Fermion coupling to loop quantum gravity: Canonical formulation

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In the model of a fermion field coupled to loop quantum gravity, we consider the Gauss and the Hamiltonian constraints. According to the explicit solutions to the Gauss constraint, the fermion spins and the gravitational spin networks intertwine with each other so that the fermion spins contribute to the volume of the spin network vertices. For the Hamiltonian constraint, the regularization and quantization procedures are presented in detail. By introducing an adapted vertex Hilbert space to remove the regulator, we propose a diffeomorphism covariant graph-changing Hamiltonian constraint operator of the fermion field. This operator shows how fermions move in the loop quantum gravity spacetime and simultaneously influences the background quantum geometry. Moreover, as an innovation of our work, introducing the vertex Hilbert space also fixes issues so that a densely defined symmetric Hamiltonian constraint operator can be obtained.

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I. INTRODUCTION

The real physical world consists of spacetime and matter. According to general relativity, "spacetime tells matter how to move; matter tells spacetime how to curve" [1], which should also be carried out in the quantum theory. Loop quantum gravity (LQG) [2–6], as a background-independent and nonperturbative quantum gravity theory, sets the stage for incorporating matters into quantum spacetime. In [7–11], the Brown-Kuchar model of gravity coupled to dust as well as the Rovelli-Smolin model of gravity coupled to massless Klein-Gordon field is quantized. In [12,13], a minimal coupling of fermions and Yang-Mills fields to covariant LQG dynamics is proposed. The quantum theory of spinor fields coupled to LQG is well understood [14–20]. In [21], a systematic procedure to couple the standard model to the canonical LQG is proposed and further developed in [17,22,23]. With the present paper, by employing the procedure in [21], we investigate the Gauss and the Hamiltonian constraint in the model of fermion field coupled to LQG. In particular, the Gauss constraint is solved explicitly and the Hamiltonian constraint is regularized and quantized by introducing the so-called vertex Hilbert space to remove the regulator.

In the classical model of gravity coupled to the fermion field. The gravity action S_G can be formulated optionally with the first-order formulation (see, e.g., [2,24] for the Palatini-Host action) or the second-order formulation (see, e.g., [5,25] for Hilbert-Einstein action). In the pure gravity case, these two formulations are equivalent to each other up to boundary terms, while for the case with the fermion field coupled, the equivalency is no longer valid. For the first-order formulation, S_G is a functional of an $\mathrm{SL}(2,\mathbb{C})$ connection, and the fermion field will be coupled to it directly. As a consequence, the fermion field will result in an on shell torsion term in the connection. However, for the second-order formulation, the fermion field will be coupled to the torsion-free spin connection compatible with the tetrad. Thus, there is no torsion involved in this formulation. In the current paper, even though we adopt the second-order formulation for discussion, the results for the first-order formulation can be obtained analogously. Moreover, since no extra field is introduced for deparametrization in our model, the dynamics will be governed by the Hamiltonian constraint $H[N]$ with lapse functions N rather than the physical Hamiltonian. Then a problem arises: the Hamiltonian constraint operator cannot be defined in the diffeomorphism invariant space. This problem will be solved by, for instance, the master constraint framework [26,27] or the deparametrization framework [10,11]. In these frameworks, one finally needs to consider the Hamiltonian constraint operator with a constant lapse function or some dynamical lapse function. These operators can be constructed directly with the Hamiltonian constraint operators $\widehat{H[N]}$. Thus, one can only focus on how to define well the Hamiltonian constraint operators $\widehat{H[N]}$ without loss of generality, which is a main task of the current work.

The phase space of the fermion field coupled to gravity is composed of fields $(A_a^i, E_j^b, \Psi, \Pi)$ on the spatial manifold Σ, where A_a^i is an SU(2) connection, E_j^b, the canonical conjugate to A_a^i, is a densitized triad field, Ψ denotes the

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fermion field, and Π is the canonical momentum conjugate to Ψ [2,28,29]. With the variables (A_a^i, E_j^b), the gravitational Hilbert space is constructed by using the spin networks [30–32]. Therein the Hamiltonian constraint of pure gravity as well as variables representing the geometric observables are regularized and promoted to operators, and properties of the operators are well studied [26,27,33–43]. The pure-gravity Hamiltonian constraint comprises of the curvature of the connection A. The curvature, as an operator, will attach loops on graphs of the spin network states. In some regularization strategies, the final operator is the limit as the loops shrink to a point. Then, the vertex Hilbert space is necessarily introduced to define the limit [44–48]. In general, a vertex Hilbert space is a Hilbert space averaged with diffeomorphisms preserving some particular vertices. Thus, elements in a vertex Hilbert space are partially diffeomorphism invariant. In the current work, the Hamiltonian constraint of fermion is also regularized by introducing some regulator so that the fermion Hamiltonian constraint is the limit of the regularized version as the regulator approaches 0. In order to define this limit, we also need to introduce an adapted vertex Hilbert space. The vertex Hilbert space can do more than just defining limit in this work. As shown in Sec. IV B, the regularized fermion Hamiltonian constraint is split into $H_{\mathcal{C}_\epsilon}$ adding its complex conjugate. Then, $H_{\mathcal{C}_\epsilon}$ can be quantized into an operator $\widehat{H(\delta)}$ which is cylindrical consistent and diffeomorphism covariant. However, the adjoint of $\widehat{H(\delta)}$, denoted by $\widehat{H(\delta)}^\dagger$, is not densely defined. Actually, as shown in Sec. IV B, acting on some particular states associated to a graph γ, the operator $\widehat{H(\delta)}$ can change γ by erasing a segment $e(\delta)$ of an edge $e \subset \gamma$. Using γ' to denote the graph of γ with $e(\delta)$ erased by $\widehat{H(\delta)}$, $\widehat{H(\delta)}^\dagger$, acting on a state $\Psi_{\gamma'}$ associated to γ', will add an edge $\tilde{e}(\delta)$ to γ', where $\tilde{e}(\delta)$ is arbitrary edge connecting the end points of $e(\delta)$. Thus, there will be infinitely many ways to add $\tilde{e}(\delta)$ to γ', and the resulting states are orthogonal to each other. Consequently, the result of $\widehat{H(\delta)}^\dagger \Psi_{\gamma'}$ is not normalizable, i.e., $\Psi_{\gamma'}$ is not in the domain of $\widehat{H(\delta)}^\dagger$. This problem will be fixed by introducing the vertex Hilbert space to define limit. In the vertex Hilbert space, $\lim_{\delta\to 0}\widehat{H(\delta)}$ will be defined properly as an operator $\hat{A}_F^\dagger$. The action of $\hat{A}_F^\dagger$ performs in a way that a projection is left multiplied in $\hat{A}_F^\dagger$ as a factor. If $\hat{A}_F^\dagger$, acting on these particular states, erases segments, then the projection will annihilate the resulting states, so that the adjoint of $\hat{A}_F^\dagger$ is densely defined. Actually, the same issue also exists in defining the gravitational Hamiltonian constraint operator with certain regularization strategy, and can be fixed with the same mechanism [41].

The usual Hamiltonian analysis tells $\Pi^\dagger = \sqrt{q}\Psi$ with $q \equiv |\det(E)|$. In the quantum theory, this equation is expected to be realized in an appropriate form. In other words, one might require that the adjoint of the operator $\hat{\Pi}$ is related to the operator $\hat{\Psi}$ via $\hat{\Pi}^\dagger = \widehat{\sqrt{q}}\,\hat{\Psi}$. Then, contradiction appears because in our quantum theory $\sqrt{q}$ will become an operator rather than some background c-number. This can be seen as follows. Considering a nontrivial real-valued function $f(A)$ of the connection A_a^i, one has $0 = [\hat{\Pi}, \widehat{f(A)}]$ on one hand, but, on the other hand, $[\hat{\Pi}, \widehat{f(A)}]^\dagger = [\widehat{f(A)}, \hat{\Pi}^\dagger] = [\widehat{f(A)}, \hat{\sqrt{q}}\,\hat{\Psi}] \neq 0$. The two results do not coincide. To overcome this inconsistency, the author in [21] introduces the Grassman-valued half-densities $\tilde{\Psi} := \sqrt[4]{q}\Psi$ and $\tilde{\Psi}^\dagger$ to define the phase space of fermion. Moreover, in order to do the quantization, the smeared version of $\tilde{\Psi}$ by $\sqrt{\delta(x,y)}$, i.e., $\int \mathrm{d}^3y\sqrt{\delta(x,y)}\tilde{\Psi}(y)$, is also introduced by [21]. With the smeared variables, the fermion sector is quantized and the resulting quantum theory carries out the diffeomorphsm-invariance feature.

It will be useful to compare our work with the known models of fermions in LQG. Among these models, a typical one is to introduce the path observables (see, e.g., [14–16]) each of which combines the holonomy along a path α and the fermions at the ends of α. These path variables form a closed algebra under the Poisson bracket so that the quantum theory is obtained by quantizing this algebra. Another typical model is the one employed in the current work (see, e.g., [17]). As aforementioned, this model considers the algebra formed by $\int \mathrm{d}^3y\sqrt{\delta(x,y)}\tilde{\Psi}(y)$, which leads to a fermionic Fock space as the Hilbert space for fermions. Our work develops this model from the following aspects. At first, our work studies the Gauss constraint in details. We not only give the exact solutions to the Gauss constraint but also study the physical pictures resulting from these solutions. Second, for the Hamiltonian constraint of the fermion field, we propose a different regularization strategy than that in [21]. Additionally, the vertex Hilbert space is introduced, not only for removing the regulator in the regularized Hamiltonian constraint but also for fixing the issues on the adjoint of $\widehat{H(\delta)}$, so that a densely defined symmetric Hamiltonian constraint operator can be obtained.

This paper is organized as follows. In Sec. II the classical theory of gravity coupled to fermion field is introduced briefly. In Sec. III we introduce some basic notions of the kinematical Hilbert space of pure gravity and revisit the construction of the fermion kinematical Hilbert space. In Sec. IV, the Gauss constraint and the Hamiltonian constraint are regularized and quantized, where the adapted vertex Hilbert space is introduced and some physical results are discussed. Finally, in Sec. V, we summarize the remarkable results and propose some outlooks for further works.

II. CLASSICAL THEORY OF GRAVITY COUPLED TO FERMION

Let $\mathcal{M}$ denote the spacetime manifold, which is homeomorphism to $\mathbb{R}\times\Sigma$ with Σ being the spatial manifold. Given a four-dimensional vector space V, let η_{IJ} be the Minkowski metric on it. A tetrad field e^I_μ gives the metric $g_{\mu\nu}=\eta_{IJ}e^I_\mu e^J_\nu$ on $\mathcal{M}$. The curvature of $g_{\mu\nu}$ defines the Einstein-Hilbert action in terms of the tetrad fields,

$$S_H[e]=\frac{1}{2\kappa}\int_{\mathcal{M}}\mathrm{d}^4xR[e], \tag{2.1}$$

where $\kappa=8\pi G$. Let $\Gamma^I_{\mu J}$ denote the spin connection compatible with the tetrad so that

$$\mathrm{d}e^I+\Gamma^I{}_J\wedge e^J=0. \tag{2.2}$$

The model of gravity coupled to the fermion field is described by the action

$$S[e,\Psi]=S_H[e]-\frac{i}{2}\int_{\mathcal{M}}\mathrm{d}^4xe(\bar{\Psi}\gamma^I e^\mu_I\nabla_\mu\Psi-\mathrm{c.c.}), \tag{2.3}$$

where γ^I denotes the gamma matrices satisfying $\gamma^I\gamma^J+\gamma^J\gamma^I=2\eta^{IJ}\mathbb{1}$, and the covariant derivative of Ψ is

$$\nabla_\mu\Psi=\partial_\mu\Psi-\frac{1}{4}\Gamma^I_{\mu J}\gamma^I\gamma^J\Psi. \tag{2.4}$$

Performing the $3+1$ decomposition and doing canonical transformation [5], we get the gravitational canonical pair (A^i_a,E^a_i). The Poisson brackets between them are

$$\{A^i_a(x),E^b_j(y)\}=\kappa\beta\delta^b_a\delta^i_j\delta(x,y), \tag{2.5}$$

where β is the Barbero-Immirzi parameter. For the fermion field, we split the Dirac fermion Ψ into its chiral components and follow the argument in [21] to introduce the half densities on Σ,

$$\xi\coloneqq\sqrt[4]{q}\Psi_-,\qquad \nu=\sqrt[4]{q}\Psi_+, \tag{2.6}$$

with $\Psi_\pm=\frac{1\pm\gamma^5}{2}\Psi$. Detailed Hamiltonian analysis (see Appendix A) tells us that the conjugate moments to ξ and ν are their complex conjugates, and the anti-Poisson brackets are

$$\begin{aligned}\{\xi_A(x),\xi^\dagger_B(y)\}_+&=-i\delta_{AB}\delta(x,y),\\ \{\nu_A(x),\nu^\dagger_B(y)\}_+&=-i\delta_{AB}\delta(x,y)\end{aligned} \tag{2.7}$$

for all $A,B=\pm1/2$.

The dynamics of this model is encoded in the Gauss constraint G_m, the diffeomorphism constraint H_a and the Hamiltonian constraint H, which are

$$\begin{aligned}G_m&=\frac{1}{\kappa\beta}D_aE^a_l+\frac{1}{2}(\xi^\dagger\sigma_m\xi+\nu^\dagger\sigma_m\nu),\\ H_a&=\frac{1}{\kappa\beta}E^b_iF^i_{ab}+\frac{i}{2}\{\xi^\dagger D_a\xi-(D_a\xi)^\dagger\xi+\nu^\dagger D_a\nu-(D_a\nu)^\dagger\nu\}+\beta K^m_aG_m,\\ H&=H_G+\frac{1}{2\sqrt{q}}\Big[i(\xi^\dagger E^a_i\sigma^iD_a\xi-(D_a\xi)^\dagger E^a_i\sigma^i\xi)-\beta E^a_iK^i_a\xi^\dagger\xi-\frac{1}{\beta}(1+\beta^2)D_aE^a_i\xi^\dagger\sigma^i\xi-\beta E^a_iD_a(\xi^\dagger\sigma^i\xi)\\ &\quad-i(\nu^\dagger E^a_i\sigma^iD_a\nu-(D_a\nu)^\dagger E^a_i\sigma^i\nu)+\beta E^a_iK^i_a\nu^\dagger\nu-\frac{1}{\beta}(1+\beta^2)D_aE^a_i\nu^\dagger\sigma^i\nu-\beta\frac{1}{\sqrt{q}}E^a_iD_a(\nu^\dagger\sigma^i\nu)\Big].\end{aligned} \tag{2.8}$$

Here, H_G denotes the scalar constraint of pure gravity,

$$H_G=\frac{1}{2\kappa\sqrt{q}}E^a_iE^b_j(F^m_{ab}\epsilon_m{}^{ij}-2(1+\beta^2)K^i_{[a}K^j_{b]}). \tag{2.9}$$

III. LOOP QUANTIZATION OF THE THEORY: KINEMATICS

A. The kinematical Hilbert space of pure gravity

In LQG, besides a fixed differentiability class C^m with $m\geq1$, a semianalytic structure on Σ is also necessary [49]. Then all local maps, differmorphisms, submanifolds, and functions thereon are assumed to be C^m and semianalytic. Particularly, an edge is a semianalytic curve embedded in Σ. A graph is a collection of edges $\{e_1,\ldots,e_n\}$ where these e_k intersect each other at most at the ending points. Given a graph $\gamma\subset\Sigma$, let $E(\gamma)$ denote the set of its edges and $V(\gamma)$, its vertices. The number of elements in $E(\gamma)$ [$V(\gamma)$, respectively] is denoted by $|E(\gamma)|$ [$|V(\gamma)|$, respectively]. A cylindrical function Ψ of the Ashtekar connection A is a function that can be written in the form

$$\Psi(A)=\psi_\gamma(h_{e_1}(A),\ldots,h_{e_n}(A)) \tag{3.1}$$

where $\psi_\gamma\colon \mathrm{SU}(2)^{|E(\gamma)|}\to\mathbb{C}$ is a complex function on $\mathrm{SU}(2)^{|E(\gamma)|}$ and $h_e(A)\in\mathrm{SU}(2)$ is the parallel transport along an edge e with respect to a given connection A,

$$h_e(A) = \mathcal{P}\exp\left(-\int_e A\right) = 1 + \sum_{n=1}^{\infty}(-1)^n \int_0^1 \mathrm{d}t_n \times \int_0^{t_n} \mathrm{d}t_{n-1} \cdots \int_0^{t_2} \mathrm{d}t_1 A(t_1)\cdots A(t_n). \quad (3.2)$$

Given a cylindrical function Ψ with respect to a graph γ, it can always be rewritten via another graph $\gamma' \supset \gamma$. Therefore, for two cylindrical functions $\Psi^{(1)}$ and $\Psi^{(2)}$ with respect to graphs γ_1 and γ_2, respectively, one can always find another graph γ_3 with $\gamma_3 \supset \gamma_1$, $\gamma_3 \supset \gamma_2$, and rewrite $\Psi^{(1)}$ and $\Psi^{(2)}$ by some functions $\psi^{(1)}_{\gamma_3}$ and $\psi^{(2)}_{\gamma_3}$, respectively, on $\mathrm{SU}(2)^{|E(\gamma_3)|}$. Then the inner product of $\Psi^{(1)}$ and $\Psi^{(2)}$ is

$$\langle \Psi^{(1)}|\Psi^{(2)}\rangle = \int_{\mathrm{SU}(2)^n} \mathrm{d}\mu_H(g)\overline{\psi^{(1)}_{\gamma_3}(g_1,\ldots,g_n)}\psi^{(2)}_{\gamma_3}(g_1,\ldots,g_n), \quad (3.3)$$

where $n = |E(\gamma_3)|$ and $\mathrm{d}\mu_H$ is the Haar measure on $\mathrm{SU}(2)^n$. Equation (3.3) defines a measure $\mathrm{d}\mu_{\mathrm{AL}}$ on the quantum configuration space $\mathcal{A}$. Thus (3.3) is always rewritten as

$$\langle \Psi^{(1)}|\Psi^{(2)}\rangle = \int_{\mathcal{A}} \mathrm{d}\mu_{\mathrm{AL}}(A)\overline{\Psi^{(1)}(A)}\Psi^{(2)}(A). \quad (3.4)$$

The space of cylindrical functions is denoted by **Cyl**. The Hilbert space $\mathcal{H}_G$ of the pure gravity is the completion of **Cyl** with the inner product define in (3.4).

Given a graph γ, the space of cylindrical functions with respect to γ is denoted by $\mathbf{Cyl}_\gamma$. The Cauchy completion of $\mathbf{Cyl}_\gamma$ with respect to (3.3) is denoted by $\mathcal{H}_{G,\gamma}$. As shown in [3], $\mathcal{H}_{G,\gamma}$ admits the spin network decomposition

$$\mathcal{H}_{G,\gamma} = \bigoplus_{\vec{j},\vec{l}} H^{\vec{j},\vec{l}}_{G,\gamma}, \quad (3.5)$$

where $\vec{j} = \{j_1, j_2, \ldots, j_{|E(\gamma)|}\}$ assigns to each edge of γ an irreducible representation of SU(2), and $\vec{l} = \{l_1, \ldots, l_{|V(\gamma)|}\}$, to each vertex of γ an irreducible representation. Consider the assignments $\vec{j}'$ such that each representation is nontrivial. Besides, let $\vec{l}'$ denote assignments of irreducible representations to vertices of γ, which are nontrivial at each spurious vertex of γ, where a vertex v is spurious if it is bivalent, and if the edges e_i and e_{i+1}, which meet at v are such that $e_i \circ e_{i+1}$ is itself an semianalytic (i.e., v just serves to split and edge). Then, we define $\mathcal{H}'_{G,\gamma}$ as

$$\mathcal{H}'_{G,\gamma} = \bigoplus_{\vec{j}',\vec{l}'} \mathcal{H}^{(\vec{j}',\vec{l}')}_{G,\gamma}. \quad (3.6)$$

Thanks to $\mathcal{H}'_{G,\gamma}$, the space **Cyl** can be decomposed as[1]

$$\mathbf{Cyl} = \bigoplus_{\gamma} \mathcal{H}'_{G,\gamma} \oplus \mathbb{C}. \quad (3.7)$$

The multiplication operator $D^\iota_{ab}(h_e)$ acts on a cylindrical function $\Psi(A) = \psi_\gamma(h_{e_1}(A), \ldots, h_{e_n}(A))$ as

$$(D^\iota_{ab}(h_e)\Psi)(A) = D^\iota_{ab}(h_e(A))\psi_\gamma(h_{e_1}(A), \ldots, h_{e_n}(A)), \quad (3.8)$$

where $D^\iota_{ab}(h_e(A))$ denotes the Wigner-D matrix of $h_e(A) \in \mathrm{SU}(2)$. Moreover, the derivative operators $\hat{J}^{v,e}_i$, for $i = 1, 2, 3$, and v being source point s_e of e or the target point t_e of e, act on Ψ as

$$(\hat{J}^{v,e}_j\Psi)(A) = \begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \psi_\gamma(h_{e_1}, \ldots, h_e e^{t\tau_j}, \ldots, h_{e_n}), & v = s_e, \\ i\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \psi_\gamma(h_{e_1}, \ldots, e^{-t\tau_j} h_e, \ldots, h_{e_n}), & v = t_e, \end{cases} \quad (3.9)$$

where $\tau_j = -i\sigma_j/2$ with σ_j being the Pauli matrices. With the operator $\hat{J}^{v,e}_j$, we can define an operator $\hat{J}^{x,[e]}_j$ such that

$$\hat{J}^{x,[e]}_j\Psi = \sum_{e' \in [e]} \hat{J}^{x,e'}_j\Psi, \quad \forall\, \Psi \in \mathbf{Cyl}, \quad (3.10)$$

where $[e]$ is a maximal family of curves beginning at $x \in \Sigma$ such that each two curves overlap on a connected initial segment containing x.

B. Loop quantization of the fermion field

From now on, we will only focus on the single Weyl component ξ. However, everything works similarly for the other chiral component ν. To quantize the fermion field, we follow [21] to use the modified symplectic structure

$$\{\theta_A(x), \theta^\dagger_B(y)\}_+ = -i\delta_{AB}\delta_{x,y}, \qquad A, B = \pm\frac{1}{2}, \quad (3.11)$$

where, with comparison to (2.7), the Dirac delta $\delta(x, y)$ is changed to the Kronecker delta $\delta_{x,y}$. This change indicates the following canonical transformation from ξ to θ:

$$\theta(x) = \int_\Sigma \mathrm{d}^3y\sqrt{\delta(x,y)}\xi(y), \qquad \xi(x) = \sum_{y\in\Sigma} \sqrt{\delta(x,y)}\,\theta(y). \quad (3.12)$$

[1]For elements in an infinite direct sum, we require that all but finitely many components are zero.

To prove the relation (3.12), one used the function $f_\epsilon(x,y) := \chi_\epsilon(x,y)/\epsilon^3$ to regularize the Dirac delta function [21], where χ_ϵ is

$$\chi_\epsilon(x,y) := \begin{cases} 1, & \sum_{a=1}^{3}(x^a)^2 + (y^a)^2 \le \left(\frac{\epsilon}{2}\right)^2, \\ 0, & \text{otherwise.} \end{cases}$$

According to (3.12), $\xi(x)$ will be singular for regular $\theta(x)$. This scenario contradicts that $\xi(x)$ is smooth as a classical field. To have a consistent understanding of this formulation, one interprets the singular fields $\xi(x)$ as defining the quantum configuration space of the fermion field so that $\theta(x)$ is a regular-field coordinate of this quantum configuration space.

As in pure LQG, the quantization starts by introducing the cylindrical functions. Before doing so, we will first introduce a convenient field ζ_A as

$$\zeta_A(x) = \frac{1}{\sqrt{\hbar}}\theta_A(x), \qquad A = \pm\frac{1}{2}. \tag{3.13}$$

1. The cylindrical functions of the fermion field

A fermionic graph γ_F is a finite subset of Σ with $|\gamma_F|$ elements. Elements in γ_F are called fermionic vertices. An orientation of γ_F is a surjection $n \mapsto v_F^{(n)} \in \gamma_F$ with $1 \le n \le |\gamma_F|$. The surjection endows the elements in γ_F with an order. Given an oriented graph γ_F, we have a family of Grassmann numbers $\{\zeta^\dagger_{\pm\frac{1}{2}}(v_F)\}_{v_F\in\gamma_F}$, which will be renamed to $\rho_n^\dagger$ with $1 \le n \le 2|\gamma_F|$ such that

$$\rho^\dagger_{2i-1} \equiv \zeta^\dagger_{\frac{1}{2}}(v_F^{(i)}), \qquad \rho^\dagger_{2i} \equiv \zeta^\dagger_{-\frac{1}{2}}(v_F^{(i)}), \qquad 1 \le i \le |\gamma_F|. \tag{3.14}$$

Then a function Ψ of $\zeta^\dagger$ with respect to γ_F takes the general form

$$\Psi(\zeta^\dagger) = f_0 + \sum_{n=1}^{2|\gamma_F|} \sum_{1\le i_1<i_2<\cdots<i_n\le 2|\gamma_F|} f_{i_1\cdots i_n}\rho^\dagger_{i_1}\rho^\dagger_{i_2}\cdots\rho^\dagger_{i_n}, \tag{3.15}$$

where f_0 and $f_{i_1\cdots i_n}$ are complex numbers. Functions taking the form (3.15) are called the (fermionic) cylindrical functions of $\zeta^\dagger$. The space of cylindrical functions of $\zeta^\dagger$ will be denoted by $\mathbf{Cyl_F}$.

According to (3.15), each cylindrical function with respect to γ_F can be identical with a vector

$$f_{\gamma_F} = (f_0, \{f_{i_1\cdots i_n}\}_{1\le i_1<i_2<\cdots<i_n\le 2|\gamma_F|}) \in \mathbb{C}^{2^{2|\gamma_F|}}.$$

Moreover, for a cylindrical function Ψ expressed via $f_{\gamma_F} \in \mathbb{C}^{2^{2|\gamma_F|}}$, there always exists a lager graph $\gamma'_F \supset \gamma_F$ such that Ψ is rewritten via some $f'_{\gamma'_F} \in \mathbb{C}^{2^{2|\gamma'_F|}}$. Thus, given two functions $\Psi_1, \Psi_2 \in \mathbf{Cyl_F}$ on γ_{F1} and γ_{F2}, respectively, we can find another graph γ'_F containing both γ_{F1} and γ_{F2} to rewrite Ψ_1 and Ψ_2 with $f^{(1)}_{\gamma'_F}, f^{(2)}_{\gamma'_F} \in \mathbb{C}^{2^{2|\gamma'_F|}}$. Then the inner product of Ψ_1 and Ψ_2 is given by

$$\langle\Psi_1,\Psi_2\rangle = \int \mathrm{d}\mu(\rho_1\rho_1^\dagger)\mathrm{d}\mu(\rho_2\rho_2^\dagger)\cdots\mathrm{d}\mu(\rho_{2|\gamma'_F|}\rho^\dagger_{2|\gamma_F|'})\Psi_1^\dagger\Psi_2, \tag{3.16}$$

with $\mathrm{d}\mu(\rho_n\rho_n^\dagger) = \mathrm{d}\rho_n^\dagger\mathrm{d}\rho_n e^{\rho_n\rho_n^\dagger}$. By applying (3.16), one can verify

$$\langle\Psi_1,\Psi_2\rangle = (f^{(1)}_{\gamma'_F})^\dagger f^{(2)}_{\gamma'_F}. \tag{3.17}$$

Moreover, even though the graphs γ'_F containing both γ_{F1} and γ_{F2} are not unique, Eq. (3.16) is independent of the choice of γ'_F since $\int \mathrm{d}\mu(\rho_n\rho_n^\dagger) = 1$. Indeed, Eq. (3.16) defines a measure $\mathrm{d}\mu_H(\zeta^\dagger\zeta)$ on $\mathbf{Cyl_F}^\dagger \otimes \mathbf{Cyl_F}$, and will be rewritten as

$$\langle\Psi_1,\Psi_2\rangle = \int \mathrm{d}\mu_H(\zeta\zeta^\dagger)\Psi_1^\dagger\Psi_2. \tag{3.18}$$

The fermion Hilbert space $\mathcal{H}_F$ is the completion of $\mathbf{Cyl_F}$ with the inner product defined by (3.18), i.e.,

$$\mathcal{H}_F = \overline{\mathbf{Cyl_F}}. \tag{3.19}$$

On $\mathbf{Cyl_F}$, a type of operators are the multiplication operators $\widehat{\zeta^\dagger_{v_F,A}}$ given by

$$(\widehat{\zeta^\dagger_{v_F,A}}\Psi)(\zeta^\dagger) = \zeta^\dagger_A(v_F)\Psi(\zeta^\dagger), \qquad \forall\,\Psi \in \mathbf{Cyl_F}. \tag{3.20}$$

Another type of operators are the derivative operators $\hat{\zeta}_{v_F,A}$, acting on $\Psi \in \mathbf{Cyl_F}$ as

$$(\hat{\zeta}_{v_F,A}\Psi)(\zeta^\dagger) = \left(\frac{\partial}{\partial\zeta^\dagger_A(v_F)}\Psi\right)(\zeta^\dagger), \qquad \forall\,\Psi \in \mathbf{Cyl_F}. \tag{3.21}$$

It is easy to verify that $\widehat{\zeta^\dagger_{v_F,A}}$ and $\hat{\zeta}_{v_F,A}$ are adjoint to each other, i.e.,

$$\widehat{\zeta^\dagger_{v_F,A}} = \hat{\zeta}^\dagger_{v_F,A}, \tag{3.22}$$

which realizes the real condition. Moreover, one has

$$[\hat{\zeta}_{v_F,A}, \hat{\zeta}^\dagger_{v'_F,B}]_+ = \hat{\zeta}_{v_F,A}\hat{\zeta}^\dagger_{v'_F,B} + \hat{\zeta}^\dagger_{v'_F,B}\hat{\zeta}_{v_F,A} = \delta_{AB}\delta_{v_F v'_F}, \tag{3.23}$$

which implements the Poisson brackets (3.11) by defining

$$\hat{\theta}_A(v_F) = \sqrt{\hbar}\hat{\zeta}_{v_F,A}. \tag{3.24}$$

2. The spin network states of fermion field

Given a graph γ_F, the space of the fermionic cylindrical functions with respect to γ_F is a finite-dimensional Hilbert space, denoted by $\mathcal{H}_{F,\gamma_F}$. Considering a graph $\gamma_F = \{v_F\}$ which is a component of a single vertex, one has the space $\mathcal{H}_{F,\{v_F\}} \equiv \mathcal{H}_{v_F}$ consisting of functions

$$\Psi(\zeta^\dagger) = a_{00} + a_{10}\zeta^\dagger_{\frac{1}{2}}(v_F) + a_{01}\zeta^\dagger_{-\frac{1}{2}}(v_F) + a_{11}\zeta^\dagger_{\frac{1}{2}}(v_F)\zeta^\dagger_{-\frac{1}{2}}(v_F). \tag{3.25}$$

The inner product of $\Psi^{(i)}(\zeta^\dagger) = a^{(i)}_{00} + a^{(i)}_{10}\zeta^\dagger_{\frac{1}{2}}(v_F) + a^{(i)}_{01}\zeta^\dagger_{-\frac{1}{2}}(v_F) + a^{(i)}_{11}\zeta^\dagger_{\frac{1}{2}}(v_F)\zeta^\dagger_{-\frac{1}{2}}(v_F)$ with $i = 1, 2$ is

$$\langle \Psi^{(1)}, \Psi^{(2)}\rangle = \sum_{i,j\in\{0,1\}} (a^{(1)}_{ij})^* a^{(2)}_{ij}. \tag{3.26}$$

For convenience, we introduce the Dirac bra-ket notation $|i,j\rangle_{v_F}$ $(i, j = 0, 1)$ to denote the state Ψ_{ij}, where Ψ_{ij} are the states given by $\Psi_{00}(\zeta^\dagger) = 1$, $\Psi_{10}(\zeta^\dagger) = \zeta^\dagger_{\frac{1}{2}}(v_F)$, $\Psi_{01}(\zeta^\dagger) = \zeta^\dagger_{-\frac{1}{2}}(v_F)$ and $\Psi_{11}(\zeta^\dagger) = \zeta^\dagger_{\frac{1}{2}}(v_F)\zeta^\dagger_{-\frac{1}{2}}(v_F)$. Then the states $|i,j\rangle_{v_F}$ form an orthonormal basis of $\mathcal{H}_{v_F}$, i.e.,

$${}_{v_F}\langle i_1, j_1|i_2, j_2\rangle_{v_F} = \delta_{i_1 i_2}\delta_{j_1 j_2}. \tag{3.27}$$

The action of $\hat{\zeta}_{v_F,A}$ and $\hat{\zeta}^\dagger_{v_F,A}$ for $A = \pm\frac{1}{2}$ on $\mathcal{H}_{v_F}$ reads

$$\begin{aligned}
&\hat{\zeta}^\dagger_{v_F,\frac{1}{2}}|0,i_2\rangle_{v_F} = |1,i_2\rangle_{v_F}, && \hat{\zeta}^\dagger_{v_F,\frac{1}{2}}|1,i_2\rangle_{v_F} = 0, && \forall\, i_2 = 0,1,\\
&\hat{\zeta}_{v_F,\frac{1}{2}}|0,i_2\rangle_{v_F} = 0, && \hat{\zeta}_{v_F,\frac{1}{2}}|1,i_2\rangle_{v_F} = |0,i_2\rangle_{v_F}, && \forall\, i_2 = 0,1,\\
&\hat{\zeta}^\dagger_{v_F,-\frac{1}{2}}|i_1,0\rangle_{v_F} = (-1)^{i_1}|i_1,1\rangle_{v_F}, && \hat{\zeta}^\dagger_{v_F,-\frac{1}{2}}|i_1,1\rangle_{v_F} = 0, && \forall\, i_1 = 0,1,\\
&\hat{\zeta}_{v_F,-\frac{1}{2}}|i_1,0\rangle_{v_F} = 0, && \hat{\zeta}_{v_F,-\frac{1}{2}}|i_1,1\rangle_{v_F} = (-1)^{i_1}|i_1,0\rangle_{v_F}, && \forall\, i_1 = 0,1.
\end{aligned} \tag{3.28}$$

For the general case where $\gamma_F \subset \Sigma$ consists of more than one vertices, we first associate to each $v_F \in \gamma_F$ the Hilbert space $\mathcal{H}_{v_F}$. Then, fixing an orientation of γ_F, one has the tensor product space

$$\mathcal{H}_{F,\gamma_F} = \mathcal{H}_{v_F^{(1)}} \otimes \mathcal{H}_{v_F^{(2)}} \otimes \cdots \otimes \mathcal{H}_{v_F^{|\gamma_F|}}. \tag{3.29}$$

An orthonormal basis of $\mathcal{H}_{F,\gamma_F}$ is composed of the vectors

$$|i_1, i_2, \ldots, i_{2|\gamma_F|}\rangle := |i_1, i_2\rangle_{v_F^{(1)}} \otimes |i_3, i_4\rangle_{v_F^{(2)}} \otimes \cdots \otimes |i_{2|\gamma_F|-1}, i_{2|\gamma_F|}\rangle_{v_F^{(|\gamma_F|)}}, \tag{3.30}$$

with $i_k \in \{0,1\}$ for all $1 \le k \le 2|\gamma_F|$. Indeed, the vector $|i_1, i_2, \ldots, i_{2|\gamma_F|}\rangle$ refers to the cylindrical function $\Psi_{\vec{i}} = \rho^\dagger_{i_1}\rho^\dagger_{i_2}\cdots\rho^\dagger_{i_n}$ with respect to γ_F. It is worth noting that the definition of $\mathcal{H}_{F,\gamma_F}$ depends on the orientation of γ_F, since $\zeta^\dagger_A(v_F)$ are Grassmann numbers. This fact can be illustrated more explicitly with the following example. Consider another orientation $n \to \tilde{v}_F^{(n)}$ of γ_F such that

$$\tilde{v}_F^{(1)} = v_F^{(2)}, \quad \tilde{v}_F^{(2)} = v_F^{(1)}, \quad \tilde{v}_F^{(k)} = v_F^{(k)}, \quad \forall\, k \ge 3. \tag{3.31}$$

Then, under the new orientation, we have the tensor-product Hilbert space $\tilde{\mathcal{H}}_{F,\gamma_F}$ possessing the basis

$$|i_1, i_2, \widetilde{\ldots}, i_{2|\gamma_F|}\rangle := |i_1, i_2\rangle_{\tilde{v}_F^{(1)}} \otimes |i_3, i_4\rangle_{\tilde{v}_F^{(2)}} \otimes \cdots \otimes |i_{2|\gamma_F|-1}, i_{2|\gamma_F|}\rangle_{\tilde{v}_F^{(\gamma_F)}}. \tag{3.32}$$

By definition, $|i_1, i_2, \widetilde{\ldots}, i_{2|\gamma_F|}\rangle$ refers to the cylindrical function

$$\tilde{\Psi}_{\vec{i}} = \rho^\dagger_{i_3}\rho^\dagger_{i_4}\rho^\dagger_{i_1}\rho^\dagger_{i_2}\cdots\rho^\dagger_{i_n} = (-1)^{(i_1+i_2)(i_3+i_4)}\Psi_{\vec{i}'}, \tag{3.33}$$

where $\vec{i}' = \{i_3, i_4, i_1, i_2, \ldots, i_{2|\gamma_F|}\}$. Equation (3.33) gives the equivalence relation between $\mathcal{H}_{F,\gamma_F}$ and $\tilde{\mathcal{H}}_{F,\gamma_F}$

$$|i_1, i_2, \widetilde{\ldots}, i_{2|\gamma_F|}\rangle = (-1)^{(i_1+i_2)(i_3+i_4)}|i_3, i_4, i_1, i_2, \ldots, i_{2|\gamma_F|}\rangle. \tag{3.34}$$

This equivalence relation can be analogously defined between the tensor product spaces with different orientations. The fermionic Hilbert space with respect to γ_F is actually space of equivalence classes associated with this equivalence relation.

Indeed, the extra sign in (3.34) can be systematically obtained by introducing the notion of graded objects. One can refer to Appendix B and the reference therein for more details on the this notion. In our work, the Hilbert spaces $\mathcal{H}_{v_F}$ are graded. The degree $\mathfrak{d}(i_1, i_2)$ of each $|i_1, i_2\rangle_{v_F}$ is

$$\mathfrak{d}(i_1, i_2) = i_1 + i_2 \mod 2. \tag{3.35}$$

The operator algebra on $\mathcal{H}_{v_F}$ is also graded. By definition, the degrees of the operators $\hat{\zeta}_{v_F,A}$ and $\hat{\zeta}^\dagger_{v_F,A}$ are

$$\mathfrak{d}(\hat{\zeta}_{v_F,A}) = 1 = \mathfrak{d}(\hat{\zeta}^\dagger_{v_F,A}). \tag{3.36}$$

A general principle to deal with these graded objects is that, whenever we swap two items, an additional sign appears by the rule $xy = (-1)^{\mathfrak{d}(x)\mathfrak{d}(y)}yx$. Following this rule, we can obtain the identity (3.34) manifestly.

Given a vertex v_F, let $\mathcal{H}^{\rm irr}_{v_F}$ be the subspace of $\mathcal{H}_{v_F}$ spanned by $|0,1\rangle_{v_F}$, $|1,0\rangle_{v_F}$ and $|1,1\rangle_{v_F}$. Then the Hilbert space $\mathcal{H}^{\rm irr}_{F,\gamma_F}$ with respect to γ_F is defined as

$$\mathcal{H}^{\rm irr}_{F,\gamma_F} = \bigotimes_{v_F\in\gamma_F} \mathcal{H}^{\rm irr}_{v_F}, \tag{3.37}$$

such that $\mathbf{Cyl}_F$ can be decomposed as

$$\mathbf{Cyl}_F = \bigoplus_{\gamma_F} \mathcal{H}^{\rm irr}_{F,\gamma_F} \oplus \mathbb{C}. \tag{3.38}$$

The kinematical Hilbert space $\mathcal{H}$ of the entire system is the tensor product of $\mathcal{H}_G$ and $\mathcal{H}_F$, i.e.,

$$\mathcal{H} = \overline{\mathcal{H}_G \otimes \mathcal{H}_F}. \tag{3.39}$$

A densely subspace $\mathbf{Cyl_{tot}}$ of $\mathcal{H}$ is

$$\mathbf{Cyl_{tot}} = \mathbf{Cyl} \otimes \mathbf{Cyl_F}. \tag{3.40}$$

The states in $\mathbf{Cyl_{tot}}$ will be called the cylindrical states. To obtain a cylindrical state, one needs a graph $\gamma = \gamma_G \cup \gamma_F$, where the gravitational graph γ_G is constituted of edges and their ending points as vertices, and the fermionic graph γ_F contains only vertices. To define a state with respect to γ, besides the data for a (gauge variant) LQG spin network state, one also needs to assign to each fermionic vertex v_F a state $|i_1(v_F), i_2(v_F)\rangle_{v_F}$ with $i_1(v_F), i_2(v_F) \in \{0,1\}$. In principle, a fermionic vertex v_F can be located anywhere, regardless of the given gravitational graph γ_G. However, if v_F is chosen as point in $e \in E(\gamma_G)$ but $v_F \notin V(\gamma_G)$, then we can always split e at v_F to define a new graph $\tilde{\gamma}_G$. Then v_F becomes a vertex of $\tilde{\gamma}_G$. Moreover, because of $\gamma_G \subset \tilde{\gamma}_G$, every cylindrical function with respect to γ_G can be rewritten by using $\tilde{\gamma}_G$. Thus, it is sufficient to consider those graphs $\gamma = \gamma_G \cup \gamma_F$ where each fermionic vertex v_F satisfies either $v_F \in V(\gamma_G)$ or $v_F \notin \gamma_G$. Then, in $V(\gamma_G)$, there could be bivalent vertices which is a fermionic vertex. These vertices serve to split edge. Thus, by (3.6), they are just spurious vertices. However, in contrast to (3.6), we can put trivial SU(2) representation at these vertices to decompose the Hilbert space with respect to graphs. More precisely, let Γ_o be the set of graphs $\gamma = \gamma_G \cup \gamma_F$ where each fermionic vertex v_F satisfies either $v_F \in V(\gamma_G)$ or $v_F \notin \gamma_G$. Consider the assignments $\vec{j}'$ to $E(\gamma)$ such that each representation is nontrivial. Besides, let $\vec{l}'$ denote assignments of irreducible representations to vertices of γ that are nontrivial at each fake vertex of γ_G where a vertex $v \in V(\gamma_G)$ is fake if $v \notin \gamma_F$ and it is spurious as defined in (3.6). Then, we define

$$\mathcal{H}^{\rm irr}_{G,\gamma_G} := \bigoplus_{\vec{j}',\vec{l}'} \mathcal{H}^{\vec{j}',\vec{l}'}_{G,\gamma_G}, \tag{3.41}$$

which gives us the Hilbert space $\mathcal{H}^{\rm irr}_{\gamma}$ as

$$\mathcal{H}^{\rm irr}_{\gamma} = \mathcal{H}^{\rm irr}_{G,\gamma_G} \otimes \mathcal{H}^{\rm irr}_{F,\gamma_F}. \tag{3.42}$$

Then we have the decomposition

$$\mathbf{Cyl_{tot}} = \bigoplus_{\gamma\in\Gamma_o} \mathcal{H}^{\rm irr}_{\gamma} \oplus \mathbb{C}. \tag{3.43}$$

IV. THE CONSTRAINT OPERATORS FOR GRAVITY COUPLED TO FERMION FIELD

A. The Gauss constraint

Classically, the Gauss constraint $G[\lambda]$ reads

$$G[\lambda] = \int_\Sigma \mathrm{d}^3x \lambda^m \left(\frac{1}{\kappa\beta} D_a E^a_m + \frac{1}{2}\xi^\dagger \sigma_m \xi\right). \tag{4.1}$$

It is straightforward to quantize it as the operator

$$\widehat{G[\lambda]} = \sum_v \lambda^m(v) \hat{G}_{v,m}, \tag{4.2}$$

with

$$\hat{G}_{v,m} = \hbar \sum_{[e]} \hat{J}^{v,[e]}_m + \hbar \hat{\zeta}^\dagger_{v,A} \frac{(\sigma_m)_{AB}}{2} \hat{\zeta}_{v,B}. \tag{4.3}$$

Let us use $\hat{\mathcal{J}}_{v,m}$ to denote the second term in (4.3), namely

$$\hat{\mathcal{J}}_{v,m} = \hat{\zeta}^\dagger_{v,A} \frac{(\sigma_m)_{AB}}{2} \hat{\zeta}_{v,B}. \tag{4.4}$$

On the fermionic Hilbert space $\mathcal{H}_v$ at vertex v, the action of $\hat{\mathcal{J}}_{v,m}$ reads

$$\hat{\mathcal{J}}_{v,m}|0,0\rangle_v = 0, \quad \hat{\mathcal{J}}_{v,m}|1,1\rangle_v = 0,$$
$$\hat{\mathcal{J}}_{v,m}(|1,0\rangle_v, |0,1\rangle_v) = (|1,0\rangle_v, |0,1\rangle_v)\frac{\sigma_m}{2}. \tag{4.5}$$

According to (4.5), the operators $\hat{\mathcal{J}}_{v,m}$ for all $m = 1, 2, 3$ behave as the angular moment operators. Thus, the operator $\hat{\mathcal{J}}_{v,m}$ generates an SU(2) action on $\mathcal{H}_v$ as

$$u \triangleright |\phi\rangle_v = (\,|1,0\rangle_v, |0,1\rangle_v\,) u \begin{pmatrix} \phi_{10} \\ \phi_{01} \end{pmatrix} + \phi_{00}|0,0\rangle_v + \phi_{11}|1,1\rangle_v. \tag{4.6}$$

where $|\phi\rangle_v = \sum_{ij} \phi_{ij}|i,j\rangle_v$ and $u \in \mathrm{SU}(2)$. Therefore, $\mathcal{H}_v$ becomes a reducible representation space of SU(2). The one-dimensional space spanned by either $|0,0\rangle_v$ or $|1,1\rangle_v$ is the trivial representation space, and the two-dimensional space spanned by $|0,1\rangle_v$ and $|1,0\rangle_v$ is the 1/2-representation space where $|0,1\rangle_v$ and $|1,0\rangle_v$ serve as the standard basis according to (4.5). This fact leads to the decomposition

$$\mathcal{H}_v \equiv \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{1/2}, \tag{4.7}$$

where $\mathcal{H}_j$ denotes the j-representation space of SU(2).

For a graph $\gamma = \gamma_G \cup \gamma_F$, a spin j_e is assigned to the edge $e \subset \gamma_G$. Then at each vertex $v \in V(\gamma)$ there is the Hilbert space

$$\mathcal{H}_v^{\mathrm{tot}} = \bigotimes_{e \text{ starts from } v} \mathcal{H}_{j_e} \otimes \bigotimes_{e' \text{ targets } v} \mathcal{H}^*_{j_{e'}} \otimes \mathcal{H}_v, \tag{4.8}$$

where $\mathcal{H}_j^*$ denotes the dual space of $\mathcal{H}_j$. On $\mathcal{H}_v^{\mathrm{tot}}$, the infinitesimal SU(2) action gives the Gauss constraint. Thus, the solution space to the Gauss constraint is

$$\mathcal{H}^{\mathrm{Gau}} = \bigotimes_{x \in V(\gamma)} \mathrm{Inv}(\mathcal{H}_v^{\mathrm{tot}}), \tag{4.9}$$

where $\mathrm{Inv}(\mathcal{H}_v^{\mathrm{tot}}) \subset \mathcal{H}_v^{\mathrm{tot}}$ is the SU(2)-invariant subspace. To see $\mathrm{Inv}(\mathcal{H}_v^{\mathrm{tot}})$ more precisely, let us assume all edges at v are outgoing without loss of generality. Then we have

$$\mathcal{H}_v^{\mathrm{tot}} = \bigotimes_{e \text{ at } v} \mathcal{H}_{j_e} \otimes \mathcal{H}_v. \tag{4.10}$$

Given an order of the edges at v, one can choose an orthonormal basis of $\bigotimes_{e \text{ at } v} \mathcal{H}_{j_e}$ composed of vectors $|k_2, k_3, \ldots, k_n, M\rangle$ satisfying

$$\begin{aligned}
\sum_{i=1}^{3} (L_i^{(l)})^2 |k_2, k_3, \ldots, k_n, M\rangle &= k_l(k_l+1)|k_2, k_3, \ldots, k_n, M\rangle, \quad \forall\, l = 2, \ldots, n \\
L_3^{(n)}|k_2, k_3, \ldots, k_n, M\rangle &= M|k_2, k_3, \ldots, k_n, M\rangle, \\
L_1^{(n)}|k_2, k_3, \ldots, k_n, M\rangle &= \sum_{s=\pm 1} \frac{1}{2}\sqrt{(k_n - sM)(k_n + sM + 1)}|k_2, k_3, \ldots, k_n, M+s\rangle, \\
L_2^{(n)}|k_2, k_3, \ldots, k_n, M\rangle &= \sum_{s=\pm 1} \frac{-is}{2}\sqrt{(k_n - sM)(k_n + sM + 1)}|k_2, k_3, \ldots, k_n, M+s\rangle,
\end{aligned} \tag{4.11}$$

with $\hat{L}_i^{(l)} := \sum_{k=1}^{l} \hat{J}_i^{v,e_k}$. Let us define $\mathrm{Inv}(\mathcal{H}_G^{(v)}) \subset \bigotimes_{e \text{ at } v} \mathcal{H}_{j_e}$ as the subspace spanned by $|k_2, k_3, \ldots, k_{n-1}, 0, 0\rangle$ for all possible $k_2, k_3, \ldots, k_{n-1}$. One has

$$L_i^{(n)}|k_2, k_3, \ldots, k_{n-1}, 0, 0\rangle = 0, \quad \forall\, i = 1, 2, 3. \tag{4.12}$$

Moreover, with the vectors $|k_2, k_3, \ldots, k_{n-1}, 1/2, M\rangle$, we define

$$|k_2, k_3, \ldots, k_{n-1}\rangle_{\mathrm{tot}} = \frac{1}{\sqrt{2}}\left|k_2, k_3, \ldots, k_{n-1}, \frac{1}{2}, \frac{1}{2}\right\rangle \otimes |0,1\rangle_v - \frac{1}{\sqrt{2}}\left|k_2, k_3, \ldots, k_{n-1}, \frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1,0\rangle_v. \tag{4.13}$$

Then one has

$$(L_i^{(n)} + \hat{\mathcal{J}}_{v,i})|k_2, k_3, \ldots, k_{n-1}\rangle_{\mathrm{tot}} = 0, \quad \forall\, i = 1, 2, 3. \tag{4.14}$$

Let $\mathcal{H}_{\mathrm{inv}} \subset \mathcal{H}_v^{\mathrm{tot}}$ denote the subspace spanned by $|k_2, k_3, \ldots, k_{n-1}\rangle_{\mathrm{tot}}$ for all possible $k_2, k_3, \ldots, k_{n-1}$. Then $\mathrm{Inv}(\mathcal{H}_v^{\mathrm{tot}})$ can be decomposed as

$$\mathrm{Inv}(\mathcal{H}_v^{\mathrm{tot}}) = (\mathrm{Inv}(\mathcal{H}_G^{(v)}) \otimes |0,0\rangle_v) \oplus (\mathrm{Inv}(\mathcal{H}_G^{(v)}) \otimes |1,1\rangle_v) \oplus \mathcal{H}_{\mathrm{inv}}, \tag{4.15}$$

where $\mathrm{Inv}(\mathcal{H}_G^{(v)}) \otimes |i_1, i_2\rangle_v$ is the space composed of vectors $|\psi\rangle \otimes |i_1, i_2\rangle$ for all $|\psi\rangle \in \mathrm{Inv}(\mathcal{H}_G^{(v)})$.

Let v be a n-valence gauge invariant fermionic vertex, where the ith edge e_i is assigned to a spin j_i. According to the decomposition (4.15), the gauge invariant Hilbert space $\mathrm{Inv}(\mathcal{H}_v^{\mathrm{tot}})$ contains a subspace $\mathcal{H}_{\mathrm{inv}}$, isometric to the gauge invariant Hilbert space of a $(n+1)$-valence pure-gravity vertex where the ith edge for $1 \le i \le n$ is assigned to spin j_i,

and the $(n+1)$th, the spin $1/2$. Then once we consider the volume operator at v, this extra spin $1/2$ will also have contribution. Since the extra spin $1/2$ originates from the fermion filed, one gets an intuitive picture that fermion field contributes to the volume of a vertex. Moreover, a n-valence vertex in pure-LQG is always regarded as a polyhedron whose faces are dual to the edges. The flux operators $\hat{J}_i^{v,e}$ associated to each edge e have the geometric interpretation of the area vector of the dual face. Then the pure-LQG Gauss constraint is just the closure condition $\sum_e \hat{J}_i^{v,e} = 0$ ensuring that the faces can form a closed polyhedron. Now, the fermion field is involved. Then the Gauss constraint (4.3) implies

$$\sum_e \hat{J}_i^{v,e} = -\hat{\mathcal{J}}_{v,i}, \tag{4.16}$$

where the right-hand side does not vanish in general. Thus, the faces dual to the edges could not give a closed polyhedron for states in $\mathcal{H}_{\rm inv}$. By (4.16), the area defect of this unclosed polyhedron is filled by $\hat{\mathcal{J}}_{v,i}$, i.e., the fermion spin at the vertex (see [22] for more details on the fermion spin). A direct consequence of the above discussion is that the volume of a three-valence vertex with fermion does not vanish any more for states in $\mathcal{H}_{\rm inv}$. Let j_i with $i = 1, 2, 3$ be the spins on the edges. Then the states in $\mathcal{H}_{\rm inv}$ are spanned by $|k\rangle_{\rm tot} \equiv |k\rangle$ with $k = j_3 \pm 1/2$. The action of the operator $\hat{q}_{123}$, the operator proportional to the square of the volume operator [3], on $|k\rangle$ reads

$$\begin{aligned}\langle k|\hat{q}_{123}|k+1\rangle = {} & \frac{-i}{4\sqrt{(2k+1)(2k+3)}}\sqrt{(j_1-j_2+k+1)(-j_1+j_2+k+1)(j_1+j_2-k)(j_1+j_2+k+2)} \\ & \times\sqrt{\left(j_3-\frac{1}{2}+k+1\right)\left(-j_3+\frac{1}{2}+k+1\right)\left(j_3+\frac{1}{2}-k\right)\left(j_3+\frac{1}{2}+k+2\right)}.\end{aligned} \tag{4.17}$$

Then we have

$$\begin{aligned}\left\langle j_3-\frac{1}{2}\middle|\hat{q}_{123}\middle|j_3+\frac{1}{2}\right\rangle = {} & i\frac{1}{16}\sqrt{(2j_1+2j_2-2j_3+1)(2j_1-2j_2+2j_3+1)} \\ & \times\sqrt{(-2j_1+2j_2+2j_3+1)(2j_1+2j_2+2j_3+3)}.\end{aligned} \tag{4.18}$$

Since the associated Hilbert space is two dimensional, the whole Hilbert space is the eigenspace of the volume operator $\frac{\kappa_0\beta\ell_p^{3/2}}{2\sqrt{2}}\sqrt{|\hat{q}_{123}|}$ with eigenvalue

$$V_v = \frac{\kappa_0\beta\ell_p^{3/2}}{2\sqrt{2}}\sqrt{\left|\left\langle j_3-\frac{1}{2}\middle|\hat{q}_{123}\middle|j_3+\frac{1}{2}\right\rangle\right|}. \tag{4.19}$$

B. The Hamiltonian constraint

As discussed in [17,21], the smeared Hamiltonian constraint in terms of $\theta_A(x)$ is

$$H_F[N] := \sum_{x\in\Sigma} N(x)H_F(x), \tag{4.20}$$

where $H_F(x)$ is given by

$$\begin{aligned}H_F = {} & i\frac{1}{2\sqrt{q}}(\theta^\dagger E_i^a\sigma^i D_a\theta - (D_a\theta)^\dagger E_i^a\sigma^i\theta) \\ & -\beta\frac{1}{2\sqrt{q}}E_i^a K_a^i\theta^\dagger\theta - \frac{1+\beta^2}{\beta}\frac{1}{2\sqrt{q}}D_a E_i^a\theta^\dagger\sigma^i\theta \\ & -\beta\frac{1}{2\sqrt{q}}E_i^a D_a(\theta^\dagger\sigma^i\theta).\end{aligned} \tag{4.21}$$

Fix a coordinate system x^a on Σ and a positive number ϵ. Divide Σ into a family $\mathcal{C}_\epsilon$ of cells such that each cell $C \in \mathcal{C}_\epsilon$ is cubic with the coordinate volume less than ϵ^3, and that different cells can only share points on their boundaries. Given a graph $\gamma = \gamma_G \cup \gamma_F$, for each cell $C \in \mathcal{C}_\epsilon$, let γ_C denote $\gamma \cap C$. Since the limit $\epsilon \to 0$ will be considered eventually, we will assume that ϵ is small enough such that $\gamma_C \neq \varnothing$ is one of the following types (see Fig. 1):

(i) the type-A graph: γ_C is composed of a single edge;
(ii) the type-B graph: γ_C is composed of a single fermionic vertex without connecting any edges;
(iii) the type-C graph: γ_C is composed of edges intersecting a single vertex.

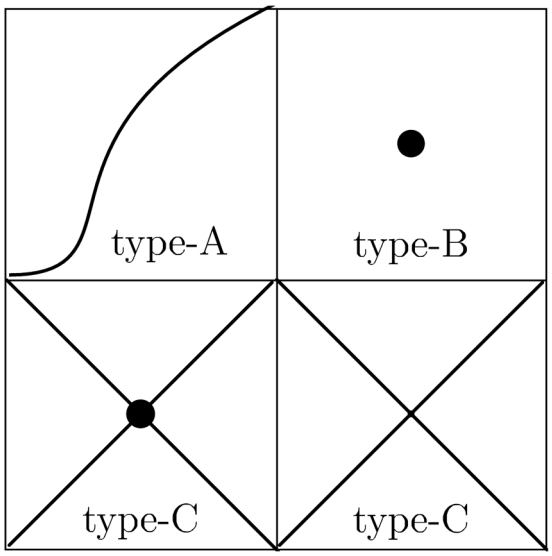


FIG. 1. Different types of graphs where in the lower left panel is a type-C graph with a fermionic vertex and, the lower right panel, a type-C graph with a gravitational vertex.

For each cell $C \in \mathcal{C}_\epsilon$, let us define

$$H_C^{(1)} = \int_C \mathrm{d}^3x N(x)(D_a\theta)^\dagger(x)E_i^a(x)\sigma^i\theta(x),$$
$$H_C^{(2)} = \int_C \mathrm{d}^3x N(x)E_i^a(x)K_a^i(x)\theta^\dagger(x)\theta(x),$$
$$H_C^{(3)} = \int_C \mathrm{d}^3x N(x)D_aE_i^a(x)\theta^\dagger(x)\sigma^i\theta(x), \tag{4.22}$$

and introduce

$$H_{\mathcal{C}_\epsilon} = \sum_{C\in\mathcal{C}_\epsilon}\frac{1}{V_C}\left(-iH_C^{(1)} - \frac{\beta}{2}H_C^{(2)} - \frac{1+\beta^2}{2\beta}H_C^{(3)} - \beta H_C^{(1)}\right), \tag{4.23}$$

where the volume V_C of C is $V_C = \int_C \mathrm{d}^3x\sqrt{|\det(E)|}$. Then $H_F[N]$ is the limit of $\frac{1}{2}H_{\mathcal{C}_\epsilon}$ adding its complex conjugate as $\epsilon \to 0$, i.e.,

$$H_F[N] = \frac{1}{2}\lim_{\epsilon\to 0}(H_{\mathcal{C}_\epsilon} + \text{c.c.}). \tag{4.24}$$

To quantize $H_{\mathcal{C}_\epsilon}$, we need to promote

$$\tilde{H}_C^{(i)} = \frac{1}{V_C}H_C^{(i)}, \quad i = 1,2,3, \quad C \in \mathcal{C}_\epsilon, \tag{4.25}$$

to an operator. As known in LQG, the volume operator is not invertible. We thus need to regularize V_C^{-1} as in [39] to be $V_C^{-1} = (\sqrt{V_C^{-1}})^2$ with

$$\sqrt{V_C^{-1}} = \kappa_0'\frac{4\times 8}{6}\left(\frac{2}{\kappa\beta}\right)^3 \sum_{e,e',e'' \text{ at } v}\epsilon(e,e',e'')\mathrm{tr} \times (h_e^{-1}\{h_e, V_C\}h_{e'}^{-1}\{h_{e'}, V_C\}h_{e''}^{-1}\{h_{e''}, V_C\}), \tag{4.26}$$

where $\epsilon(e,e',e'') = 0, \pm 1$ depending on the orientation of $e \wedge e' \wedge e''$, and κ_0' is a constant to remove the dependence of partition. Then, the operator $\widehat{\tilde{H}_C^{(i)}}$ is given by

$$\widehat{\tilde{H}_C^{(i)}} = \widehat{\sqrt{V_C^{-1}}}\hat{H}_C^{(i)}\widehat{\sqrt{V_C^{-1}}}, \tag{4.27}$$

where $\hat{H}_C^{(i)}$ will be discussed latter, and the operator $\widehat{\sqrt{V_C^{-1}}}$ reads

$$\widehat{\sqrt{V_C^{-1}}} = \kappa_0'\frac{4\times 8}{6}\left(\frac{2}{i\hbar\kappa\beta}\right)^3 \sum_{v \text{ in } C}\sum_{e,e',e'' \text{ at } v}\epsilon(e,e',e'') \times \mathrm{tr}(h_e^{-1}[h_e, \hat{V}_C]h_{e'}^{-1}[h_{e'}, \hat{V}_C]h_{e''}^{-1}[h_{e''}, \hat{V}_C]). \tag{4.28}$$

Here there are several issues on $\widehat{\sqrt{V_C^{-1}}}$. At first, $\widehat{\sqrt{V_C^{-1}}}$ actually acts on states ψ_{γ_C} with respect to the graph $\gamma_C = \gamma \cap C$. Given a state ψ_{γ_C}, according to (3.43), there is a unique graph $\mathbf{P}(\gamma_C) \subset \gamma_C$ such that $\psi_{\gamma_C} \in \mathcal{H}^{\text{irr}}_{\mathbf{P}(\gamma_C)}$. The summation in (4.28) over $v \in C$ actually means the summation over $v \in V(\mathbf{P}(\gamma_C))$, and the summation over e, e', e'' means the summation over $e, e', e'' \in E(\mathbf{P}(\gamma_C))$. As a consequence, for γ_C being a type-A or type-B graph, one has $\widehat{\sqrt{V_C^{-1}}}\psi_{\gamma_C} = 0$, and for γ_C being a type-C graph with the vertex v_C, one has

$$\widehat{\sqrt{V_C^{-1}}}\psi_{\gamma_C} = \kappa_0'\frac{4\times 8}{6}\left(\frac{2}{i\hbar\kappa\beta}\right)^3 \sum_{e,e',e''\in E(\mathbf{P}(\gamma_C))}\epsilon(e,e',e'')\mathrm{tr} \times (h_e^{-1}[h_e, \hat{V}_{v_C}]h_{e'}^{-1}[h_{e'}, \hat{V}_{v_C}]h_{e''}^{-1}[h_{e''}, \hat{V}_{v_C}])\psi_{\gamma_C}, \tag{4.29}$$

where $\hat{V}_{v_C}$ is the volume operator introduced in [35]. According to this result, the operator $\widehat{\sqrt{V_C^{-1}}}$ at the most right of $\widehat{\tilde{H}_C^{(i)}}$ will annihilate the states with respect to the type-A and type-B graphs. Hence, only the states on type-C graphs is needed to be considered. From now on, γ_C will be referred to as the type-C graphs until otherwise stated. The vertex for the edges in γ_C intersecting will be denoted by v_C.

Let us begin with the operator $\widehat{\tilde{H}_C^{(1)}}$. Replacing $E_i^a(x)$ by $-i\kappa\hbar\beta\delta/\delta A_a^i(x)$ in $H_C^{(1)}$ in (4.22), we can quantize $H_C^{(1)}$ as

$$\hat{H}_C^{(1)} = -i\kappa\hbar\beta \int_C \mathrm{d}^3x N(x)(\widehat{D_a\theta})^\dagger(x)\sigma^i\hat{\theta}(x)\frac{\delta}{\delta A_a^i(x)}. \quad (4.30)$$

Note that we put the derivative $\frac{\delta}{\delta A_a^i(x)}$ at the most right so that the resulting operator is cylindrical consistent, i.e., the results of states acted by the resulting operator do not depend on the edges taking trivial representation. Given an edge $e\colon [0,\delta] \to C$ of γ_C with $e(0) = v_C$, let $U_e(t,0,A)$ denote the parallel transport from $e(0)$ to $e(t)$ along e. Thus, $U_e(t,0,A)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U_e(t,0;A) = -A_a(e(t))\dot{e}^a(t)U(t,0;A), \quad \text{and} \quad U(0,0;A) = I. \quad (4.31)$$

Defining $h_e \coloneqq U(\delta,0;A)$, one has, for any $f_a^i(x)$,

$$\int \mathrm{d}^3x f_a^i(x)\frac{\delta}{\delta A_a^i(x)}h_e = -\int_0^\delta \mathrm{d}\tau f_a^i(e(\tau))\dot{e}^a(\tau)U_e(\delta,\tau,A)\tau_i U_e(\tau,0,A). \quad (4.32)$$

Thus, we get

$$-i\kappa\hbar\beta\int_C \mathrm{d}^3x N(x)\widehat{D_a\theta}^\dagger(x)\sigma^i\hat{\theta}(x)\frac{\delta}{\delta A_a^i(x)}h_e = \kappa\hbar\beta N(s_e)(\hat{\theta}^\dagger(t_e)h_e - \hat{\theta}^\dagger(s_e))\sigma^i\hat{\theta}(s_e)\hat{J}_i^{v,e}h_e, \quad (4.33)$$

where we used

$$\delta \times \dot{e}^a(t_1)(D_a\theta)^\dagger(e(t_1)) = \theta^\dagger(e(t_1+\delta))U_e(t_1+\delta,t_1;A) - \theta^\dagger(e(t_1)) + O(\delta^2). \quad (4.34)$$

Then, for a state Ψ_C with respect to γ_C, one has

$$\hat{H}_C^{(1)}\Psi_C = \kappa\hbar\beta N(v_C)\sum_{e\in\gamma_C}(\hat{\theta}^\dagger(t_e)h_e - \hat{\theta}^\dagger(v_C))\sigma^i\hat{\theta}(v_C)\hat{J}_i^{v_C,e}\Psi_C, \quad (4.35)$$

which gives $\widehat{\tilde{H}_C^{(1)}}$ acting on Ψ_C as

$$\widehat{\tilde{H}_C^{(1)}}\Psi_C = \kappa\hbar\beta N(v_C)\sum_{e\in\gamma_C}\sqrt{\widehat{V_C^{-1}}}(\hat{\theta}^\dagger(t_e)h_e - \hat{\theta}^\dagger(v_C))\sigma^i\hat{\theta}(v_C)\hat{J}_i^{v_C,e}\sqrt{\widehat{V_C^{-1}}}\Psi_C. \quad (4.36)$$

For the second term $\widehat{\tilde{H}_C^{(2)}}$, one has

$$\hat{H}_C^{(2)} = \int_C \mathrm{d}^3x N(x)E_i^a(x)K_a^i(x)\theta^\dagger(x)\theta(x). \quad (4.37)$$

Taking advantage of the Thiemann's trick to quantize the pure-gravity Hamiltonian constraint in LQG [5], one has

$$\int_C \mathrm{d}^3x f(x)E_i^a(x)K_a^i(x) = \frac{1}{2\kappa\beta^2}\left\{\int_C \mathrm{d}^3x f(x)H_E(x), V_C\right\}, \quad (4.38)$$

where $H_E(x)$ is the Euclidean part of the pure-gravity Hamiltonian constraint, i.e.,

$$H_E(x) = \frac{\epsilon_{ijk}F_{ab}^i(x)E_j^a(x)E_k^b(x)}{\sqrt{q(x)}}.$$

Thus, $H_C^{(2)}$ is quantized as

$$\hat{H}_C^{(2)} = \frac{1}{2i\kappa\hbar\beta^2}[N(v_C)\hat{H}_{E,v_C}\hat{\theta}^\dagger(v_C)\hat{\theta}(v_C), \hat{V}_{v_C}], \quad (4.39)$$

where $\hat{H}_{E,v_C}$ denotes the Euclidean part of the pure-gravity Hamiltonian constraint operator at v_C, and $\hat{V}_{v_C}$, the volume operator at v_C. Equation (4.39) leads to the operator $\widehat{\tilde{H}_C^{(2)}}$ given by

$$\widehat{\tilde{H}_C^{(2)}} = \frac{1}{2i\kappa\hbar\beta^2}N(v_C)\sqrt{\widehat{V_{v_C}^{-1}}}(\hat{H}_{E,v_C}\hat{V}_{v_C} - \hat{V}_{v_C}\hat{H}_{E,v_C})\hat{\theta}^\dagger(v_C) \times \hat{\theta}(v_C)\sqrt{\widehat{V_{v_C}^{-1}}}. \quad (4.40)$$

Finally, for the third term $\widehat{\tilde{H}_C^{(3)}}$, we have

$$\hat{H}_C^{(3)}\Psi_C = \kappa\hbar\beta N(v_C)\left(\sum_{e \text{ at } v_C} \hat{J}_i^{v_C,e}\right)\hat{\theta}^\dagger(v_C)\sigma^i\hat{\theta}(v_C)\Psi_C, \tag{4.41}$$

by taking advantage of the Gauss constraint operator of the pure gravity. Thus $\widehat{\tilde{H}_C^{(3)}}$ reads

$$\widehat{\tilde{H}_C^{(3)}}\Psi_C = \kappa\hbar\beta N(v_C)\widehat{\sqrt{V_C^{-1}}}\left(\sum_{e \text{ at } v_C} \hat{J}_i^{v_C,e}\right) \times \hat{\theta}^\dagger(v_C)\sigma^i\hat{\theta}(v_C)\widehat{\sqrt{V_C^{-1}}}\Psi_C. \tag{4.42}$$

With (4.36), (4.40), and (4.42), the operator $\widehat{H_{\mathcal{C}_\epsilon}}$ is

$$\hat{H}_{\mathcal{C}_\epsilon} = \sum_{C\in\mathcal{C}_\epsilon^{(3)}} -i\widehat{\tilde{H}_C^{(1)}} - \frac{\beta}{2}\widehat{\tilde{H}_C^{(2)}} - \frac{1+\beta^2}{2\beta}\widehat{\tilde{H}_C^{(3)}} - \beta\widehat{\tilde{H}_C^{(1)}}, \tag{4.43}$$

where

$$\mathcal{C}_\epsilon^{(3)} = \{C \in \mathcal{C}_\epsilon, \gamma \cap C \text{ is type-C}\}.$$

Even though $\widehat{H_{\mathcal{C}_\epsilon}}$ in (4.43) is defined with a partition structure $\mathcal{C}_\epsilon$ on Σ, this partition structure is indeed not necessary. One can define an operator equivalent to $\widehat{H_{\mathcal{C}_\epsilon}}$ with Σ endowed with another structure, which is more convenient for the further study. To this end, let us first introduce the following notion.

Definition IV.1 (removable vertex). A vertex v of a graph γ is removable if it satisfies the following conditions:

(i) v is a bivalence vertex connecting e_1 and e_2.

(ii) The composition of e_1 and e_2 as a curve is C^m and semianalytic.

Given a graph γ, one can obtain another graph $\ker(\gamma)$ by removing all of its removable vertices. $\ker(\gamma)$ will be called the kernel of γ. Let Γ_{ker} be the collection of the kernels of all graphs in Γ_o. Fix once and for all a parametrization for each $\gamma \in \Gamma_{\text{ker}}$, where a parametrization of a graph is an assignment to each edge $e \in E(\gamma)$ a parametrization $[0,1] \ni t \to e(t) \in \Sigma$. For an edge e of a kernel in Γ_{ker} taking v as an end point, we can define $e(v,\delta) \subset e$ as the segment starting from v and ending at either $e(\delta)$ for $v = e(0)$ or $e(1-\delta)$ for $v = e(1)$. Given a graph $\gamma = \gamma_G \cup \gamma_F$, for each $v \in V(\gamma_G)$, mimicking the operator $\widehat{\tilde{H}_C^{(i)}}$ in (4.36), (4.40), and (4.42), we define

$$\widetilde{H^{(i)}(v;\vec{\delta}_v)} = \widehat{\sqrt{V_v^{-1}}}\,\widehat{H^{(i)}(v;\vec{\delta}_v)}\,\widehat{\sqrt{V_v^{-1}}}, \quad i = 1,2,3, \tag{4.44}$$

with

$$\begin{aligned}
\widehat{H^{(1)}(v;\vec{\delta}_v)} &= \kappa\hbar\beta N(v)\sum_{e \text{ at } v}(\hat{\theta}^\dagger(t_{e(v,\delta_{v,e})})h_{e(v,\delta_{v,e})}\sigma^i\hat{\theta}(v) - \hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v))\hat{J}_i^{v,e},\\
\widehat{H^{(2)}(v;\vec{\delta}_v)} &= \frac{1}{2i\kappa\hbar\beta^2}N(v)(\hat{H}_{E,v}\hat{V}_v - \hat{V}_v\hat{H}_{E,v})\hat{\theta}^\dagger(v)\hat{\theta}(v),\\
\widehat{H^{(3)}(v;\vec{\delta}_v)} &= \kappa\hbar\beta N(v)\left(\sum_{e \text{ at } v}\hat{J}_i^{v,e}\right)\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v),
\end{aligned} \tag{4.45}$$

where $\vec{\delta}_v = \{\delta_{v,e}\}_{e \text{ at } v}$ is a vector of real numbers for each vertex. Taking advantage of these operators, we define

$$\begin{aligned}
\widehat{H(\delta)} &:= \sum_{v\in V(\gamma_G)} -i\widetilde{H^{(1)}(v;\vec{\delta}_v)} - \frac{\beta}{2}\widetilde{H^{(2)}(v;\vec{\delta}_v)} - \frac{1+\beta^2}{2\beta}\widetilde{H^{(3)}(v;\vec{\delta}_v)} - \beta\widetilde{H^{(1)}(v;\vec{\delta}_v)},\\
&= \sum_{v\in V(\gamma)} -i\widetilde{H^{(1)}(v;\vec{\delta}_v)} - \frac{\beta}{2}\widetilde{H^{(2)}(v;\vec{\delta}_v)} - \frac{1+\beta^2}{2\beta}\widetilde{H^{(3)}(v;\vec{\delta}_v)} - \beta\widetilde{H^{(1)}(v;\vec{\delta}_v)},
\end{aligned} \tag{4.46}$$

where $\delta \equiv \{\vec{\delta}_v\}_{v\in V(\gamma)}$ and the second step is a consequence of the operator $\widehat{\sqrt{V_v^{-1}}}$ in $\widetilde{H^{(i)}(v;\vec{\delta}_v)}$. Actually, due to the operator $\widehat{\sqrt{V_v^{-1}}}$ in $\widetilde{H^{(i)}(v;\vec{\delta}_v)}$, $\widetilde{H^{(i)}(v;\vec{\delta}_v)}$ with $v \notin \gamma_G$ vanish for all $i = 1, 2, 3$. Comparing the operators $\widehat{H(\delta)}$ and $\widehat{H_{\mathcal{C}_\epsilon}}$, one can verify easily that $\widehat{H(\delta)} = \widehat{H_{\mathcal{C}_\epsilon}}$ with a suitably chosen δ. Hence, we can use $\widehat{H(\delta)}$ instead of $\widehat{H_{\mathcal{C}_\epsilon}}$ for our further study. In the rest of the paper, we will assume that $\delta_{v,e}$ is a constant for all v and e for convenience. The discussion for nonconstant δ can be discussed similarly.

When the limit of $\widehat{H(\delta)}$ as $\delta \to 0$ is taken, the only nontrivial term in (4.45) one needs to consider is

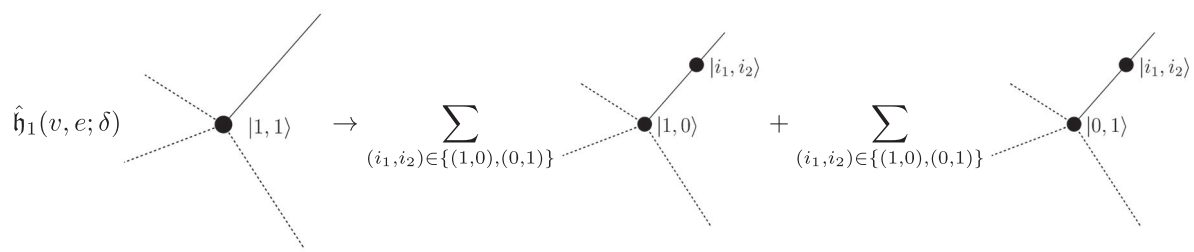
$$\hat{\mathfrak{h}}_1(v,e;\delta) := \hat{\theta}^\dagger(t_{e(v,\delta)})h_{e(v,\delta)}\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e}.$$

By definition, we have

$$\begin{aligned}
&\hat{\mathfrak{h}}_1(v,e;\delta)(D^j_{mn}(h_{e(v,\delta)}) \otimes |k_1,k_2\rangle_v) \\
&= -2W_jW_{\frac{1}{2}}\sum_{J=j\pm\frac{1}{2}}\sum_{A,C}(-1)^{m+A-n-C}\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ j & j & J\end{bmatrix}\begin{pmatrix} j & \frac{1}{2} & J\\ m & A & -(m+A)\end{pmatrix}\begin{pmatrix} J & \frac{1}{2} & j\\ -(n+C) & C & n\end{pmatrix} \\
&\quad \times D^J_{m+A,n+C}(h_{e(v,\delta)}) \otimes \hat{\theta}_C(v)|k_1,k_2\rangle_v \otimes \hat{\theta}^\dagger_A(t_{e(v,\delta)})|0,0\rangle_{t_{e(v,\delta)}},
\end{aligned} \tag{4.47}$$

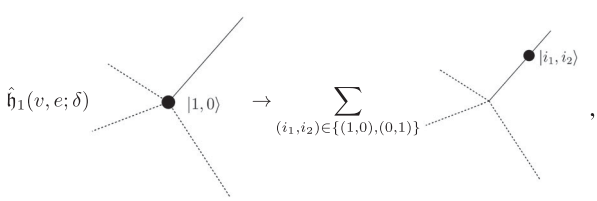
where $W_j = \sqrt{j(j+1)(2j+1)}$, $\left[\begin{smallmatrix} J & j & \frac{1}{2}\\ \frac{1}{2} & 1 & j\end{smallmatrix}\right]$ is the $6j$ symbol and $\left(\begin{smallmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{smallmatrix}\right)$ denotes the $3j$ symbol. Equation (4.47) gives us the picture that the operator $\hat{\mathfrak{h}}_1(v,e;\delta)$ moves the fermionic vertex at v to $t_{e(v,\delta)} \in e$, and simultaneously change the spin on $e(v,\delta)$. To illustrate this statement, let us use a disk • to represent a fermionic vertex and a solid line to represent the edge e. Then, by omitting the explicit coefficients in the right hand of (4.47), one has

(1) for $k_1 = 1 = k_2$,



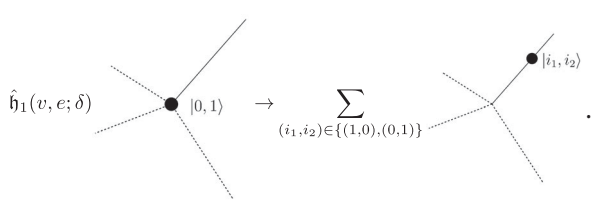
(4.48)

(2) for $k_1 = 1$ and $k_2 = 0$



(4.49)

(3) for $k_1 = 0$ and $k_2 = 1$



(4.50)

According to (4.48)–(4.50), one gets $\langle\hat{\mathfrak{h}}_1(v,e;\delta)\psi|\hat{\mathfrak{h}}_1(v,e;\delta')\psi\rangle = 0$ for $\delta \neq \delta'$, which implies that the limit of $\hat{\mathfrak{h}}_1(v,e;\delta)$ as $\delta \to 0$ does not exist. As a consequence, the limit of $\widehat{H(\delta)}$ as $\delta \to 0$ does not exist, too. However, as final operator should be define as $\lim_{\delta\to 0}\widehat{H(\delta)}$, we need to introduce the vertex Hilbert space $\mathcal{H}_{\rm vtx}$ which is in the dual space of $\mathbf{Cyl_{tot}}$. Once $\mathcal{H}_{\rm vtx}$ is introduced, $\widehat{H(\delta)}$ can be promoted to an operator $\widehat{H(\delta)}^*$ in $\mathcal{H}_{\rm vtx}$ by the duality such that $\widehat{H(\delta)}^* = \widehat{H(\delta')}^*$ for $\delta \neq \delta'$. Then, the limit of $\widehat{H(\delta)}^*$ as $\delta \to 0$ exists. It will be seen below that $\mathcal{H}_{\rm vtx}$ is defined to be the kinematical Hilbert space averaged by diffeomorphisms preserving some particular vertices.

Another issue motivating us to introduce the vertex Hilbert space is that the adjoint operator to $\widehat{H(\delta)}$ is not densely defined. To see this, let us investigate the adjoint of $\widehat{H^{(1)}(v;\delta)}$, a term of $\widehat{H(\delta)}$. By definition, for a state $\Psi_\gamma \in \mathcal{H}^{\rm irr}_\gamma$ with γ taking v as a vertex, the adjoint $\widehat{H^{(1)}(v;\delta)}^\dagger$ acts on it such that for all $\Phi_{\gamma'} \in \mathcal{H}$,

$$\langle \widehat{H^{(1)}(v;\delta)}^{\dagger}\Psi_\gamma|\Phi_{\gamma'}\rangle = \langle \Psi_\gamma|\widehat{H^{(1)}(v;\delta)}\Phi_{\gamma'}\rangle = \sum_{\substack{e\in E(\gamma')\\ e\,\mathrm{at}\,v}} \kappa\hbar\beta N(v)\langle(\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger - \hat{\theta}(v)\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e})\Psi_\gamma|\Phi_{\gamma'}\rangle, \quad (4.51)$$

with $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger = \hat{J}_i^{v,e}\hat{\theta}^\dagger(v)\sigma^i h^{-1}_{e(v,\delta)}\hat{\theta}(t_{e(v,\delta)})$. The subtlety arises due to the summation over edges of γ' rather than γ. Let us consider a state Ψ_γ with γ shown in Fig. 2. Moreover, the fermion state of Ψ_γ at the fermionic vertex $t_{e(v,\delta)}$ is chosen to be $a|0,1\rangle_{t_{e(v,\delta)}} + b|1,0\rangle_{t_{e(v,\delta)}}$ for some $a,b\in\mathbb{C}$. Then, for each graph γ' obtained from γ by using an edge $e(v,\delta)$ to connect the vertex $t_{e(v,\delta)}$ and v, dropping the fermionic vertex $t_{e(v,\delta)}$ and adding a fermionic vertex at v (see the two examples in Fig. 2), one can find a state $\Phi_{\gamma'}\in\mathcal{H}^{\mathrm{irr}}_{\gamma'}$ such that $\langle(\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger - \hat{\theta}(v)\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e})\Psi_\gamma|\Phi_{\gamma'}\rangle \neq 0$, where $e\in\gamma'$ is the edge containing $e(v,\delta)$ as a segment. Because there exist uncountably infinitely many such graphs γ' and the states $\Phi_{\gamma'}$ associated to different graphs are orthogonal to each other, Ψ_γ is not in the domain of $\widehat{H^{(1)}(v;\delta)}^{\dagger}$. A similar argument can be made for all of states in $\mathcal{H}^{\mathrm{irr}}_\gamma$ (including those taking $|1,1\rangle_{t_{e(v,\delta)}}$ at $t_{e(v,\delta)}$), which implies that the entire Hilbert space $\mathcal{H}^{\mathrm{irr}}_\gamma$ is not contained in the domain of $\widehat{H^{(1)}(v;\delta)}^{\dagger}$. Besides, it is manifest that there are infinitely many such Hilbert spaces that are excluded from the domain of $\widehat{H^{(1)}(v;\delta)}^{\dagger}$. Actually, according to (4.47), the operator $\hat{\mathfrak{h}}_1(v,e;\delta)$ will change the state taking spin $j=\frac{1}{2}$ on $e(v,\delta)$ to a superposition of states taking spins $j=0$ and 1 on $e(v,\delta)$. In other words, the operator $\hat{\mathfrak{h}}_1(v,e;\delta)$ can erase the segment $e(v,\delta)\subset e$, which is indeed the essential reason for the problem of defining $\widehat{H^{(1)}(v;\delta)}^{\dagger}$. It will be shown below that this problem can also be fixed by introducing the vertex Hilbert space to define the limit. After taking the limit as $\delta\to 0$ in the vertex Hilbert space, we will get a projection $\chi_+(|\hat{J}^{v,e}|^2)$ [see (4.84)]. This projection is left multiplied to the operator corresponding to $\hat{\mathfrak{h}}_1(v,e;\delta)$ in $\mathcal{H}_{\mathrm{vtx}}$, and kill the state with $j=0$ on $e(v,\delta)$.

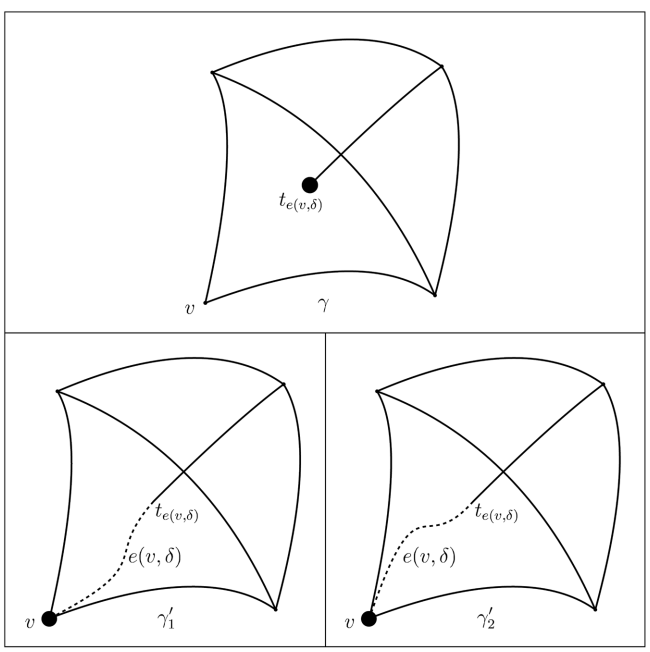


FIG. 2. An example of a graph γ (top panel) such that $\mathcal{H}^{\mathrm{irr}}_\gamma$ is not in the domain of $\widehat{H^{(1)}(v;\delta)}^{\dagger}$, and examples of graphs γ' (bottom panel) such that $\langle \widehat{H^{(1)}(v;\delta)}^{\dagger}\Psi_\gamma|\Phi_{\gamma'}\rangle \neq 0$. The black disks represent fermionic vertices.

Finally, let us discuss why we choose $H_{\mathcal{C}_e}$ rather than its complex conjugate for quantization. Suppose that one chooses the complex conjugate of $H_{\mathcal{C}_e}$ to do the above quantization. Then, in the resulting operator, one will get a term involving $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger = \hat{J}_i^{v,e}\hat{\theta}^\dagger(v)\sigma^i h^{-1}_{e(v,\delta)}\hat{\theta}(t_{e(v,\delta)})$. In contrast to $\hat{\mathfrak{h}}_1(v,e;\delta)$, $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$ moves the fermionic vertex at $t_{e(v,\delta)}\in e$ to v. Now let us consider the states $\hat{\mathfrak{h}}_1(v,e;\delta)|\psi\rangle$ and $\hat{\mathfrak{h}}_1(v,e;\delta')|\psi\rangle$ with $\delta\neq\delta'$ for $|\psi\rangle$, says, being $D^j_{mn}(h_e)\otimes|k_1,k_2\rangle_v$. Then, one gets $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\hat{\mathfrak{h}}_1(v,e;\delta')|\psi\rangle)=0$ but $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\hat{\mathfrak{h}}_1(v,e;\delta)|\psi\rangle)\neq 0$. This result means that $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$ is not diffeomorphism covariant. Namely, diffeomorphism equivalent states, i.e., the states $\hat{\mathfrak{h}}_1(v,e;\delta)|\psi\rangle$ and $\hat{\mathfrak{h}}_1(v,e;\delta')|\psi\rangle$, could be no longer diffeomorphism equivalent after acted by $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$. This facts leads to a problem that $\sum_{e\,\mathrm{at}\,v}\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$ cannot be promoted to a well-defined operator in the diffeomorphism invariant Hilbert space.

1. The vertex Hilbert space $\mathcal{H}_{\mathrm{vtx}}$

Given a graph γ, we will consider the subgroup of C^m semianalytic diffeomorphisms which act on $V(\ker(\gamma))$ trivially, i.e., the diffeomorphisms preserving every vertex of $\ker(\gamma)$. This subgroup will be denoted by $\mathrm{Diff}_{V(\ker(\gamma))}$. There are two subgroups of $\mathrm{Diff}_{V(\ker(\gamma))}$. The first one, denoted by Diff_γ, preserves γ. The other one, denoted by Diff^0_γ, preserves every edge of γ. Hence Diff^0_γ is a subgroup of Diff_γ. The quotient

$$\mathrm{GS}_\gamma = \mathrm{Diff}_\gamma/\mathrm{Diff}^0_\gamma \quad (4.52)$$

is the group of graph symmetries of γ. GS_γ is a finite group with order $|\mathrm{GS}_\gamma|$. Given a state Ψ_γ with respect to γ and a diffeomorphism ϕ, $\phi\star\Psi_\gamma$ denotes the pullback of Ψ under ϕ. The averaging with respect to GS_γ defines a projection $P_\gamma\colon \mathcal{H}^{\mathrm{irr}}_\gamma\to\mathcal{H}^{\mathrm{irr}}_\gamma$,

$$P_\gamma\colon\ \Psi_\gamma \mapsto \frac{1}{|\mathrm{GS}_\gamma|}\sum_{\phi\in\mathrm{GS}_\gamma}\phi\star\Psi_\gamma. \tag{4.53}$$

The averaging with respect to the remaining diffeomorphisms $\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}_\gamma$ defines

$$\begin{aligned}\eta(\Psi_\gamma) &:= \sum_{\phi\in\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}_\gamma}\phi\star P_\gamma(\Psi_\gamma)\\ &= \frac{1}{|\mathrm{GS}_\gamma|}\sum_{\phi\in\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}^0_\gamma}\phi\star\Psi_\gamma.\end{aligned} \tag{4.54}$$

Obviously, $\eta(\Psi_\gamma)$ belongs to the algebraic dual space $\mathbf{Cyl_{tot}}^*$ of $\mathbf{Cyl_{tot}}$. Taking advantage the decomposition (3.43), one can extend η to a well-defined operation on $\mathbf{Cyl_{tot}}$. Let $\eta(\mathbf{Cyl_{tot}})\subset\mathbf{Cyl_{tot}}^*$ denote the image of $\mathbf{Cyl_{tot}}$ under η. The group averaging procedure naturally endows $\eta(\mathbf{Cyl_{tot}})$ with an inner product

$$(\eta(\Psi_\gamma),\eta(\Psi'_{\gamma'})) = (\eta(\Psi_\gamma)|\Psi'_{\gamma'}\rangle, \tag{4.55}$$

where $(\eta(\Psi_\gamma)|\Psi'_{\gamma'}\rangle$, the action of $\eta(\Psi_\gamma)\in\mathbf{Cyl_{tot}}^*$ on $\Psi_{\gamma'}$, is given by

$$(\eta(\Psi_\gamma)|\Psi'_{\gamma'}\rangle = \sum_{\phi\in\mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}^0_\gamma}\langle\phi\star\Psi_\gamma,\Psi'_{\gamma'}\rangle. \tag{4.56}$$

Here, $\langle\cdot,\cdot\rangle$ is the inner product in $\mathcal{H}$. Then the vertex Hilbert space $\mathcal{H}_{\mathrm{vtx}}$ is the completion of $\eta(\mathbf{Cyl_{tot}})$, i.e.,

$$\mathcal{H}_{\mathrm{vtx}} = \overline{\eta(\mathbf{Cyl_{tot}})}. \tag{4.57}$$

Given a finite subset $W\subset\Sigma$, let $\Gamma_{\ker}(W)$ be the collection of graphs γ satisfying $V(\ker(\gamma))=W$. Two graphs $\gamma,\gamma'\in\Gamma_{\ker}(W)$ are said to be equivalent, denoted by $\gamma\sim_d\gamma'$, if there exists a diffeomorphism $\phi\in\mathrm{Diff}_W$, i.e., a diffeomorphism preserving W, such that $\phi(\gamma)=\gamma'$. Let $[\Gamma_{\ker}(W)]$ be the quotient space $\Gamma_{\ker}(W)/\sim_d$, $[\gamma]\in[\Gamma_{\ker}(W)]$ be the equivalence class of γ, and $\mathcal{S}^{\mathrm{irr}}_\gamma$ be the image of P_γ. Then $\eta\colon\mathcal{S}^{\mathrm{irr}}_\gamma\to\mathcal{H}_{\mathrm{vtx}}$ is isometric. By $\eta(\mathcal{S}^{\mathrm{irr}}_\gamma)$ denoting the image of $\mathcal{S}^{\mathrm{irr}}_\gamma$ under η, one has

$$\eta(\mathcal{S}^{\mathrm{irr}}_\gamma) = \eta(\mathcal{S}^{\mathrm{irr}}_{\gamma'}),\qquad\forall\,\gamma,\qquad\gamma'\in[\gamma]. \tag{4.58}$$

Thus, we can define

$$\eta(\mathcal{S}^{\mathrm{irr}}_{[\gamma]}) := \eta(\mathcal{S}^{\mathrm{irr}}_{\gamma'}) \tag{4.59}$$

by choosing arbitrary $\gamma'\in[\gamma]$. Let $\mathrm{FS}(\Sigma)$ be the set of finite subsets of Σ. Then we have

$$\eta(\mathbf{Cyl_{tot}}) = \bigoplus_{W\in\mathrm{FS}(\Sigma)}\ \bigoplus_{[\gamma]\in[\Gamma_{\ker}(W)]}\eta(\mathcal{S}^{\mathrm{irr}}_{[\gamma]})\oplus\mathbb{C}. \tag{4.60}$$

The factor $1/|\mathrm{GS}_\gamma|$ in (4.55) ensures that η is an isometric between $\mathcal{H}^{\mathrm{irr}}_\gamma$ and $\eta(\mathcal{H}^{\mathrm{irr}}_\gamma)$, i.e.,

$$\eta(\mathbf{Cyl_{tot}}) \cong \bigoplus_{W\in\mathrm{FS}(\Sigma)}\ \bigoplus_{[\gamma]\in[\Gamma_{\ker}(W)]}\mathcal{S}^{\mathrm{irr}}_{\sigma([\gamma])}\oplus\mathbb{C}, \tag{4.61}$$

where $\sigma([\gamma])\in[\gamma]$ is a representative of $[\gamma]$ fixed once and for all.

2. The Hamiltonian operator on $\mathcal{H}_{\mathrm{vtx}}$

Given $\Psi_\gamma\in\mathcal{S}^{\mathrm{irr}}_\gamma$, let $\widehat{H(\delta)}^*$ be the dual of $\widehat{H(\delta)}$ acting on $\eta(\Psi_\gamma)$, i.e.,

$$(\widehat{H(\delta)}^*\eta(\Psi_\gamma)|\Phi_{\gamma'}\rangle = (\eta(\Psi_\gamma)|\widehat{H(\delta)}\Phi_{\gamma'}\rangle,\qquad\forall\,\gamma'\in\Gamma_o,\qquad\Phi_{\gamma'}\in\mathcal{H}^{\mathrm{irr}}_{\gamma'}. \tag{4.62}$$

The operator $\widehat{H(\delta)}$ takes the form $\widehat{H(\delta)}=\sum_v N(v)\widehat{\sqrt{V_v^{-1}}}\,\hat{O}(v)\widehat{\sqrt{V_v^{-1}}}$ with $\hat{O}(v)$ representing some operator. Due to the inverse volume operators, we have

$$\lim_{\delta\to0}(\widehat{H(\delta)}^*\eta(\Psi_\gamma)|\Phi_{\gamma'}\rangle = \lim_{\delta\to0}\sum_{v\in V(\ker(\gamma))}N(v)(\eta(\Psi_\gamma)|\widehat{\sqrt{V_v^{-1}}}\,\hat{O}(v)\widehat{\sqrt{V_v^{-1}}}\Phi_{\gamma'}\rangle. \tag{4.63}$$

Recalling (4.46), we will consider the right-hand term by term.

Let us begin with the operator $\widehat{H^{(1)}(v;\delta)}$ with $v\in V(\ker(\gamma))$. By (4.63), we need to investigate

$$I_v(\gamma',\delta) = (\eta(\Psi_\gamma)|\widehat{H^{(1)}(v;\delta)}\Phi_{\gamma'}\rangle \tag{4.64}$$

for all graphs γ' taking v as a vertex. By definition, we can split $\widehat{H^{(1)}(v;\delta)}$ into two parts such that

$$\widehat{H^{(1)}(v;\delta)}\Phi_{\gamma'} = \kappa\hbar\beta N(v)\hat{\mathfrak{h}}_1(v;\delta)\Phi_{\gamma'} - \kappa\hbar\beta N(v)\hat{\mathfrak{h}}_2(v)\Phi_{\gamma'} \tag{4.65}$$

with $\hat{\mathfrak{h}}_1(v;\delta)\Phi_{\gamma'} = \sum_{e \text{ at } v} \hat{\mathfrak{h}}_1(v,e;\delta)\Phi_{\gamma'}$ and $\hat{\mathfrak{h}}_2(v) = \sum_{[e] \text{ at } v} \hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\hat{\mathcal{J}}_i^{v,[e]}$. Substituting (4.65) into (4.64), we will be concerned about the term

$$I_v^{(1)}(\gamma',\delta) = (\eta(\Psi_\gamma)|\hat{\mathfrak{h}}_1(v;\delta)\Phi_{\gamma'}\rangle = \sum_{\substack{e\in E(\gamma')\\ e \text{ at } v}} (\eta(\Psi_\gamma)|\hat{\mathfrak{h}}_1(v,e;\delta)\Phi_{\gamma'}\rangle. \quad (4.66)$$

Substituting (4.54) into (4.66), we get

$$I_v^{(1)}(\gamma',\delta) = \sum_{\substack{e\in E(\gamma')\\ e \text{ at } v}} \sum_{\phi\in \mathrm{Diff}'} \langle \hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi\star\Psi_\gamma)|\Phi_{\gamma'}\rangle, \quad (4.67)$$

with $\mathrm{Diff}' := \mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}_\gamma$ for abbreviation, where we employ $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger = \hat{\mathcal{J}}_i^{v,e}\hat{\theta}^\dagger(v)\sigma^i h_{e(v,\delta)}^{-1}\hat{\theta}(t_{e(v,\delta)})$. Note that the summation in the right-hand side of (4.67) is over edges of γ', while the operator $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$ acts on states associated to graphs diffeomorphism to γ. Thus, we cannot get $I_v^{(1)}(\gamma',\delta) = (\hat{\mathfrak{h}}_1(v;\delta)^\dagger\eta(\Psi_\gamma)|\Phi_{\gamma'}\rangle$ naively from (4.67).

Consider the action of $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$ on $D^{j_e}_{m_e n_e}(h_{e(v,\delta)}) \otimes D^{j'_e}_{m'_e n'_e}(h_{e\backslash e(v,\delta)}) \otimes |k_1,k_2\rangle_{t_{e(v,\delta)}}$, a general state associated to the graph $\{e\}\cup\{t_{e(v,\delta)}\}$, where $e\backslash e(v,\delta)$ is the other segment of e complementing $e(v,\delta)$. Here and in the rest, by a state associated to a graph α, we refer to a state in $\mathcal{H}_\alpha^{\mathrm{irr}}$. Acting on the state, the holonomy $h_{e(v,\delta)}^{-1}$ in $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger$ changes the spin j_e on $e(v,\delta)$ to $j_e \pm \frac{1}{2}$. Then, for $j_e = \frac{1}{2}$, the spin on $e(v,\delta)$ is changed to 0, i.e., the segment $e(v,\delta)$ is erased. Thus, for $j_e = \frac{1}{2}$, acted by the holonomy, the state becomes a combination of one state associated $\{e\}$ and the other state associated to $\{e\backslash e(v,\delta)\}$. However, the operator $\hat{\mathcal{J}}_i^{v,e}$ in $\hat{\mathfrak{h}}_1(v;\delta)^\dagger$ will annihilate this one associated to $\{e\backslash e(v,\delta)\}$. Consequently, acted by $\hat{\mathfrak{h}}_1(v;\delta)^\dagger$, the resulting state is still a state associated to $\{e\}$, regardless of the change of the fermionic vertices by $\hat{\theta}^\dagger(v)$ and $\hat{\theta}(t_{e(v,\delta)})$.

$\phi_o\star\Psi_\gamma$ for some diffeomorphism ϕ_o is a state associated to the graph $\phi_o(\gamma)$. Let $\tilde{\gamma}_G$ and $\tilde{\gamma}_F$ denote the gravitational sector and the fermionic sector of $\phi_o(\gamma)$, i.e., $\phi_o(\gamma) = \tilde{\gamma}_G \cup \tilde{\gamma}_F$. Then, the discussion in the last paragraph tells that $\hat{\mathfrak{h}}_1(v;\delta)^\dagger\phi_o\star\Psi_\gamma$ is a state associated to some graph whose gravitational sector is $\tilde{\gamma}_G \cup e(v,\delta)$. Thus, $\langle\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle \neq 0$ for $\Phi_{\gamma'} \in \mathcal{H}_{\gamma'}^{\mathrm{irr}}$ implies $e \subset \tilde{\gamma}_G \cup e(v,\delta)$, which means either $e \subset \phi_o(\gamma)$, or $e(v,\delta)\not\subset\phi_o(\gamma)$ but $e\backslash e(v,\delta) \subset \phi_o(\gamma)$.

Let us first consider the case where $e(v,\delta)\not\subset\phi_o(\gamma)$ but $e\backslash e(v,\delta) \subset \phi_o(\gamma)$ such that $\langle\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle \neq 0$. The hypothesis $\langle\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle \neq 0$ implies $\tilde{\gamma}_G \cup e(v,\delta) = \gamma'_G$ where γ'_G is the gravitational sector of γ'. Together with $e(v,\delta)\not\subset\phi_o(\gamma)$, one gets $\tilde{\gamma}_G = \gamma'_G\backslash e(v,\delta)$. Since $e(v,\delta)$ is a segment of an edge $e \subset \gamma'_G$, $t_{e(v,\delta)}$ is a one-valence vertex in $\tilde{\gamma}_G$. Furthermore, $t_{e(v,\delta)}$ is a one-valence fermionic vertex because of $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma) \neq 0$. Thus, $t_{e(v,\delta)}$ is not a removable vertex; i.e., all of the diffeomorphisms in Diff' preserve this vertex. Now, let us come to $\delta' < \delta$. If $\sum_{\phi\in\mathrm{Diff}'}\langle\hat{\mathfrak{h}}_1(v,e;\delta')^\dagger(\phi\star\Psi_\gamma)|\Phi_{\gamma'}\rangle = \sum_{\phi\in\mathrm{Diff}'}\langle\hat{\mathfrak{h}}_1(v,e;\delta')^\dagger(\phi\star\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle$ still does not vanish, then there has to be another diffeomorphism ϕ'_o that can move a fermionic vertex in $\phi_o(\gamma)$ to $t_{e(v,\delta')}$ such that $\hat{\mathfrak{h}}_1(v,e;\delta')^\dagger(\phi'_o\star\phi_o\star\Psi_\gamma) \neq 0$, and simultaneously preserve all of the other vertices but not necessarily $t_{e(v,\delta)}$ in $\phi_o(\gamma)$ such that $\langle\hat{\mathfrak{h}}_1(v,e;\delta')^\dagger(\phi'_o\star\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle \neq 0$. This can be done only if ϕ'_o can move $t_{e(v,\delta)}$ to $t_{e(v,\delta')}$, which is impossible because $t_{e(v,\delta)}$ is preserved by all diffeomorphisms in Diff'. Therefore, $\sum_{\phi\in\mathrm{Diff}'}\langle\hat{\mathfrak{h}}_1(v,e;\delta')^\dagger(\phi\star\Psi_\gamma)|\Phi_{\gamma'}\rangle = 0$ for all $\delta' < \delta$. This discussion tells that for sufficiently small δ, the case with $e(v,\delta)\not\subset\phi_o(\gamma)$ but $e\backslash e(v,\delta) \subset \phi_o(\gamma)$ can be excluded. Because we finally need to consider the limit as $\delta \to 0$, choosing a sufficiently small δ can be done without loss of generality.

For the case with $e \subset \phi_o(\gamma)$, $\langle\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle \neq 0$ requires that the graph of $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)$ is equal to γ'. Since δ is chosen to be sufficiently small, in the graph γ' there is no fermionic vertex on the segment $e(v,\delta) \subset e \subset \gamma'$, and $t_{e(v,\delta)} \in \gamma'$ is also not a fermionic vertex. Therefore, for $\langle\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)|\Phi_{\gamma'}\rangle \neq 0$, there cannot exist any fermionic vertices in $e(v,\delta)\cup\{t_{e(v,\delta)}\}$, where $e(v,\delta)$ and $t_{e(v,\delta)}$ now are thought of as a segment and a vertex in the graph of $\hat{\mathfrak{h}}_1(v,e;\delta)^\dagger(\phi_o\star\Psi_\gamma)$. This conclusion has two meanings. At first, $\phi_o \in \mathrm{Diff}'$ must be the diffeomorphism that moves the closest fermionic vertex to v in $e \in \gamma$ to $t_{e(v,\delta)}$ so that $\hat{\theta}(t_{e(v,\delta)})$ can kill it. The closest fermionic vertex to v in $e \in \gamma$ will be denoted by $v_F^{(v,e)}$, and Fig. 3 gives an illustration of $v_F^{(v,e)}$. Second, it means that the fermion state at $v_F^{(v,e)}$ cannot be $|1,1\rangle_{v_F^{(v,e)}}$ because it

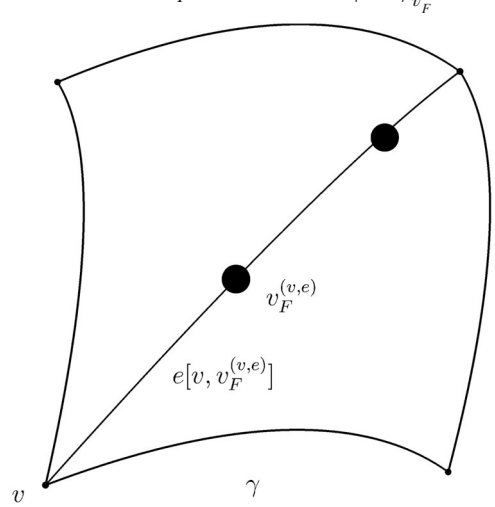


FIG. 3. An example of graphs γ with $v_F^{(v,e)}$ being the closest fermionic vertex to v in $e \in E(\ker(\gamma))$, and $e[v_F^{(v,e)}, v] \subset e$ is the edge of γ from v to $v_F^{(v,e)}$.

cannot be annihilated completely by $\hat{\theta}(t_{v,\delta})$. For the second point, we introduce a projection $\hat{\mathbb{P}}_{v_F}$ on the fermion Hilbert space $\mathcal{H}_{v_F}$ such that

$$\hat{\mathbb{P}}_{v_F}|i_1, i_2\rangle = \begin{cases} 0, & i_1 = 1 = i_2, \\ |i_1, i_2\rangle, & \text{otherwise.} \end{cases} \tag{4.68}$$

The first point makes us decompose ϕ_o as follows. Let $\gamma_\star$ be the graph of γ' with $v_F^{(v,e)}$ dropped, and $v \in \gamma$ promoted to a new fermionic vertex if it is not. Then, we find a diffeomorphism $\phi_1 \in \mathrm{Diff}_{V(\ker(\gamma))}/\mathrm{Diff}_{\gamma_\star} = \mathrm{Diff}_{V(\ker(\gamma_\star))}/\mathrm{Diff}_{\gamma_\star}$ such that $\phi_1(\gamma_\star) = \gamma'$. With ϕ_1, we find $\phi_2 \in \mathrm{Diff}_{\phi_1(\gamma_\star)}/\mathrm{Diff}_\gamma$ which moves $v_F^{(v,e)}$ to $t_{e(v,\delta)}$. One thus gets $\phi_o = \phi_2 \circ \phi_1$. Introducing $\hat{\mathfrak{H}}_v(e)$ as

$$\hat{\mathfrak{H}}_v(e) := \begin{cases} \hat{J}_i^{v,e}\hat{\theta}^\dagger(v)\sigma^i h^{-1}_{e[v_F^{(v,e)},v]}\hat{\theta}(v_F^{(v,e)})\hat{\mathbb{P}}_{v_F^{(v,e)}}, & \text{if } v_F^{(v,e)} \text{exists} \\ 0 & \text{otherwise} \end{cases}, \tag{4.69}$$

where $e[v_F^{(v,e)}, v]$ denotes the edge from v to $v_F^{(v,e)}$. We thus get

$$\langle \hat{\mathfrak{h}}_1(v, e; \delta)^\dagger(\phi_o \star \Psi_\gamma)|\Phi_{\gamma'}\rangle = \langle \phi_1 \star (\hat{\mathfrak{H}}_v(e)\Psi_\gamma)|\Phi_{\gamma'}\rangle, \tag{4.70}$$

where the right-hand side is independent of δ, and $\phi_2(\gamma') = \gamma'$ is used. Using (4.70), we finally get

$$\lim_{\delta\to 0} I_v^{(1)}(\gamma', \delta) = \sum_{\phi \in \mathrm{Diff}_{V(\gamma)}/\mathrm{Diff}_{\gamma_\star}} \langle \phi \star \hat{\mathfrak{H}}_v \Psi_\gamma | \Phi_{\gamma'}\rangle, \tag{4.71}$$

with

$$\hat{\mathfrak{H}}_v \Psi_\gamma = \sum_{e \text{ at } v} \hat{\mathfrak{H}}_v(e)\Psi_\gamma, \quad \forall\, \Psi_\gamma \in \mathcal{S}_\gamma^{\mathrm{irr}}. \tag{4.72}$$

Equation (4.71) leads to

$$\lim_{\delta\to 0}(\hat{\mathfrak{h}}_1(v;\delta)^* \eta(\Psi_\gamma)| = (\eta(\hat{\mathfrak{H}}_v \Psi_\gamma)|. \tag{4.73}$$

Finally, taking advantage of $\lim_{\delta\to 0}\hat{\mathfrak{h}}_2^* \eta(\Psi_\gamma) = \eta(\hat{\mathfrak{h}}_2 \Psi_\gamma)$, we have

$$\lim_{\delta\to 0}(\widehat{H^{(1)}(v;\delta)})^* \eta(\Psi_\gamma) = \kappa\hbar\beta N(v)\eta\left[\left(\hat{\mathfrak{H}}_v - \sum_{e \text{ at } v}\hat{J}_i^{v,e}\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\right)\Psi_\gamma\right]. \tag{4.74}$$

Now let us consider the operator $\widehat{H^{(2)}(v;\delta)}$. An issue on defining $\lim_{\delta\to 0}\widehat{H^{(2)}(v;\delta)}$ is the operator $\hat{H}_{E,v}$ comprising $\widehat{H^{(2)}(v;\delta)}$. In this paper, we will employ the work [45] to define $\hat{H}_{E,v}$ in $\mathcal{H}_{\mathrm{vtx}}$. More precisely, $\hat{H}_{E,v}$ is given by $\hat{H}_{E,v} = \widehat{\sqrt{V_v^{-1}}}(\hat{F}_v + \hat{F}_v^\dagger)\widehat{\sqrt{V_v^{-1}}}$ with

$$\hat{F}_v = -2\sum_{e,e' \text{ at } v} \epsilon(\dot{e}, \dot{e}')\epsilon^{ijk}\mathrm{tr}(h_{\alpha_{ee'}}\tau_k)\hat{J}_i^{v,e}\hat{J}_k^{v,e'}, \tag{4.75}$$

where $\alpha_{ee'}$ is a loop tangent to the two edges e and e' at the vertex v up to orders $k_e + 1$ and $k_{e'} + 1$, respectively, with k_e and $k_{e'}$ being, respectively, the orders of tangentiality of e and e' at v (see [45] for more details.) By this definition, $\hat{F}_v$ will change the graph by adding a loop at v and, thus, $\hat{F}_v^\dagger$, by removing a loop.

A subtlety here is that the vertex Hilbert space defined in [45] is a little different from ours. In [45], the authors defines the vertex Hilbert space with diffeomorphisms preserving $V(\gamma)$ while our work considers the diffeomorphisms preserving $V(\ker(\gamma))$. Regardless of this difference, the operator $\hat{H}_{E,v}$ introduced in [45], for $v \in V(\ker(\gamma))$, is well defined in our vertex Hilbert space $\mathcal{H}_{\mathrm{vtx}}$. Indeed, due to the operator $\widehat{\sqrt{V_v^{-1}}}$ in $\widehat{\tilde{H}^{(2)}(v;\delta)}$, we do not need to consider $\hat{H}_{E,v}$ for removable v, i.e., $v \notin V(\ker(\gamma))$. According to this discussion, for $v \in V(\ker(\gamma))$, we have

$$\lim_{\delta\to 0} H^{\widehat{(2)}(v;\delta)}\eta(\Psi_\gamma) = \frac{\kappa_1(v)}{2i\kappa\hbar\beta^2}N(v)\eta[(\hat{H}_{E,v}\hat{V}_v - \hat{V}_v\hat{H}_{E,v})\hat{\theta}^\dagger(v)\hat{\theta}(v)\Psi_\gamma], \tag{4.76}$$

where $\kappa_1(v)$ is introduced in [45] to remove the dependence on the partition.

Finally, for the operator $H^{\widehat{(3)}(v;\delta)}$, since $H^{\widehat{(3)}(v;\delta)}$ for $v \in V(\ker(\gamma))$ is independent of δ and $\mathrm{Diff}_{V(\ker(\gamma))}$ invariant. We have

$$\lim_{\delta\to 0}(H^{\widehat{(3)}(v;\delta)})^* \eta(\Psi_\gamma)| = (\eta(\hat{H}_F^{(3)}(v)\Psi_\gamma)| \tag{4.77}$$

with

$$\hat{H}_F^{(3)}(v) := \kappa\hbar\beta N(v)\left(\sum_{e \text{ at } v_C}\hat{J}_i^{v_C,e}\right)\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v). \tag{4.78}$$

Let us summarize our results. According to the above discussion, we have

$$\hat{A}_F\eta(\Psi_\gamma) := \lim_{\delta\to 0}\widehat{H(\delta)}^*\eta(\Psi_\gamma) = \sum_{v\in V(\gamma)}\eta\left(\widehat{\sqrt{V_v^{-1}}}\hat{H}_v\widehat{\sqrt{V_v^{-1}}}\Psi_\gamma\right), \tag{4.79}$$

where $\hat{H}_v$ is given by

$$\hat{H}_v = i\widehat{H^{(1)}(v)} - \frac{\beta}{2}\widehat{H^{(2)}(v)} - \frac{1+\beta^2}{2\beta}\widehat{H^{(3)}(v)} - \beta\widehat{H^{(1)}(v)}, \tag{4.80}$$

with

$$\begin{aligned}
\widehat{H^{(1)}(v)} &= \kappa_2(v)\ell_p^2\beta N(v)\left(\hat{\mathfrak{H}}_v - \sum_{e\text{ at }v}\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e}\right),\\
\widehat{H^{(2)}(v)} &= \frac{\kappa_1(v)}{2i\ell_p^2\beta^2}N(v)(\hat{H}_{E,v}\hat{V}_v - \hat{V}_v\hat{H}_{E,v})\hat{\theta}^\dagger(v)\hat{\theta}(v),\\
\widehat{H^{(3)}(v)} &= \kappa_3(v)\ell_p^2\beta N(v)\left(\sum_{e\text{ at }v}\hat{J}_i^{v,e}\right)\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v),
\end{aligned} \tag{4.81}$$

according to (4.74), (4.76), and (4.77), respectively. Here we again introduce the parameters κ_2 and κ_3 as in [35] in order to remove the dependence on the partition. Finally, because of (4.24), we finally define the fermion Hamiltonian operator $\hat{H}_F[N]$ on $\mathcal{H}_{\text{vtx}}$ as

$$\hat{H}_F[N] = \frac{1}{2}(\hat{A}_F + \hat{A}_F^\dagger). \tag{4.82}$$

Let $\hat{\mathfrak{H}}'_v$ on $\mathcal{H}_{\text{vtx}}$ be the operator defined by

$$\hat{\mathfrak{H}}'_v\eta(\Psi_\gamma) = \eta(\hat{\mathfrak{H}}_v\Psi_\gamma). \tag{4.83}$$

Then, the only nontrivial term in $A_F^\dagger$ is the operator $(\hat{\mathfrak{H}}'_v)^\dagger$. By the definition, $\hat{\mathfrak{H}}_v$ is composed of $\hat{\mathfrak{H}}_v(e)$, which annihilates the fermionic vertex contained in e and closest to v. Thus, the operator $(\hat{\mathfrak{H}}'_v)^\dagger$ contains the operators, each of which is associated to an edge e at v and promotes a point $v_F \in e$ to a fermionic vertex such that v_F becomes the closest to v among the fermionic vertices contained in e. To be more precise, let us define an operator $\hat{\mathfrak{H}}_v^\#(e)$ [refer to (4.69)] as

$$\hat{\mathfrak{H}}_v^\#(e) = \chi_+(|\hat{J}^{v,e}|^2)\hat{\theta}^\dagger(\mathfrak{v}_{e,v})h_{e(\mathfrak{v}_{e,v})}\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e}, \tag{4.84}$$

where χ_+ denotes the characteristic function of $[\frac{3}{4},\infty)$, $|\hat{J}^{v,e}|^2 = \delta^{kl}\hat{J}_k^{v,e}\hat{J}_l^{v,e}$, $\mathfrak{v}_{e,v} \in e$ is an arbitrary point between v and the closest fermionic vertex to v in e, and $e(\mathfrak{v}_{e,v}) \subset e$ is the segment starting from v and ending at $\mathfrak{v}_{e,v}$. Here the factor $\chi_+(|\hat{J}^{v,e}|^2)$ ensures that $\hat{\mathfrak{H}}_v^\#(e)$ cannot erase the segment $e(\mathfrak{v}_{e,v}) \subset e$. In other words, $\hat{\mathfrak{H}}_v^\#(e)$, acting on a state with spin $\frac{1}{2}$ on $e(\mathfrak{v}_{e,v})$, will only change the spin $\frac{1}{2}$ to 1. Moreover, $\mathfrak{v}_{e,v}$ is defined such that it becomes closest to v among the fermionic vertices in e after the action of $\hat{\mathfrak{H}}_v^\#(e)$. With $\hat{\mathfrak{H}}_v^\#(e)$, $(\hat{\mathfrak{H}}'_v)^\dagger$ is given by

$$(\hat{\mathfrak{H}}'_v)^\dagger\eta(\Psi_\gamma) = \eta\left(\sum_{e\text{ at }v}\hat{\mathfrak{H}}_v^\#(e)\Psi_\gamma\right). \tag{4.85}$$

Note that even though $\mathfrak{v}_{e,v}$ is not uniquely determined by its definition, the operator $(\hat{\mathfrak{H}}'_v)^\dagger$ on $\mathcal{H}_{\text{vtx}}$ is independent of the choice of $\mathfrak{v}_{e,v}$, due to the averaging operation η. Moreover, $\chi_+(|\hat{J}^{v,e}|^2)$ is introduced in (4.84) because (4.72) implies

$$\hat{\mathfrak{H}}_v = \sum_{e\text{ at }v}\hat{\mathfrak{H}}_v(e)\chi_+(|\hat{J}^{v,e}|^2). \tag{4.86}$$

Applying $\hat{\mathfrak{H}}_v^\#(e)$, $\hat{A}_F^\dagger$ reads

$$\hat{A}_F^\dagger\eta(\Psi_\gamma) = \sum_{v\in V(\gamma)}\eta\left(\widehat{\sqrt{V_v^{-1}}}\hat{H}_v^\#\widehat{\sqrt{V_v^{-1}}}\Psi_\gamma\right), \tag{4.87}$$

where $\hat{H}_v^\#$ is

$$\hat{H}_v^\# = -i\widehat{H^{(1)}(v)}^\# - \frac{\beta}{2}\widehat{H^{(2)}(v)} - \frac{1+\beta^2}{2\beta}\widehat{H^{(3)}(v)} - \beta\widehat{H^{(1)}(v)}^\#, \tag{4.88}$$

with $\widehat{H^{(2)}v)}$ and $\widehat{H^{(3)}(v)}$ given in (4.81), and $\widehat{H^{(1)}(v)}^\#$ given by

$$\begin{aligned}
\widehat{H^{(1)}(v)}^\# &= \kappa_2(v)\ell_p^2\beta N(v)\\
&\quad\times\left(\sum_{e\text{ at }v}\hat{\mathfrak{H}}_v^\#(e) - \sum_{e\text{ at }v}\hat{\theta}^\dagger(v)\sigma^i\hat{\theta}(v)\hat{J}_i^{v,e}\right).
\end{aligned} \tag{4.89}$$

Let us complete this section with a discussion on the intuitive picture led by the action of $\hat{\mathfrak{H}}_v(e)$ and $\hat{\mathfrak{H}}_v^\#(e)$. By definition, $\hat{\mathfrak{H}}_v^\#(e)$ creates a fermionic vertex $v_F \in e$ so that v_F will be closest to v and carries states $a|1,0\rangle + b|0,1\rangle$. Simultaneously, $\hat{\mathfrak{H}}_v^\#(e)$ changes the fermion state $|i_1,i_2\rangle_v$ at v in such a way that $|1,1\rangle_v$ becomes $c|1,0\rangle_v + d|1,0\rangle_v$ for some constant c and d, and $\alpha|1,0\rangle_v + \beta|0,1\rangle_v$ for arbitrary α and β is changed to $|0,0\rangle_v$. Moreover, because of the holonomy operator $h_{e(\mathfrak{v}_{e,v})}$ and flux operator $\hat{J}_i^{v,e}$ in $\hat{\mathfrak{H}}_v^\#(e)$, the spin on the segment $e(\mathfrak{v}_{e,v}) \subset e$ and the intertwiner at v are changed. These results can be summarized as that the operator $\hat{\mathfrak{H}}_v^\#(e)$ moves a fermion at v to v_F, and changes the geometry around v simultaneously. For the operator $\hat{\mathfrak{H}}_v(e)$, it reverses this procedure. $\hat{\mathfrak{H}}_v(e)$ moves a fermion at the fermionic vertex $v_F^{(v,e)}$, i.e., the closest fermionic vertex to v, to the vertex v, and changes the geometry around v simultaneously. Moreover, because of the projection operator $\hat{\mathbb{P}}_{v_F}$ in $\hat{\mathfrak{H}}_v(e)$ [see (4.69)], when the fermionic vertex $v_F^{(v,e)}$ carries a fermion state $|1,1\rangle_{v_F^{(v,e)}}$, the fermion at $v_F^{(v,e)}$ cannot be moved by the operator $\hat{\mathfrak{H}}_v(e)$. To see the consequence of this fact, let us imagine an edge e with both the source s_e and the middle point $v_F^{(v,e)}$ being the fermionic vertices, where $v_F^{(v,e)}$ carries a fermion state, say,

$|1,0\rangle_{v_F^{(v,e)}}$. Acted by $\hat{\mathfrak{H}}^{\#}_{s_e}(e)$, the fermion at s_e will be moved to some point v'_F between s_e and $v_F^{(v,e)}$. Then, acted by $\hat{\mathfrak{H}}_{t_e}(e)$ twice, the fermion at v'_F will be moved to t_e. Now, suppose that $v_F^{(v,e)}$ carries the state $|1,1\rangle_{v_F^{(v,e)}}$. Then, acted by $\hat{\mathfrak{H}}^{\#}_{s_e}(e)$, the fermion at s_e will be again moved to v'_F. However, the fermion at v'_F cannot be moved to t_e after acted by $\hat{\mathfrak{H}}_{t_e}(e)$, due to the operator $\hat{\mathbb{P}}_{v_F^{(v,e)}}$ in $\hat{\mathfrak{H}}_{t_e}(e)$. Intuitively, in the second situation, the fermion at s_e is confined around s_e by the fermion state $|1,1\rangle_{v_F^{(v,e)}}$ at $v_F^{(v,e)}$. The above picture tells how a fermion moves in loop quantum spacetime and influences the background quantum geometry in the LQG framework.

V. SUMMATION AND OUTLOOK

This work is concerned about the model of fermion field coupled to LQG. The Gauss and Hamiltonian constraints in this model are studied in details. In the solution to the Gauss constraint, fermion spins and the gravitational spin network intertwine with each other so that the fermion spins contribute to the volume of the spin network vertices. Consequently, the closure condition encoded in the Gauss constraint will no longer be satisfied for the gauge invariant state with nonvanishing fermion spins. In other words, the faces dual to the edges at a fermionic vertex with a nonvanishing fermion spin could not form a closed polyhedron, and the area defect of this unclosed polyhedron is filled by the fermion spin. Consequently, in contract to pure-gravity case, a three-valence gauge invariant vertex with nonvanishing fermion spin will get nonvanishing volume from the fermion spin. The volume of this type of vertices is computed in details.

For the Hamiltonian constraint, the regularization and quantization procedures are presented in details. There are several remarkable issues on the Hamiltonian constraint operator. At first, in order to take the limit of the regularized expression as the regulator approaches 0, we introduce the vertex Hilbert space. By definition, the vertex Hilbert space is the space of cylindrical functions averaged with the diffeomorphisms preserving the unremovable vertices. Thus, the states in the vertex Hilbert space are partially diffeomorphism invariant. The vertex Hilbert space is the dual space to the space of cylindrical functions. Then, the regularized Hamiltonian operator $\widehat{H(\delta)}$ can be promoted as an operator $\widehat{H(\delta)}^*$ therein by duality. Due to the diffeomorphism invariant feature of the vertex Hilbert space, the operators $\widehat{H(\delta)}^*$ for different values of the regulator δ are identical. Thus, the limit of $\widehat{H(\delta)}^*$ as δ approaches 0 can be taken. Moreover, by introducing the vertex Hilbert space, we fix several other problematic issues. Classically, the Hamiltonian can be divided into two parts as $H_{\mathcal{C}_e}$ adding its complex conjugate $\overline{H_{\mathcal{C}_e}}$. $H(\delta)$ can be promoted to the operator $\widehat{H(\delta)}$ which is cylindrical consistent and diffeomorphism covariant. However, its adjoint $\widehat{H(\delta)}^\dagger$, as a candidate of the operator corresponding to $\overline{H_{\mathcal{C}_e}}$, is not densely defined, because $\widehat{H(\delta)}$ could change graphs by erasing some segments of edges in graphs. These problems on $\widehat{H(\delta)}^\dagger$ are finally solved by introducing the vertex Hilbert space to define limit. In the vertex Hilbert space, the operator $\hat{A}_F^\dagger$, as the limit of $H(\delta)$, performs in a way such that a projection is left multiplied in $\hat{A}_F^\dagger$ as a factor. Then, if $\hat{A}_F^\dagger$ erases segments of edges, the projection will annihilate the resulting state, so that the adjoint of $\hat{A}_F^\dagger$ is densely defined. Finally, in the Hamiltonian constraint operator, there are the operators $\hat{\mathfrak{H}}^{\#}_v(e)$ and $\hat{\mathfrak{H}}_v(e)$ involved. These two operators tell how a fermion moves in LQG spacetime and influences the background loop quantum geometry. According to our results, the operator $\hat{\mathfrak{H}}^{\#}_v(e)$ moves a fermion at the vertex v to a point $v_F \in e$ so that v_F becomes the closest fermionic vertex to v, and simultaneously changes the spin on the segment connecting v to v_F of e as well as the intertwiner at v. This procedure will be reversed by the operator $\hat{\mathfrak{H}}_v(e)$, which moves a fermion, located at the fermionic vertex $v_F^{(v,e)}$ closest to v in e, to the vertex v, and simultaneously changes the spin on the segment connecting v to $v_F^{(v,e)}$ in e as well as the intertwiner at v. In addition, $\hat{\mathfrak{H}}_v(e)$ is defined to contain a projection operator $\hat{\mathbb{P}}_{v_F^{(e,v)}}$. As a consequence of this operator, the fermion located at, say, s_e will be confined around s_e by the state $|1,1\rangle_{v_F^{(v,e)}}$ located at $v_F^{(v,e)} \in e$.

Even though the current work is concerned about the graph changing feature, the framework can be easily adapted to define a graph preserving version of the Hamiltonian constraint operator. Then one can apply this graph preserving operator to the lattice model of fermion coupled to LQG, so that some open issues in lattice fermion field theory can be employed and studied. Moreover, the properties of the fermion Hamiltonian operator are still not well understood although we have discussed some of them. All of these will be left as our future works.

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APPENDIX A: HAMILTONIAN ANALYSIS FOR A FERMION FIELD

Define $P_\pm = \frac{1\pm\gamma^5}{2}$, one has

$$P_\pm^2 = P_\pm, \qquad P_+P_- = P_-P_+ = 0, \qquad P_\pm\gamma^\mu = \gamma^\mu P_\mp. \tag{A1}$$

Therefore, we have

$$\bar{\Psi}\gamma^I e_I^\mu P_\pm \nabla_\mu \Psi = (\Psi_\pm)^\dagger \gamma^0 \gamma^I e_I^\mu \Psi_\pm + \frac{1}{4}(\Psi_\pm)^\dagger \gamma^0 \gamma^I e_I^\mu \omega_{\mu KL}\gamma^K\gamma^L\Psi_\pm, \tag{A2}$$

with $\Psi_\pm := P_\pm\Psi$. Let us choose the Weyl basis of the γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ i\mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & i\sigma^k \\ -i\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{A3}$$

Then $\Psi_\pm$ take the form $\Psi_- = (\psi, 0)^T$ and $\Psi_+ = (0, \eta)^T$. We thus get

$$\bar{\Psi}\gamma^I e_I^\mu \nabla_\mu \Psi = -\psi^\dagger \bar{\sigma}^I e_I^\mu \partial_\mu \psi + \frac{1}{4}\psi^\dagger \bar{\sigma}^I e_I^\mu \omega_{\mu KL}\sigma^K\bar{\sigma}^L\psi - \eta^\dagger \sigma^I e_I^\mu \eta + \frac{1}{4}\eta^\dagger \sigma e_I^\mu \omega_{\mu KL}\bar{\sigma}^K\sigma^L\eta, \tag{A4}$$

with $\sigma^I = (\mathbb{1}, \text{Pauli matrix}^i)$ and $\bar{\sigma}^I = (\mathbb{1}, -\text{Pauli matrix}^i)$.

Performing the $3+1$ decomposition $\mathcal{M} = \mathbb{R}\times\Sigma$, one has

$$e_I^\mu = e_I^\nu q_\nu^\mu - n^\mu n_I, \tag{A5}$$

where q_ν^μ is the projection to Σ and $n^\mu = (t^\mu - N^\mu)/N$ with N and N^μ being the lapse function and the shift vector, respectively, and t^μ being some time evolution vector field given by $t^\mu\partial_\mu t = 1$. Substituting (A5) into (A4), we have

$$\bar{\Psi}e_I^\mu\gamma^I\nabla_\mu\Psi = (\psi^\dagger e_i^a\sigma^i\mathcal{D}_a^+\psi - \eta^\dagger e_i^a\mathcal{D}_a^-\eta) - \frac{1}{N}(t^\mu - N^\mu)(\psi^\dagger\mathcal{D}_\mu^+\psi + \eta^\dagger\mathcal{D}_\mu^-\eta), \tag{A6}$$

where we defined

$$\mathcal{D}_a^\pm = \partial_a + (\Gamma_a^m \mp K_a^m)\tau_m =: \partial_a + \mathcal{A}_a^\pm. \tag{A7}$$

Defining

$$\nabla_a = \partial_a + \Gamma_a^m\tau_m \tag{A8}$$

we can express the action of the fermion field in terms of K_a^i and ∇_a explicitly, which reads

$$\begin{aligned} S_F &= -\frac{i}{2}\int_{\mathcal{M}} \mathrm{d}^4x(\bar{\Psi}\gamma^I e_I^\mu\nabla_\mu\Psi - \text{c.c.}), \\ &= \frac{i}{2}\int \mathrm{d}^4x\sqrt{q}\Big((\psi^\dagger\partial_t\psi + \eta^\dagger\partial_t\eta - \text{c.c.}) + 2\Gamma_{tm}(\psi^\dagger\tau^m\psi + \eta^\dagger\tau^m\eta) - N^a(\psi^\dagger\nabla_a\psi + \eta^\dagger\nabla_a\eta - (\nabla_a\psi)^\dagger\psi - (\nabla_a\eta)^\dagger\eta) \\ &\quad - \frac{N}{\sqrt{q}}[\psi^\dagger E_i^a\sigma^i\nabla_a\psi - (\nabla_a\psi)^\dagger E_i^a\sigma^i\psi + 2\psi^\dagger[E^a, K_a]\psi - \eta^\dagger E_i^a\sigma^i\nabla_a\eta + (\nabla_a\eta)^\dagger E_i^a\sigma^i\eta + 2\eta^\dagger[E^a, K_a]\eta]\Big). \end{aligned} \tag{A9}$$

Define $\xi_A = \sqrt[4]{q}\psi_A$ and $\nu_A = \sqrt[4]{q}\eta_A$ with $A = \pm\frac{1}{2}$. Equation (A9) implies the following nonvanishing anti-Poisson bracket,

$$\{\xi_A(x), \xi_B^\dagger(y)\}_+ = -i\delta_{AB}\delta(x,y), \qquad \{\nu_A(x), \nu_B^\dagger(y)\}_+ = -i\delta_{AB}\delta(x,y). \tag{A10}$$

For the gravitational parts, the action is

$$S_H = \frac{1}{\kappa}\int \mathrm{d}^4x\Big(E_i^a\mathcal{L}_t K_a^i + \frac{1}{2\sqrt{q}}NE_i^aE_j^b\Omega_{ab}^{ij} + (t\cdot\Gamma)_m\epsilon^{klm}K_{ak}E_l^a + 2N^bE_i^a\nabla_{[a}K_{b]}^i\Big). \tag{A11}$$

Substituting the expression (A9) and (A11) into the total action $S = S_H + S_F$, one can obtain the constraints governing the classical dynamics which are expressed in terms of $\nabla_a = \partial_a + \Gamma_a^m\tau_m$. Then taking advantage of $A_a^i = \Gamma_a^i + \beta K_a^i$, one can simplify these constraints in terms of the derivative $D_a = \partial_a + A_a^i\tau_i$. The results are listed as follows. The total action reads

$$\begin{aligned} S &= S_G + S_F \\ &= \int \mathrm{d}^4x(\text{symplectic structure terms} - \lambda^m G_m - N^aH_a - NH). \end{aligned} \tag{A12}$$

The Gaussian constraint is

$$G_m = \frac{1}{\kappa\beta} D_a E^a_l + \frac{1}{2}\sqrt{q}(\psi^\dagger \sigma_m \psi + \eta^\dagger \sigma_m \eta). \tag{A13}$$

The vector constraints is

$$H_a = \frac{1}{\kappa\beta} E^b_i F^i_{ab} + \frac{i}{2}\sqrt{q}\{\psi^\dagger D_a \psi - (D_a\psi)^\dagger \psi + \eta^\dagger D_a \eta - (D_a \eta)^\dagger \eta\} + \beta K^m_a G_m. \tag{A14}$$

The scalar constraint is

$$\begin{aligned} H = H_G + \frac{1}{2}\Big[& i(\psi^\dagger E^a_i \sigma^i D_a \psi - (D_a\psi)^\dagger E^a_i \sigma^i \psi) - \beta E^a_i K^i_a \psi^\dagger \psi - \frac{1}{\beta}(1+\beta^2) D_a E^a_i \psi^\dagger \sigma^i \psi - \beta \frac{1}{\sqrt{q}} E^a_i D_a(\sqrt{q}\psi^\dagger \sigma^i \psi) \\ & - i(\eta^\dagger E^a_i \sigma^i D_a \eta - (\nabla_a \eta)^\dagger E^a_i \sigma^i \eta) + \beta E^a_i K^i_a \eta^\dagger \eta - \frac{1}{\beta}(1+\beta^2) D_a E^a_i \eta^\dagger \sigma^i \eta - \beta \frac{1}{\sqrt{q}} E^a_i D_a(\sqrt{q}\eta^\dagger \sigma^i \eta)\Big], \end{aligned} \tag{A15}$$

where H_G denotes the scalar constraint of pure gravity

$$H_G = \frac{1}{2\kappa\sqrt{q}} E^a_i E^b_j (F^m_{ab} \epsilon_m{}^{ij} - 2(1+\beta^2) K^i_{[a} K^j_{b]}). \tag{A16}$$

APPENDIX B: GRADED VECTOR SPACE AND GRADED ALGEBRA

We follow the notions given in [50]. A vector space V over $\mathbb{R}$ or $\mathbb{C}$ is graded (over $\mathbb{Z}_2$) if there are fixed subspaces V_0 and V_1 such that $V = V_0 \oplus V_1$. An element $v \in V$ is homogeneous if v is either in V_0 or in V_i. For all $v \in V_i$ with $v \neq 0$, we define their degree as

$$\mathfrak{d}(v) = i. \tag{B1}$$

Given two graded vector space V and W, the space $\mathrm{Hom}(V, W)$ of homomorphism from V to W is graded. An element $\alpha \in \mathrm{Hom}(V, W)$ is said to be homogeneous and of $\mathfrak{d}(\alpha)$ provided

$$\alpha[V_i] \subset W_{i+\mathfrak{d}(\alpha) \bmod 2}, \tag{B2}$$

with $\alpha[V_i]$ denotes the image of α acting on V_i.

An algebra $(A, \cdot)$ is a graded algebra if A is a graded vector space and $A_i \cdot A_j \subset A_{i+j \bmod 2}$, where $A_i \cdot A_j$ denotes the space of elements $a_i \cdot a_j$ for all $a_i \in A_i$ and $a_j \in A_j$. A graded algebra A is a graded commutative algebra if the product satisfies

$$x \cdot y = (-1)^{\mathfrak{d}(x)\mathfrak{d}(y)} y \cdot x, \tag{B3}$$

where $x, y \in A$ are homogeneous. Any commutative algebra $\mathbb{A}$ is a graded commutative algebra with the grade $\mathbb{A}_1 = \mathbb{A}$ and $\mathbb{A}_0 = \{0\}$. An example of the graded commutative algebra is the exterior algebra of some finite vector space V, i.e.,

$$A = \mathbb{R} \oplus V \oplus (V \wedge V) \oplus (V \wedge V \wedge V) \oplus \cdots \oplus \bigwedge^n V. \tag{B4}$$

A is graded as

$$A_0 = \bigoplus_{k=0} \bigwedge^{2k} V, \qquad A_0 = \bigoplus_{k=0} \bigwedge^{2k+1} V. \tag{B5}$$

A graded algebra $(\mathfrak{a}, [\cdot,\cdot])$ is a graded Lie algebra if the Lie bracket satisfies

(1) $[x, y] = (-1)^{1+\mathfrak{d}(x)\mathfrak{d}(y)}[y, x]$;
(2) $(-1)^{\mathfrak{d}(x)\mathfrak{d}(z)}[[x, y], z] + (-1)^{\mathfrak{d}(y)\mathfrak{d}(x)}[[y, z], x] + (-1)^{\mathfrak{d}(z)\mathfrak{d}(y)}[[z, x], y] = 0$.

An operation ∂ on a graded algebra A is called a derivative if it satisfies

$$\partial(xy) = (\partial x)y + (-1)^{\mathfrak{d}(\partial)\mathfrak{d}(x)} x(\partial y), \tag{B6}$$

where $\mathfrak{d}(\partial)$ is defined by thinking of it as a homomorphism on A. It can be checked that the operator $[x, \cdot]$ on a graded Lie algebra A for all $x \in A$ is a derivative.

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