Unifying inflation with early and late dark energy with multiple fields: Spontaneously broken scale invariant two measures theory

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A unified multiscalar field model with three flat regions is discussed. The three flat regions are the inflation, early- and late dark energy epochs. The potential is obtained by a spontaneous breaking of scale invariance generated by non-Riemannian measures of integration (or two measures theories (TMT)). We define the scale invariant couplings of the scalar fields to the different measures through exponential potentials. Spontaneous breaking of scale invariance takes place when integrating the fields that define the measures. When going to the Einstein frame we obtain: (i) An effective potential for the scalar fields with three flat regions which allows for a unified description of both early Universe inflation (in the higher-energy density flat region) as well as of present dark energy epoch which can be realized with a double phase, i.e., in two flat regions. (ii) In the slow-roll inflation, only one field combination, the "dilaton," which transforms under scale transformations, has nontrivial dynamics; the orthogonal one, which is scale invariant, remains constant. The corresponding perturbations of the dilaton are calculated. (iii) For a reasonable choice of the parameters the present model perturbations conforms to the Planck Collaboration data. (iv) In the late Universe we define scale-invariant couplings of Dark Matter to the dilaton. These couplings define a matter-induced potential for the dilaton and extremizing this potential determines the scale-invariant scalar field, while all exotic noncanonical behavior of the Dark Matter as well as any possible fifth force disappear. (v) We calculate the evolution of the late Universe under these conditions with the realization of two different possible realizations of Λ cold dark matter-type scenarios depending on the flat region in the late Universe. These two phases could appear at different times in the history of the Universe. (vi) From the Planck data, we find the constraints on the parameters during the inflationary epoch and these values are used to obtain constraints relevant to the present epoch.

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I. INTRODUCTION

In the "standard cosmological" framework for the early Universe [1–9] the Universe starts with a period of exponential expansion called "inflation." At the same time, after the discovery of the accelerating Universe [10–13], we have now a late Universe standard cosmological framework for the late Universe, the Λ Cold Dark Matter (ΛCDM) picture [14,15], consisting of a cosmological constant, Dark Matter, and ordinary visible matter, the Universe being now dominated by the Cosmological Constant or Dark Energy (DE) and the Dark Matter (DM). This simple ΛCDM is now being somewhat challenged by the discovery of several cosmological tensions, the H_0 tension [16–19] and the σ_8 tension [20–26]. This suggests that the introduction of a cosmological term to describe the DE and the addition of DM may be a too simple description of the late Universe. In the inflationary period also primordial density perturbations are generated [2]. The inflation is followed by particle creation, where the observed matter and radiation

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were generated, and finally the evolution arrives at a present phase of slowly accelerating Universe. In this standard model, however, at least two fundamental questions remain unanswered:

(i) The early inflation, although solving many cosmological puzzles, like the horizon and flatness problems, cannot address the initial singularity problem; exponential expansion with such wildly different scales—the inflationary phase and the present phase of slowly accelerated expansion of the Universe.

The best-known mechanism for generating a period of accelerated expansion is through the presence of some vacuum energy. In the context of a scalar-field theory, vacuum energy density appears naturally when the scalar field acquires an effective potential $U_{\rm eff}$ which has flat regions so that the scalar field can "slowly roll" [7,8,27,28] and its kinetic energy can be neglected, resulting in an energy-momentum tensor $T_{\mu\nu} \simeq -g_{\mu\nu} U_{\rm eff}$.

The possibility of continuously connecting an inflationary phase to a slowly accelerating Universe through the evolution of a single scalar field—the *quintessential inflation scenario*—has been first studied in Ref. [29]. Also, $F(R)$ models can yield both an early-time inflationary epoch and a late-time de Sitter phase with vastly different values of effective vacuum energies [30–32]. For a recent proposal of a quintessential inflation mechanism based on the k-essence framework, see Ref. [33]. For another recent approach to quintessential inflation based on the "variable gravity" model [34,35] and for an extensive list of references to earlier work on the topic, see Ref. [36]. Other ideas based on the so-called α attractors [7,37–51], which use noncanonical kinetic terms, have been studied. Finally, a quintessential inflation based on a Lorentzian Slow Roll ansatz [52] which automatically gives two flat regions.

In previous papers [53–55] we have studied a unified scenario where both an inflation and a slowly accelerated phase for the universe can appear naturally from the existence of two flat regions in the effective scalar-field potential which we derive systematically from a Lagrangian action principle. Namely, we started with a new kind of globally Weyl-scale invariant gravity-matter action within the first-order (Palatini) approach formulated in terms of two different non-Riemannian volume forms (integration measures) [55]. In this new theory there is a single scalar field with kinetic terms coupled to both non-Riemannian measures, and in addition to the scalar curvature term R also an R^2 term is included (which is similarly allowed by global Weyl-scale invariance). Scale invariance is spontaneously broken upon solving part of the corresponding equations of motion due to the appearance of two arbitrary dimensionful integration constants.

Let us briefly recall the origin of current approach. The main idea comes from Refs. [56–58], where some of us have proposed a new class of gravity-matter theories based on the idea that the action integral may contain a new metric-independent generally covariant integration measure density, i.e., an alternative non-Riemannian volume form on the space-time manifold defined in terms of an auxiliary antisymmetric gauge field of maximal rank. The originally proposed modified-measure gravity-matter theories [56–58] contained two terms in the pertinent Lagrangian action—one with a non-Riemannian integration measure and a second one with the standard Riemannian integration measure (in terms of the square root of the determinant of the Riemannian space-time metric). An important feature was the requirement for global Weyl-scale invariance which subsequently underwent dynamical spontaneous breaking [56–59]. The second action term with the standard Riemannian integration measure might also contain a Weyl-scale symmetry-preserving R^2 term [58].

The latter formalism yields various new interesting results in all types of known generally covariant theories:

(i) $D =$ four-dimensional models of gravity and matter fields containing the new measure of integration appear to be promising candidates for resolution of the dark energy and dark matter problems, the fifth force problem, and a natural mechanism for spontaneous breakdown of global Weyl-scale symmetry [56,59–65].

(ii) Study of reparametrization-invariant theories of extended objects (strings and branes) based on employing of a modified non-Riemannian world-sheet/world-volume integration measure [66–68] leads to dynamically induced variable string/brane tension and to string models of non-Abelian confinement, interesting consequences from the modified measures spectrum [69], and construction of new braneworld scenarios [70]. Recently [71], this formalism was generalized to the case of string and brane models in curved supergravity background.

(iii) Study in [72,73] of modified supergravity models with an alternative non-Riemannian volume form on the space-time manifold produces some outstanding new features: (a) This new formalism applied to minimal $N = 1$ supergravity naturally triggers the appearance of a dynamically generated cosmological constant as an arbitrary integration constant, which signifies a new explicit mechanism of spontaneous (dynamical) breaking of supersymmetry; (b) Applying the same formalism to anti–de Sitter supergravity allows us to appropriately choose the above-mentioned arbitrary integration constant so as to obtain simultaneously a very small effective observable cosmological constant as well as a very large physical gravitino mass.

In this paper we will study a quintessential scenario where we will be driven from inflation to a slowly accelerated phase describing our Universe using a scale-invariant two-field model. Multifield inflation has been studied by several authors; see for example [74–78]. In the context of modified measures formalism, the ratio of two measures can become an additional scalar field if we use the second-order formalism [79,80]. In the present paper we will consider only the first-order formulation, however, and the measure field remains nondynamical, determined by a constraint, and therefore they do not introduce new degrees of freedom. Introducing two fields gives rise to very interesting new possibilities. This is also the case when we consider multifield scale-invariant inflationary models leading to DE/DM for the late Universe, where interesting new features appear for both the inflationary phase and for the DE/DM late Universe phase; in particular, we will see that the late Universe acquires a fine structure with two possible vacuums for the late Universe that can occur at different times in the late evolution of the Universe.

The plan of the present paper is as follows. In the next section (Sec. II) we describe in some detail the general formalism for the new class of gravity-matter systems defined in terms of two independent non-Riemannian integration measures. In Sec. III we describe the properties of the three flat regions in the Einstein-frame effective scalar potential, one corresponding to the evolution of the early inflation and the other two for the late Universe. We also present in this section the relevant solutions for the slow-roll inflation. In Sec. IV we present a numerical analysis, for a reasonable choice of the parameters, of the resulting ratio of tensor-to-scalar perturbations and show that the present model conforms to the Planck Collaboration data. In Sec. V we study how the model can describe Dark Matter in a scale-invariant fashion in the late Universe, what are the conditions for avoiding 5th force problem, or what is equivalent for the dust Dark Matter to behave canonically. We find that in the two flat regions of the late Universe the Dark Energy and Dark Matter can acquire different parameters. In Sec. VI we find the different values of the particle masses in the relevant vacuums of the two flat regions relevant to the late Universe where the 5th force is eliminated, the dust is canonical, etc. The dynamical connection between these two phases requires a noncanonical dust- and dark-energy behavior transition, since particle masses have to change when transitioning between these two states. This has not been studied in full detail yet. We conclude in Sec. VIII with some discussions. For simplicity we will use units where the Newton constant is taken as $G_{\mathrm{Newton}} = 1/16\pi$.

II. GRAVITY-MATTER FORMALISM WITH TWO INDEPENDENT NON-RIEMANNIAN VOLUME FORMS

In this section, we shall consider the following non-standard gravity-matter system with an action involving two independent non-Riemannian integration measure densities generalizing the model analyzed in [55]. In this form, the action is given by

$$S = \int d^4x \Phi_1(A)[R + L^{(1)}] + \int d^4x \Phi_2(B)\left[L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}}\right], \tag{1}$$

where the following notations are used:

(i) The quantities $\Phi_1(A)$ and $\Phi_2(B)$ are two independent non-Riemannian volume forms, i.e., generally covariant integration measure densities on the underlying space-time manifold and are given by

$$\Phi_1(A) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu A_{\nu\kappa\lambda}, \qquad \Phi_2(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda}, \tag{2}$$

defined as a function of field strengths of two auxiliary 3-index antisymmetric tensor gauge fields.[1] The functions $\Phi_{1,2}$ take over the role of the standard Riemannian integration measure density defined as $\sqrt{-g} \equiv \sqrt{-\det\|g_{\mu\nu}\|}$ and it is expressed in terms of the space-time metric $g_{\mu\nu}$.

(ii) [The functions $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ and $R_{\mu\nu}(\Gamma)$ correspond to the scalar curvature and the Ricci tensor in the first-order (Palatini) formalism, where the affine connection $\Gamma^\mu_{\nu\lambda}$ is *a priori* independent of the metric $g_{\mu\nu}$. Also, we have added in the second action term an R^2 gravity term (again in the Palatini form). We mention that $R + R^2$ gravity within the second-order formalism (which was the first inflationary model) was originally analyzed in Ref. [6].

(iii) The quantities $L^{(1,2)}$ denote two different Lagrangians of two scalar matter fields φ_1 and φ_2 in analogy to Refs. [56,59]. These Lagrangians are defined as

$$L^{(1)} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_1\partial_\nu\varphi_1 - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_2\partial_\nu\varphi_2 - V(\varphi_1,\varphi_2), \tag{3}$$

$$L^{(2)} = U(\varphi_1,\varphi_2), \tag{4}$$

where the scalar potential V is given by

$$V(\varphi_1,\varphi_2) = f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2}, \tag{5}$$

and another scalar potential is defined as

[1] In general for the D space-time dimensions one can always represent a maximal rank antisymmetric gauge field $A_{\mu_1\ldots\mu_{D-1}}$ as a function of D auxiliary scalar fields ϕ^i $(i = 1,\ldots,D)$ as $A_{\mu_1\ldots\mu_{D-1}} = \frac{1}{D}\varepsilon_{ii_1\ldots i_{D-1}}\phi^i\partial_{\mu_1}\phi^{i_1}\ldots\partial_{\mu_{D-1}}\phi^{i_{D-1}}$, so that its (dual) field strength $\Phi(A) = \frac{1}{D!}\varepsilon_{i_1\ldots i_D}\varepsilon^{\mu_1\ldots\mu_D}\partial_{\mu_1}\phi^{i_1}\ldots\partial_{\mu_D}\phi^{i_D}$.

$$U(\varphi_1, \varphi_2) = f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2}, \tag{6}$$

where the quantities f_1, f_2, g_1, g_2,α_1, and α_2 are positive parameters.

(iv) The function $\Phi(H)$ denotes the dual field strength of a third auxiliary 3-index antisymmetric tensor gauge field:

$$\Phi(H) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu H_{\nu\kappa\lambda}, \tag{7}$$

whose introduction is fundamental for nontriviality of the model.

We mention the scalar potentials V and U have been chosen in such a way that the action given Eq. (1) is invariant under global Weyl-scale transformations:

$$\begin{aligned} &g_{\mu\nu} \to \lambda g_{\mu\nu}, \qquad \Gamma^\mu_{\nu\lambda} \to \Gamma^\mu_{\nu\lambda}, \qquad \varphi_1 \to \varphi_1 + \frac{1}{\alpha_1}\ln\lambda, \\ &\varphi_2 \to \varphi_2 + \frac{1}{\alpha_2}\ln\lambda, \qquad A_{\mu\nu\kappa} \to \lambda A_{\mu\nu\kappa}, \\ &B_{\mu\nu\kappa} \to \lambda^2 B_{\mu\nu\kappa}, \qquad H_{\mu\nu\kappa} \to H_{\mu\nu\kappa}. \end{aligned} \tag{8}$$

Note that this combination is invariant, $\alpha_1\varphi_1 - \alpha_2\varphi_2 \to \alpha_1\varphi_1 - \alpha_2\varphi_2$, from Eq. (8). Additionally, we observe that the requirement about the global Weyl-scale symmetry (8) uniquely fixes the structure of the non-Riemannian-measure gravity-matter action given by Eq. (1).

In the following we will use $\epsilon = 0$ and in this case the equations of motion resulting from the variation of (1) with respect to affine connection $\Gamma^\mu_{\nu\lambda}$, are

$$\int d^4x\sqrt{-g}g^{\mu\nu}\left(\frac{\Phi_1}{\sqrt{-g}}\right)(\nabla_\kappa\delta\Gamma^\kappa_{\mu\nu} - \nabla_\mu\delta\Gamma^\kappa_{\kappa\nu}) = 0. \tag{9}$$

Therefore, $\Gamma^\mu_{\nu\lambda}$ corresponds to a Levi-Civita connection,

$$\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}(\partial_\nu\bar{g}_{\lambda\kappa} + \partial_\lambda\bar{g}_{\nu\kappa} - \partial_\kappa\bar{g}_{\nu\lambda}), \tag{10}$$

with respect to the Weyl-rescaled metric $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = \chi_1 g_{\mu\nu}, \quad \text{and} \quad \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}}. \tag{11}$$

Also, from the variation of the action (1) with respect to auxiliary tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, and $H_{\mu\nu\lambda}$ that yields the equations, we have

$$\begin{aligned} &\partial_\mu[R + L^{(1)}] = 0, \qquad \partial_\mu\left[L^{(2)} + \frac{\Phi(H)}{\sqrt{-g}}\right] = 0, \\ &\partial_\mu\left(\frac{\Phi_2(B)}{\sqrt{-g}}\right) = 0, \end{aligned} \tag{12}$$

whose solutions are given by

$$\frac{\Phi_2(B)}{\sqrt{-g}} \equiv \chi_2, \quad R + L^{(1)} = -M_1, \quad L^{(2)} + \frac{\Phi(H)}{\sqrt{-g}} = -M_2. \tag{13}$$

Here the quantities M_1, M_2, and χ_2 are integration constants. However, the constants M_1 and M_2 are arbitrary and dimensional and χ_2 arbitrary and dimensionless.

We mention that the integration constant χ_2 in Eq. (13) preserves global Weyl-scale invariance in Eq. (8), whereas the appearance of the other integration constants M_1, M_2 signifies *dynamical spontaneous breakdown* of global Weyl-scale invariance under (8) due to the scale-noninvariant solutions in Eq. (13).

Also, varying the action (1) with respect to $g_{\mu\nu}$ and using relations (13), we have

$$\chi_1\left[R_{\mu\nu} + \frac{1}{2}(g_{\mu\nu}L^{(1)} - T^{(1)}_{\mu\nu})\right] = \frac{\chi_2}{2}[T^{(2)}_{\mu\nu} + g_{\mu\nu}M_2 - 2RR_{\mu\nu}], \tag{14}$$

where χ_1 and χ_2 are defined in (11), and the quantities $T^{(1,2)}_{\mu\nu}$ correspond to the energy-momentum tensors of the scalar-field Lagrangians with the standard definitions:

$$T^{(1,2)}_{\mu\nu} = g_{\mu\nu}L^{(1,2)} - 2\frac{\partial}{\partial g^{\mu\nu}}L^{(1,2)}. \tag{15}$$

Now, taking the trace of Eq. (14) and using again the second relation of Eq. (13), we find that the scale factor χ_1 becomes

$$\chi_1 = 2\chi_2\frac{T^{(2)}/4 + M_2}{L^{(1)} - T^{(1)}/2 - M_1}, \tag{16}$$

where $T^{(1,2)} = g^{\mu\nu}T^{(1,2)}_{\mu\nu}$.

Thus, considering the second relation of Eq. (13) together with Eq. (14), we obtain the Einstein-like form

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}g_{\mu\nu}(L^{(1)} + M_1) + \frac{1}{2}(T^{(1)}_{\mu\nu} - g_{\mu\nu}L^{(1)}) \\ + \frac{\chi_2}{2\chi_1}[T^{(2)}_{\mu\nu} + g_{\mu\nu}M_2]. \end{aligned} \tag{17}$$

In this context, we can bring Eqs. (17) into the standard form of Einstein equations for the metric $\bar{g}_{\mu\nu}$, i.e., the Einstein-frame gravity equations

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) = \frac{1}{2}T^{\text{eff}}_{\mu\nu}, \tag{18}$$

in with the energy-momentum tensor [analogously to (15)]:

$$T^{\rm eff}_{\mu\nu} = g_{\mu\nu} L_{\rm eff} - 2\frac{\partial}{\partial g^{\mu\nu}} L_{\rm eff}, \tag{19}$$

where the effective Einstein-frame scalar-field Lagrangian

$$L_{\rm eff} = \frac{1}{\chi_1}\left\{L^{(1)} + M_1 + \frac{\chi_2}{\chi_1}[L^{(2)} + M_2]\right\}, \tag{20}$$

where $L^{(1,2)}$ represent Lagrangian densities defined as

$$L^{(1)} = \chi_1(X_1 + X_2) - V, \qquad L^{(2)} = U, \tag{21}$$

with the potentials V and U as in relations (3) and (4). Also, to write $L_{\rm eff}$ in terms of the Einstein-frame metric $\bar{g}_{\mu\nu}$ we consider the short-hand notation for the kinetic terms:

$$X_1 \equiv -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi_1\partial_\nu\varphi_1, \qquad X_2 \equiv -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi_2\partial_\nu\varphi_2. \tag{22}$$

By combining Eqs. (16) and (19), and taking into account (21), we obtain

$$\chi_1 = \frac{2\chi_2[U + M_2]}{(V - M_1)}. \tag{23}$$

From Eqs. (23) and (20), we find at the explicit form for the Einstein-frame scalar Lagrangian $L_{\rm eff}$

$$L_{\rm eff} = X_1 + X_2 - U_{\rm eff}(\varphi_1, \varphi_2), \tag{24}$$

in which the effective scalar potential $U_{\rm eff}(\varphi_1, \varphi_2)$ becomes

$$\begin{aligned} U_{\rm eff}(\varphi_1, \varphi_2) &\equiv \frac{(V - M_1)^2}{4\chi_2[U + M_2]} \\ &= \frac{(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} - M_1)^2}{4\chi_2[f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2} + M_2]}. \end{aligned} \tag{25}$$

We note that choosing the "wrong" sign of the scalar potential U [Eq. (4)] in the initial non-Riemannian-measure gravity-matter action (1) is necessary to end up with the right sign in the effective potential (25) associated with scalar fields φ_1 and φ_2 in the physical Einstein-frame effective gravity-matter action given by Eq. (24). On the other hand, the overall sign of the other initial scalar potential V [Eq. (4)] is in fact irrelevant since changing its sign does not alter the positivity of effective potential given by Eq. (25).

III. FLAT REGIONS OF THE EFFECTIVE SCALAR POTENTIAL

A. Flat region values

We mention that the important feature of the effective potential $U_{\rm eff}$ [see Eq. (25)] is the presence of three infinitely large flat regions—for large positive values of the fields φ_1 and φ_2. For large positive values of φ_1 and φ_2, the effective potential reduces to

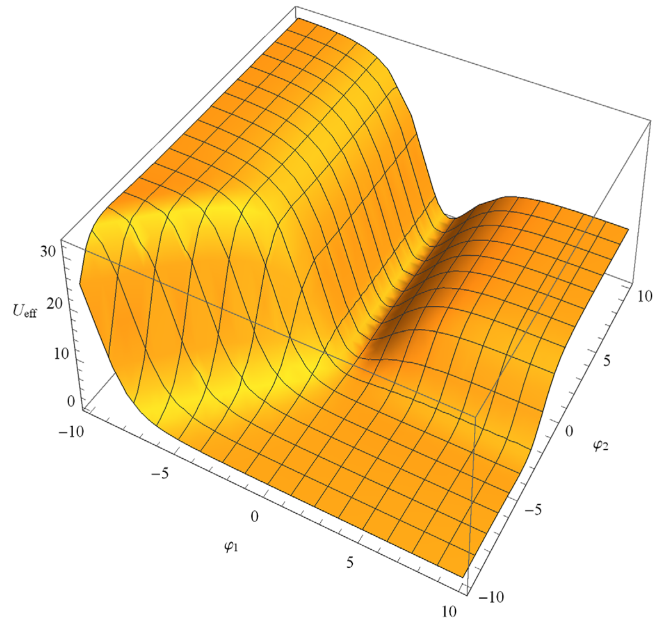

FIG. 1. The effective potential with three flat regions. One flat region refers to the inflationary phase and the other region refers to dark energy. The third could be another early dark-energy phase. Here, we have used a positive value for M_1.

$$U_{\rm eff}(\varphi_1, \varphi_2) \simeq U_{(++)} \equiv \frac{M_1^2}{4\chi_2 M_2}. \tag{26}$$

For the case in which we only have large negative φ_1:

$$U_{\rm eff}(\varphi_1, \varphi_2) \simeq U_{(\varphi_1\to-\infty)} \equiv \frac{f_1^2}{4\chi_2 f_2}. \tag{27}$$

In the other flat region in which we only have large negative φ_2:

$$U_{\rm eff}(\varphi_1, \varphi_2) \simeq U_{(\varphi_2\to-\infty)} \equiv \frac{g_1^2}{4\chi_2 g_2}. \tag{28}$$

Figure 1 shows a qualitative example for the three fat regions. The flat regions (26), (27), and (28) correspond to the evolution of the early and the late Universe, respectively, provided we choose the ratio of the coupling constants in the original scalar potentials versus the ratio of the scale-symmetry breaking integration constants to obey

$$\frac{M_1^2}{M_2} \gg \frac{f_1^2}{f_2}, \quad \text{and} \quad \frac{M_1^2}{M_2} \gg \frac{g_1^2}{g_2}, \tag{29}$$

which makes the vacuum energy density of the early Universe $U_{(++)}$ much bigger than that of the late Universe.

On the other hand, from the cosmological perturbations together with the Planck data [81–86], we have that the first flat region of the effective potential is approximately

$$U_{(++)} \sim M_1^2/\chi_2 M_2 \sim 6\pi^2 r \mathcal{P}_S \sim 10^{-8}, \tag{30}$$

(in units of $M_{\rm Pl}^4$), where the r denotes the tensor-to-scalar ratio and $\mathcal{P}_S$ corresponds to the scalar power perturbation. Let us recall that, since we are using units where $G_{\rm Newton} = 1/16\pi$, in the present case the Planck mass $M_{\rm Pl} = \sqrt{1/8\pi G_{\rm Newton}} = \sqrt{2}$.

In order to study the dynamics of the Universe, we consider that the metric corresponds to the standard flat Friedmann-Lemaitre-Robertson-Walker space-time metric given by

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{31}$$

where $a(t)$ denotes the scale factor. Thus, the associated Friedmann equations (recall the presently used units $G_{\rm Newton} = 1/16\pi$) result:

$$\frac{\ddot{a}}{a} = -\frac{1}{12}(\rho + 3p), \qquad H^2 = \frac{1}{6}\rho, \qquad H \equiv \frac{\dot{a}}{a}, \tag{32}$$

where H is the Hubble parameter. Also, the quantities ρ and p are defined as

$$\rho = \frac{1}{2}\dot{\varphi}_1^2 + \frac{1}{2}\dot{\varphi}_2^2 + U_{\rm eff}(\varphi_1, \varphi_2), \tag{33}$$

$$p = \frac{1}{2}\dot{\varphi}_1^2 + \frac{1}{2}\dot{\varphi}_2^2 - U_{\rm eff}(\varphi_1, \varphi_2), \tag{34}$$

and denote the total energy density and pressure of the scalar fields $\varphi_1 = \varphi_1(t)$ and $\varphi_2 = \varphi_2(t)$, respectively. In the following, we will consider that the dots indicate derivatives with respect to the time t.

In relation to the scalar equations of motion for the scalar field φ_1 and φ_2, we have

$$\ddot{\varphi}_1 + 3H\dot{\varphi}_1 + \partial U_{\rm eff}/\partial\varphi_1 = 0, \tag{35}$$

and

$$\ddot{\varphi}_2 + 3H\dot{\varphi}_2 + \partial U_{\rm eff}/\partial\varphi_2 = 0. \tag{36}$$

From these equations it is useful to track the behavior of the solution for different values of the initial condition. From comparing the potential derivatives into zero, $\partial U_{\rm eff}/\partial\varphi_1 = \partial U_{\rm eff}/\partial\varphi_2 = 0$, we get few points or paths. One path reads

$$f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} = M_1, \tag{37}$$

with $U_{\rm eff} = 0$, which is possible if $M_1 > 0$—the case we focus on. The path is a minimum from one side. The other points has infinite eigenvalues so we do not take them into account.

Since the potential has three different flat regions that give $\partial U_{\rm eff}/\partial\varphi_1 = \partial U_{\rm eff}/\partial\varphi_2 = 0$, the asymptotic behavior of the quintessential inflationary solution is quantified by these areas. In early times the potential begins at $U_{(++)}$ and finishes at the lower value of the late dark energy.

B. Slow-roll approximation

In the context of the slow-roll inflation, we can introduce the standard slow-roll parameters [27,28]:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}, \qquad \eta_1 \equiv -\frac{\ddot{\varphi}_1}{H\dot{\varphi}_1}, \quad \text{and} \quad \eta_2 \equiv -\frac{\ddot{\varphi}_2}{H\dot{\varphi}_2}, \tag{38}$$

and under the slow-roll approximation ε, η_1, and $\eta_2 \ll 1$; thus, one ignores the terms with $\ddot{\varphi}_{1,2}$, so that the φ_1, φ_2-equations of motion together with the second Friedmann equation (32) simplify to

$$3H\dot{\varphi}_1 + \partial U_{\rm eff}/\partial\varphi_1 \simeq 0, \qquad 3H\dot{\varphi}_2 + \partial U_{\rm eff}/\partial\varphi_2 \simeq 0,$$
$$H^2 \simeq \frac{1}{6}U_{\rm eff}. \tag{39}$$

Since now the fields φ_1 and φ_2 evolve on the first flat region of $U_{\rm eff}$ for large positive values (26), we can consider that the effective potential during inflationary scenario can be approximated to

$$U_{\rm eff}(\varphi_1, \varphi_2) \simeq \frac{M_1^2 - 2M_1(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2})}{4\chi_2 M_2}. \tag{40}$$

Here, we have used the expansion of the effective potential given Eq. (25).

In the following we will introduce the number of e-folds N defined as $N = \ln(a/a_f)$, where a_f corresponds to the scale factor at the end of the inflation, that is, at the end of inflation $N = 0$. Thus, Eqs. (39) and (40) can be rewritten as

$$\frac{d\varphi_1}{dN} = \frac{6M_1\alpha_1 f_1 e^{-\alpha_1\varphi_1}}{[M_2^2 - 2M_1(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2})]}, \tag{41}$$

and

$$\frac{d\varphi_2}{dN} = \frac{6M_1\alpha_2 g_1 e^{-\alpha_2\varphi_2}}{[M_2^2 - 2M_1(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2})]}. \tag{42}$$

Dividing these two equations we get a relation between the scalar fields φ_1 and φ_2 given by

$$e^{\alpha_1\varphi_1} d\varphi_1 = \frac{f_1\alpha_1}{g_2\alpha_2} e^{\alpha_2\varphi_2} d\varphi_2. \tag{43}$$

Notice that the symmetry-breaking constants M_1 and M_2 dropped from this equation. The integration of this equation introduces a new constant of integration C:

$$e^{\alpha_1\varphi_1} = \frac{f_1\alpha_1^2}{g_1\alpha_2^2} e^{\alpha_2\varphi_2} + C. \tag{44}$$

In the following we will consider that the integration constant $C = 0$.

Now, we can redefine two new scalar fields ϕ_1 and ϕ_2, in terms of the scalar fields φ_1 and φ_2, such that

$$\phi_1 = \frac{\alpha_1\varphi_1 - \alpha_2\varphi_2}{\sqrt{\alpha_1^2+\alpha_2^2}}, \quad \text{and} \quad \phi_2 = \frac{\alpha_2\varphi_1 + \alpha_1\varphi_2}{\sqrt{\alpha_1^2+\alpha_2^2}}. \tag{45}$$

Thus, this transformation is orthogonal, $\dot{\phi}_1^2 + \dot{\phi}_2^2 = \dot{\varphi}_1^2 + \dot{\varphi}_2^2$, where ϕ_1 is invariant and ϕ_2 transforms under a scale transformation.

Notice that in this case, the scale-invariant combination $\alpha_1\varphi_1 - \alpha_2\varphi_2$ gets determined, which corresponds to fixing the scalar field ϕ_1 defined in (45); this scalar field is scale invariant and is given by

$$\phi_1 = \frac{1}{\sqrt{\alpha_1^2+\alpha_2^2}} \ln\left[\frac{f_1\alpha_1^2}{g_1\alpha_2^2}\right] = \text{constant}. \tag{46}$$

However, the scalar field ϕ_2 defined also in Eq. (45), evolves in time. This means that although we have broken the scale invariance, through the integration constants M_1 and M_2, some of the remaining equations recall such scale invariance. As we have noticed in particular, the integration constants M_1 and M_2 dropped from such equation. That is indeed the reason that the equation that relates the two scalars retains the scale invariance, which is not true for other equations. We can now go back to the fields φ_1 and φ_2; in particular, we have that the relation between the scalar field φ_2 and the number of e-folds N becomes

$$\frac{A_2}{\alpha_2} e^{\alpha_2\varphi_2} + A_3\varphi_2 = A_1 N + cte, \tag{47}$$

and we can obtain $\varphi_2 = \varphi(N)$ using the ProductLog function. In mathematics, the product logarithm, also called the Omega function or Lambert W function, is a multivalued function, namely the branches of the converse relation of the function $f(w) = we^w$, see Ref. [87]. Using this definition, we find that the scalar field φ_2 in terms of the number of e-folds results:

$$\varphi_2(N) = (A_1N + C_1)/A_3 - \alpha_2^{-1}\text{ProductLog}[(A_2/A_3)e^{\alpha_2(A_1N+C_1)/A_3}], \tag{48}$$

where C_1 corresponds to another integration constant and the quantities A_1, A_2, and A_3 are defined as

$$A_1 = 6M_1\alpha_2 g_1, \quad A_2 = M_2^2, \quad A_3 = -2M_1\left[\frac{g_1\alpha_2^2}{\alpha_1^2} + g_1\right].$$

In order to obtain a real solution for the scalar field φ_2 it is necessary that the argument of the function ProductLog satisfies the condition in which the quantities $(A_2/A_3)e^{\alpha_2(A_1N+C_1)/A_3} > -e^{-1}$; see Ref. [88].

From Eqs. (44) and (45) we find that the new scalar field ϕ_2 can be written as

$$\phi_2 = \sqrt{\left[\left(\frac{\alpha_2}{\alpha_1}\right)^2 + 1\right]}\varphi_2 + C_2, \tag{49}$$

where C_2 is a constant defined as

$$C_2 = \frac{\alpha_2}{\alpha_1\sqrt{\alpha_1^2+\alpha_2^2}} \ln\left[\frac{f_1\alpha_1^2}{f_2\alpha_2^2}\right].$$

Now, the effective potential associated with the new field ϕ_2 becomes

$$U_{\text{eff}}(\phi_2) \simeq \frac{M_1^2 - 2M_1 g_1[(\frac{\alpha_2}{\alpha_1})^2 + 1]e^{\frac{-\alpha_1\alpha_2(\phi_2 - C_2)}{\sqrt{\alpha_1^2+\alpha_2^2}}}}{4\chi_2 M_2}. \tag{50}$$

In this way, the inflationary scenario reduces to a single field ϕ_2, such that the new equations are $6H^2 = \frac{\dot{\phi}_2^2}{2} + U_{\text{eff}}(\phi_2)$ and $\ddot{\phi}_2 + 3H\dot{\phi}_2 + \partial U_{\text{eff}}(\phi_2)/\partial\phi_2 = 0$.

The new slow-roll parameters ϵ and η associated with the scalar field ϕ_2 are defined as in the standard case:

$$\epsilon \simeq \left(\frac{\partial U_{\text{eff}}/\partial\phi_2}{U_{\text{eff}}}\right)^2, \quad \text{and} \quad \eta \simeq 2\left(\frac{\partial^2 U_{\text{eff}}/\partial\phi_2^2}{U_{\text{eff}}}\right). \tag{51}$$

By considering the effective potential given by Eq. (50) we obtain that the slow-roll parameters result:

$$\epsilon \simeq \left[\frac{4g_1^2\alpha_2^2(\alpha_1^2+\alpha_2^2)}{M_1^2\alpha_1^2}\right] e^{\frac{-2\alpha_1\alpha_2(\phi_2-C_2)}{\sqrt{\alpha_1^2+\alpha_2^2}}},$$
$$\eta \simeq -\left[\frac{4g_1\alpha_2^2}{M_1}\right] e^{\frac{-\alpha_1\alpha_2(\phi_2-C_2)}{\sqrt{\alpha_1^2+\alpha_2^2}}}. \tag{52}$$

Here, we have considered that the effective potential $U_{\text{eff}} \sim M_1^2/(4\chi_2 M_2)$.

Additionally, we can obtain the value of ϕ_2 at the end of the slow-roll regime $\phi_{2\text{end}}$ and it is determined from the condition $\epsilon = 1$ which through (52) becomes

$$\phi_{2\text{end}} = \frac{\sqrt{\alpha_1^2+\alpha_2^2}}{2\alpha_1\alpha_2} \ln\left[\frac{4g_1^2\alpha_2^2(\alpha_1^2+\alpha_2^2)}{M_1^2\alpha_1^2}\right] + C_2. \tag{53}$$

Also, considering Eq. (48) we have that the value of scalar field ϕ_2 at the end of inflation occurs when the number of e-folds $N = 0$, and then

$$\phi_{2\mathrm{end}} = C_2 + \sqrt{\left(\frac{\alpha_2^2}{\alpha_1^2}\right)^2 + 1} \times \left[\frac{C_1}{A_3} - \frac{1}{\alpha_2}\mathrm{ProducLog}[(A_2/A_3)e^{\alpha_2 C_1/A_3}]\right]. \quad (54)$$

Thus, from the above equations, we obtain that the constant C_1 is given by

$$C_1 \approx \frac{A_3}{\alpha_2}\left[\frac{1}{2}\ln\left(\frac{4g_1^2\alpha_2^2(\alpha_1^2+\alpha_2^2)}{M_1^2\alpha_1^2}\right) + 1\right]. \quad (55)$$

Here, we have considered that the term ProductLog is a function that does not change very much and is of order 1.

IV. PERTURBATIONS

In this section we will describe the scalar and tensor perturbations during the inflationary stage for our model of the single field ϕ_2. Following Refs. [89,90] the power spectrum of the scalar perturbation $\mathcal{P}_S$ under the slow-roll approximation is defined as

$$\mathcal{P}_S = \left(\frac{H^2}{2\pi\dot{\phi}_2}\right)^2 \simeq \left(\frac{1}{96\pi^2}\frac{U_{\mathrm{eff}}^3}{(\partial U_{\mathrm{eff}}/\partial\phi_2)^2}\right). \quad (56)$$

The scalar spectral index n_s is given by

$$n_s - 1 = \frac{d\ln\mathcal{P}_S}{d\ln k} = -6\epsilon + 2\eta, \quad (57)$$

where the slow-roll parameters ϵ and η are defined by Eq. (52).

On the other hand, it is well known that the generation of tensor perturbations in the scenario of inflation would generate gravitational waves. In this context, the spectrum of the tensor perturbations $\mathcal{P}_T$ is defined as [89,90]

$$\mathcal{P}_T = \left(\frac{H}{\pi}\right)^2 \simeq \frac{U_{\mathrm{eff}}}{6\pi^2}. \quad (58)$$

Also, the tensor spectral index n_T can be expressed in terms of the slow parameter ϵ as $n_T = \frac{d\ln\mathcal{P}_T}{d\ln k} = -2\epsilon$.

Additionally, an important observational quantity is the tensor-to-scalar ratio $r = \frac{\mathcal{P}_T}{\mathcal{P}_S}$. We mention that these observational quantities should be evaluated when the cosmological scale exits the horizon. In what follows, the subscript $*$ is utilized to indicate the epoch in which the cosmological scale exits the horizon.

Considering the slow-roll approximation, the power spectrum of the scalar perturbation $\mathcal{P}_S$ from Eq. (56) can be written as

$$\mathcal{P}_S * \simeq k_1 e^{\frac{2\alpha_1\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2}}(\phi_{2*}-C_2)}, \quad (59)$$

where the constant k_1 is given by

$$k_1 = \left(\frac{1}{1536\pi^2}\right)\left(\frac{M_1^4}{\chi_2 M_2 g_1^2\alpha_2^2[(\alpha_2/\alpha_1)^2+1]}\right).$$

From Eq. (57), the scalar spectral index n_s becomes

$$n_s * \simeq 1 - \frac{8g_1\alpha_2^2}{M_1}\left[\frac{3g_1(\alpha_1^2+\alpha_2^2)}{M_1\alpha_1^2}e^{-\frac{\alpha_1\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2}}(\phi_{2*}-C_2)} + 1\right] \times e^{-\frac{\alpha_1\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2}}(\phi_{2*}-C_2)}. \quad (60)$$

From Eq. (59) we find that the quantity $\chi_2 M_2 g_1^4/M_1^6$ as a function of the power spectrum and the number of e-folds can be written as

$$\frac{\chi_2 M_2 g_1^4}{M_1^6} = \left(\frac{g_1}{M_1}\right)^4\left(\frac{1}{4U_{(++)}}\right) = \left(\frac{1}{6144\pi^2}\right)\left(\frac{\alpha_1^4}{\alpha_2^2(\alpha_1^2+\alpha_2^2)^2\mathcal{P}_S}\right)e^{\frac{6\alpha_2^2}{\sqrt{(\alpha_2^2/\alpha_1^2)+1}}N_*}. \quad (61)$$

Also, considering Eq. (60) we obtain that the ratio g_1/M_1 has four solutions and the real and positive solution is given by

$$\frac{g_1}{M_1} = \frac{\alpha_1^{3/2}}{\sqrt{12}\alpha_2^{1/2}(\alpha_1^2+\alpha_2^2)^{3/4}} \times \left[1 + \sqrt{1 + \frac{3}{2}\frac{(\alpha_1^2+\alpha_2^2)}{\alpha_1^2\alpha_2^2}(1-n_s)}\right]^{1/2} e^{\frac{3\alpha_1\alpha_2^2}{2\sqrt{\alpha_1^2+\alpha_2^2}}N_*}. \quad (62)$$

Additionally, we find that the tensor-to-scalar ratio r as a function of the number of e-folds N can be written as

$$r(N = N_*) = r_* = \left(\frac{2g_1}{M_1}\right)^4\left[\frac{\alpha_2^4(\alpha_1^2+\alpha_2^2)^2}{\alpha_1^4}\right]e^{\frac{-6\alpha_1^2\alpha_2^2}{(\alpha_1^2+\alpha_2^2)}N_*}, \quad (63)$$

where we have used Eqs. (58) and (59).

By combining Eqs. (61) and (63), we find an upper bond for the parameter α_2 given by

$$\alpha_2 < r_*^{1/2}\left(\frac{6144\pi^2\mathcal{P}_{S*}}{2^6 U_{(++)}}\right)^{1/2}. \quad (64)$$

Also, from Eqs. (61) and (62) we can obtain an equation that gives a relation between α_1 and α_2 given by

$$\left(\frac{3\gamma^2(\alpha_1^2+\alpha_2^2)^{1/2}}{\alpha_1}-1\right)^2 = 1+\frac{3}{2}(1-n_s)\left[\frac{(\alpha_1^2+\alpha_2^2)}{\alpha_1^2\alpha_2^2}\right], \tag{65}$$

where γ is defined as $\gamma = 2[U_{(++)}/(1536\pi^2\mathcal{P}_S)]^{1/4}$.

In particular, for the case in which the tensor-to-scalar ratio takes the value $r_* = 0.036$, $\mathcal{P}_{S*} \simeq 2.2\times 10^{-9}$, and the vacuum energy $U_{(++)} \simeq 6\pi^2 r_* \mathcal{P}_{S_*} \simeq 10^{-8}$ [see Eq. (30)] from Refs. [81–86,91], we obtain from Eq. (64) that the upper limit for the parameter α_2 becomes $\alpha_2 < 2.74$. Now, using this upper bound for $\alpha_2 = 2.74$ and $n_{s_*} = 0.967$, we find from Eq. (65) that the real solution for α_1 is given by $\alpha_1 = 0.24$. In the case in which $r_* = 0.01$, we find that the upper bound for $\alpha_2 \sim 1.44$ and $\alpha_1 \sim 0.07$.

Additionally, in order to find a constraint for the ratio g_1/M_1, we can consider Eq. (61) [or (62)], obtaining that the ratio g_1/M_1 for the special case in which $\alpha_1 = 0.24$, $\alpha_2 = 2.74$, and $N_* = 60$ becomes $g_1/M_1 \simeq 7\times 10^{25}$, and for $\alpha_1 = 0.07$, $\alpha_2 = 1.44$ we get $g_1/M_1 \simeq 4800$.

Notice that the slow-roll trajectory defined by (46) for a given constant defines a straight line in the (φ_1, φ_2) plane in the top vacuum and for another constant defines another parallel line in the top vacuum. We can then choose the line we desire so as to fall in one of the two lower vacuua from the top vacuum.

V. EVOLUTION TO DARK ENERGY AND DARK MATTER

In this section we will analyze the evolution of the dark energy and dark matter as a remnant of the early Universe. After the inflation period has ended there must be a period of particle creation that will produce dark matter as well as ordinary matter; this can be achieved in many different ways, even in the case of one scalar field coupled to different measures [92]. In this section we add now a dark-matter particle contribution, defined in a scale-invariant form by the matter action defined as

$$S_m = \int (\Phi_1 + b_m e^{\kappa_1\phi_2}\sqrt{-g})L_m d^4x, \tag{66}$$

where b_m is a constant that defines the strength to the coupling of ϕ_2 to $\sqrt{-g}$; coupling to Φ_2 does not give a physically different situation, since still Φ_2 and $\sqrt{-g}$ are proportional. Also, the matter Lagrangian density L_m is given by

$$L_m = -\sum_i m_i \int e^{\kappa_2\phi_2}\sqrt{g_{\alpha\beta}\frac{dx_i^\alpha}{d\lambda}\frac{dx_i^\beta}{d\lambda}}\frac{\delta^4(x-x_i(\lambda))}{\sqrt{-g}}d\lambda, \tag{67}$$

where the constants κ_1 and κ_2 satisfy the condition of scale invariance and the quantity m_i denotes the mass parameter of the "ith" particle. This invariance determines the coupling constants to be equal to $\kappa_1 = -\frac{\alpha_1\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2}}$ and $\kappa_2 = -\frac{1}{2}\kappa_1$.

Under these conditions the presence of matter induces a potential for the scalar field ϕ_2 since there is a scalar-field dependence ϕ_2 which multiplies a "density of matter" contribution which is ϕ_2 independent. The scalar field ϕ_2 dependence is of the form

$$\left(e^{-\frac{1}{2}\kappa_1\phi_2}\Phi_1 + b_m e^{\frac{1}{2}\kappa_1\phi_2}\sqrt{-g}\right). \tag{68}$$

Such potential is extremized by the condition

$$\Phi_1 - b_m e^{\kappa_1\phi_2}\sqrt{-g} = 0. \tag{69}$$

Interestingly enough the same condition eliminates all kind of noncanonical anomalous effects, like the appearance of pressure in the contribution to the energy momentum from the particles; see Sec. VII. Also, the constraint equation that was used to determine the ratio of the measures Φ_1 and $\sqrt{-g}$ becomes unaffected by the presence of the dust when the condition above (69) is satisfied (see Sec. VII), so we can use Eq. (23) and in the late Universe, neglecting M_1 and M_2, we obtain an equation that determines ϕ_1. Analogous effects were recognized in a scale-invariant two measure model of gravity, matter, and one scalar field in [93] to obtain the avoidance of the fifth force problem, which the ϕ_2, the dilaton, could possibly cause, since it is a massless field. Here, the avoidance of the fifth force problem is also achieved and we can arrange for this to happen when the scalar field ϕ_1 adjusts itself so as to satisfy the above equation. In this context, we find that the equation for ϕ_1 is given by

$$2\chi_2 f_2 e^{-\frac{\alpha_1^2}{\sqrt{\alpha_1^2+\alpha_2^2}}\phi_1} + 2\chi_2 g_2 e^{\frac{\alpha_1^2}{\sqrt{\alpha_1^2+\alpha_2^2}}\phi_1} = b_m f_1 + b_m g_1 e^{\sqrt{\alpha_1^2+\alpha_2^2}\phi_1}. \tag{70}$$

Thus, Eq. (70) determines the value of ϕ_1 to be a given constant; solving this equation and then the velocity of the scalar field ϕ_1 is zero, i.e., $\dot{\phi}_1 = 0$. In order to determine the value of the scalar field ϕ_1 we consider $x = e^{\frac{\alpha_1^2\phi_1}{\sqrt{\alpha_1^2+\alpha_2^2}}}$; then, Eq. (70) can be rewritten as

$$2\chi_2 g_2 x^2 - b_m g_1 x^{\frac{2\alpha_1^2+\alpha_2^2}{\alpha_1^2}} - b_m f_1 x + 2\chi_2 f_2 = 0. \tag{71}$$

Interestingly enough, the field ϕ_2 drops from this equation. This is quite reasonable since the field ϕ_2 undergoes a shift under the scale transformation, so if we were to determine the field ϕ_2, that would correspond to a breaking of scale invariance, but now we are working in a phase with exact scale invariance, since we are neglecting the scale symmetry-breaking constants M_1 and M_2. The field ϕ_2 is decoupled from matter, which is a consequence of the elimination of the 5th force.

In order to obtain a solution for the scalar field ϕ_1 from Eqs. (70) or (71), we consider that for very large value of ϕ_1 or equivalently $x \to \infty$ the dominate terms of Eq. (71) are

$$2\chi_2 g_2 x^2 - b_m g_1 x^{\frac{2\alpha_1^2+\alpha_2^2}{\alpha_1^2}} \sim 0, \quad \text{then } x \sim \left(\frac{2\chi_2 g_2}{g_1 b_m}\right)^{(\alpha_1/\alpha_2)^2}, \tag{72}$$

where for consistency, we must choose the quantity $(\chi_2 g_2/g_1 b_m) \to \infty$. Here, the value of the scalar field ϕ_1 at this point is

$$\phi_{1(+)} \sim \frac{\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_2^2} \ln\left[\frac{2\chi_2 g_2}{f_1 b_m}\right]. \tag{73}$$

Now, in the region in which the scalar field $\phi_1 \to -\infty$ or $x \to 0$ we have that the dominant terms are

$$-b_m f_1 x + 2\chi_2 f_2 \sim 0, \quad \text{and} \quad x \sim \left(\frac{2\chi_2 f_2}{f_1 b_m}\right) \to 0, \tag{74}$$

and the value of the scalar field at this point is

$$\phi_{1(-)} \sim \frac{\sqrt{\alpha_1^2+\alpha_2^2}}{\alpha_1^2} \ln\left[\frac{2\chi_2 f_2}{f_1 b_m}\right]. \tag{75}$$

In what follows of this section we study the dynamics of the dark energy and as defined before, with the equations for the ratio of the two measures obtained in the absence of dark matter (23) still being valid, so we can still consider the effective potential for the dark energy by Eq. (25) and the dark matter is described as a dust since all noncanonical effects disappear when $\Phi_1 - b_m e^{\kappa_1\phi_2}\sqrt{-g} = 0$ is satisfied. When we also work in the very flat region, there is also no inconsistency with ϕ_1 being a constant.

The flat-Friedmann equation for this stage is given by

$$6H^2 = \rho_{\varphi_1,\varphi_2} + \rho_m, \tag{76}$$

where the energy density $\rho_{\varphi_1,\varphi_2}$ associated with the scalar fields φ_1 and φ_2 is

$$\rho_{\varphi_1,\varphi_2} = \frac{\dot{\varphi}_1^2}{2} + \frac{\dot{\varphi}_2^2}{2} + U_{\text{eff}}(\varphi_1,\varphi_2). \tag{77}$$

For the energy density of the dark matter ρ_m we have

$$\dot{\rho}_m + 3H\rho_m = 0, \quad \text{then } \rho_m(a) \propto \left(\frac{1}{a}\right)^3.$$

From Eq. (25) and considering the region in which $f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2} \gg M_1$ and $f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2} \gg M_2$, the effective potential reduces to

$$U_{\text{eff}}(\varphi_1,\varphi_2) = \frac{(f_1 e^{-\alpha_1\varphi_1} + g_1 e^{-\alpha_2\varphi_2})^2}{4\chi_2(f_2 e^{-2\alpha_1\varphi_1} + g_2 e^{-2\alpha_2\varphi_2})}. \tag{78}$$

From Eq. (45) we have that the effective potential given by Eq. (78) can be rewritten in terms of the single scalar field ϕ_1 in which

$$U_{\text{eff}}(\phi_1) = \frac{(f_1 e^{-\sqrt{\alpha_1^2+\alpha_2^2}\phi_1} + g_1)^2}{4\chi_2(f_2 e^{-2\sqrt{\alpha_1^2+\alpha_2^2}\phi_1} + g_2)}. \tag{79}$$

As we have seen before, the condition that the matter-induced potential of the scalar field ϕ_2 is extremized requires the scalar field ϕ_1 to be fixed at a very well-specified point and now given the scalar-field potential above, the equation of motion of ϕ_1 requires that this constant value be located at one of the two flat regions of the above potential at $\phi_{1(+)}$ and $\phi_{1(-)}$; see Eqs. (73) and (75). Thus, the energy density associated with the dark energy can be written as

$$\rho_{\varphi_1,\varphi_2} = \rho_{\phi_1,\phi_2} = \frac{\dot{\phi}_1^2}{2} + \frac{\dot{\phi}_2^2}{2} + U_{\text{eff}}(\phi_1) = \frac{\dot{\phi}_2^2}{2} + U_{\text{eff}}(\phi_1), \tag{80}$$

where now the effective potential U_{eff} depends only on the scalar field ϕ_1. Also, as we have seen, the scalar field ϕ_1 has been fixed to a constant because of the extremization of the ϕ_2 matter-induced potential, so we take $\dot{\phi}_1 = 0$; see Eq. (70).

In order to study the evolution of the our model, we can choose the first flat region after inflation for the effective potential given by Eq. (79) assuming a very large scalar field ϕ_1 given by $U_{\text{eff}+} = g_1^2/(4\chi_2 g_2)$, where the value of the scalar field ϕ_1 is fixed in this flat region by Eq. (72) $x \sim (2\chi_2 g_2/g_1 b_m)^{(\alpha_1/\alpha_2)^2} = (g_1/[2b_m U_{\text{eff}+}])^{(\alpha_1/\alpha_2)^2}$ or equivalently $\phi_{1(+)}$ defined by Eq. (73). For the second flat region after the inflation we can consider $\phi_1 \to -\infty$ where the effective potential in this region is $U_{\text{eff}-} = (f_1^2/4\chi_2 f_2)$. Here, the value of the scalar field ϕ_1 is fixed at $x \sim (2\chi_2 f_2/f_1 b_m) = (f_1/2b_m U_{\text{eff}-})$ or Eq. (75).

Additionally, we can note that the scalar field ϕ_2 corresponds to a massless field. In this way, the evolution of the scalar field ϕ_2 as a function of the scale factor results,

$$\ddot{\phi}_2 + 3H\dot{\phi}_2 = 0, \to \quad \dot{\phi}_2 = \frac{B_1}{a^3} = \dot{\phi}_{2+}\left(\frac{a_+}{a}\right)^3, \tag{81}$$

where B_1 denotes an integration constant. By convenience $B_1 = \dot{\phi}_{2+} a_+^3$, where a_+ and $\dot{\phi}_{2+}$ correspond to the scale factor and the velocity of the scalar field in the first flat regime of the effective potential $U_{\mathrm{eff}(+)} = g_1^2/(4\chi_2 g_2)$.

The evolution of the scalar field ϕ_2 as a function of the scale factor can be obtained considering that $\dot{\phi}_2 = aH da/dt$; then, Eq. (81) can be rewritten as

$$\frac{d\phi_2}{da} = \frac{B_1}{a^4 H}, \tag{82}$$

and the Hubble parameter in terms of the scalar field is given by

$$H = \frac{1}{\sqrt{6}}\left[\frac{B_1^2}{2a^6} + U_{\mathrm{eff}(+)} + \frac{B_2}{a^3}\right]^{1/2}, \quad \text{with} \quad B_2 = \rho_{m_+} a_+^3, \tag{83}$$

where ρ_{m+} is the energy density associated with the dark matter in the first flat region of the effective potential $U_{\mathrm{eff}(+)}$. In particular, we have that in the first region the quantities ρ_{m+} and $\dot{\phi}_{2+}$ become

$$\begin{aligned} \rho_{m+} &= 6H_+^2 \Omega_{m+}, \\ \dot{\phi}_{2+} &= [2(6H_+^2 \Omega_{\phi_{+1},\phi_{+2}} - U_{\mathrm{eff}(+)})]^{1/2}. \end{aligned} \tag{84}$$

In this way, we find that the evolution of the scalar field as a function of the scale factor becomes

$$\begin{aligned} \phi_2(a) = \phi_{2_+} &+ \frac{2}{\sqrt{3}}\left[\mathrm{Arctanh}\left(\frac{B_1^2 + B_2 a_+^3}{B_1\sqrt{B_1^2 + 2a_+^3(B_2 + U_{\mathrm{eff}(+)} a_+^3)}}\right)\right] \\ &- \frac{2}{\sqrt{3}}\left[\mathrm{Arctanh}\left(\frac{B_1^2 + B_2 a^3}{B_1\sqrt{B_1^2 + 2a^3(B_2 + U_{\mathrm{eff}(+)} a^3)}}\right)\right]. \end{aligned} \tag{85}$$

Also, we can determine the equation of state (EoS) or EoS parameter w associated with the scalar fields in terms of the scale factor given by

$$w(a) = \frac{\frac{\dot{\phi}_2^2}{2U_{\mathrm{eff}(+)}} - 1}{\frac{\dot{\phi}_2^2}{2U_{\mathrm{eff}(+)}} + 1} = \frac{\left(\frac{2\chi_2 g_2 B_1^2}{g_1^2}\right)a^{-6} - 1}{\left(\frac{2\chi_2 g_2 B_1^2}{g_1^2}\right)a^{-6} + 1}. \tag{86}$$

Additionally, the total EoS parameter w_T associated with dark matter and scalar fields becomes

$$w_T = \frac{w}{(1 + \rho_m/\rho_{\phi_1,\phi_2})}, \tag{87}$$

and in terms of the scale factor the EoS parameter $w_T(a)$ is given by

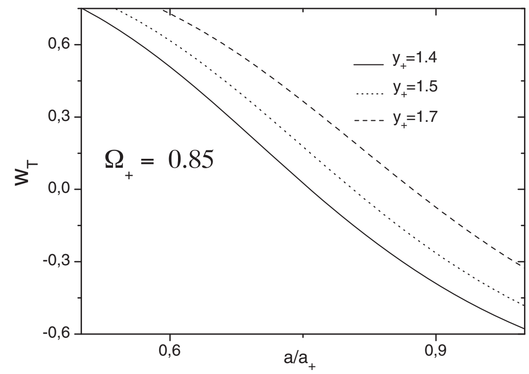

FIG. 2. In this plot we show the evolution of the total EoS parameter as a function of the scale factor $\tilde{a} = a/a_+$ in the first flat region of the potential $U_{\mathrm{eff}(+)} = g_1^2/(4\chi_2 g_2)$, for different values of the ratio $y_+ = 6H_+^2/U_{\mathrm{eff}(+)}$; see Eq. (89). In the first flat region we have used that the density parameter associated with the dark energy corresponds to $\Omega_+ = 0.85$, in order to satisfy the constraint from nucleosynthesis.

$$w_T(a) = \left[\frac{(\frac{2\chi_2 g_2 B_1^2}{g_1^2})a^{-6} - 1}{(\frac{2\chi_2 g_2 B_1^2}{g_1^2})a^{-6} + 1}\right] \times \left(1 + \frac{B_2 a^{-3}}{(B_1^2/2)a^{-6} + (g_1^2/4\chi_2 g_2)}\right)^{-1}. \tag{88}$$

We note that Eq. (88) can be rewritten in terms of the density parameter Ω_+, by considering the Friedmann equation in which $1 = \Omega_+ + \Omega_{m+}$, where Ω_+ and Ω_{m+} denote the densities parameters of different components in the first flat region and then the EoS parameter becomes

$$w_T(a) = \left[\frac{(\Omega_+ y_+ - 1)\tilde{a}^{-6} - 1}{(\Omega_+ y_+ - 1)\tilde{a}^{-6} + 1}\right] \times \left(1 + \frac{y_+(1 - \Omega_+)\tilde{a}^{-3}}{(\Omega_+ y_+ - 1)\tilde{a}^{-6} + 1}\right)^{-1}, \tag{89}$$

where the new scale factor $\tilde{a}$ is defined as $\tilde{a} = a/a_+$ and the quantity y_+ corresponds to the rate $y_+ = 6H_+^2/U_{\mathrm{eff}(+)}$ and H_+ is the Hubble parameter in the first flat region. As the kinetic energy is defined as positive, then the condition for the quantity y_+ is $y_+ > 1/\Omega_+$.

In Fig. 2 we show the development of the total EoS parameter w_T versus the scale factor $\tilde{a} = a/a_+$, in the first flat region of the effective potential $U_{\mathrm{eff}(+)}$ for different values of the ratio $y_+ = 6H_+^2/U_{\mathrm{eff}(+)} > 1/\Omega_+$. We choose that the value of the density parameter of the dark energy in the flat region is $\Omega_+ = 0.85$. From the plot we observe that when we increase the ratio y_+ the total EoS parameter w_T also increases. Also, we note that for values of the scale factor $a < a_+$, the Universe does not present an accelerated phase, since the total EoS parameter w_T approaches positive values. However, for values of $a \sim a_+$, we observe that the total EoS parameter is $w_T < -0.3$ and the universe shows an accelerated expansion for values of y_+ near to $1/\Omega_+$.

In addition, it is interesting to analyze the evolution of the barotropic parameter w associated with the dark energy. Following Refs. [94–97], we can distinguish two categories from the behavior of $(dw/d\phi_2)$: the tracking freezing $(dw/d\phi_2 < 0)$ and thawing $(dw/d\phi_2 > 0)$ models. Thus, in our case for the first flat region and considering that $dw/d\phi_2 = (dw/d\tilde{a})(d\tilde{a}/d\phi_2)$, we find

$$\begin{aligned}\frac{dw}{d\phi_2} &= \frac{dw}{d\tilde{a}}\frac{d\tilde{a}}{d\phi_2}\\ &= -12\sqrt{12}\left(\frac{[\Omega_+ y_+ - 1]}{1 + y_+\Omega_+\tilde{a}^{-3} + [y_+ - 1 - \Omega_+]\tilde{a}^{-6}}\right)^{1/2}\\ &\quad\times\left(\frac{[\Omega_+ y_+ - 1]}{[\Omega_+ y_+ - 1 + \tilde{a}^6]^2}\right)\tilde{a} < 0. \end{aligned} \tag{90}$$

Here, we have used Eqs. (82) and (86). In this form, we can infer that our model has a behavior of freezing model, since $dw/d\phi_2$ is negative. This behavior of the model occurs because that the ratio $dw/d\tilde{a}$ results negative, i.e., $dw/d\tilde{a} < 0$, and then we have a tracking freezing model. In this sense, we have that in general for these tracking freezing models, are initially characterized by $w > -1$ and $adw/da = dw/d\ln a < 0$ [98]. Thus, the tracker fields are characterized by having the attractor-like solutions that converge to a common cosmic evolutionary track from different initial condition [99,100]. This suggests that the cosmology of the late time is independent of initial condition due to behavior of w given by Eq. (86).

A new and important constraint on the density energy associated with the dark energy during the radiation stage results from the nucleosynthesis. It is well known that the quintessence scalar field modifies expansion of the Universe at a given temperature and in particular during the nucleosynthesis where the temperature $T \sim 1$ MeV, see [101,102]. Following Ref. [101], the energy density of the scalar field during this scenario can be constrained to $\Omega_\phi(T \sim 1\text{ MeV}) < 7\Delta N_{\rm eff}/4/(10.75 + 7\Delta N_{\rm eff}/4)$, where the value 10.75 corresponds to the effective number of standard model degrees of freedom and the quantity $\Delta N_{\rm eff}$ denotes the additional relativistic degrees of freedom. In relation to the additional relativistic degrees of freedom in the literature it is considered as $\Delta N\text{eff} \simeq 1.5$ [103] (see also Ref. [104] where $\Delta N_{\rm eff} \simeq 0.9$). Thus, considering $\Delta N_{\rm eff} \simeq 1.5$, then any quintessences models require to satisfy $\Omega_\phi(T \sim 1\text{ MeV}) < 0.2$ during the nucleosynthesis.

For our model we find that the density energy associated with dark energy $\Omega_{\phi_1,\phi_2} = \rho_{\phi_1,\phi_2}/6H^2$ can be written

$$\Omega_{\phi_1,\phi_2}(a) = \left(1 + \frac{([\Omega_+ y_+ - 1]\tilde{a}^{-6} + 1)\tilde{a}^3}{y_+(1 - \Omega_+)}\right)^{-1}. \tag{91}$$

Thus, in order to satisfy the constraint imposed by the nucleosynthesis at the temperature $T \sim 1$ MeV, we obtain the following bounds:

$$\frac{4}{5} < \Omega_+ < 1, \qquad 0 < \tilde{a}^3_{T_*} < \frac{4 - 4\Omega_+}{\Omega_+},$$

and

$$y_+ > \frac{1 - \tilde{a}^6_{T_*}}{\Omega_+(1 + 4\tilde{a}^3_{T_*}) - 4\tilde{a}^3_{T_*}}, \tag{92}$$

where $\tilde{a}_{T_*}$ corresponds to the scale factor evaluated at the temperature $T_* = 1$ MeV i.e., $\tilde{a}_{T_*} = a(T_* = 1\text{ MeV})/a_+$. Here, we note that the nucleosynthesis epoch imposes a strong condition on the density parameter Ω_+ and ratio $6H_+^2/U_{\text{eff}(+)} = y_+$ in the flat region.

On the other hand, during the second flat regime associated with the effective potential $U_{\text{eff}(-)}$, the evolution of the scalar field ϕ_2 as a function of the scale factor can be obtained considering as before that $\dot{\phi}_2 = aHda/dt$; then, Eq. (81) can be rewritten as

$$\frac{d\phi_2}{da} = \frac{\tilde{B}_1}{a^4 H}, \quad \text{where } \tilde{B}_1 = \dot{\phi}_{02}a_0^3, \tag{93}$$

where $\dot{\phi}_{02}$ and a_0 denote the velocity of the scalar field and the scale factor at the present epoch.

As before, the Hubble parameter in terms of the scale factor in this region is given by

$$H = \frac{1}{\sqrt{6}}\left[\frac{\tilde{B}_1^2}{2a^6} + U_{\text{eff}(-)} + \frac{\tilde{B}_2}{a^3}\right]^{1/2}, \quad \text{with} \quad \tilde{B}_2 = \rho_{m0}a_0^3, \tag{94}$$

where $U_{\text{eff}(-)}$ corresponds to the effective potential for very negative large scalar field ϕ_1 and it is defined as $U_{\text{eff}(-)} = f_1^2/(4\chi_2 f_2)$, from Eq. (79). Also, the value ρ_{m0} corresponds to the dark energy of the matter at the present epoch in which the scale factor $a = a_0 = 1$. From the Friedmann equation we have $1 = \Omega_{\phi_1,\phi_2} + \Omega_m$, where Ω_{ϕ_1,ϕ_2} and Ω_m denote the densities parameters of the different components. In particular, from this equation we obtain that at present era the quantities ρ_{m0} and $\dot{\phi}_{0_2}$ become

$$\begin{aligned}\rho_{m0} &= 6H_0^2\Omega_{m0},\\ \dot{\phi}_{0_2} &= [2(6H_0^2\Omega_{\phi_{01},\phi_{0_2}} - U_{\text{eff}(-)})]^{1/2}, \end{aligned} \tag{95}$$

where from the observational data we have $\Omega_{m0} \simeq 0.3$ and $\Omega_{\phi_{01},\phi_{02}} \simeq 0.7$.

Also, we obtain that the evolution of the scalar field as a function of the scale factor during this second scenario results:

$$\phi_2(a)=\phi_{20}+\frac{2}{\sqrt{3}}\left[\mathrm{Arctanh}\left(\frac{\tilde{B}_1^2+\tilde{B}_2a_0^3}{\tilde{B}_1\sqrt{\tilde{B}_1^2+2a_0^3(\tilde{B}_2+U_{\mathrm{eff}(-)}a_0^3)}}\right)\right]-\frac{2}{\sqrt{3}}\left[\mathrm{Arctanh}\left(\frac{\tilde{B}_1^2+\tilde{B}_2a^3}{\tilde{B}_1\sqrt{\tilde{B}_1^2+2a^3(\tilde{B}_2+U_{\mathrm{eff}(-)}a^3)}}\right)\right]. \quad (96)$$

As before, we can determine the EoS parameter w associated with the scalar fields in terms of the scale factor during this second flat region:

$$w(a)=\frac{\frac{\dot{\phi}_2^2}{2U_{\mathrm{eff}-}}-1}{\frac{\dot{\phi}_2^2}{2U_{\mathrm{eff}-}}+1}=\frac{\left(\frac{2\chi_2f_2\tilde{B}_1^2}{f_1^2}\right)a^{-6}-1}{\left(\frac{2\chi_2f_2\tilde{B}_1^2}{f_1^2}\right)a^{-6}+1}. \quad (97)$$

Also, we find that the total EoS parameter $w_T=w_T(a)$ associated with dark matter and scalar fields during this scenario results:

$$w_T(a)=\left[\frac{\left(\frac{2\chi_2f_2\tilde{B}_1^2}{f_1^2}\right)a^{-6}-1}{\left(\frac{2\chi_2f_2\tilde{B}_1^2}{f_1^2}\right)a^{-6}+1}\right]\times\left(1+\frac{\tilde{B}_2a^{-3}}{(\tilde{B}_1^2/2)a^{-6}+(f_1^2/4\chi_2f_2)}\right)^{-1}. \quad (98)$$

Also, we can rewrite Eq. (98) in terms of the density parameter at present epoch $\Omega_{\phi_{01},\phi_{02}}=\Omega_-$, and then the total EoS parameter becomes

$$w_T(a)=\left[\frac{(\Omega_-y_--1)a^{-6}-1}{(\Omega_-y_--1)a^{-6}+1}\right]\times\left(1+\frac{y_-(1-\Omega_-)a^{-3}}{(\Omega_-y_--1)a^{-6}+1}\right)^{-1}, \quad (99)$$

with the scale factor $a/a_0=a$ and the quantity y_- corresponds to the rate $y_-=6H_0^2/U_{\mathrm{eff}(-)}$. As the kinetic energy is positive, then we determine that the condition for the parameter $y_->1/\Omega_-$. In particular, we have that the density parameter at the present associated with dark energy $\Omega_-\simeq 0.7$, such that $y_->10/7$.

In Fig. 3 we show the evolution of the total EoS parameter w_T versus the scale factor $a/a_0=a$ for different values of the ratio $y_-=6H_0^2/U_{\mathrm{eff}(-)}>1/\Omega_-$. From the observational data we have considered that the density parameter associated with the dark energy at the present era $\Omega_-=0.7$. As before in Fig. 2, from the plot we note that when we increase the ratio y_-, the total EoS parameter w_T also grows. We observe that for values of the ratio $y_-\gg 1/\Omega_-$, the Universe does not present an accelerated phase until now, since $w_T>-1/3$.

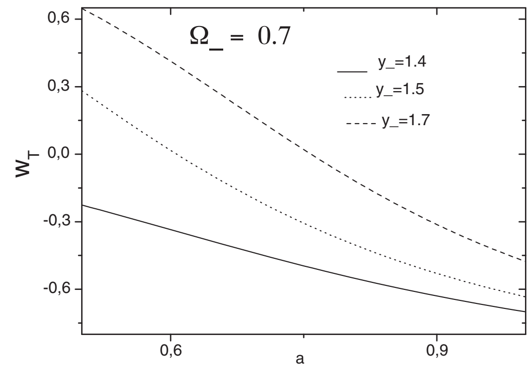

FIG. 3. In this plot we show the evolution of the total EoS parameter w_T as a function of the scale factor $a/a_0=a$ in the second flat region of the effective potential $U_{\mathrm{eff}(-)}=f_1^2/(4\chi_2f_2)$. Here, we have considered different values of the ratio $y_-=6H_0^2/U_{\mathrm{eff}(-)}$, in Eq. (99). At the present time we have used that the density parameter associated with the dark energy is $\Omega_-=0.7$ and the scale factor $a_0=1$.

On other hand, we can obtain some estimates and constraints on the parameter space of our model. For the second flat region of the effective potential $U_{\mathrm{eff}(-)}$, we can choose that the scales of the scale symmetry-breaking integration constants $f_1\sim M_{\mathrm{EW}}^4$ and $\chi_2f_2\sim M_{\mathrm{Pl}}^4$, where $M_{\mathrm{EW}}, M_{\mathrm{Pl}}$ are the electroweak and Plank scales, respectively. In this case, we have a very small vacuum energy density $U_{(\phi_1\to-\infty)}=U_{\mathrm{eff}(-)}\sim f_1^2/\chi_2f_2$ given by

$$U_{\mathrm{eff}(\phi_1\to-\infty)}=U_{\mathrm{eff}(-)}\sim M_{\mathrm{EW}}^8/M_{\mathrm{Pl}}^4\sim 10^{-120}M_{\mathrm{Pl}}^4, \quad (100)$$

where the mass $M_{\mathrm{EW}}\sim 10^{-15}M_{\mathrm{Pl}}$ and Eq. (100) corresponds to the right order of magnitude for the present epoch's vacuum energy density; see Ref. [105]. Thus, we can assume that the parameter $f_1\sim 10^{-60}$ (in units of Planck mass to the fourth power).

In order to transfer the information of the inflationary stage to the present epoch, we can consider the constraints from inflationary scenario. In this context, we can utilize the constraint from inflation for the ratio $g_1/M_1\sim 10^{26}$ for the special case in which $r_*=0.036$. In this way, we find that the effective potential in the first flat region during the late Universe $U_{\mathrm{eff}(+)}$ can be written as

$$U_{\mathrm{eff}(\phi_1\to\infty)}=U_{\mathrm{eff}(+)}\simeq\frac{g_1^2}{4\chi_2g_2}\sim 10^{52}\frac{M_2U_{(++)}}{g_2}\sim 10^{44}\frac{M_2}{g_2}>U_{\mathrm{eff}(-)}. \quad (101)$$

Here, we have considered that during inflation the energy density is $U_{(++)}\simeq 10^{-8}$. Also, as we have assumed that the effective potentials in the flat regions satisfied the condition

$U_{\mathrm{eff}(+)} > U_{\mathrm{eff}(-)} \sim 10^{-120}$, then we find that lower bound for the ratio M_2/g_2 becomes $M_2/g_2 > 10^{-164}$. In this form, we obtain that the ratio between the parameters associated with inflation (M_1 and M_2) and the first dark-energy region (g_1 and g_2) results,

$$\frac{M_2}{M_1} > 10^{-138} \frac{g_2}{g_1}. \tag{102}$$

Here, we have used that the ratio $g_1/M_1 \sim 10^{26}$.

VI. DEPENDENCE OF THE POINT-PARTICLE MASSES ON THE SCALAR FIELD ϕ_1 AND ITS CONSEQUENCES

One particular aspect that should be studied is the dependence of the point-particle masses on the scalar field ϕ_1 and its consequences. We study this field dependence when the condition (69) is satisfied, which implies certain values of the scalar field ϕ_1 are allowed. In this case we can solve the measure Φ_1 using (69) and then considering the action in the Einstein frame. In such situation, a straightforward calculation shows that the masses of particles depend only on the scalar field ϕ_1 in the following way:

$$m_{i\text{th-part}}(\phi_1) = 2m_i b_m \sqrt{\frac{f_1 e^{-\sqrt{\alpha_1^2+\alpha_2^2}\phi_1} + g_1}{2\chi_2(f_2 e^{-2\sqrt{\alpha_1^2+\alpha_2^2}\phi_1} + g_2)}} e^{-\frac{\alpha_2^2\phi_1}{2\sqrt{\alpha_1^2+\alpha_2^2}}}. \tag{103}$$

As we can see from this equation, the particles in the solution with large ϕ_1, which correspond to the larger dark energy, will have a much smaller mass than the same particle when located at the vacuum with a much smaller value of ϕ_1. In a possible transition of these states, which will necessarily break condition (69), their DE and DM component will behave therefore in an opposite way after the process is completed and at the end point (69) is restored again, so, as a result, when DE decreases, the DM component masses increase; of course, the DM component is still being diluted by the expansion of the Universe, but enhanced by their increase in particle masses. As long as the particles remain in the states that satisfy (69), the masses are fixed of course and the dust behaves canonically as described in the previous section. The discussion here concerns a transition between the two states that we have found that satisfy (69), and the masses displayed by (103) concern masses only for such states. During the transition itself, the condition given by Eq. (69) must be violated, since this condition allows only a discrete set of values, like those provided in (72) and (74) only.

VII. CONDITIONS FOR CANONICAL DUST BEHAVIOR BEYOND THE BACKGROUND CASE

In our previous considerations we have only considered cases where the scalar fields and the dust are distributed homogeneously in the Universe and we have also chosen the scalar field ϕ_1 by the observation that the presence of matter induces a potential for the scalar field ϕ_2 since there is a scalar-field dependence ϕ_2 which multiplies a density of matter contribution which is ϕ_2 independent and the result of such minimization led us to a value of ϕ_1 defined by Eq. (69), which in turn led us to a dust behavior for our model of point particles coupled in a scale-invariant fashion. Here, we will go a bit deeper, following the method studied in [93] for a single scalar field (for earlier treatments of the 5th force problems for more field-theoretical models of matter rather than for point-particle models of matter; see [106,107]), and establish the more detailed conditions where this procedure can be more rigorously justified, we consider $\chi_2 = 1$, since this constant can be reabsorbed into the definitions of particle densities, etc. In this case, for the purpose of this paper we restrict ourselves to a zero-temperature gas of particles, i.e., we will assume that $d\vec{x}_i/d\lambda \equiv 0$ for all particles, which can be interpreted as the particles moving as comoving; similar conclusions are easily derived without this assumption nevertheless. It is convenient to proceed in the frame where $g_{0l} = 0$, $l = 1, 2, 3$. Then, the particle density is defined by

$$n(\vec{x}) = \sum_i \frac{1}{\sqrt{-g_{(3)}}} \delta^{(3)}(\vec{x} - \vec{x}_i(\lambda)), \tag{104}$$

where $g_{(3)} = \det(g_{kl})$. We transform to the Einstein frame where this transformation causes the transformation of the particle density:

$$\bar{n}(\vec{x}) = (\chi_1)^{-3/2} n(\vec{x}). \tag{105}$$

The gravitational equations take the standard general relativity (GR) form:

$$G_{\mu\nu}(\bar{g}_{\alpha\beta}) = \frac{\kappa}{2} T^{\mathrm{eff}}_{\mu\nu}, \tag{106}$$

where $G_{\mu\nu}(\bar{g}_{\alpha\beta})$ is the Einstein tensor in the Riemannian space-time with the metric $\bar{g}_{\mu\nu}$. The components of the effective energy-momentum tensor are as follows:

$$T^{\mathrm{eff}}_{00} = (\dot{\phi_1}^2 - \bar{g}_{00} X_1) + (\dot{\phi_2}^2 - \bar{g}_{00} X_2) + \bar{g}_{00}\left[U_{\mathrm{eff}}(\phi_1, \phi_2; \chi_1, M_1, M_2) + \frac{3\chi_1 e^{-\frac{\kappa_1 \phi_2}{2}} + b_m e^{\frac{\kappa_1 \phi_2}{2}}}{2\sqrt{\chi_1}} m\bar{n}\right], \tag{107}$$

and

$$T_{ij}^{\rm eff}=(\phi_{1,k}\phi_{1,l}-\bar{g}_{kl}X_1)+(\phi_{2,k}\phi_{2,l}-\bar{g}_{kl}X_2)+\bar{g}_{kl}\left[U_{\rm eff}(\phi_1,\phi_2,\chi_1,M_1,M_2)+\frac{\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}}{2\sqrt{\chi_1}}m\bar{n}\right]. \quad (108)$$

Here, the following notations have been used:

$$X_1\equiv-\frac{1}{2}\bar{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}\quad\text{and}$$
$$X_2\equiv-\frac{1}{2}\bar{g}^{\alpha\beta}\phi_{2,\alpha}\phi_{2,\beta}, \quad (109)$$

and the function $U_{\rm eff}$ is defined by

$$U_{\rm eff}(\phi_1,\phi_2;\chi_1)=\frac{1}{\chi_1}[M_1-V]+\frac{\chi_2}{\chi_1^2}(U+M_2), \quad (110)$$

where χ_1 has to be solved now for the case particles are present, which may differ somewhat from the solution in vacuum. The dilaton ϕ_2 field equation is sourced by matter particles and in the Einstein frame is as follows:

$$\frac{1}{\sqrt{-\bar{g}}}\partial_\mu[\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu\phi_2]+\frac{\partial U_{\rm eff}}{\partial\phi_2}=\kappa_1\frac{\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}}{2\sqrt{\chi_1}}m\bar{n}. \quad (111)$$

In the above equations, the scalar field χ_1 is determined as a function $\chi_1(\phi_1,\phi_2,\bar{n})$ by means of the following constraint:

$$\frac{\chi_1(M_1+V)-2\chi_2(U+M_2)}{(\chi_1)^2}=\frac{\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}}{2\sqrt{\chi_1}}m\bar{n}. \quad (112)$$

In summary, a "miracle" takes place here; the same combination $\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}$ appears in the right-hand side of Eqs. (112), (111) and in the anomalous pressure contribution produced by the dust displayed in (108). The vanishing of $\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}$ was also obtained in our simplified considerations in Eq. (69) from the condition of minimization of the matter-induced potential for ϕ_2, which (111) expresses in its full generality.

The 5th force resolution for dense matter: Notice that in parallel to the idea of minimizing a matter-induced potential, which gave us the vanishing of the right-hand side of Eq. (111), we can look at Eq. (112) as consisting of two parts: the right-hand side can be compared with the energy density of the scalar fields (110), so we can indeed say that this side is of the order of magnitude of this DE, but the other side on the other hand is proportional to the energy density of matter and for matter in ordinary state, which has energy density much larger than the vacuum energy of the Universe, the only way to have consistently is to have the coefficient $\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}$ that appears in the right-hand side of Eqs. (112) to be very close to zero. This coefficient represents the strength of the coupling of the scalar field ϕ_2 to matter.

The next important issue to take notice of is that once $\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}$ is taken to be zero, because this minimizes the matter-induced potential for ϕ_2, this leads us the vanishing of the right-hand side of (112) and as a consequence to the same solution for χ_2 as we obtained in vacuum [Eq. (23)], which means that we can use the expressions for the effective potential in vacuum, now in the presence of dust. The dust is now totally canonical, as we have assumed in sections above, provided $\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}=0$, which determine spacial values for ϕ_1 in each of the flat regions of the effective potential as we have seen.

Finally, the resulting effective potential in these flat regions is absolutely independent of ϕ_2 as we have seen, so $\frac{\partial U_{\rm eff}}{\partial\phi_2}=0$ and furthermore there is no source since $\chi_1 e^{-\frac{\kappa_1\phi_2}{2}}-b_m e^{\frac{\kappa_1\phi_2}{2}}$ is taken to be zero, so that indeed, (111) implies then that $\frac{1}{\sqrt{-\bar{g}}}\partial_\mu[\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu\phi_2]=0$ as we have assumed.

Let us analyze consequences of this wonderful coincidence in the case when the matter energy density (modeled by dust) is much larger than the dilaton contribution to the dark-energy density in the space region occupied by this matter. Evidently this is the condition under which all tests of Einstein's GR, including the question of the fifth force, are fulfilled. If the dust is in the normal conditions there is a possibility to provide the desirable feature of the dust in GR: it must be pressureless. This is realized provided that in normal conditions (n.c.) the following equality holds with extremely high accuracy:

$$\chi_1^{(n.c.)}\approx b_m e^{\kappa_1\phi_2}. \quad (113)$$

Remember that we have assumed $b_m>0$, Inserting the above equation in the last term of Eq. (107) we obtain the effective dust-energy density in n.c., where the dependence on ϕ_2 has disappeared, as it should be for acceptable resolution of the 5th force problem:

$$\rho_m^{(n.c.)}=2\sqrt{b_m}m\tilde{n}. \quad (114)$$

When we get only a slight deviation of from χ_1 from $b_m e^{\kappa_1\phi_2}$, when the matter-energy density is many orders of magnitude larger than the dilaton contribution to the dark-energy density, we obtain an effective 5th force coupling f. For this, look at the ϕ equation in the form (111) and estimate the Yukawa-type coupling constant in the rhs of this equation. In fact, using the constraint (112) and representing the particle density in the form $\tilde{n} \approx N/\upsilon$, where N is the number of particles in a volume υ, one can make the following estimation for the effective dilaton to matter-coupling "constant" f defined by the Yukawa-type interaction term $f\bar{n}\phi$ [if we were to invent an effective action whose variation with respect to ϕ would result in Eq. (111)]:

$$f \equiv \kappa_1 \frac{\chi_1 e^{-\kappa_1\phi_2/2} - b_m e^{\kappa_1\phi_2/2}}{2\sqrt{\chi_1}} \approx \kappa_1 \frac{\rho_{vac}}{\tilde{n}} \approx \kappa_1 \frac{\rho_{vac}\upsilon}{N}. \quad (115)$$

If we consider that κ_1 is a number divided by the Planck mass, then f becomes less than the ratio of the "mass of the vacuum" in the volume occupied by the matter to the Planck mass. The model yields this kind of "Archimedes law" without any special (intended for this) choice of the underlying action and without fine tuning of the parameters. The model not only explains why all attempts to discover a scalar-force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter), and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

Finally, we want to point out fundamental differences of our solution of the fifth force force problem to the Chameleon approach. The important point to make is that we are talking of totally different mechanisms; in the Chameleon model, the proposed quintessential scalar, the Chameleon field has a mass in vacuum which is very small, of the order of the Hubble parameter for example (or in any case very very small). The Chameleon scalar however becomes massive in presence of dense matter, in compact objects, like Earth; a typical number for this mass has been cited, $m^{-1} \sim 60_{mm}$ [108]. This is why a quanta of this scalar field can penetrate only into a thin shell of the body in the depth about 60 micrometer, and the fifth force acts only on the thin shell. This is a way the Chameleon model is argued to explain the smallness of the fifth force. In our case there is no mass generation whatsoever since for our dilaton field what happens here is the vanishing of the effective coupling constant between the dilaton field and the dense matter, while the dilaton keeps its mass zero or very close to zero. The elimination of interaction between our dilaton field and dense matter is total and absolute. In comparison, a Chameleon wave can suffer a total reflection from a dense-matter region; in such a situation it will not be a total elimination of the fifth force, but it may be hard indeed to prepare such an experiment. The elimination of the fifth force in the Chameleon model is argued to exist because in a spherically symmetric static configuration of a macroscopic object only a very small shell of the object can be a source of the Chameleon scalar, while in our case there would be no source for the scalar, not even the edge or surface of the dense object or at any place of the dense object. Higher-order theories of gravity also have been studied in connection with fifth force suppression and have been shown to produce an explicit realization of the Chameleon scenario from first principles [109–113].

VIII. DISCUSSION

In the present paper we have constructed a new kind of gravity-matter theory defined in terms of two different non-Riemannian volume forms (generally covariant integration measure densities) on the space-time manifold. We also introduced two scalar fields in a scale-invariant way. The integration of the equations of motion of the degrees of freedom that define the measures provides the constants of integration M_1 and M_2 which provide us with the spontaneous breaking of scale invariance. In the early Universe inflation M_1 and M_2 play an important role, determining the scale of the inflationary energy density and defining the slow-roll features in the inflationary epoch. In the slow-roll solutions we have studied one linear combination of the scalar fields φ_1 and φ_2, which we have called ϕ_1, that remains constant during the inflationary phase. This combination is invariant under scale transformations; see Eq. (8).

The dynamics of inflation reduces to that of only one scalar field (ϕ_2), but the full range of parameters obtained from the original two scalar-field couplings, which have different couplings to the different measures, plays a fundamental role. The parameter range allowed from observations is studied. This study of allowed parameter ranges in inflation imposes constraints on the parameter ranges in the late Universe, where DM in addition to DE has to be considered. We have recognized also under which conditions we will fall from inflation to one of the two possible low-vacuum energy DE states, since the slow-roll trajectory defined by (46) for a given constant defines a straight line in the (φ_1, φ_2) plane and for another constant defines another parallel line. We can then choose the line we desire (corresponding to a choice of initial conditions) to fall in one of the two lower vacua from the top vacuum. The DE/DM sector in the late Universe is determined by a dynamics where the constants of integration M_1 and M_2, which provide us with the spontaneous breaking of scale invariance, can be ignored. In this situation the scalar-field potential that depends only on ϕ_1 allows two different flat regions for possible dark-energy sectors. In each of these sectors there are particular values of ϕ_1 where the matter-induced potential for ϕ_2 is stabilized. At those points the matter behaves canonically, i.e., the dust does not produce pressure, etc., but in these two different regions the point-particle masses are different. The scalar field ϕ_2 remains a massless field in the two flat regions. Notice that in the present treatment, DE and DM are not unified; although there is the possibility of unifying also DE and DM [64], such unification has not been studied here in the context of the quintessential inflation and transition to a slowly accelerated phase, but it may be a possible generalization in a future research. What has been done here however is to introduce the dark matter in a completely scale-invariant form and we have shown explicitly the conditions under which this DM behaves as canonical dust, which is not trivial because of the couplings to the scalar field ϕ_2, which is massless and when possible 5th force effects from this massless field disappear in each of the two flat regions that can describe DE.

The above implies that the two flat regions at the values of ϕ_1 where the matter behaves canonically contain the following three elements: a constant DE, a DM component, and a massless scalar field; these components differ in the two different regions. For these regions in the later Universe, we have chosen the first flat region for the effective potential given by $U_{\mathrm{eff}(+)}$ that corresponds to large scalar field ϕ_1, i.e., $U_{\mathrm{eff}(\phi_1\to\infty)} = U_{\mathrm{eff}(+)}$. For the vacuum-energy density at the present epoch, we have chosen the effective potential $U_{\mathrm{eff}(\phi_1\to-\infty)} = U_{\mathrm{eff}(-)}$, such that $U_{\mathrm{eff}(+)} > U_{\mathrm{eff}(-)}$. For this scenario in which $U_{\mathrm{eff}(+)} > U_{\mathrm{eff}(-)}$ it is reasonable to consider that the scalar field ϕ_1 that remains fixed is $\phi_{1(+)}$; see Eq. (73). Also, for both regions, we have found analytically the evolution of the scalar field as a function of the scale factor and also the total EoS parameter in terms of the scale factor, i.e., $w_T = w_T(a)$. From the total EoS parameter, we have observed that for values of the ratio $y_\pm$ much bigger than the density parameter associated with dark energy $\Omega_\pm$, the Universe does not present an accelerated phase and then the model does not work. However, for values of $y_\pm \sim \Omega_\pm$, we have found that in both scenarios in which the effective potential corresponds to a flat region, the Universe presents an accelerated expansion, since the total EoS parameter $w_T < -0.3$. Also, we have found from the barotropic parameter associated with the dark energy that our model corresponds to a tracking freezing model and this indicates that the cosmology of the late time is independent of the initial condition, once that one of the different slow-roll trajectories is defined so to fall into one of the two lower vacuua from the top vacuum. To complement the constraints on the parameter space during epoch dominated by dark energy, we have considered the constraint from the nucleosynthesis, which imposes a strong condition on the density parameter and the ratio between the Hubble parameter and the effective potential in the flat region.

Another possibility that could occur is the inverse situation in which $U_{\mathrm{eff}(-)} > U_{\mathrm{eff}(+)}$ and the scalar field ϕ_1 in this scenario should be $\phi_{1(-)}$. Also, an interesting situation that could take place is that the second flat region of the effective potential associated with the dark energy will be in the future and has not yet been part of the history of the Universe. Also, we have found from Planck data the different constraints on the parameters associated with our model during the inflationary stage and these values are considered to obtain constraints relevant to the DE/DM epoch. The dynamical connection between these two regions of the late Universe may provide interesting clues concerning cosmological puzzles like the H_0 tension [114–116].

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