Generalized nonconservative gravitational field equations from Herglotz action principle

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We formulate a nonconservative gravitational theory based on the Herglotz variational principle in a tensorial covariant form. The model presented here may be seen as an improvement of the theory proposed in [Lazo *et al.*, Phys. Rev. D **95**, 101501 (2017)], whose resulting field equations are meaningful just in particular coordinate systems. The new theory we report in this work is free from such a restriction. In comparison to the standard general relativity, both theories based on the Herglotz principle contain an extra vector field, the Herglotz field, but the new theory is formulated by taking advantage of the restricted equivalence between Lagrangian functions in the scope of the Herglotz action principle. The more restricted class of equivalent Lagrangian functions, in comparison with the Hamilton variational principle, is the key point to finding a Lagrangian that generates a new alternative gravitational theory in a covariant form. Once the equations that govern the dynamics of the gravitational field are obtained, a few simple cosmological models are investigated. It is found that the Herglotz vector field reduces to a single function that, under certain conditions, plays the role of the cosmological constant in general relativity, turning unnecessary the introduction of dark energy to explain the accelerated expansion of the Universe. The linearized version of the theory is also investigated and it is verified that the theory shows a dissipative character regarding the propagation of gravitational perturbations. From observational data, in both scenarios, the magnitude of the Herglotz field is estimated.

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I. INTRODUCTION

The principle of stationary action, together with important theorems such as the Noether theorem, that relates the symmetry properties of the action to some conserved physical quantities of a given system, form the basis of modern theoretical physics. Despite its remarkable contribution to the progress of theoretical physics, such a principle leaves behind some gaps regarding the description of all possible physical systems. An important gap is the lack of a general formulation for dissipative systems. Alternative formulations have been tried over the years with relative success, at least for some nonconservative mechanical systems [1] (see also, e.g., [2–4] and references therein). An interesting alternative is a formulation due to Herglotz [5–8], a variational principle that appropriately describes mechanical systems with damping forces. The Herglotz principle was extended to classical fields [7–9] allowing us to obtain, e.g., the electromagnetic field equations in dissipative media [8].

Interestingly, in Ref. [10] a nonconservative gravity based on the Herglotz variational problem theory was proposed. However, the resulting field equations of this theory are not manifestly covariant since they involve some extra terms that depend on the frame choice. In fact, the proposed theory introduces a new vector field λ_μ which couples to the action-density field s_μ. The noncovariance is a consequence of choosing a nonscalar Lagrangian density for the geometry sector so that the additional terms in the field equations involving λ_μ are not tensorial functions. Despite this fact, recent results show that such a nonconservative gravity is a promising alternative theory to dark energy [11–13]. In particular, Ref. [11] investigates the correspondence between a cosmological solution within such a modified gravity theory and the universe filled with a viscous cosmological fluid.

Since its recent formulation, the nonconservative gravity obtained through the Herglotz variational principle [10] has been considered in several contexts [11–23]. Besides the applications to mimic the dark energy in cosmological models mentioned above [11–13], other interesting applications have been made, e.g., to build models for compact objects [18], in braneworld gravity [21], and to model cosmic string configurations [14]. In Ref. [18] (see also [17]), the authors analyze the existence of compact object solutions, a work that motivated investigations also on wormhole solutions [15]. Additionally, in Ref. [20] the possible correspondence between scalar-tensor gravity theories of Brans-Dicke type and Lagrangian descriptions of dissipation such as the gravity theory derived from the Herglotz principle [10] was evidenced. These, and other studies on this subject, are now motivating further research on generalized nonconservative gravity theories.

Nonconservative gravity theories have been proposed over the years for a variety of reasons. For instance, Rastall [24] argued that energy-momentum conservation could be a valid phenomenon only in flat spacetime, and proposed a modified theory of gravity that appears to be nonconservative. See, however, Ref. [25] for a criticism of this and other similar theories, and Refs. [26,27] for further considerations and other references on this subject. See also Ref. [28] for a recent review on nonconservative gravity theories.

Pursuing the idea of formulating a consistent nonconservative gravity theory, we follow a similar path as done in [10] but keeping control of the assumptions to obtain a theory whose field equations are formulated in a manifestly covariant form. Actually, in the present work, we show that the Herglotz action principle introduced in [10] provides two direct possibilities to formulate nonconservative gravity theories. The first possibility is by considering only first-order derivatives of the metric tensor in the Lagrangian function, as done in [10] and that leads to a noncovariant and nonconservative theory of gravity. The second possibility, that we consider in the present work, is to consider up to second-order derivatives of the metric tensor in the Lagrangian. As we are going to show, this second approach yields a covariant nonconservative gravity theory.

The present work is structured as follows. The Herglotz variational problem is presented in the next section, where some of its features of interest for the present study are briefly discussed. Section III is devoted to the formulation of a covariant nonconservative gravity theory based on the Herglotz problem. In Sec. IV a few cosmological solutions are presented and analyzed, and an estimate for the Herglotz parameter is given. The linear approximation of the theory is obtained in Sec. V, where the damping effects on the propagation of gravitational perturbations are confirmed. In Sec. VI we make further considerations about the properties of the formulated theory and conclude. Some additional analysis regarding the first-order perturbation theory is performed in Appendixes A and B.

II. AN ACTION PRINCIPLE FOR NONCONSERVATIVE SYSTEMS

A. The Herglotz variational principle

In recent works, a physically meaningful action principle for nonconservative systems was proposed [8,10]. It was first employed to obtain a nonconservative gravity theory [10], and then it was extended to other fields and systems [8]. This action principle is a generalization of the Herglotz variational problem in order to include fields as a function of several independent variables. The original formulation of such a principle was introduced in 1930 by Herglotz [5–7], and since then it has been applied mainly to mechanical systems. Here we briefly review the formulation of such a variational problem to emphasize some aspects that are relevant for the present study.

The basic idea of the Herglotz variational problem is to consider a Lagrangian function that, besides depending on the usual dynamical variables, depends also on the action itself. The original formulation [5] applies to the classical dynamics and consists in the problem of determining the function $x(t)$ that extremizes the functional $S(b)$, where the action $S(t)$, with $t \in [a, b] \in \mathbb{R}$, is a solution of the problem

$$\dot{S}(t) = L(t, x(t), \dot{x}(t), S(t)), \tag{1}$$

under the boundary conditions

$$S(a) = S_a, \quad x(a) = x_a, \quad x(b) = x_b, \tag{2}$$

with the overdot standing for the total derivative with respect to the parameter t. It is important to stress that $S(t)$ is a functional since, for each function $x(t)$, it follows a different differential equation. Therefore, $S(t)$ depends on the function $x(t)$. Furthermore, the Herglotz variational problem (1)–(2) reduces to the classical fundamental problem of the calculus of variations when the Lagrangian function L does not depend on $S(t)$. In this particular case, integrating (1) results in the classical variational problem, which consists of extremizing the functional

$$S(b) = \int_a^b \bar{L}(t, x(t), \dot{x}(t))dt, \tag{3}$$

where $a < b$, $x(a) = x_a$, $x(b) = x_b$ are fixed endpoints, and

$$\bar{L}(t, x(t), \dot{x}(t)) = L(t, x(t), \dot{x}(t)) + \frac{S_a}{b - a}. \tag{4}$$

This is, of course, equivalent to the Hamilton variational principle.

Herglotz [5,6] proved that a necessary condition for a function $x(t)$ to yield an extreme for the variational problem (1)–(2) is to satisfy the generalized Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial S}\frac{\partial L}{\partial \dot{x}} = 0. \tag{5}$$

It is clear that in the case where $\partial L/\partial S = 0$, as in the classical problem (3), the differential equation (5) reduces to the usual Euler-Lagrange equation, which is obtained from the Hamilton principle.

The potential of the Herglotz problem for applications to nonconservative systems is evident even in the simplest case, where the dependence of the Lagrangian function on the action is linear [8]. For instance, the Lagrangian functional

$$L = \frac{m\dot{x}^2}{2} - U(x) - \frac{\gamma}{m} S \tag{6}$$

describes a dissipative system of a pointlike particle of mass m under the potential $U(x)$ and submitted to a viscous force proportional to the velocity. In fact, the resulting equation of motion that follows from Eqs. (6) and (5),

$$m\ddot{x} + \gamma\dot{x} = F, \tag{7}$$

where $\ddot{x}$ is the particle acceleration and $F = -dU/dx$ is the external force, includes the well-known dissipative term proportional to the velocity $\dot{x}$, and whose resistance coefficient is γ. In this context, the linear term $-\gamma S/m$ in the Lagrangian (6) can be interpreted as a potential function for the nonconservative force, see [8]. Furthermore, the Lagrangian given by (6) is physical in the sense that it provides us with physically meaningful relations for the momentum and the Hamiltonian (see, e.g., [8,29,30]).

The formulation of an action principle in terms of the Herglotz variational problem (1)–(2), instead of the traditional calculus of variation problem (3), has two direct justifications. The first one is the fact that the Herglotz problem enables us to formulate a physically meaningful Lagrangian problem for nonconservative systems governed by forces depending linearly on the velocity, like the frictional force in (7). The second justification is the fact that, in any physical theory, the Lagrangian function which defines the action is constructed from the scalars (invariant quantities) of the theory. Consequently, since the action itself is a scalar, the most general Lagrangian may itself be a function of the action [10].

B. Equivalence of Lagrangians according to the Herglotz principle

In the classical Hamilton action principle, the Lagrangian describing a physical system is not uniquely defined. Two Lagrangian functions L and $\tilde{L}$ are said to be equivalent if they establish the same Euler-Lagrange equations. However, in general, this equivalence does not hold in the context of the Herglotz action principle.

To verify this fact we consider the Herglotz problem with the Lagrangian $\tilde{L} = L + \dot{f}$. It thus consists in extremizing the functional $\tilde{S}(b)$, but now the action $\tilde{S}(t)$ is such that

$$\dot{\tilde{S}}(t) = L(t, x(t), \dot{x}(t), \tilde{S}(t)) + \dot{f}(t, x(t)), \tag{8}$$

subject to the same boundary conditions as in Eq. (2), with fixed $\tilde{S}(a) = \tilde{S}_a$. Notice that the function f does not depend on the variables $\dot{x}$ and $\tilde{S}$, since the total Lagrangian for the standard Herglotz problem (1) may depend only on t, $x(t)$, $\dot{x}(t)$, and $S(t)$.

Now, from (5) the Herglotz problem with the Lagrangian function $\tilde{L} = L + \dot{f}$ yields the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial \dot{x}}\frac{\partial L}{\partial S}\frac{\partial S}{\partial \tilde{S}} + \frac{\partial f}{\partial x}\frac{\partial L}{\partial \tilde{S}} = 0, \tag{9}$$

where we have used the identities

$$\frac{d}{dt}\frac{\partial \dot{f}}{\partial \dot{x}} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\dot{x}\right] = \frac{d}{dt}\frac{\partial f}{\partial x} = \frac{\partial \dot{f}}{\partial x}. \tag{10}$$

Equation (9), in general, is different from the Euler-Lagrange equation for the Herglotz problem with Lagrangian L, Eq. (5), unless $\partial L/\partial \tilde{S} = 0$, where we recovered the classical variational problem, or the conditions

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial S}{\partial \tilde{S}} = 1, \tag{11}$$

satisfy simultaneously.

Therefore, it is clear that, in general, Lagrangian functions differing from each other by a total derivative are not equivalent in the context of the Herglotz variational problem.

III. ACTION AND EQUATIONS OF MOTION OF A GENERALIZED NONCONSERVATIVE GRAVITY

Let us start this section by reviewing the equivalence between Lagrangian functions in the context of the classical (Hamilton) action principle for the general theory of relativity. As it is well known, in such a context there are two mostly used equivalent Lagrangian densities for gravity. The first is the Einstein-Hilbert Lagrangian $\mathcal{L}_g$ given by

$$\begin{aligned}\mathcal{L}_g(x^\mu, g_{\rho\sigma}, g_{\rho\sigma,\mu}, g_{\rho\sigma,\mu\nu}) = g^{\mu\nu}R_{\mu\nu} &\equiv \tilde{\mathcal{L}} - \mathcal{L}_{ef} \\ &= g^{\mu\nu}(\Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\nu,\sigma}) \\ &\quad - g^{\mu\nu}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}),\end{aligned} \tag{12}$$

where $g_{\mu\nu}$ is metric tensor, $R_{\mu\nu}$ is the Ricci tensor, and we defined $\tilde{\mathcal{L}} = g^{\mu\nu}(\Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\nu,\sigma})$ and $\mathcal{L}_{ef} = g^{\mu\nu}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho})$. The second commonly used Lagrangian is $\mathcal{L}$ itself. The equivalence between these two Lagrangian functions is verified by noting that it holds the relation $\mathcal{L}_g = 2\mathcal{L}_{ef} + \nabla_\mu J^\mu$, where $J^\mu = g^{\mu\nu}\Gamma^\sigma_{\nu\sigma} - g^{\sigma\nu}\Gamma^\mu_{\sigma\nu}$. Consequently,

$\mathcal{L}_g$ and $2\mathcal{L}$ differ from a divergence term, and by integrating over a given subset $\mathcal{V}$ of the n-dimensional spacetime manifold $\mathcal{M}$ it gives, $\int_\mathcal{V}\mathcal{L}_g\sqrt{-g}d^nx = 2\int_\mathcal{V}\mathcal{L}_{ef}\sqrt{-g}d^nx$ plus a surface term (see, e.g., [31]), where g is the determinant of the metric, demonstrating that $\mathcal{L}_g$ and $2\mathcal{L}_{ef}$ are equivalent Lagrangian functions according to the Hamilton variational principle.

Now, since Lagrangian functionals differing from a total derivative are not equivalent in the Herglotz variational problem (and consequently are not equivalent when differing from a divergence term), we have two simple possibilities to formulate a dissipative gravitational theory in such a context. The first possibility, investigated in Ref. [10], is by taking $\mathcal{L}_{ef}$ as the gravitational part of the Lagrangian. The second possibility, that we explore in the present work, is by choosing $\mathcal{L}_g$, with $\mathcal{L}_g$ given in Eq. (12).

The approach considered in [10] has the interesting mathematical advantage of the Lagrangian to be a function depending only on first-order derivatives of the metric tensor. It considers a Lagrangian function given by $\mathcal{L} = \mathcal{L}_{ef} + \lambda_\nu s^\nu$, where $\lambda_\mu = \lambda_\mu(x)$ is an arbitrary vector field,[1] and s^μ is the action-density vector field (see [8,10]). However, this approach has the physical disadvantage that the Lagrangian $\mathcal{L}_{ef}$ is not a scalar density and, consequently, the resulting field equations for the theory are valid only in a specific set of referential frames fixed it *a priori*.

Inspired by Ref. [10], here we consider an alternative proposal by taking the Lagrangian as $\mathcal{L} = \mathcal{L}_g + \lambda_\nu s^\nu$. Since $\mathcal{L}_g$ is an invariant (scalar density), the field equations of our theory will be given by truly tensorial equations and it is not necessary to fix a preferential coordinate system *a priori* as in [10]. However, since $\mathcal{L}_g$ has second-order derivatives of the metric tensor it is necessary to impose additional boundary conditions on the metric to fix the variational problem solution. The derivation of the field equations from such a Lagrangian is presented next.

Let the spacetime be defined as an n-dimensional smooth manifold $\mathcal{M}$ endowed with a Lorentzian metric $g_{\mu\nu}$. Now let $\mathcal{V}$ be a subset of $\mathcal{M}$ with boundary Ω, which is considered as a Jordan surface whose unit normal vector is denoted by n_μ. Then, the generalized action principle may be stated in terms of the functional $S(\Omega)$ given by (see also [10])

$$S(\Omega) = \int_\Omega n_\mu s^\mu\sqrt{|h|}d^{n-1}x = \int_\mathcal{V} s^\mu{}_{;\mu}d^nx,$$
$$s^\mu{}_{;\mu} = \mathcal{L}(x^\nu, g_{\rho\sigma}, g_{\rho\sigma,\nu}, g_{\rho\sigma,\tau\nu}, s^\nu), \tag{13}$$

where s^μ is a differentiable action-density vector field, the semicolon (;) stands for covariant derivative, and h is the determinant of the induced metric on Ω. The boundary conditions we impose, in order to close the variational problem, is by keeping both the metric $g_{\mu\nu}$ and its derivatives $g_{\mu\nu,\gamma}$ fixed on Ω.

We consider a generalized Lagrangian given by

$$\mathcal{L} = \mathcal{L}_g + \lambda_\mu s^\mu + F\mathcal{L}_m, \tag{14}$$

where $\mathcal{L}_m$ stands for the standard matter Lagrangian, and $F = F(x)$ is a coupling factor that may be a function of the coordinates. With this Lagrangian and from Eq. (13), it follows that the action density s^μ is subjected to the additional condition

$$(s^\mu\sqrt{-g})_{,\mu} = \sqrt{-g}(R_{\mu\nu}g^{\mu\nu} + \lambda_\mu s^\mu + F\mathcal{L}_m), \tag{15}$$

where the comma index $()_{,\mu}$ indicates a partial derivative with respect to the coordinate x^μ.

Our goal is to obtain the field equations for $g_{\mu\nu}$ whose solutions make the functional $S(\Omega)$ stationary under the condition (15). Taking the variation of (13) and (15) with respect to $g^{\mu\nu}$ it gives, respectively,

$$\delta S(\Omega) = \int_\Omega n_\mu\delta(s^\mu\sqrt{h})d^{n-1}x = 0,$$
$$\zeta^\mu{}_{,\mu} = \delta(R_{\mu\nu}g^{\mu\nu}\sqrt{-g} + F\mathcal{L}_m\sqrt{-g}) + \lambda_\mu\zeta^\mu, \tag{16}$$

where $\zeta^\mu = \delta(s^\mu\sqrt{-g})$. As considered in [10], since the hypersurface Ω and, consequently, $\sqrt{h}$ are fixed, i.e., they do not depend on the metric variation, we obtain from the first equation in (16) that $\delta s^\mu = 0$ on Ω. On the other hand, the last equation in (16) can be written as

$$(\zeta^\mu e^{-\phi})_{,\mu} = e^{-\phi}\delta(R_{\mu\nu}g^{\mu\nu}\sqrt{-g} + F\mathcal{L}_m\sqrt{-g}), \tag{17}$$

with $\phi = \int\lambda_\mu(x)dx^\mu$. Integrating the left-hand side of (17) over $\mathcal{V}$, and then working out the variation of $s^\mu\sqrt{g}$, it follows

$$\int_\mathcal{V}(\zeta^\mu e^{-\phi})_{,\mu}d^nx$$
$$= \int_\mathcal{V}\left[\sqrt{-g}\left(\delta s^\mu - \frac{s^\mu}{2}g_{\nu\sigma}\delta g^{\nu\sigma}\right)e^{-\phi}\right]_{,\mu}d^nx$$
$$= \int_\Omega n_\mu\left(\delta s^\mu - \frac{s^\mu}{2}g_{\nu\sigma}\delta g^{\nu\sigma}\right)e^{-\phi}\sqrt{h}d^{n-1}x = 0, \tag{18}$$

where we have used the fact that $\delta s^\mu = 0$ on Ω, and we imposed the usual condition in the variational procedure that the metric field is fixed on the boundary Ω, i.e., since $g_{\mu\nu}(\Omega)$ is fixed then $\delta g^{\nu\sigma}$ vanishes on Ω. Consequently, the last integral in Eq. (18) gives zero. Thus, taking these results back into Eq. (17), after integration, we find

[1]We name it the Herglotz vector field, or the Herglotz parameter.

$$\int_{\mathcal{V}} e^{-\phi}[g^{\mu\nu}\delta R_{\mu\nu} + G_{\mu\nu}\delta g^{\mu\nu}]\sqrt{-g}d^n x + \int_{\mathcal{V}} e^{-\phi}F\delta(\mathcal{L}_m\sqrt{-g})d^n x = 0, \tag{19}$$

where $G_{\mu\nu}$ is the Einstein tensor.

Let us now consider the first term in the last integral. Using the definition of the Ricci tensor in terms of the metric, we get (see, e.g., [32])

$$\begin{aligned}\int_{\mathcal{V}} e^{-\phi} g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^n x &= \int_{\mathcal{V}} e^{-\phi}[(g_{\mu\nu}g^{\sigma\gamma}(\delta g^{\mu\nu})_{;\gamma} - (\delta g^{\sigma\gamma})_{;\gamma})\sqrt{-g}]_{,\sigma}d^n x \\ &= \int_{\mathcal{V}} e^{-\phi}[(g_{\mu\nu}g^{\sigma\gamma}(\delta g^{\mu\nu})_{;\gamma} - (\delta g^{\sigma\gamma})_{;\gamma})\sqrt{-g}]\lambda_\sigma d^n x,\end{aligned} \tag{20}$$

where an integration by parts was performed, and we consider the additional boundary condition imposed in the problem that $g_{\mu\nu,\gamma}(\Omega)$ is fixed [and consequently $(\delta g^{\mu\nu})_{;\gamma}$ vanishes on the boundary Ω].

Let us now work out the integral terms on the rhs of the last relation in Eq. (20). The first integral term reads

$$\begin{aligned}\int_{\mathcal{V}} e^{-\phi} g_{\mu\nu}g^{\sigma\gamma}(\delta g^{\mu\nu})_{;\gamma}\lambda_\sigma\sqrt{-g}d^n x &= \int_{\mathcal{V}} e^{-\phi} g_{\mu\nu}\lambda^\gamma(\delta g^{\mu\nu}_{,\gamma} + \Gamma^\nu_{\sigma\gamma}\delta g^{\mu\sigma} + \Gamma^\mu_{\sigma\gamma}\delta g^{\sigma\nu})\sqrt{-g}d^n x \\ &= \int_{\mathcal{V}} e^{-\phi} g_{\mu\nu}(\lambda^\sigma\lambda_\sigma - \lambda^\rho_{,\rho} + \Gamma^\sigma_{\gamma\rho}\lambda^\rho)\delta g^{\mu\nu}\sqrt{-g}d^n x,\end{aligned} \tag{21}$$

where an integration by parts was performed, and the boundary terms were neglected once again. Applying the same procedure to the second term on the rhs of Eq. (20) one gets

$$\begin{aligned}\int_{\mathcal{V}} e^{-\phi}(\delta g^{\sigma\gamma})_{;\gamma}\lambda_\sigma\sqrt{-g}d^n x &= \int_{\mathcal{V}} e^{-\phi}(\delta g^{\sigma\gamma}_{,\gamma} + \Gamma^\gamma_{\mu\gamma}\delta g^{\sigma\mu} + \Gamma^\sigma_{\mu\gamma}\delta g^{\mu\gamma})\lambda_\sigma\sqrt{-g}d^n x \\ &= \int_{\mathcal{V}} e^{-\phi}(\lambda_\mu\lambda_\nu - \lambda_{\mu,\nu} + \Gamma^\sigma_{\mu\nu}\lambda_\sigma)\delta g^{\mu\nu}\sqrt{-g}d^n x.\end{aligned} \tag{22}$$

Now we put the results given by Eqs. (21) and (22) back into (20) to obtain

$$\int_{\mathcal{V}} e^{-\phi}(g^{\mu\nu}\delta R_{\mu\nu} - K_{\mu\nu}\delta g^{\mu\nu})\sqrt{-g}d^n x = 0, \tag{23}$$

where we introduced the tensor $K_{\mu\nu}$ given by

$$K_{\mu\nu} = \Lambda_{\mu\nu} - g_{\mu\nu}\Lambda, \tag{24}$$

with $\Lambda_{\mu\nu}$ being the symmetric tensor

$$\Lambda_{\mu\nu} = \frac{1}{2}(\lambda_{\mu;\nu} + \lambda_{\nu;\mu}) - \lambda_\mu\lambda_\nu, \tag{25}$$

and $\Lambda = \Lambda^\mu_\mu$ is its trace.

Finally, from (19) and (23) we find the generalized field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + K_{\mu\nu} = \frac{F}{2}T_{\mu\nu}, \tag{26}$$

where $F = F(x)$ is a non-negative arbitrary function that, in the conservative Einstein-Hilbert action, plays the role of the (Newtonian) gravitational coupling constant, and the energy-momentum tensor is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}}. \tag{27}$$

The usual Einstein field equations are recovered in the case $\lambda_\mu = 0$, as long as we take $F = 16\pi G/c^4$, with G being the universal gravitational constant.

Since λ_μ is a tensor (a vector) by definition, and the covariant derivative of a vector field is also a tensor, it becomes clear that $\Lambda_{\mu\nu}$ is a tensor. Thus, Eq. (26) is written in a manifestly covariant form.

It is worth noticing that in an empty spacetime region, i.e., for $T_{\mu\nu} = 0$, or by assuming conservation of the quantity on the rhs of Eq. (26), i.e., by imposing $(FT^\nu_\mu)_{;\nu} = 0$, the Bianchi identity implies that the tensor $K_{\mu\nu} = \Lambda_{\mu\nu} - g_{\mu\nu}\Lambda$ also satisfies the conservation condition $K^\nu_{\mu;\nu} = 0$. However, the Herglotz principle is adapted to dissipative systems, for which the energy momentum tensor does not satisfy the conservation equation, and so the Bianchi identity gives $(FT^\nu_\mu)_{;\nu} = \Lambda^\nu_{\mu;\nu} - \delta^\nu_\mu\Lambda_{,\nu}$. Note that even by assuming energy-momentum conservation, tensor $K_{\mu\nu}$ may be a nonconserved quantity because of the presence of the function $F(x)$, i.e., $K^\nu_{\mu;\nu} = T^\nu_\mu F_{,\nu}$. This fact was explored in Ref. [10], where the accelerating effect of dark energy in standard cosmological models was simulated by a time-dependent coupling function $F = F(t)$, t being the cosmological time (see also Ref. [11]).

At this point, a comment on the choice of the coupling factors in the Lagrangian function (14) is in order. In the formulation of general relativity, the usual choice is a total Lagrangian in the form $\mathcal{L}_{RG} = \frac{1}{8\pi G}\mathcal{R} + \mathcal{L}_m$. Therefore, when allowing the gravitational coupling G to be a scalar function, one obtains alternative theories of gravity as the Brans-Dicke or similar scalar-tensor theories. Notice, however, that the factor F in (14) couples directly to the matter Lagrangian, and then it does not generate extra terms to the field equations as it happens, e.g., in the Brans-Dicke theory, and the freedom introduced by such a factor can be seen as a positive aspect of the theory. Moreover, a further natural assumption that F is a constant factor sets the theory free of the potential problems that a varying coupling may bring with it. With this in mind, in the present work we assume the usual form $F = 16\pi G/c^4$, but for the sake of generality we observe that in the present theory the factor G may be an arbitrary function of the coordinates.

IV. APPLICATION TO COSMOLOGY

A. The modified Friedmann equations

In order to investigate the consequences of the Herglotz vector field λ_μ, we analyze some cosmological models filled with a perfect fluid. The energy-momentum tensor is of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{28}$$

where ρ is the energy density, p is pressure, and u^μ is the four-velocity of the fluid. For now, we consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric with zero space curvature, which may be written as

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \tag{29}$$

where $a(t)$ is the scale factor, t being the comoving time coordinate, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit sphere.

Given that the metric (29) represents a spatially homogeneous and isotropic spacetime, it admits a set of Killing vectors that generate the isometries. These vectors are useful for fixing the general form of the Herglotz vector field λ_μ in this spacetime. For that, the λ_μ vector needs to satisfy the Killing equation

$$\mathcal{L}_\chi \lambda_\mu = 0, \tag{30}$$

where χ stands for the set of Killing vectors of the metric (29). After a detailed analysis, we realize that the most general vector λ_μ that satisfies Eq. (30) is of the form

$$\lambda_\mu = (\phi(t), 0, 0, 0), \tag{31}$$

with ϕ being a smooth function of the time t only.

Substituting the vector field (31), the metric (29), and the energy-momentum tensor (28) into Eq. (26), and using (25), we get the modified Friedmann equations for the scale factor

$$3\left(\frac{\dot{a}}{a}\right)^2 - 3\frac{\dot{a}}{a}\phi = 8\pi G\rho, \tag{32}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\dot{a}}{a}\phi + \phi^2 - \dot{\phi} = -8\pi G p. \tag{33}$$

This is a system of two equations for four unknown functions. Hence, even after establishing, as usual, an equation of state for the cosmological fluid, additional conditions are needed.

It is worth emphasizing that the function $\phi(t)$ is arbitrary and hence a new cosmological model is built for every choice of that function. In fact, this freedom may be somehow fixed by imposing some physical conditions required by the cosmological model under construction. For instance, considering an expanding cosmological model ($\dot{a}/a > 0$), in order to guarantee the non-negativity of energy density, the constraint

$$\frac{\dot{a}}{a} - \phi \geq 0 \tag{34}$$

must be obeyed at least for sufficiently large times. In the following, we analyze some particular simple cosmological models emerging from the present theory that satisfy such a constraint.

Using the Bianchi identities and proceeding with the idea that G may not be a constant (see, e.g., [33]) it follows just one nontrivial relation, namely,

$$3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\phi\right)\phi + 8\pi\dot{G}\rho = -8\pi G\left[\dot{\rho} + 3\frac{\dot{a}}{a}(p+\rho)\right]. \tag{35}$$

The right-hand side of Eq. (35) represents the covariant divergence of the energy-moment tensor, coupled to gravity through the function G. Note that, as expected, in the present theory the energy-momentum conservation may be violated even in the case of nonconstant G. By assuming energy-momentum conservation, it follows

$$3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\phi\right)\phi + 8\pi\dot{G}\rho = 0. \tag{36}$$

As seen from this relation, the present model admits conservative solutions even in the presence of the Herglotz parameter. Two particular situations promptly come out. One is for varying G, while the other, more interesting one, is for constant G. To simplify the analysis, and also to avoid the apparent conflict of a time-dependent gravitational coupling with observational data, we restrict the analysis from now on just to the case of constant G.

B. A conservative cosmological model

Although the theory introduces a nonconservative geometric gravitational aspect, in this section we explore the existence of solutions that deviate from general relativity in cases where the energy-momentum tensor is conserved and the coupling strength G is constant.

Considering energy-momentum conservation and the constancy of G, Eq. (36) results in

$$\phi = -\frac{\ddot{a}}{\dot{a}}. \tag{37}$$

After introducing this result into (32), the energy density reads

$$8\pi G\rho = 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{\ddot{a}}{a}. \tag{38}$$

Now, by analyzing Eq. (38) one concludes that any solution with accelerated expansion ($\ddot{a}/a > 0$) provides a positive definite energy density.

On the other hand, by substituting (37) into (33), the expression for the pressure results in the form

$$8\pi Gp = -\left(\frac{4\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{a^{(3)}}{\dot{a}}\right), \tag{39}$$

where $a^{(3)}$ stands for the third-order derivative of the scale factor with respect to the time t.

The system of equations to be solved is now formed by Eqs. (37)–(39), with three equations for four unknowns. The usual strategy to close the system is to pick up an equation of state for the cosmic fluid. However, in the present case, such a strategy leads to a nonlinear third-order differential equation for the scale factor which has no solution in closed form. Hence, for simplicity, and since it is not our objective here to consider the most general solution for this system, we follow a simpler road by choosing the explicit form of one of the unknown functions. The first choice is a power-law function for the scale factor, i.e., $a(t) \sim t^\alpha$. In this case, we get

$$a(t) = a_0\left(\frac{t}{t_0}\right)^\alpha, \tag{40}$$

$$\phi(t) = \frac{1-\alpha}{t}, \tag{41}$$

$$8\pi G\rho(t) = \frac{3\alpha(2\alpha - 1)}{t^2}, \tag{42}$$

$$8\pi Gp(t) = \frac{\alpha(7 - 6\alpha) - 2}{t^2}, \tag{43}$$

where $a_0 > 0$ and α are constant parameters.

Assuming that the scale factor increases with time ($\alpha > 0$), the non-negativity of the energy density ρ implies the constraint $\alpha \geq 1/2$. The cosmic fluid is well defined for all values of α in the interval $1/2 \leq \alpha < \infty$, with the ratio $\mathcal{R} = p(t)/\rho(t)$ being independent of time and varying with α from $\mathcal{R} = 1/3$ (for α close to $1/2$) to $\mathcal{R} = -1$ (in the limit $\alpha \to \infty$).

For α smaller than $1/2$ the energy density and the pressure assume only negative values.

Taking $\alpha = 1/2$, $a(t) = a_0(t/t_0)^{1/2}$, the energy density and pressure vanish, $p(t) = \rho(t) = 0$, and ϕ reduces exactly to the Hubble function, that is, $\phi = \dot{a}/a = 1/2t$. This case reproduces exactly the same behavior of the spatially flat FLRW model dominated by radiation in general relativity. Thus, in vacuum or when the field ϕ dominates, the Herglotz field ϕ behaves like a fictitious radiation component, thinking of ϕ as a fictitious source, it simulates a cosmological model in general relativity with a perfect fluid whose effective energy density and pressure obey the equation of state $p_{ef} = \rho_{ef}/3 \sim a^{-4}(t) \sim t^{-2}$.

By taking $\alpha = 2/3$ it follows that $a(t) = a_0(t/t_0)^{2/3}$, and $p = 0$. This solution is known as the Einstein-de Sitter universe with cold dark matter (CDM). The equivalent (effective) in general relativity is a cosmic fluid obeying the equation of state $p_{ef} = 0$.

For α larger than $2/3$ the pressure assumes only negative values.

Therefore, under the assumption that the energy-momentum tensor is conserved we get cosmological models which are equivalent to general relativity models.

In the case where ϕ is constant, Eq. (37) can be integrated for the scale factor yielding

$$a(t) = a_0(e^{-\phi t} - 1), \tag{44}$$

where a_0 is the integration constant and the big bang condition $[a(t) = 0]$ was chosen at $t = 0$. It is clear that the constant ϕ must be negative ($\phi < 0$) so that the scale factor is in accordance with the present observational data.

For the scale factor (44), energy density and pressure are given, respectively, by

$$8\pi G\rho = \frac{3\phi^2(2 - e^{\phi t})}{(1 - e^{\phi t})^2},$$
$$8\pi Gp = \frac{\phi^2(6e^{\phi t} - e^{2\phi t} - 6)}{(1 - e^{\phi t})^2}. \tag{45}$$

Note that, although the energy density is non-negative, the pressure is always negative for any cosmological time. The ratio $\mathcal{R}(t) = p(t)/\rho(t)$ varies with time from $\mathcal{R}(t) = -1/3$ (for $t \to 0$) to $\mathcal{R}(t) = -1$ (for $t \to \infty$). Therefore, applying this solution to the beginning of times, the result is an inflationary model governed by a fluid of cosmic strings $p \simeq -\rho/3$. On the other hand, applying the solution to very late times, the result is an accelerated expansion driven by a cosmological constant. In fact, for very large times, the components of the fluid (45) reduce to $8\pi Gp = -8\pi G\rho = -6\phi^2$, which is the same equation of state for a fluid represented by the cosmological constant Λ in general relativity, with $\Lambda = 6\phi^2$.

C. A nonconservative model: An accelerated expanding phase dominated by cold dark matter

Sticking to the case of constant G, here we investigate the possibility of building models for accelerated expansion within the present theory without recurring to the mysterious dark energy content. To take the simplest road, we put the pressure to zero, $p = 0$, so it results in a cold dark matter dominated phase. Even after substituting the ansatz (46) into Eqs. (32) and (33), 1 degree of freedom is available. Again aiming at a simple model, let us assume that, during a given phase of the cosmic expansion, the scale factor $a(t)$ may be approximate by a growing exponential function,

$$a(t) = a_0 e^{ht}, \tag{46}$$

with a_0 and h being constant parameters, and with $h > 0$. After this choice, Eq. (33) provides

$$\phi(t) = h + \sqrt{2}h \tan[\sqrt{2}h(t - t_1)], \tag{47}$$

where t_1 is an integration constant. Substituting (47) into (32) it follows

$$\rho(t) = \frac{3\sqrt{2}}{8\pi G} h^2 \tan[\sqrt{2}h(t_1 - t)], \tag{48}$$

while the pressure is zero. This phase of accelerated expansion is generated by a CDM model.

Since t_1 is an arbitrary integration constant, its value may be adjusted so that the accelerated expansion phase lasts long enough to conform the present observational data. However, to guarantee the non-negativity of the energy density, in the present case, the accelerated expansion cannot last forever after. There must be a mechanism to turn on the field $\phi(t)$ at the time $t \equiv t_0 = t_1 - \pi/2\sqrt{2}h$, and to turn it off just before the time $t = t_1$. Once this mechanism is activated, its duration is at most a time interval given by $\Delta t = \pi/(2\sqrt{2}h)$ until it is turned off. It is still necessary to adjust the constant t_1 in favor of explaining the accelerated expansion at the current cosmological time.

D. Interpreting and estimating the Herglotz field from cosmological data

We first comment on the possible physical interpretation of the Herglotz vector field within the FLRW cosmological models. In such models, the isometries of the spacetime reduces the vector field λ_μ to just one component, exactly the timelike component of the four-vector, i.e., $\lambda_\mu = \phi(t)\delta^t_\mu$, with δ^ν_μ being the Kronecker tensor. The field $\phi(t)$ depends only on the cosmic time and its effects on the field equations may be compared to the Hubble factor. This can be seen by considering a simple case of Eq. (32), e.g., the case with $\rho = 0$ for which (32) reduces to $(\dot{a}/a - \phi)\dot{a}/a = 0$. The nontrivial solution to this equation is $\phi = \dot{a}/a = H$, with H being the Hubble parameter. In this particular case, the presence of the Herglotz field ϕ induces global expansion when $\phi > 0$, and global contraction when $\phi < 0$.

According to Eq. (32), the contribution of the Herglotz field to the energy density is through the term $-3\phi\dot{a}/a$ which, in an expanding phase, is positive for a negative ϕ and is negative for a positive ϕ. In turn, according to Eq. (33), the contribution of the Herglotz field to the pressure is through the terms $2\phi\dot{a}/a - \phi^2 + \dot{\phi}$. It is reasonable to think of ϕ as a function of time proportional to $\dot{a}/a$, i.e., $\phi = k\dot{a}/a$ with k being small compared to unity. In such a case, the contribution to the pressure is mostly negative, unless the model is in a highly accelerating phase. This example shows that the Herglotz field may induce similar situations to the dark energy or phantom matter models in general relativity. Hence, the overall effect of the Herglotz field in cosmological models is to cause the (accelerated) spacetime expansion (or contraction) even in the absence of ordinary matter.

The intricate role of the field ϕ is seen in the simple cases as, for instance, by taking $\phi =$ constant and solving Eq. (33) for zero pressure. The result is an oscillatory function for the scale factor $a(t)$. On the other hand, as seen in Sec. IV C, an oscillatory behavior of $\phi(t)$ results in a monotonic behavior of the scale factor $a(t)$. Therefore, the physical interpretation of the Herglotz field is not a simple task in the cosmological models. However, the asymptotic limit of solutions like (44) shows a behavior similar to the vacuum solution with a cosmological constant (dark energy component) in general relativity. This solution allows us to estimate the value of the Herglotz field ϕ at the present epoch. In fact, in a FLRW model (within general relativity) dominated by the cosmological constant one has $H_0^2 = \Lambda/3$. Hence, taking into account that the present solution gives $\phi^2 = \Lambda/6$, we get $|\phi| = \sqrt{2}H_0/2 \simeq 0.71H_0$. Now, by using the value of H_0 obtained, for instance, from Ref. [34], $H_0 \sim 70$ km/s/Mpc, it follows $|\phi| \simeq 50$ km/s/Mpc $\simeq 1.6 \times 10^{-18}$ s^{-1}.

V. LINEAR APPROXIMATION

A. Linearized theory

Here we consider the metric resulting from a small perturbation around the Minkowski spacetime, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \|h_{\mu\nu}\| \ll 1, \tag{49}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor, and the quantities $h_{\mu\nu}$ are perturbation functions.

Proceeding with the linearization of the field equations, it is well known that, at first order in $h_{\mu\nu}$ and its derivatives, the Einstein tensor reads

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = -\frac{1}{2}\Box\gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma^{\rho\sigma}_{,\rho\sigma} + \gamma^{\rho}_{(\mu,\nu)\rho}, \quad (50)$$

where $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and with the parentheses in indexes indicating symmetrization. This notation will be adopted throughout the text.

The next step is to obtain the series expansion of the tensor $K_{\mu\nu}$, cf. Eqs. (24) and (25), up to first order in perturbations. For this we first have to consider how to deal with the Herglotz vector field λ_μ in the perturbation theory. Since the beginning, λ_μ has been treated as a nondynamical field, and then a natural procedure is to keep λ_μ as a fixed (unperturbed) quantity. Moreover, taking into account the results from the previous section, cosmological observations imply that the background λ_μ assumes very small values and can be considered as a perturbation itself. In this sense, it is a reasonable choice to neglect any possible perturbation on such a field.

Therefore, the perturbations on $K_{\mu\nu}$ come from the metric perturbations alone and, up to first order in $h_{\mu\nu}$, such a tensor is given by

$$\begin{aligned} K_{\mu\nu} &= \bar{\Lambda}_{\mu\nu} - \eta_{\mu\nu}\bar{\Lambda} - \bar{\lambda}^{\rho}\gamma_{\rho(\nu,\mu)} + \frac{1}{2}\bar{\lambda}^{\rho}\gamma_{\mu\nu,\rho} \\ &\quad + \eta_{\mu\nu}\left(\bar{\Lambda}_{\rho\sigma}\gamma^{\rho\sigma} + \bar{\lambda}^{\rho}\gamma^{\sigma}_{\rho,\sigma} - \frac{1}{4}\bar{\lambda}^{\sigma}\gamma_{,\sigma}\right) \\ &\quad + \frac{1}{2}\bar{\lambda}_{(\mu}\gamma_{,\nu)} - \gamma_{\mu\nu}\bar{\Lambda}, \end{aligned} \quad (51)$$

where we have defined

$$\bar{\Lambda}_{\mu\nu} = \bar{\lambda}_{(\mu,\nu)} - \bar{\lambda}_{\mu}\bar{\lambda}_{\nu}, \quad (52)$$

with $h \equiv h^{\mu}_{\mu}$ and $\bar{\Lambda} \equiv \bar{\Lambda}^{\mu}_{\mu}$.

The energy-momentum tensor is also perturbed and may be split as

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \tau_{\mu\nu}, \quad (53)$$

where $\bar{T}_{\mu\nu}$ is the background energy-momentum tensor and $\tau_{\mu\nu}$ stands for the respective perturbation tensor, with all $\tau_{\mu\nu}$ being small quantities when compared to the nonzero components of $\bar{T}_{\mu\nu}$ for all μ, ν.

The zeroth order approximation of Eq. (26) results in

$$\bar{K}_{\mu\nu} = \bar{\Lambda}_{\mu\nu} - \eta_{\mu\nu}\bar{\Lambda} = 8\pi G\bar{T}_{\mu\nu}, \quad (54)$$

where it was used the fact that the unperturbed Einstein tensor $\bar{G}_{\mu\nu}$ vanishes, i.e., one has $\bar{G}_{\mu\nu} = 0$. Relation (54) defines the background energy-momentum tensor in flat spacetimes that is, in general, different from zero. In the present theory, the flat Minkowski spacetime is fulfilled by a nonisotropic energy-momentum tensor given by $8\pi G\bar{T}_{\mu\nu} = \bar{\Lambda}_{\mu\nu} - \eta_{\mu\nu}\bar{\Lambda}$. The important point here is that, since this energy-momentum tensor does not affect the geometry and, then, the trajectory of geodesic particles are straight lines, it cannot be detected by local experiments. On the other hand, the hypothesis $\bar{T}_{\mu\nu} = 0$ requires the background vector $\bar{\lambda}_\mu$ must satisfy the condition $\bar{\lambda}_{(\mu,\nu)} - \bar{\lambda}_\mu\bar{\lambda}_\nu = 0$. In this case, $\bar{\lambda}_\mu$ is given by $\bar{\lambda}_\mu = b_0\partial_\mu \ln(t + x + y + z)$, where b_0 is an integration constant. Both cases, the vanishing $\bar{T}_{\mu\nu}$ or satisfying (54), are consistent with the analysis presented in the present section.

The linearized version of the field equations is obtained after replacing expressions (50) and (51) into Eq. (26), i.e.,

$$-\frac{1}{2}\Box\gamma_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}\gamma^{\sigma\rho}_{,\rho\sigma} + \gamma^{\sigma}_{(\mu,\nu)\sigma} = 8\pi G\tau_{\mu\nu} - k_{\mu\nu}, \quad (55)$$

where Eq. (54) has been used, G is taken as a constant parameter, and we defined

$$\begin{aligned} k_{\mu\nu} &= -\bar{\lambda}^{\rho}\gamma_{\rho(\mu,\nu)} + \frac{1}{2}\bar{\lambda}^{\rho}\gamma_{\mu\nu,\rho} + \frac{1}{2}\bar{\lambda}_{(\mu}\gamma_{,\nu)} - \gamma_{\mu\nu}\bar{\Lambda} \\ &\quad + \eta_{\mu\nu}\left(\bar{\Lambda}_{\rho\sigma}\gamma^{\rho\sigma} + \bar{\lambda}^{\rho}\gamma^{\sigma}_{\rho,\sigma} - \frac{1}{4}\bar{\lambda}^{\sigma}\gamma_{,\sigma}\right). \end{aligned} \quad (56)$$

In order to determine the physical properties of the metric perturbations in the present theory, we proceed as usual and consider the infinitesimal diffeomorphism generated by a vector field ξ^μ, which gives rise to the coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x). \quad (57)$$

Taking notice that ξ^μ is an infinitesimal generator, it follows that the metric perturbations $h_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ transform respectively as (see Appendix A for more details)

$$\begin{aligned} \delta h_{\mu\nu} &= \xi_{\mu,\nu} + \xi_{\nu,\mu}, \\ \delta T_{\mu\nu} &= \xi^{\rho}\bar{T}_{\mu\nu,\rho} + \xi^{\rho}_{,\mu}\bar{T}_{\rho\nu} + \xi^{\rho}_{,\nu}\bar{T}_{\mu\rho}. \end{aligned} \quad (58)$$

From Eqs. (24) and (57) we obtain the transformation of the tensor $K_{\mu\nu}$ in the form

$$\delta K_{\mu\nu} = \xi^{\rho}\bar{K}_{\mu\nu,\rho} + \xi^{\rho}_{,\nu}\bar{K}_{\mu\rho} + \xi^{\rho}_{,\mu}\bar{K}_{\nu\rho}. \quad (59)$$

Now, by using Eqs. (54), (58), and (59) it follows

$$\delta K_{\mu\nu} = 8\pi G\delta T_{\mu\nu}. \quad (60)$$

This relation states that, since the Einstein tensor $G_{\mu\nu}$ is invariant by the gauge transformation (57), the field equations (55) are also invariant under transformations (58).

Now, due to the diffeomorphism invariance in a background spacetime region where Eq. (54) is obeyed, we are free to make a gauge choice. In the present case of unperturbed λ_μ, we choose the modified gauge condition

$$\gamma^{\rho}_{\mu,\rho} - \bar{\lambda}^{\rho}\gamma_{\mu\rho} + \frac{1}{2}\bar{\lambda}_{\mu}\gamma = 0, \tag{61}$$

to simplify the field equations (55). With such a choice, the perturbation equations are cast as

$$\Box\gamma_{\mu\nu} - \bar{\lambda}^{\rho}\gamma_{\mu\nu,\rho} + 2\bar{\Lambda}\gamma_{\mu\nu} + \bar{\lambda}_{(\mu,\nu)}\gamma - 2\bar{\lambda}^{\sigma}_{(,\mu}\gamma_{\nu)\sigma} + \eta_{\mu\nu}\left(\bar{\lambda}_{\sigma,\rho}\gamma^{\sigma\rho} - \frac{1}{2}\bar{\lambda}^{\sigma}_{,\sigma}\gamma - \bar{\lambda}^{\rho}\gamma^{\sigma}_{\rho,\sigma} - 2\bar{\Lambda}^{\rho\sigma}\gamma_{\rho\sigma}\right) = -16\pi G\tau_{\mu\nu}. \tag{62}$$

Condition (61) implies that 4 metric degrees of freedom are fixed, as in standard general relativity. Some of the remaining 6 degrees of freedom may not be physical, or may not propagate, and further detailed analysis is necessary to answer this question. This is an important study that we prefer not to present here to avoid a too lengthy text.

B. Plane wave decomposition of the gravitational perturbations

For simplicity, we now assume that $\bar{\lambda}_{\mu}$ is a constant vector and that the perturbed energy-momentum tensor is zero, $\tau_{\mu\nu} = 0$. We then look for solutions to the last equations in the plane wave form

$$\gamma_{\mu\nu} = A_{\mu\nu}e^{ik_{\sigma}x^{\sigma}}, \tag{63}$$

where $A_{\mu\nu}$ is a constant and symmetric tensor, and k_{μ} is the wave vector. By taking the expression (63) into (61) and (62) it follows

$$\begin{aligned} A\bar{\lambda}_{\mu} + 2(ik^{\nu} - \bar{\lambda}^{\nu})A_{\mu\nu} &= 0, \\ k^2 + i\bar{\lambda}_{\mu}k^{\mu} + 2\bar{\lambda}^2 &= 0, \\ A_{\mu\nu}\bar{\lambda}^{\nu}(ik^{\mu} + 2\bar{\lambda}^{\mu}) &= 0. \end{aligned} \tag{64}$$

Note that, the admission of a plane wave solution for a constant Herglotz vector field λ_{μ}, together with the gauge condition (61), imposes five restrictions to the amplitude tensor $A_{\mu\nu}$, namely, the first (four) and the last (one) relations in (64). Hence, five components are left undetermined, and a deeper analysis to get the physical interpretation of them is necessary. As commented above, this study is beyond the scope of the preset work.

Now, the constancy restriction on $\bar{\lambda}_{\mu}$ would pick out a preferential direction in spacetime. In other words, the local Lorentz symmetry is broken and the wave propagation may not be isotropic even in flat spacetime.

Additionally, since the Herglotz vector is arbitrary, we may make further assumptions to simplify the analysis. Here we assume that second-order terms on λ_{μ} such as $\bar{\lambda}^2$ may be neglected, what is exactly true in the case $\bar{\lambda}_{\mu}$ is a lightlike vector, and it is a good approximation in cases where $|\lambda^2|$ is much smaller than $|k^2|$ and $|\lambda_{\mu}k^{\mu}|$ in the second relation of (64).

C. Damping gravitational perturbations

As it is well known, the complex form of the dispersion relation, as the second relation in (64), leads to dissipative effects on the wave propagation. Indeed, the components of the wave vector k^{μ} assume complex values and the imaginary parts contribute to the damping or amplification of the wave amplitude, depending on the vector $\bar{\lambda}_{\mu}$. To explore this dependence, we split the timelike and spacelike components of vectors $\bar{\lambda}_{\mu}$ and k_{μ}, respectively, as $\bar{\lambda}_{\mu} = (-\bar{\lambda}_t, \bar{\boldsymbol{\lambda}})$ and $k_{\mu} = (-\omega, \boldsymbol{k})$. Therefore, from the second equation in (64), and keeping in mind that here we neglected the $\bar{\lambda}^2$ term, i.e., here we assume $\bar{\lambda}^2 = \bar{\lambda}_{\mu}\bar{\lambda}^{\mu} = 0$, one finds the relation

$$\omega = \frac{1}{2}[\alpha_+ - i(\pm\alpha_- + \bar{\lambda}_t)], \tag{65}$$

with $\alpha_{\pm}$ defined by

$$\alpha_{\pm} = \left[\sqrt{(2|\boldsymbol{k}|^2 - \bar{\lambda}_t^2/2)^2 + (2\boldsymbol{k}\cdot\bar{\boldsymbol{\lambda}})^2} \pm (2|\boldsymbol{k}|^2 - \bar{\lambda}_t^2/2)\right]^{1/2}, \tag{66}$$

where the dot $(\cdot)$ stands for the scalar product, and we have chosen the solution for which the real part of ω is nonnegative. After using (65), the exponential part of the solution in (63) goes as $\exp[-(\pm\alpha_- + \bar{\lambda}_t)t/2]$ times an oscillatory function of time. This indicates that the wave may be damped (or amplified) while traveling throughout spacetime.

The effects of the Herglotz field λ_{μ} in the wave propagation are more easily identified in two particular cases, namely, the case where the wave vector $\boldsymbol{k}$ is orthogonal to $\bar{\boldsymbol{\lambda}}$ and the case where these two spatial vectors are parallel to each other.

Taking the case where $\boldsymbol{k}$ is parallel to $\bar{\boldsymbol{\lambda}}$ it follows $\alpha_+ = 2|\boldsymbol{k}|$, $\alpha_- = |\bar{\lambda}_t|$, and $\omega = |\boldsymbol{k}| - i\bar{\lambda}_t$. This result follows by noticing that, without loss of generality, we may choose coordinate axes so that the wave and Herglotz vectors take respectively the forms $k_{\mu} = (-\omega, 0, 0, k_z)$ and $\bar{\lambda}_{\mu} = (-\bar{\lambda}_t, 0, 0, \pm\bar{\lambda}_t)$, with $k_z = \pm|\boldsymbol{k}|$ and where the assumption $\bar{\lambda}^2 = 0$ was used to write $\bar{\lambda}_z = \pm\bar{\lambda}_t$. Hence, one has $\boldsymbol{k}\cdot\bar{\boldsymbol{\lambda}} = \pm k_z\bar{\lambda}_t$ and from Eqs. (65) and (66) the just stated result follows. As a consequence, the time dependence of the wave function (63) becomes $\exp[-(i|\boldsymbol{k}| + \bar{\lambda}_t)t]$, so that the wave amplitude varies with the exponential factor $\exp[-\bar{\lambda}_t t]$. Therefore the wave perturbation is damped in the case $\bar{\lambda}_t > 0$ and it is amplified in the case $\bar{\lambda}_t < 0$.

Now choosing the particular case where $\boldsymbol{k}$ is orthogonal to $\bar{\boldsymbol{\lambda}}$, i.e., with $\boldsymbol{k}\cdot\bar{\boldsymbol{\lambda}}=0$, two distinct situations come out. The first situation is $\alpha_+ = 2\sqrt{|\boldsymbol{k}|^2-\bar{\lambda}_t^2/2}$, $\alpha_-=0$, for $|\boldsymbol{k}|^2-\bar{\lambda}_t^2/2>0$. In this case it follows $\omega=(\alpha_+-i\bar{\lambda}_t)/2$, which implies the wave amplitude varies with time as $\exp[-\bar{\lambda}_t t/2]$. Therefore, as in the preceding case, the wave is damped in the case $\bar{\lambda}_t>0$ and it is amplified in the case $\bar{\lambda}_t<0$. The second situation is for $|\boldsymbol{k}|^2-\bar{\lambda}_t^2/2<0$, which gives $\alpha_+=0$ and $\alpha_-=2\sqrt{\bar{\lambda}_t^2/2-|\boldsymbol{k}|^2}$. In this case it follows $\omega=-i(\pm\alpha_-+\bar{\lambda}_t)/2$, which implies the wave amplitude varies with time as $\exp[-(\pm\alpha_-+\bar{\lambda}_t)t/2]$, and since one has $0\le\alpha_-\le|\bar{\lambda}_t|$, once again the wave is damped in the case $\bar{\lambda}_t>0$ and it is amplified in the case $\bar{\lambda}_t<0$. The interesting new feature here is that the long wavelength modes, for which $|\boldsymbol{k}|^2-\bar{\lambda}_t^2/2<0$, do not propagate.

D. The wave speed

The speed of the perturbation waves may be determined from the above results. We start by studying the phase speed v_f, defined by $v_f=|\Re(k_t)|/|\boldsymbol{k}|=\Re(\omega)/|\boldsymbol{k}|$, where $\Re$ represents the real part of a complex number. Using relations (65) and (66) we see that the phase speed depends on the wavelength, on the propagation direction, and on the Herglotz vector field λ_μ. The dependence of the phase speed on the propagation direction is seen more clearly by taking two particular cases, namely, the case where $\boldsymbol{k}$ is parallel and the case where it is perpendicular to $\bar{\boldsymbol{\lambda}}$, for which the dispersion relation turns out simple.

The first interesting particular case occurs for $\boldsymbol{k}$ parallel to $\bar{\boldsymbol{\lambda}}$, where one has $\omega=|\boldsymbol{k}|-i\bar{\lambda}_t$. In this case the phase speed of the wave is given by

$$v_f=\frac{\Re(\omega)}{|\boldsymbol{k}|}=1, \tag{67}$$

which is exactly the speed of light.

A second simple case is when the propagation is orthogonal to the spacelike part of the Herglotz vector, i.e., for $\boldsymbol{k}\cdot\bar{\boldsymbol{\lambda}}=0$. Here, it follows

$$v_f=\sqrt{1-\frac{\lambda_t^2}{2|\boldsymbol{k}|^2}},\qquad |\boldsymbol{k}|^2-\bar{\lambda}_t^2/2>0,$$
$$v_f=0,\qquad |\boldsymbol{k}|^2-\bar{\lambda}_t^2/2\le 0. \tag{68}$$

This result implies that the Herglotz parameter $\bar{\lambda}_t$ imposes a cutoff for the propagation of plane waves, no propagation for low wave number (large wavelength) values compared to λ_t. For large wave number (low wavelength) values, the phase speed approaches the speed of light.

Additionally, one must also consider the group velocity in the present theory. A simple calculation by using the definition $v_g=\partial\Re(\omega)/\partial k$ provides the group speed, and the resulting expressions show that the group velocity may be larger than the speed of light. However, a deeper analysis is necessary to investigate whether the energy transported by gravitational waves really may travel faster than the speed of light, but this analysis is beyond the goals of the present work.

E. Interpreting and estimating the Herglotz parameter from gravitational waves data

A possible interpretation of the Herglotz vector in the linear regime of the theory may be obtained by analyzing the dispersion relations resulting from the plane wave solutions investigated in Secs. V B and V C, in particular by considering the relations given in Eq. (64). As commented in Sec. V C, the true dispersion relation is given by the real part of ω, while the imaginary part of ω gives the damping (or amplification) of the wave amplitude with time, both parts depend upon the Herglotz vector. In complement to the analysis presented in Sec. V C, where the Herglotz vector λ_μ was assumed to be a lightlike vector, here we consider two different particular cases, namely, the case of a spacelike Herglotz vector ($\lambda_t=0$ and $\bar{\boldsymbol{\lambda}}\neq 0$), and the case of a timelike Herglotz vector ($\bar{\lambda}_t\neq 0$ and $\bar{\boldsymbol{\lambda}}=0$).

(i) For $\bar{\lambda}_t=0$ and $\bar{\boldsymbol{\lambda}}\neq 0$:

Taking $\bar{\lambda}_t=0$ into the second relation in Eq. (65) and solving for ω one finds

$$\omega=\frac{1}{2}[\alpha_+\mp i\alpha_-], \tag{69}$$

with $\alpha_\pm$ defined by

$$\alpha_\pm=\left[\sqrt{(2|\boldsymbol{k}|^2+4|\bar{\boldsymbol{\lambda}}|^2)^2+(2\boldsymbol{k}\cdot\bar{\boldsymbol{\lambda}})^2}\pm(2|\boldsymbol{k}|^2+4|\bar{\boldsymbol{\lambda}}|^2)\right]^{1/2}. \tag{70}$$

To simplify the analysis we consider small $|\bar{\boldsymbol{\lambda}}|$ when compared to $|\boldsymbol{k}|$ and expand the relation (70) in powers of $|\bar{\boldsymbol{\lambda}}\cdot\boldsymbol{k}|/|\boldsymbol{k}|^2$ to get, to the second order in $\bar{\boldsymbol{\lambda}}$, $\omega=|\boldsymbol{k}|+|\bar{\boldsymbol{\lambda}}|^2/|\boldsymbol{k}|+\bar{\lambda}_\parallel^2/8|\boldsymbol{k}|)\pm i|\bar{\lambda}_\parallel|/2$, where $\bar{\lambda}_\parallel$ is the component of $\bar{\lambda}$ along the wave vector $\boldsymbol{k}$. This regime can be compared to the Eikonal limit of some quasioscillatory systems such as quasinormal modes of black holes, wave propagation in viscous fluids, among others. The real part of the frequency shows a deviation from the linear dependence on the wave number just at the second order on the Herglotz vector components. On the other hand, the imaginary part of the frequency is a first-order correction to the nondamped propagating wave, and the damping (amplification) of the waves goes as $\exp[\mp|\bar{\lambda}_\parallel|/2]$. This behavior is essentially the same as in the case of lightlike Herglotz vector studied in Sec. V C.

(ii) For $\bar{\lambda}_t \neq 0$ and $\bar{\boldsymbol{\lambda}} = 0$:

In cases where the spatial part of the Herglotz vector $\bar{\boldsymbol{\lambda}}$ may be neglected, the dispersion relation reduces to

$$\omega = \sqrt{|\boldsymbol{k}|^2 - \frac{9\bar{\lambda}_t^2}{4}} - \frac{i}{2}\bar{\lambda}_t. \tag{71}$$

Considering the expansion for small $\bar{\lambda}_t$, we get $\omega = |\boldsymbol{k}| - 9\bar{\lambda}_t^2/(8|\boldsymbol{k}|) - i\bar{\lambda}_t/2$. Again the damping (amplification) of the wave is of the exponential form, i.e., the amplitude goes as $\exp[-\bar{\lambda}_t/2]$, the same behavior as in the case investigated in Sec. V C.

As we have just seen, the Herglotz vector field couples to the wave vector in such a way to modify the dispersion relations of propagating waves with subsequent effects on the phase and group velocity, and also damping (or amplifying) the amplitude of the waves. In this way, the additional field may be interpreted as introducing a dispersive media throughout the spacetime. The damping effect on the wave propagation is similar to what happens in a dissipating media. In the present theory, the dissipation and dispersion effects may be thought of as properties of the spacetime itself.

The recent data from gravitational waves detection may be used to estimate the parameter $\bar{\lambda}_t$ in the cases of timelike and lightlike Herglotz vector, or the component parallel to the wave number, $\bar{\boldsymbol{\lambda}}_{\parallel}$, in the case of a spacelike Herglotz vector. Taking for instance the event GW170817 [35], which is located at the distance of about 130 million light years from Earth, using the result (67) and the fact that the wave speed is of the order of the speed of light, we find the time travel τ of waves produced at that event is of about 100 million years. Let R be the relation between the supposedly damped wave and the amplitude of the corresponding nondamped wave, both observed at the Earth. The theoretical prediction for R is obtained from the analysis of the last section, that is $R = \exp[-\bar{\lambda}_t\tau]$, or equivalently $|\bar{\lambda}_t\tau| \simeq |\ln R|$. Assuming further that the amplitude damping due to dissipative effects is of the order of the initial amplitude we have $R = 1/2$ and it follows $|\bar{\lambda}_t| \sim 10^{-16}\ \mathrm{s}^{-1}$. This is to be considered as an upper bound for the parameter $|\lambda_t|$. The more distant the gravitational source is, the sharper is the upper bound on $|\bar{\lambda}_t|$. In fact, using the event of Ref. [36] which is estimated to have occurred at the redshift of the order of $z = 0.1$, which means that the wave has traveled about 1.3 billion years to reach Earth, we find $|\bar{\lambda}_t| \sim 10^{-17}\ \mathrm{s}^{-1}$, yet 1 order of magnitude larger than what is estimated from cosmology (see Sec. IV D).

VI. FINAL REMARKS

We have considered the Herglotz variational principle to propose a covariant formulation for a nonconservative gravity theory. As a result, by using the usual gravity Lagrangian density and introducing an arbitrary background vector field (the Herglotz field), we obtained the modified gravitational field equations that present a totally tensorial structure. Therefore, the nontensorial character of the theory obtained in [10] is solved.

When the theory was put to the test within the scope of cosmology using the FLRW geometry, different types of solutions were obtained by assuming the conservation and nonconservation of the energy-momentum tensor. In the case of conserved energy-momentum tensors, it results in solutions for the scale factor such as power-law and exponential forms, the last form being appropriate for both inflation with big bang and a late phase of accelerated expansion. In this case, the extra (Herglotz) vector field, which in FRLW spacetimes has only one nontrivial component ϕ, plays a role similar to the cosmological constant in general relativity, and an estimate for its numerical value at present time was obtained by considering the present value of the Hubble parameter. In the case of nonconserved energy-moment tensors, we have found the inflation standard solution among other interesting solutions. Among these, the solution obtained in Sec. IV C stands out. It represents an accelerated universe filled with dust (cold dark matter), over a period of time, thus avoiding the introduction of a dark energy component (or a dilaton field) to explain the accelerated expansion (or the inflationary) period.

Despite a variety of cosmological solutions that can be found in view of the arbitrariness of the Herglotz field ϕ, the simplest cosmological models have some issues to be considered and further investigated. One of them is the existence of solutions with negative energy density, leading us to restrict the choices for ϕ that result in non-negative energy density. Another issue is the same as it happens in general relativity. Due to the restriction imposed by choosing an equation of state in the form $p = \omega\rho$, with ω being a constant parameter, the theory is unable to provide an accelerated expansion in late cosmological times without the aid of an exotic material known as dark energy. However, in the present theory, these apparent flaws may be remedied by introducing some kind of mechanism to select the appropriate $\phi(t)$ for each phase of the universal expansion.

The linear regime of the theory was also studied and wavelike solutions were shown to exist for the metric perturbations. As expected, this study revealed the dissipative behavior of wave propagation, which can be damped or forced depending on the Herglotz vector λ_μ. In the present theory, the phase speed of the plane waves may be different from the speed of light, but further investigation is needed to determine the speed of energy propagation, and also to establish the real number of independent propagation modes. Again, considering a plane wave propagating along a specific direction, a numerical estimate for the extra (Herglotz) parameter was obtained by using the recent data on gravitational waves.

The applications presented in the present work should be considered as a preliminary analysis, so that further and deeper studies are necessary to test the theory against observational data. Our immediate interest is to investigate the existence of solutions representing compact objects in this nonconservative gravity theory.

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APPENDIX A: PERTURBATIONS AROUND MINKOWSKI: UNPERTURBED HERGLOTZ FIELD

1. Preliminary remarks

Let us consider a perturbed metric in the form given by Eq. (49), i.e., $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ being considered as a perturbation over the background metric $\bar{g}_{\mu\nu}$. Up to the first order in $h_{\mu\nu}$ and its derivatives, the Einstein tensor may be split as $G_{\mu\nu} = \bar{G}_{\mu\nu} + \mathcal{G}_{\mu\nu}$, where $\mathcal{G}_{\mu\nu}$ stands for the first-order correction on the background (unperturbed) Einstein tensor $\bar{G}_{\mu\nu}$. Similarly, we may write $T_{\mu\nu} = \bar{T}_{\mu\nu} + \tau_{\mu\nu}$, $K_{\mu\nu} = \bar{K}_{\mu\nu} + k_{\mu\nu}$. Therefore, the filed equations (26) are satisfied order-by-order and separate into two equations,

$$\bar{G}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu} - \bar{K}_{\mu\nu}, \tag{A1}$$

$$\mathcal{G}_{\mu\nu} = 8\pi G \tau_{\mu\nu} - k_{\mu\nu}. \tag{A2}$$

Taking the divergence of Eqs. (A1) and (A2) it follows

$$8\pi G \bar{T}^{\nu}_{\mu;\nu} - \bar{K}^{\nu}_{\mu;\nu} = 0, \tag{A3}$$

$$8\pi G \tau_{\mu\ ;\nu}^{\ \nu} - k_{\mu;\nu}^{\ \nu} = 0. \tag{A4}$$

It is then clear that, in general, the perturbation quantities $\tau_{\mu\nu}$ and $k_{\mu\nu}$ form a divergence free tensor just when combined in the form $8\pi G \tau_{\mu\nu} - k_{\mu\nu}$. In fact, as we show next, in general the quantities $k_{\mu\nu}$ and $\tau_{\mu\nu}$ are not tensorial objects, they define a tensorial function just when combined as in Eq. (A4).

2. Perturbed quantities and gauge transformations

Here we investigate the perturbation equations in the case where the Herglotz field λ_μ is kept fixed, while the metric and other fields are perturbed around the flat spacetime, i.e., we may write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \|h_{\mu\nu}\| \ll 1, \qquad \lambda_\mu = \bar{\lambda}_\mu, \tag{A5}$$

where $\bar{\lambda}_\mu$ is the background Herglotz vector.

Let us start by investigating the behavior of the tensor $\Lambda_{\mu\nu}$ under the perturbations (A5). Such a tensor is defined in terms of the metric and the vector λ_μ in Eq. (25). In the case of unperturbed λ_μ and up to the first order in $h_{\mu\nu}$, the tensor $\Lambda_{\mu\nu}$ reads

$$\Lambda_{\mu\nu} = \bar{\Lambda}_{\mu\nu} - \Gamma^{\sigma}_{\mu\nu}\bar{\lambda}_\sigma, \tag{A6}$$

where

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}), \tag{A7}$$

and $\bar{\Lambda}_{\mu\nu}$ is defined in Eq. (52).

Now, by performing an infinitesimal coordinate transformation $x'^{\mu} = x^\mu + \xi^\mu(x)$, see Eq. (57), we find the transformation for the metric perturbation tensor

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}, \\ h'^{\mu\nu} &= h^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu}. \end{aligned} \tag{A8}$$

Similarity, we obtain the transformed Herglotz vector field,

$$\lambda'_\mu = \lambda_\mu + \xi^\sigma \bar{\lambda}_{\mu,\sigma} + \xi_{\sigma,\mu}\bar{\lambda}^\sigma. \tag{A9}$$

Therefore, using the first relation in (A7) and (A8) we get the transformation on the Christoffel symbol as $\delta\Gamma^{\sigma}_{\mu\nu} = \xi^{\sigma}_{,\mu\nu}$. From this result and (A6), we calculate the variation of $\Lambda_{\mu\nu}$ in (A6) at first order, i.e.,

$$\delta\Lambda_{\mu\nu} = \xi^\rho \bar{\Lambda}_{\mu\nu,\rho} + \xi^{\rho}_{,\nu}\bar{\Lambda}_{\mu\rho} + \xi^{\rho}_{,\mu}\bar{\Lambda}_{\nu\rho}. \tag{A10}$$

So now the calculation of the change in the tensor $K_{\mu\nu} = \Lambda_{\mu\nu} - g_{\mu\nu}\Lambda$ is obtained straightforwardly. Indeed,

$$\begin{aligned} \delta K_{\mu\nu} &= \delta\Lambda_{\mu\nu} - \delta g_{\mu\nu}\Lambda - g_{\mu\nu}\delta(\Lambda_{\sigma\rho}g^{\sigma\rho}) \\ &= \delta\Lambda_{\mu\nu} - \delta g_{\mu\nu}\Lambda - \eta_{\mu\nu}(\delta\Lambda_{\sigma\rho}\eta^{\sigma\rho} + \Lambda_{\sigma\rho}\delta g^{\sigma\rho}), \end{aligned} \tag{A11}$$

where we eliminate all the second-order terms. Finally, by using (A8) and (A10), it follows

$$\begin{aligned} \delta K_{\mu\nu} &= \xi^\rho(\bar{\Lambda}_{\mu\nu,\rho} - \eta_{\mu\nu}\bar{\Lambda}_{,\rho}) + \xi^{\rho}_{,\nu}(\bar{\Lambda}_{\mu\rho} - \eta_{\mu\rho}\bar{\Lambda}) \\ &\quad + \xi^{\rho}_{,\mu}(\bar{\Lambda}_{\nu\rho} - \eta_{\nu\rho}\bar{\Lambda}) \\ &= \xi^\rho \bar{K}_{\mu\nu,\rho} + \xi^{\rho}_{,\nu}\bar{K}_{\mu\rho} + \xi^{\rho}_{,\mu}\bar{K}_{\nu\rho}, \end{aligned} \tag{A12}$$

where

$$\bar{K}_{\mu\nu} = \bar{\Lambda}_{\mu\nu} - \eta_{\mu\nu}\bar{\Lambda}, \tag{A13}$$

and $\bar{\Lambda}_{\mu\nu}$ is defined in Eq. (52). This result confirms that, in the case the Herglotz vector λ_μ is not perturbed, the complete perturbed tensor $K_{\mu\nu}$ transforms appropriately, and because of the gauge invariance of the Einstein tensor, the combination $8\pi G T_{\mu\nu} - K_{\mu\nu}$ is such that $8\pi G \delta T_{\mu\nu} - \delta K_{\mu\nu} = 0$.

Notice that, although the full object $K_{\mu\nu}$ does transform as a tensor under the gauge transformation (A8), the background quantity $\bar{K}_{\mu\nu}$ does not obey the same transformation law. As a consequence, both objects $\bar{T}_{\mu\nu}$ and $\tau_{\mu\nu}$ are affected in the same way as $\bar{K}_{\mu\nu}$ and $k_{\mu\nu}$ [see Eq. (56)], since $\bar{K}_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ are linked by Eq. (54), i.e.,

$$\bar{K}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}. \tag{A14}$$

Indeed, by using the expressions (A6) and (A10) it is found

$$\delta\bar{K}_{\mu\nu} = \xi^\rho \bar{K}_{\mu\nu,\rho} + \xi^\rho_{,\nu}\bar{K}_{\mu\rho} + \xi^\rho_{,\mu}\bar{K}_{\nu\rho} + \xi^\sigma_{,\mu\nu}\bar{\lambda}_\sigma + 2\bar{\Lambda}\xi_{(\mu,\nu)} - \eta_{\mu\nu}(\bar{\lambda}_\sigma \Box \xi^\sigma + 2\xi_{\sigma,\rho}\bar{\Lambda}^{\sigma\rho}). \tag{A15}$$

With this result, Eq. (A14) lead us to the following relation:

$$\delta\bar{T}_{\mu\nu} = \xi^\rho \bar{T}_{\mu\nu,\rho} + \xi^\rho_{,\nu}\bar{T}_{\mu\rho} + \xi^\rho_{,\mu}\bar{T}_{\nu\rho} + \frac{1}{8\pi G}\xi^\sigma_{,\mu\nu}\bar{\lambda}_\sigma + \frac{1}{8\pi G}[2\bar{\Lambda}\xi_{(\mu,\nu)} - \eta_{\mu\nu}(\bar{\lambda}_\sigma \Box \xi^\sigma + 2\xi_{\sigma,\rho}\bar{\Lambda}^{\sigma\rho})]. \tag{A16}$$

Furthermore, since $T_{\mu\nu} = \bar{T}_{\mu\nu} + \tau_{\mu\nu}$ is a tensor, it transforms as

$$\delta T_{\mu\nu} = \delta\bar{T}_{\mu\nu} + \delta\tau_{\mu\nu} = \xi^\rho \bar{T}_{\mu\nu,\rho} + \xi^\rho_{,\nu}\bar{T}_{\mu\rho} + \xi^\rho_{,\mu}\bar{T}_{\nu\rho}. \tag{A17}$$

Therefore, by comparing Eqs. (A16) and (A17), it follows

$$8\pi G \delta\tau_{\mu\nu} = -\xi^\sigma_{,\mu\nu}\bar{\lambda}_\sigma - 2\bar{\Lambda}\xi_{(\mu,\nu)} + \eta_{\mu\nu}(\bar{\lambda}_\sigma \Box \xi^\sigma + 2\xi_{\sigma,\rho}\bar{\Lambda}^{\sigma\rho}), \tag{A18}$$

which shows explicitly that $\tau_{\mu\nu}$ is not a tensor. However, the combination of Eqs. (A16)–(A18) confirms that the quantity $8\pi G \tau_{\mu\nu} - k_{\mu\nu}$ transforms as a second rank tensor, as expected.

3. Gauge compatibility

Let us now consider the changes in the field equations for the metric perturbations and for the gauge choice under the coordinate transformations (57), $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$. Performing the gauge the transformation and using the results of the last section into the equations of motion (62), at first order in the perturbations, we get

$$(\Box\xi_\nu - \bar{\lambda}^\rho \xi_{\nu,\rho})_{,\mu} + (\Box\xi_\mu - \bar{\lambda}^\rho \xi_{\mu,\rho})_{,\nu} + \eta_{\mu\nu}(\bar{\lambda}^{\sigma,\rho}\xi_{\sigma,\rho} + \bar{\lambda}^{\sigma,\rho}\xi_{\rho,\sigma} - 4\bar{\Lambda}_{\sigma\rho}\xi^{\sigma,\rho}) + 4\bar{\Lambda}\xi_{(\mu,\nu)} - 2\lambda_{\sigma,(\mu}\xi^\sigma_{,\nu)} - \eta_{\mu\nu}(\Box\xi^\sigma_{,\sigma} - \bar{\lambda}^\rho \xi^\sigma_{,\rho\sigma} + \bar{\lambda}^\sigma \Box \xi_\sigma) = -16\pi G \delta\tau_{\mu\nu}. \tag{A19}$$

By substituting (A18) into (A19) it gives

$$(\Box\xi_\nu - \bar{\lambda}^\rho \xi_{\nu,\rho} - \bar{\lambda}^\sigma \xi_{\sigma,\nu})_{,\mu} + (\Box\xi_\mu - \bar{\lambda}^\rho \xi_{\mu,\rho} - \bar{\lambda}^\sigma \xi_{\sigma,\mu})_{,\nu} - \eta_{\mu\nu}(\Box\xi^\sigma_{,\sigma} - \bar{\lambda}^\rho \xi^\sigma_{,\rho\sigma} - \bar{\lambda}^\sigma \Box \xi_\sigma) + \eta_{\mu\nu}(\bar{\lambda}^{\sigma,\rho}\xi_{\sigma,\rho} + \bar{\lambda}^{\sigma,\rho}\xi_{\rho,\sigma}) = 0. \tag{A20}$$

Apparently, the field equations are not invariant under transformation (A8). To verify this, we follow the usual procedure as in general relativity and consider the variation of gauge choice (61) under the transformation (57), which implies in

$$\Box\xi_\nu - \bar{\lambda}^\rho \xi_{\nu,\rho} - \bar{\lambda}^\sigma \xi_{\sigma,\nu} = 0. \tag{A21}$$

This means that the gauge condition (61) obligates the coordinate system to satisfy Eq. (A21). Hence, by taking (A21) into (A20) it follows

$$\bar{\lambda}^{\sigma,\rho}(\xi_{\sigma,\rho} + \xi_{\rho,\sigma}) = 0. \tag{A22}$$

Since this condition cannot be satisfied in general, the field equations are not invariant under the gauge transformation. There is however, a particular case where (A22) becomes an identity, namely, when $\bar{\lambda}_\mu$ is a constant vector. Therefore, for consistency, the present first order metric perturbation theory requires the Herglotz vector field $\bar{\lambda}_\mu$ to be a constant vector.

APPENDIX B: PERTURBATION AROUND MINKOWSKI: PERTURBED HERGLOTZ FIELD

1. Perturbed quantities and gauge transformations

Here we assume that, besides the metric and mater fields, also the Herglotz vector field is perturbed around its background value $\bar{\lambda}_\mu$, i.e.,

$$\lambda_\mu = \bar{\lambda}_\mu + l_\mu, \tag{B1}$$

where $\bar{\lambda}_\mu$ is the solution of Eq. (26) in Minkowski spacetime, and l_μ stands for the perturbation on $\bar{\lambda}_\mu$. Therefore, up to the first order in the perturbations $h_{\mu\nu}$, l_μ, and their derivatives the tensor $K_{\mu\nu}$ defined in Eq. (24) may be split as a background and a perturbation term,

$$K_{\mu\nu} = \bar{K}_{\mu\nu} + k_{\mu\nu}, \tag{B2}$$

where $\bar{K}_{\mu\nu}$ and $k_{\mu\nu}$ are given respectively by

$$\bar{K}_{\mu\nu} = \bar{\Lambda}_{\mu\nu} - \eta_{\mu\nu}\bar{\Lambda},$$
$$k_{\mu\nu} = \ell_{\mu\nu} - \eta_{\mu\nu}\ell - h_{\mu\nu}\bar{\Lambda} + \eta_{\mu\nu}\bar{\Lambda}_{\rho\sigma}h^{\rho\sigma}, \quad \text{(B3)}$$

where $\bar{\Lambda}_{\mu\nu}$ is given by Eq. (52), and we have defined

$$\ell_{\mu\nu} = l_{(\mu,\nu)} - 2\bar{\lambda}_{(\mu}l_{\nu)} - \bar{\lambda}^{\rho}h_{\rho(\nu,\mu)} + \frac{1}{2}\bar{\lambda}^{\rho}h_{\mu\nu,\rho}, \quad \text{(B4)}$$

with $\bar{\Lambda}_{\mu\nu}$ defined in Eq. (52), and we use the same notation and conventions of the preceding sections. In terms of $\gamma_{\mu\nu}$, the perturbation tensor $k_{\mu\nu}$ reads

$$\begin{aligned} k_{\mu\nu} = {} & l_{(\mu,\nu)} - 2\bar{\lambda}_{(\mu}l_{\nu)} - \bar{\lambda}^{\rho}\gamma_{\rho(\nu,\mu)} + \frac{1}{2}\bar{\lambda}^{\rho}\gamma_{\mu\nu,\rho} \\ & - \eta_{\mu\nu}\left(l^{\sigma}_{,\sigma} - 2\bar{\lambda}_{\sigma}l^{\sigma} - \bar{\lambda}^{\rho}\gamma^{\sigma}_{\rho,\sigma} + \frac{1}{4}\bar{\lambda}^{\sigma}\gamma_{,\sigma}\right) \\ & + \frac{1}{2}\bar{\lambda}_{(\mu}\gamma_{,\nu)} + \eta_{\mu\nu}\bar{\Lambda}_{\rho\sigma}\gamma^{\rho\sigma} - \gamma_{\mu\nu}\bar{\Lambda}. \end{aligned} \quad \text{(B5)}$$

The energy-momentum tensor is also perturbed and may be split as in Eq. (53), i.e., $T_{\mu\nu} = \bar{T}_{\mu\nu} + \tau_{\mu\nu}$.

We proceed as usual and consider the infinitesimal diffeomorphism generated by a vector field ξ^{μ}, which gives rise to the coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$. It follows that the metric perturbations $h_{\mu\nu}$ transform as in Eq. (A8), while the perturbation of the Herglotz vector l_{μ} and the energy-momentum tensor $\tau_{\mu\nu}$ transform as

$$l'_{\mu} = l_{\mu} + \xi^{\rho}\bar{\lambda}_{\mu,\rho} + \bar{\lambda}_{\rho}\xi^{\rho}_{,\mu},$$
$$\tau'_{\mu\nu} = \tau_{\mu\nu} + \xi^{\rho}\bar{T}_{\mu\nu,\rho} + \xi^{\rho}_{,\mu}\bar{T}_{\rho\nu} + \xi^{\rho}_{,\nu}\bar{T}_{\mu\rho}, \quad \text{(B6)}$$

respectively.

From Eqs. (B6), (B4), and (B3) we obtain the variation of the tensor $k_{\mu\nu}$ as

$$k'_{\mu\nu} = k_{\mu\nu} + \xi^{\rho}\bar{K}_{\mu\nu,\rho} + \xi^{\rho}_{,\nu}\bar{K}_{\mu\rho} + \xi^{\rho}_{,\mu}\bar{K}_{\nu\rho}. \quad \text{(B7)}$$

Now, by substituting $\bar{K}_{\mu\nu}$ from Eq. (54) into (B7) it follows

$$k'_{\mu\nu} = k_{\mu\nu} + 8\pi G(\xi^{\rho}\bar{T}_{\mu\nu,\rho} + \xi^{\rho}_{,\nu}\bar{T}_{\mu\rho} + \xi^{\rho}_{,\mu}\bar{T}_{\nu\rho}), \quad \text{(B8)}$$

where we assumed a constant G.

Hence, given that the source on the rhs of Eq. (55) transforms according to the last relation in (58), i.e., $\delta\tau_{\mu\nu} = \xi^{\rho}\bar{T}_{\mu\nu,\rho} + \xi^{\rho}_{,\nu}\bar{T}_{\mu\rho} + \xi^{\rho}_{,\mu}\bar{T}_{\nu\rho}$, the comparison between this and (B8), by means of Eq. (60), shows that the first-order perturbation equations (55) are gauge invariant.

Now, due to the diffeomorphism invariance in a background spacetime region where Eq. (54) is obeyed, we are free to make a gauge choice. In the present case, we choose the modified gauge condition

$$\gamma^{\rho}_{\mu,\rho} - \bar{\lambda}^{\rho}\gamma_{\mu\rho} + \frac{1}{2}\bar{\lambda}_{\mu}\gamma + l_{\mu} = 0, \quad \text{(B9)}$$

to simplify the field equations (55). With such a choice, the perturbation equations are cast as

$$\begin{aligned} & \Box\gamma_{\mu\nu} - \bar{\lambda}^{\rho}\gamma_{\mu\nu,\rho} + 2\bar{\Lambda}\gamma_{\mu\nu} - 2\gamma_{\rho(\mu}\bar{\lambda}^{\rho}_{,\nu)} + \gamma\bar{\lambda}_{(\mu,\nu)} \\ & + \eta_{\mu\nu}\left(\bar{\lambda}_{\sigma,\rho}\gamma^{\sigma\rho} - \frac{1}{2}\bar{\lambda}^{\sigma}_{,\sigma}\gamma - \bar{\lambda}^{\rho}\gamma^{\sigma}_{\rho,\sigma} - 2\bar{\Lambda}^{\rho\sigma}\gamma_{\rho\sigma}\right) \\ & + \eta_{\mu\nu}(l^{\sigma}_{,\sigma} - 4\bar{\lambda}_{\sigma}l^{\sigma}) + 4\bar{\lambda}_{(\mu}l_{\nu)} = -16\pi G\tau_{\mu\nu}. \end{aligned} \quad \text{(B10)}$$

Since l_{μ} is not a dynamic field, it does not propagate through spacetime and then all degrees of freedom associated with it may be chosen arbitrarily. In particular, if one is allowed to choose $l_{\mu} = 0$, the perturbations equations result the same as the case analyzed in Sec. V.

2. Gauge compatibility

Let us now consider the changes in the field equations for the metric perturbations under the infinitesimal gauge transformation (57), $x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, for arbitrary l_{μ}. To simplify analysis, we assume a constant $\bar{\lambda}_{\mu}$ from now onward. At first order in the perturbations, transformations (B6) imply that the equations of motion (B10) change to the form

$$\begin{aligned} & (\Box\xi_{\nu} - \bar{\lambda}^{\rho}\xi_{\nu,\rho})_{,\mu} + (\Box\xi_{\mu} - \bar{\lambda}^{\rho}\xi_{\mu,\rho})_{,\nu} - \eta_{\mu\nu}(\Box\xi^{\sigma}_{,\sigma} - \bar{\lambda}^{\rho}\xi^{\sigma}_{,\rho\sigma}) \\ & - 2(\bar{K}^{\sigma}_{\mu}\xi_{\sigma,\nu} + \bar{K}^{\sigma}_{\nu}\xi_{\sigma,\mu}) = -16\pi G\delta\tau_{\mu\nu}. \end{aligned} \quad \text{(B11)}$$

After using the second relation in (B6), with $\bar{K}_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu}$, we get $\bar{K}^{\sigma}_{\mu}\xi_{\sigma,\nu} + \bar{K}^{\sigma}_{\nu}\xi_{\sigma,\mu} = 8\pi G\delta\tau_{\mu\nu}$ and, as a consequence, Eq. (B11) reduces to

$$(\Box\xi_{\nu} - \bar{\lambda}^{\rho}\xi_{\nu,\rho})_{,\mu} + (\Box\xi_{\mu} - \bar{\lambda}^{\rho}\xi_{\mu,\rho})_{,\nu} - \eta_{\mu\nu}(\Box\xi^{\sigma}_{,\sigma} - \bar{\lambda}^{\rho}\xi^{\sigma}_{,\rho\sigma}) = 0. \quad \text{(B12)}$$

Again, the equation for the metric perturbations (B10) does not appear to be invariant under the transformations (B6). In fact, this is just an illusion of the gauge choice. To see this, we just take the variation of the gauge (B9) to find

$$\Box\xi_{\mu} - \bar{\lambda}^{\rho}\xi_{\mu,\rho} = 0. \quad \text{(B13)}$$

Equation (B13) implies that (B12) is identically satisfied. Therefore the field equations (B10) are invariant under the transformations (B6) and consequently are gauge compatible.

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