple and elegant structure to their system based on directional derivatives, it becomes problematic when expressing the equations in a form suitable for integration over a finite region, which is required to express the equations in flux-conservative form and to obtain constraint-preserving discretizations as those presented here.

Lacking still a proof of the hyperbolicity of our system, the fact that it is possible to build a symmetric hyperbolic system for the coefficients of the spin connection is very promising. In fact the components of the field strength and the Nester-Witten form are linear combinations of them when expressed in the tetrad frame. The closer similarity of our system to Maxwell's equations is also encouraging, even though the relation between the constitutive equations and hyperbolicity for the latter is still a matter of study [51].

VII. CONCLUSIONS

By expressing the equations that govern spacetime dynamics in general relativity in the language of exterior calculus and projecting them onto three-dimensional space-like hypersurfaces, we have obtained a new $3+1$ formulation of the field equations of general relativity. This new formulation, which we name DGREM, shows a surprising resemblance to the equations of relativistic MHD and to electromagnetism in material media. The system, summarized in Sec. VI, consists of a set of first-order evolution equations, in conservative form, and a set of algebraic, divergence, and curl constraints, closed by a set of constitutive relations.

The similarities with $3+1$ electrodynamics make explicit some important features of general relativity, such as the global conservation of total energy-momentum currents (in analogy to that of electric current), the fact that both the gravitational and matter energy momentum act as sources of the gravitational field, as well as the energy-momentum exchange between the gravitational and matter sectors.

Additionally, the DGREM formulation exhibits several interesting properties from the point of view of numerical implementations. Being first order and flux conservative, it is suitable for the application of high-resolution shock-capturing schemes such as finite-volume and finite-element methods. In particular the formulation contains a global conservation equation for the sum of gravitational and "matter" energy-momentum in which source terms have been eliminated, and which opens the possibility of applying techniques such as first-order flux limiting to ensure positivity of energy-momentum densities.

As shown in Sec. VI G, the expression of the formulation as a set of equations in differential forms permits to integrate them over mesh zones and use Stoke's theorem to obtain a natural staggered discretization potentially suitable for machine-precision constraint-preserving schemes. One such scheme could potentially reduce the number of evolution variables to a minimum of 21, both by not requiring extra variables to clean the constraints and by making redundant some of the equations.

Although a staggered scheme would enforce at machine precision both the fulfilment of the Einstein constraints and the conservation of energy-momentum, these advantages may be limited in practice for general relativistic hydrodynamic simulations due to the adoption of a floor model as it is customarily done to handle vacuum regions.

However, these techniques could in principle also be exploited in fully general relativistic N-body simulations, which could recycle the infrastructure developed for PIC simulations of collisionless plasmas, in which both staggered schemes and divergence cleaning techniques have been successfully applied.

In the same way, it is conceivable that resemblance of the form taken by the constraints of this formulation to Gauss's laws in electromagnetism could present advantages for the computation of initial data by recycling techniques used to solve the Poisson equation.

Finally, another benefit of deriving the system as a set of equations in terms of differential forms and exterior derivatives is that they naturally give relations between quantities evolved inside mesh cells and quantities evaluated at cell boundaries, regardless of the shape of the cells. This makes them particularly suitable for simulations using non-Cartesian coordinates and unstructured meshes.

Finally, the matter sector of the Einstein field equations (including relativistic dissipative fluid dynamics) can be also formulated in the language of differential forms and exterior calculus [77,78], and thus can be relatively easily incorporated in the constrained transport computational scheme discussed in Sec. VI G.

Together with the promising properties summarized above, there are still some questions regarding DGREM that need to be answered for a successful numerical implementation. The most important one is perhaps on its hyperbolicity, and how it could depend on gauge choices and on the new degrees of freedom given by spatial rotations of the tetrads between different hypersurfaces.

Other particulars of an actual numerical implementation are still under development, and will be part of a future work.

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APPENDIX A: A SMALL PRIMER ON DIFFERENTIAL FORMS AND EXTERIOR CALCULUS

We collect here some fundamental results about differential forms and exterior calculus, necessary to follow the derivations in this work. The modern theory of differential forms and exterior calculus stems from the work of Élie Cartan in the first half of the 20th century, and the literature regarding this field is by now very extensive. For further reading we refer the reader to [43,79,80] and references therein, which are the sources this primer is based on. Note that we quote definitions and results in the form they assume in the spacetime of GR, i.e., a four-dimensional Lorentzian manifold, indicated by the symbol $\mathcal{M}$. We refer the interested reader to the literature for statements valid in more general settings.

A sum of the form

$$\boldsymbol{F} = F_a \boldsymbol{\theta}^a \tag{A1}$$

is called a 1-differential form, or simply a 1-form, and F_a are its components; 1-forms are therefore identical to covariant vectors. More generally, p-differential forms (in the following simply p-forms) are rank-p totally antisymmetric covariant tensors on $\mathcal{M}$. The differential forms of highest possible degree are 4-forms, since for higher degrees the antisymmetry requirement would make any differential form vanish identically; 0-forms are defined as scalar functions on $\mathcal{M}$ (scalar fields).

The set of p-forms at a point P of $\mathcal{M}$ forms a $\binom{4}{p}$-dimensional vector space. Therefore the dimensions of the spaces of 0-, 1-, 2-, 3- and 4-forms (and the number of components of any form in one of these spaces) are, respectively, 1, 4, 6, 4, 1.

For the rest of this section, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be generic p- and q-forms, respectively. We define an operation that acts on two such forms to produce a $(p+q)$-form. This is referred to as the exterior product or wedge product, and it is defined as

$$\boldsymbol{A} \wedge \boldsymbol{B} \coloneqq \mathrm{Alt}(\boldsymbol{A} \otimes \boldsymbol{B}), \tag{A2}$$

where $\otimes$ is the standard tensor product and $\mathrm{Alt}(\boldsymbol{T})$ denotes is the totally antisymmetric part of the tensor $\boldsymbol{T}$. The components of a the result of the wedge product are therefore

$$(\boldsymbol{A} \wedge \boldsymbol{B})_{a_1 \ldots a_{p+q}} = \frac{1}{(p+q)!} \sum_{P \in S} \mathrm{sgn}(P) A_{a_{P(1)} \ldots a_{P(p)}} B_{b_{P(p+1)} \ldots b_{P(p+q)}}, \tag{A3}$$

where S is the set of all possible permutations of $p+q$ elements, P is one such permutation and $\mathrm{sgn}(P)$ equals $+1$ for even permutations and -1 for odd ones. Using a shorthand notation common in the GR literature, this formula can be written as

$$(\boldsymbol{A} \wedge \boldsymbol{B})_{c_1 \ldots a_{p+q}} = A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}. \tag{A4}$$

The exterior product is associative, and more importantly it satisfies the relation

$$\boldsymbol{A} \wedge \boldsymbol{B} = (-1)^{pq} \boldsymbol{B} \wedge \boldsymbol{A}. \tag{A5}$$

This in particular implies that for 1-forms the exterior product is antisymmetric.

Recall that the set $\boldsymbol{\theta}^a$ is a basis of the vector space of 1-forms. Leveraging the antisymmetry of the exterior product for 1-forms, it can be seen that the set of elements of the form

$$\boldsymbol{\theta}^{a_1} \wedge \ldots \wedge \boldsymbol{\theta}^{a_p}, \tag{A6}$$

i.e., the exterior product of p elements of the basis of 1-forms, constitutes a basis for the vector space of p-forms. For example, a basis for the space of 2-forms in a four-dimensional spacetime is

$$\{\boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1, \boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^2, \boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^3, \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2, \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^3, \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}^3\}, \tag{A7}$$

which as noted above has six elements.

A 1-form defines a linear operator acting on vectors and producing a real number, so that the result of a 1-form $\boldsymbol{F}$ acting on a vector $\boldsymbol{X}$ can be written

$$\boldsymbol{F}(\boldsymbol{X}) = F_a X^a = \langle \boldsymbol{F}, \boldsymbol{X} \rangle, \tag{A8}$$

where the last equality shows that this is nothing but the interior product between vectors and their duals induced by the metric.

The *interior product* is instead an operation between a p-form and a vector $\boldsymbol{X}$, which gives as result a $(p-1)$-form according to the following definition:

$$(\iota_X A)_{a_2\ldots a_p} \coloneqq X^{a_1} A_{a_1 a_2 \ldots a_p}. \tag{A9}$$

While the inner product and the interior product should not be confused, the latter is in a sense an extension of the former, since $\iota_X \boldsymbol{F} = \langle \boldsymbol{X}, \boldsymbol{F}\rangle = \boldsymbol{F}(\boldsymbol{X})$.

As stated above, p-forms are antisymmetric $(0, p)$ tensors, and as tensors they are acted upon by the standard partial and covariant derivatives. There is however another type of derivation which affects these objects (and is instead not defined for more general tensors). This is called the exterior derivative and denoted by the symbol $\boldsymbol{d}$. It can be defined by stating that the exterior derivative of a form $\boldsymbol{A} = A_{a_1\ldots a_p}\boldsymbol{\theta}^{a_1} \wedge \ldots \wedge \boldsymbol{\theta}^{a_p}$ is

$$\boldsymbol{dA} = (\partial_b A_{a_1\ldots a_p})\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^{a_1} \wedge \ldots \wedge \boldsymbol{\theta}^{a_p}. \tag{A10}$$

Since the exterior products automatically antisymmetrize the coefficients, this definition implies that the components of the result can be written as

$$(\boldsymbol{dA})_{b a_1 \ldots a_p} = \partial_{[b} A_{a_1\ldots a_p]}. \tag{A11}$$

The exterior derivative associates to any p-form a $(p+1)$-form, and it clearly does not depend on the metric or on any other additional structure on the manifold. Despite the partial derivative being used in its definition, the components of the exterior derivative form the components of a tensor, i.e., objects obtained by applying it transform as tensors under changes of basis.

Note that as the partial derivative, the exterior derivative is a linear operation, however it exhibits a modified Leibniz rule with respect to the exterior product:

$$\boldsymbol{d}(\boldsymbol{A} \wedge \boldsymbol{B}) = \boldsymbol{dA} \wedge \boldsymbol{B} + (-1)^p \boldsymbol{A} \wedge \boldsymbol{dB}, \tag{A12}$$

where $\boldsymbol{A}$ is a p-form.

Another fundamental property of the exterior derivative, which is leveraged at several points in the present work, is its nilpotency:

$$\boldsymbol{ddA} = 0. \tag{A13}$$

Note that having defined the exterior derivative and interior product, the definition of the Lie derivative of a p-form $\boldsymbol{A}$ along a vector $\boldsymbol{X}$ becomes particularly compact and easy to recall:

$$\mathcal{L}_X \boldsymbol{A} = \boldsymbol{d}\iota_X \boldsymbol{A} + \iota_X \boldsymbol{dA}. \tag{A14}$$

This is known as "Cartan's magic formula."

There also exists a definition of a exterior covariant derivative, but to state it we need to first introduce so-called tensor-valued differential forms. So far in this section we only have used real-valued differential forms, i.e., forms that when acting upon (sets of) vectors return a real value. However in the main text we make extensive use of tensor-valued forms, which return a collection of real values instead. These forms can be seen as collections of real-valued forms, each member of the collection labeled by indices. Such objects are the connection forms $\boldsymbol{\omega}^a_b$, a collection of 1-forms, defined by

$$\nabla_{\boldsymbol{e}_a} \boldsymbol{e}_b = \boldsymbol{\omega}^c{}_b(\boldsymbol{e}_a)\boldsymbol{e}_c. \tag{A15}$$

If the connection is chosen as the usual Levi-Civita connection, then $\boldsymbol{\omega}^\mu{}_\nu = \Gamma^\mu{}_{\lambda\nu}\boldsymbol{\theta}^\lambda$ where $\Gamma^\mu{}_{\lambda\nu}$ are the usual Christoffel symbols. In general however the connection forms encode any arbitrary connection.

A few comments are in order. First of all, despite the possibly confusing notation, note that $\boldsymbol{\omega}^a{}_b$ is not a rank-2 tensor of type (1, 1). It is collection of 1-forms, which becomes apparent by noting that it is defined as the product of the basis 1-forms and a collection of numbers. Second, just as the components of the Christoffel symbols do not transform as the components of a tensor, neither do the components of the object that the connection forms yield when applied to a vector. In this sense the name "tensor-valued form" if applied to the connection forms is a misnomer, since the components of the object yielded by such a form do not, in general, transform as a tensor. The locution "collection of p-forms" while possibly less descriptive, is also more appropriate. In light of this, we refer to the indices of the connection 1-forms in (A15) as "nontensorial" indices. In the main text we deal with collection of forms, some of which are nontensorial like the connection forms and others instead are proper tensor-valued forms, i.e., their components do transform as those of tensors.

The connection 1-forms allow us to finally define the exterior covariant derivative of a tensor-valued p-form by

$$\begin{aligned}\boldsymbol{DT}^{a\ldots d}{}_{e\ldots h} = {}& \boldsymbol{dT}^{a\ldots d}{}_{e\ldots h} + \boldsymbol{\omega}^a{}_i \wedge \boldsymbol{T}^{i\ldots d}{}_{e\ldots h} + \boldsymbol{\omega}^d{}_i \wedge \boldsymbol{T}^{a\ldots i}{}_{e\ldots h} \\ & + -\boldsymbol{\omega}^i{}_e \wedge \boldsymbol{T}^{a\ldots d}{}_{i\ldots h} - \boldsymbol{\omega}^i{}_h \wedge \boldsymbol{T}^{a\ldots d}{}_{e\ldots i}.\end{aligned} \tag{A16}$$

Note however that this operation is only defined when applied on a form that is tensor valued in the strict sense, i.e., when its indices are actually tensorial and transform as the components of a tensor. Under this condition, the indices of the result of applying the covariant exterior derivative will also transform as those of a tensor.

In what follows we go back to real-valued forms. As a consequence of the antisymmetry of differential forms, all the 4-forms (i.e., the highest possible degree forms in a four-dimensional manifold) are multiples of a single 4-form, called volume form or metric volume element, and defined as

$$\boldsymbol{\varepsilon} = \sqrt{-g}\boldsymbol{\theta}^0 \wedge \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 \wedge \boldsymbol{\theta}^3. \tag{A17}$$

Its components can be written as

$$\varepsilon_{abcd} = \sqrt{-g}\epsilon_{abcd}, \quad \text{(A18)}$$

where as anticipated in Sec. II, g is the determinant of the metric and the Levi-Civita symbol ϵ_{abcd} equals $+1$ or -1 depending on whether (a, b, c, d) is an even or an odd permutation of (0, 1, 2, 3). Note also that raising the components of the volume element with the metric results in

$$\varepsilon^{abcd} = -\frac{1}{\sqrt{-g}}\epsilon^{abcd}. \quad \text{(A19)}$$

It is also useful to note these properties of the volume form and Levi-Civita symbol when restricted to purely spatial, tridimensional hypersurfaces, which are used extensively in the main text:

$$\varepsilon_{0ijk} = -\alpha\varepsilon_{ijk}, \quad \text{(A20)}$$

$$\varepsilon^{0ijk} = \frac{1}{\alpha}\varepsilon^{ijk}, \quad \text{(A21)}$$

$$\varepsilon_{ijk} = \sqrt{\gamma}\epsilon_{ijk}, \quad \text{(A22)}$$

$$\varepsilon^{ijk} = \frac{1}{\sqrt{\gamma}}\epsilon^{ijk}. \quad \text{(A23)}$$

Furthermore, we note that, in a noncoordinate, orthonormal frame, $g = -1$ so that

$$\varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \epsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \quad \text{and} \quad \varepsilon_{\hat{i}\hat{j}\hat{k}} = \epsilon_{\hat{i}\hat{j}\hat{k}}. \quad \text{(A24)}$$

Accordingly in such a case (but not in general) we can write the former for the latter and vice versa.

As outlined above the vector space of p-forms and that of $(4-p)$-forms have the same dimension. Therefore it is possible to build an isomorphism between these spaces. A very important such isomorphism is the Hodge duality, represented by the symbol $\star$. The components of the Hodge dual can be obtained as

$$(\star \boldsymbol{A})_{a_{p+1}\ldots a_4} = \varepsilon_{a_1\ldots a_p, a_{p+1}\ldots a_4} A^{a_1\ldots a_p}. \quad \text{(A25)}$$

Applying this formula to computing the Hodge dual of 0-forms, it follows in particular that $\star 1 = \boldsymbol{\epsilon}$.

An important property of the Hodge dual is that for any p-form

$$\star\star \boldsymbol{A} = (-1)^{1+p(4-p)}\boldsymbol{A}, \quad \text{(A26)}$$

which implies

$$\star^{-1}\boldsymbol{A} = (-1)^{1+p(4-p)}\star\boldsymbol{A}. \quad \text{(A27)}$$

Another property of p-forms that is fundamental for the present work is that they are natural integrands over p-dimensional (sub)manifolds of $\mathcal{M}$. In particular, if a p-dimensional submanifold of $\mathcal{M}$ is further divided into a set of nonoverlapping p-dimensional regions, a p-form $\boldsymbol{A}$ naturally establishes a map from this set to the set of real numbers. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are such regions, then

$$\boldsymbol{A}[\mathcal{S}_1] = \int_{\mathcal{S}_1}\boldsymbol{A} \quad \text{(A28)}$$

and

$$\boldsymbol{A}[\mathcal{S}_1 \cup \mathcal{S}_2] = \int_{\mathcal{S}_1}\boldsymbol{A} + \int_{\mathcal{S}_2}\boldsymbol{A}. \quad \text{(A29)}$$

Note in particular that the integral of $\boldsymbol{\epsilon}$ over a portion of $\mathcal{M}$ is nothing but the volume of that portion, hence the name volume form for $\boldsymbol{\epsilon}$.

We can then state the modern version of Stokes's theorem, which generalizes the well-known theorems of vector calculus by Green, Stokes, and Gauss. It allows us to relate integrals over a general submanifold $\mathcal{S}$ of $\mathcal{M}$ to integrals over its boundary $\partial\mathcal{S}$

$$\int_{\mathcal{S}} \boldsymbol{dA} = \int_{\partial\mathcal{S}}\boldsymbol{A}. \quad \text{(A30)}$$

Equation (A30) too has a fundamental importance for this work.

Finally, it can be useful to restate standard vector-calculus operators in terms of differential forms and exterior calculus operators, e.g.,

$$\nabla f = \overrightarrow{\boldsymbol{d}f}, \quad \text{(A31)}$$

$$\nabla \cdot (\boldsymbol{u}) = -\star^{-1}\boldsymbol{d}\star\tilde{\boldsymbol{u}}, \quad \text{(A32)}$$

$$\nabla \times (\boldsymbol{u}) = \overrightarrow{\star\boldsymbol{d}\tilde{\boldsymbol{u}}}. \quad \text{(A33)}$$

In these expressions f is a generic scalar field (or equivalently a 0-form), and $\boldsymbol{u}$ a generic vector; an arrow is used to denote the operation of transforming a differential 1-form to its dual vector, and a tilde to denote the inverse operation.

APPENDIX B: HODGE DUAL OF THE NESTER-WITTEN FORM IN TERMS OF THE CONNECTION

In order to obtain Eq. (93), we start from the definition of the Nester-Witten form (64), which can also be written as

$$\boldsymbol{u}_a = -\frac{1}{2}\omega^{bc}{}_d\boldsymbol{\theta}^d \wedge \boldsymbol{\Sigma}_{abc}. \quad \text{(B1)}$$

Using the identity (51), we obtain

$$\boldsymbol{u}_a = -\frac{3}{2}\omega^{bc}{}_d\delta^d{}_{[a}\boldsymbol{\Sigma}_{bc]}. \quad \text{(B2)}$$

Expanding the antisymmetric brackets,

$$\boldsymbol{u}_a = -\frac{1}{4}[\omega^{bc}{}_a\boldsymbol{\Sigma}_{bc} + \omega^{bd}{}_d\boldsymbol{\Sigma}_{ab} + \omega^{dc}{}_d\boldsymbol{\Sigma}_{ca} - \omega^{bc}{}_a\boldsymbol{\Sigma}_{cb} - \omega^{dc}{}_d\boldsymbol{\Sigma}_{ac} - \omega^{bd}{}_d\boldsymbol{\Sigma}_{ba}], \quad \text{(B3)}$$

and renaming indices to factor out $\boldsymbol{\Sigma}_{bc}$,

$$\boldsymbol{u}_a = -\frac{1}{2}(\omega^{[bc]}{}_a + \delta^b{}_a\omega^{[cd]}{}_d - \delta^c{}_a\omega^{[bd]}{}_d)\boldsymbol{\Sigma}_{bc}. \quad \text{(B4)}$$

From the definition of the hypersurface forms (50) and the formula to obtain the components of the Hodge dual (A25), it follows that the expression in parenthesis equals the components of $\star\boldsymbol{u}_a$, as stated in Eq. (93).

APPENDIX C: DERIVATION OF EVOLUTION EQUATION FOR $\sqrt{\gamma}$

As mentioned in Sec. V D, one can obtain an evolution equation for $\sqrt{\gamma}$ in a conservative form by taking the exterior derivative of the hypersurface form $\boldsymbol{\Sigma}_{\hat{0}}$ orthogonal to $-n_\mu$. This form is identical to the Hodge dual of $\boldsymbol{\theta}^{\hat{0}}$, and in a coordinate basis it has components

$$(\boldsymbol{\Sigma}_{\hat{\mathbf{0}}})_{\mu\nu\lambda} = -\varepsilon_{\alpha\mu\nu\lambda}n^\alpha. \quad \text{(C1)}$$

Similarly as in Sec. III, the components of its exterior derivative will take the form of a conservation equation for n^α,

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}n^\mu\right)\boldsymbol{\Sigma} = \boldsymbol{d\Sigma}_{\hat{\mathbf{0}}}. \quad \text{(C2)}$$

To find an expression for the right-hand side, we recall the definition of hypersurface forms (50) and take the exterior derivative. After applying the Leibniz rule for the exterior product (A12), we substitute the definition of $\boldsymbol{F}^{\hat{a}}$ (107) and relabel indices, which gives

$$\boldsymbol{d\Sigma}_{\hat{0}} = \frac{1}{6}\varepsilon_{\hat{0}\hat{i}\hat{j}\hat{k}}\boldsymbol{F}^{\hat{i}} \wedge \boldsymbol{\theta}^{\hat{j}} \wedge \boldsymbol{\theta}^{\hat{k}}. \quad \text{(C3)}$$

Finally, we can use the relation

$$\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\boldsymbol{\Sigma} = \boldsymbol{\theta}^{\hat{\alpha}} \wedge \boldsymbol{\theta}^{\hat{\beta}} \wedge \boldsymbol{\theta}^{\hat{\gamma}} \wedge \boldsymbol{\theta}^{\hat{\delta}}, \quad \text{(C4)}$$

which can also be derived from (50), to obtain

$$\boldsymbol{d\Sigma}_{\hat{0}} = F^{\hat{i}}{}_{\hat{0}\hat{i}}\boldsymbol{\Sigma} = -E^{\hat{i}}{}_{\hat{i}}\boldsymbol{\Sigma} = \frac{5}{2}D_{\hat{i}}{}^{\hat{i}}\boldsymbol{\Sigma}, \quad \text{(C5)}$$

where relation (127) was used. Equating (C2) and (C5), we arrive to the result shown in (121).

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