Probability distribution for black hole evaporation

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Nonthermal correction to the emission probability of particles from black holes can be obtained if the backreaction or self-gravitational effects of the emitted particles on the black hole spacetime are taken into consideration. These nonthermally emitted particles conserve the entropy of the black hole—i.e., the entropy of the system of radiated particles after complete evaporation of the black hole matches the initial entropy of the black hole. Using the nonthermal emission probability, we have determined the probability for a black hole of mass M to be completely evaporated by a given number of particles n. This is done by first evaluating the number of possible ways in which the black hole can be evaporated by emitting n number of particles, and then the total number of ways in which the black hole can be evaporated. The ratio of these two quantities gives us the desired probability. From the probability distribution, we get a displacement relation between the most probable number of particles exhausting the black hole and the temperature of the initial black hole. This relation resembles Wien's displacement law for blackbody radiation.

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I. INTRODUCTION

Unifying general relativity and quantum mechanics, two important pillars of modern physics, has been a long sought-after goal for physicists. However, despite considerable efforts, a complete theory of quantum gravity remains elusive. A relatively simpler approach to studying quantum phenomena in gravitational fields is to treat matter fields quantum mechanically and background spacetimes classically. This is a semiclassical approach, much like studying atomic physics in classical electromagnetic fields instead of using full quantum electrodynamics. An important landmark in this line of research is the discovery of Hawking radiation by S. W. Hawking [1]. Hawking showed that quantum effects of matter fields in the vicinity of black holes lead to the creation of particle-antiparticle pairs. The antiparticle tunnels through the event horizon, inside the black hole, and the particle is emitted outside. Thus, black holes act like hot bodies with temperature $T_{\rm BH} = \frac{1}{8\pi GM}$, emitting radiation. The discovery of Hawking radiation not only predicts a completely revolutionary phenomenon—i.e., the emission of particles from black holes, which were classically thought to be "regions of no return"—but it also poses some deep questions about the nature of the interplay between quantum mechanics and gravity. Consider a black hole that may have been formed due to the gravitational collapse of a star. The information about the quantum states of its forming matter, which has crossed the event horizon, is not accessible to an observer outside the event horizon. But the information is stored safely beyond the horizon (until it reaches the singularity at the center). Now, with the discovery of Hawking radiation, as the black hole emits thermal radiation, it evaporates completely, leaving behind no trace of the information about its forming matter. On the other hand, thermal radiation does not carry any information as it escapes the black hole. Thus, information seems to be lost in the black hole evaporation process [2]. This loss of information is in contradiction to the principle of "unitary evolution" in quantum mechanics. So, Hawking radiation presents a conflict of principles between general relativity and quantum mechanics. As a result of nonunitary evolution, entropy is not conserved in the evaporation process [3]. In particular, the entropy of the radiation system obtained after complete evaporation of the black hole appears to be more than the Bekenstein-Hawking entropy [4] of the initial black hole ($S_{\rm rad} = \frac{4}{3} S_{\rm BH}$). Over the past few decades, a substantial amount of work has been done on the "information loss paradox" and its possible resolution [5–20].

An important work by Parikh and Wilczek [21] showed that the radiation spectrum from a black hole is not strictly thermal if the self-gravitational effects of the emitted particles on the black hole spacetime are taken into consideration. It has been further shown that the evaporation of black holes by the emission of nonthermal radiation is consistent with the principle of unitary evolution of quantum mechanics [22]. If the evolution is unitary, one expects to get back all the information that was stored

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inside the black hole from the emitted radiation. Indeed, Zhang *et al.* [23] showed that nonthermally emitted particles share correlations between them in the form of mutual information. These correlations carry information out of the black hole, and one can get back the entire information by collecting all of these particles. As a result, entropy is conserved in the evaporation process. To show the conservation of entropy in the black hole evaporation process, Zhang *et al.* have calculated the entropy of the system of radiation obtained after the complete evaporation of the black hole, which matches the Bekenstein-Hawking entropy of the initial black hole. However, this resolution of the information loss paradox is not unanimously accepted. As pointed out in Ref. [24], although entropy is conserved in the nonthermal radiation process, it does not account for the increase of entanglement entropy between the inside and outside of the black hole during the entire period of evaporation, which is the main essence of the information loss paradox. Also, see Ref. [25]. Very recently, a new resolution to the paradox has been proposed in Refs. [19,20] which is in line with the objections raised in Ref. [24].

However, whether or not these works resolve the paradox, we can still get some further useful information regarding black hole evaporation from them. A black hole of some given mass M may be completely evaporated by the emission of an arbitrary number of particles. The number of particles that are obtained after the complete evaporation of the black hole is uncertain. Building upon the earlier works on nonthermal radiation, we have evaluated the probability that the black hole evaporates completely by emitting a given number of particles n. To do so, first we have evaluated the number of possible ways in which the black hole can be evaporated by emitting n particles. Then, we have evaluated the total number of ways in which the black hole can be evaporated. The ratio of these two quantities gives us the desired probability. From the probability distribution, we get a relation between the most probable number of particles exhausting the black hole and the temperature of the initial black hole.

The outline of this article is as follows: In Sec. II, we briefly discuss nonthermal Hawking radiation. In particular, we discuss, following the works of Parikh [22] and Zhang *et al.* [23], how nonthermal radiation is consistent with the unitarity of quantum mechanics, and how these emitted particles leak the information stored inside the black hole and conserve entropy. In Sec. III, we calculate the probability for a black hole to be completely evaporated by a given number of particles that are emitted nonthermally. From the probability distribution obtained, we get a displacement relation between the most probable number of particles exhausting the black hole and the initial black hole temperature. Then, we try to interpret the meaning of the entropy of the radiation system, obtained after the complete evaporation of the black hole. Finally, we summarize our results and conclude in Sec. IV. Throughout the paper, we use natural units ($c = \hbar = k_B = 1$) unless otherwise mentioned.

II. NONTHERMAL HAWKING RADIATION

The evolution of a black hole to a thermal state, irrespective of its initial state, is nonunitary in nature and does not preserve information. In the original work [1] on Hawking radiation, Hawking considered a static spacetime geometry for a black hole which is not perturbed by the loss of energy of the emitted particles. Since the spacetime geometry is not perturbed during the emission of a particle with energy E, the mass parameter M of the black hole spacetime remains unchanged during the emission period of that particle. This violates the principle of energy conservation. Parikh, Kraus, and Wilczek [21,26] considered the case of dynamic geometry to enforce energy conservation. The dynamic nature of the geometry is due to the varying mass parameter of the black hole spacetime. It takes into account the backreaction of the emitted particles on the spacetime. The calculation of the tunneling probability through the event horizon—of a particle from the inside of a black hole to the outside, or equivalently, of an antiparticle from the outside to the inside—in this dynamic geometry, results in nonthermal correction terms to Hawking's original calculation [21]. The modified tunneling or emission probability of a particle with energy E from a black hole of mass M, up to a constant factor, is given by

$$\begin{aligned}\Gamma(E;M) &\sim \exp\left[-8\pi GE\left(M-\frac{E}{2}\right)\right]\\ &= \exp\left[-4\pi G(M^2-(M-E)^2)\right]\\ &= \exp\left[\Delta S_{\mathrm{BH}}\right],\end{aligned} \tag{1}$$

where $S_{\mathrm{BH}} = 4\pi GM^2$ is the Bekenstein-Hawking entropy of the black hole [4]. The first term in the exponent corresponds to thermal emission, whereas the second term gives the nonthermal correction. The constant prefactor of the exponential term in Eq. (1) can be determined by including higher orders of $\hbar$ corrections [27] (quantum corrections) to the tunneling calculation done by Parikh and Wilczek [21].

In Ref. [21], the emission probability is estimated by considering modes of emitted particles that propagate from a point that is arbitrarily close to the horizon of the black hole. Due to infinite gravitational redshift near the black hole horizon, the modes can have an arbitrarily small wavelength ($\lambda \to 0$). However, there are indications from many theories of quantum gravity of the existence of an observer-independent minimum length, which is identified with the Planck length ($l_P = \sqrt{\frac{\hbar G}{c^3}}$). Using the Planck length cut to the wavelength of the modes (quantum gravity correction), the spectrum is recalculated by Arzano *et al.* [28]. This calculation again gives the prefactor of Eq. (1). However, this prefactor is quite obviously different from Ref. [27], because two different mechanisms are used. Both of these calculations also result in a logarithmic correction term to the entropy of the black hole [29].

$$S = \frac{A}{4G} + \alpha \ln \frac{A}{G}, \tag{2}$$

where $A = 16\pi G^2 M^2$ is the area of the event horizon of the black hole.

A. Nonthermal radiation is consistent with the principle of unitary evolution in quantum mechanics

In Ref. [22], it is argued that the evaporation process of a black hole by the emission of nonthermal radiation is unitary in nature. In quantum mechanics, the rate of a unitary process from an initial state i to its final state f is given by

$$\Gamma(i \to f) = |\mathcal{M}_{fi}|^2 \times (\text{phase space factor}), \tag{3}$$

where $\mathcal{M}_{fi}$ is the amplitude of the process. The phase space factor is obtained by summing over all possible final states, which is simply the exponential of the final entropy (S_f) of the system, and averaging over all possible initial states, which is the exponential of the initial entropy (S_i) of the system. Therefore,

$$\Gamma(i \to f) \sim \frac{e^{S_f}}{e^{S_i}}, = e^{\Delta S}. \tag{4}$$

Since Eq. (1) matches the above expression, one can say that the evaporation process is also unitary in nature.

B. Correlation between nonthermal Hawking quanta

Considering the nonthermal emission probability [Eq. (1)], Zhang *et al.* [23] showed that this nonthermal Hawking radiation can carry information out of a black hole in the form of correlation between sequential emissions of quanta. Consider the successive emission of two particles with energies E_1 and E_2 from a black hole of mass M. The emission probability of the first particle with energy E_1 is given by

$$\Gamma(E_1; M) = \exp\left[-8\pi G E_1 \left(M - \frac{E_1}{2}\right)\right]. \tag{5}$$

Now, after the emission of the first particle, the black hole's mass has reduced to $M - E_1$. The emission probability of the second particle of energy E_2 from the black hole of reduced mass is given by

$$\Gamma(E_2; M - E_1) = \exp\left[-8\pi G E_2 \left(M - E_1 - \frac{E_2}{2}\right)\right]. \tag{6}$$

One can see that the emission probability of the second particle depends on the energy of the first particle. This implies the existence of statistical correlations between the two emissions. The joint probability of the two emissions is

$$\begin{aligned}\Gamma(E_1, E_2) &= \Gamma(E_1; M)\Gamma(E_2; M - E_1)\\ &= \exp\left[-8\pi G\left(ME_1 - \frac{E_1^2}{2} + ME_2 - E_1E_2 - \frac{E_2^2}{2}\right)\right]\\ &= \exp\left[-8\pi G(E_1 + E_2)\left(M - \frac{E_1 + E_2}{2}\right)\right]\\ &= \Gamma(E_1 + E_2; M),\end{aligned} \tag{7}$$

where $\Gamma(E_1 + E_2; M)$ is the emission probability of a single particle with energy $E_1 + E_2$ from a black hole of mass M.

The correlation function for the two emissions is defined as [23]

$$\chi(E_1, E_2) = \ln\left(\frac{\Gamma(E_1 + E_2; M)}{\Gamma(E_1; M)\Gamma(E_2; M)}\right), \tag{8}$$

where the numerator is the probability of the emission of two particles with energy E_1 and E_2 *simultaneously*, or of a single particle with total energy $E_1 + E_2$, from a black hole of mass M, and the denominator is the product of the probabilities of emission of particles of energies E_1 and E_2, each occurring *independently* from a black hole of the same mass. Since each particle is emitted independently, their emission probabilities do not depend on the energy of the other particle. Therefore,

$$\begin{aligned}\Gamma(E_1; M) &= \exp\left[-8\pi G E_2\left(M - \frac{E_1}{2}\right)\right],\\ \Gamma(E_2; M) &= \exp\left[-8\pi G E_2\left(M - \frac{E_2}{2}\right)\right],\\ \text{and}\quad \chi(E_1, E_2) &= 8\pi G E_1 E_2.\end{aligned} \tag{9}$$

So, there exists a nonzero correlation between the two emissions. Zhang *et al.* argued that this implies that radiation can carry information out of the black hole in the form of correlations between sequential emissions. Defining the entropy of the emitted particles as [23]

$$S(E) = -\ln \Gamma(E), \tag{10}$$

we get

$$\begin{aligned}\chi(E_1, E_2) &= \ln\Gamma(E_1 + E_2; M) - \ln\Gamma(E_1; M) \\ &\quad - \ln\Gamma(E_2; M)] \\ &= S(E_1) + S(E_2) - S(E_1, E_2) \\ &= I(E_1 : E_2), \end{aligned} \tag{11}$$

which is simply the mutual information shared between the two particles. So, the statistical correlation between the two emissions is shared in the form of mutual information.

C. Entropy conservation by nonthermal radiation

Consider the following scenario: A black hole of mass M is evaporated completely by the emission of two particles with energies E_1 and E_2. Therefore, $E_1 + E_2 = M$. Now, from Eq. (10), the entropy of the first particle is

$$S(E_1) = -\ln\Gamma(E_1; M) = 8\pi G E_1\left(M - \frac{E_1}{2}\right), \tag{12}$$

and the entropy of the second particle is

$$\begin{aligned} S(E_2|E_1) &= -\ln\Gamma(E_2; M - E_1) \\ &= 8\pi G E_2\left(M - E_1 - \frac{E_2}{2}\right), \end{aligned} \tag{13}$$

where $S(E_2|E_1)$ is the conditional entropy of the second particle, provided that the first particle is emitted with energy E_1. Since the emission probability of the second particle is dependent (conditional) on the first particle, the entropy of the second particle is also conditional. Therefore, the total entropy of the system is

$$S_{\text{rad}} = S(E_1) + S(E_2|E_1) = 4\pi G M^2 = S_{\text{BH}}. \tag{14}$$

So, the total entropy of the emitted particles, after complete evaporation of the black hole, is the same as the initial Bekenstein-Hawking entropy of the black hole.

This holds, in general, for any number of particles exhausting the black hole—i.e., if the black hole is evaporated completely by the emission of n particles, then

$$\begin{aligned} S_{\text{rad}} &= S(E_1) + S(E_2|E_1) + \cdots + S(E_n|E_1, E_2, \ldots, E_{(n-1)}) \\ &= 4\pi G M^2 = S_{\text{BH}}, \end{aligned} \tag{15}$$

where $E_1 + E_2 + E_3 + \cdots + E_n = M$. Note that entropy conservation holds true independent of the individual energies of the emitted particles, and hence the sequence of emissions also.

Zhang *et al.*'s work on correlation and entropy conservation by nonthermal Hawking radiation has also been supported by Ref. [32]. The correlation and conservation of entropy has also been studied for the case where the quantum gravity correction is taken [33,34]. It has been shown that for quantum-gravity-corrected emission probability, the logarithmic corrected entropy of the black hole is conserved only when there is a black hole remnant. That means that the black hole cannot be evaporated completely if the quantum gravity correction is taken into account.

III. PROBABILITY DISTRIBUTION

A. Probability of complete black hole evaporation by emission of n particles

We have seen that, for nonthermal emissions, entropy is conserved irrespective of the number of particles emitted and the energies of the individual particles. However, it is not known for certain how many particles will be emitted before the black hole is completely evaporated. That is, the black hole can be completely evaporated by emitting a single particle, or infinitely many particles. We wish to determine the probability (q_n) that a black hole of some given mass M is completely evaporated by the emission of a given number of particles n. For this, first we have to know in how many ways (Ω_n) the black hole evaporates completely by the emission of n particles. We will clarify what these different ways correspond to in a moment. Also, we have to know the total number of ways (Ω_{total}) in which the black hole evaporates, where

$$\Omega_{\text{total}} = \sum_{n=1}^{\infty} \Omega_n. \tag{16}$$

Then, we can define the probability as

$$q_n = \frac{\Omega_n}{\Omega_{\text{total}}}. \tag{17}$$

To determine Ω_n, we consider the system of n emitted particles of total energy M (mass of the initial black hole), obtained after complete evaporation of the black hole. If this system can be obtained in Ω_n different possible ways, we may say that Ω_n is the number of microstates of the system corresponding to the macrostate defined by total energy M. Furthermore, since entropy is conserved irrespective of the individual energies and the number of emitted particles, the total entropy of this system is S_{BH} for all the Ω_n possible microstates. Therefore, the macrostate of this system can be defined by macroscopic properties like total energy, total entropy, and the number of particles (M, S_{BH}, n). Note that we do not have a well-defined volume of the system.

Having defined the macrostate of the system, now we need to define precisely what we mean by the microstates of the system—or, in other words, what the different possible ways of black hole evaporation by n particles correspond to. Since the total energy of the system is a macroscopic property which is same for all the microstates, one may consider that different possible partitions of the total energy among the individual particles form the microstates of the system. This is typical for microcanonical systems. But the system that we have considered here is different, in the sense that (i) the volume of the system does not define the macrostate, and (ii) the entropy of the system defines the macrostate of the system. Furthermore, one can show that the conservation of entropy naturally leads to the conservation of energy in the evaporation process. That is,

$$\sum_{i=1}^{n} S_i = S_{\mathrm{BH}} \Rightarrow \sum_{i=1}^{n} E_i = M, \tag{18}$$

where S_i and E_i are the entropy and energy of the ith emitted particle, respectively. Here, by conservation of energy we mean that the sum of the individual energies of the emitted particles equals the mass of the initial black hole. However, the reverse cannot be shown. So, conservation of entropy is more fundamental for our system, and we are essentially left with only two independent macroscopic properties (S_{BH}, n) that define the macrostate of the system. For this reason, we define the microstates of the system to be the different possible partitions of the total entropy S_{BH} among the individual particles. That is, every possible set of individual entropies,

$$\left\{ S_i, i = 1, 2, \ldots, n \,\middle|\, \sum_{i=1}^{n} S_i = S_{\mathrm{BH}} \right\}, \tag{19}$$

forms the microstates of the system.

Now, we need to evaluate the number of possible partitions of the total entropy S_{BH} to get Ω_n. This can be done as follows. Consider the *entropy space* formed by entropy of the individual particles. Then, the equation

$$\sum_{i=1}^{n} S_i = S_{\mathrm{BH}} \tag{20}$$

defines an $(n-1)$-dimensional hyperplane in the n-dimensional entropy space. Moreover, since the entropy of any emitted particle is always positive [see Eq. (10)], the microstates that we define lie on the region of this hyperplane which is bounded by the hyperplanes $S_i = 0, \ \forall\, i = 1, 2, \ldots, n$. The entropy space and the microstate hyperplane for three-particle evaporation of the black hole are shown in Fig. 1. Now, the number of microstates can be given by

$$\text{Number of microstates}(\Omega_n) = \frac{\text{Surface area of the microstate hyperplane}}{\text{Area of a unit cell on the microstate hyperplane}} = \frac{\mathcal{S}_{n-1}}{\mathcal{A}_{n-1}}, \tag{21}$$

where the unit cell effectively contains a single microstate.

The surface area of the $(n-1)$-dimensional microstate hyperplane defined by Eq. (20) is given by (see Appendix A)

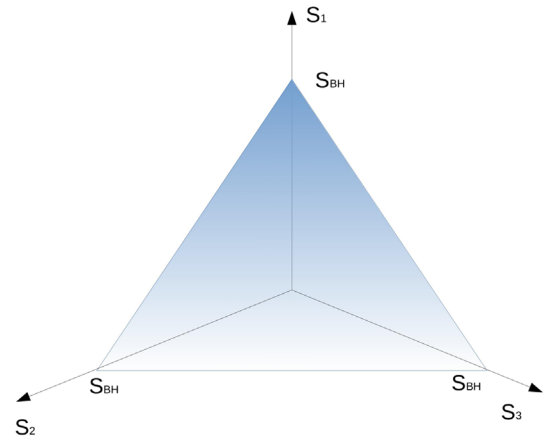

FIG. 1. Three-dimensional entropy space for three-particle evaporation of the black hole. The individual entropies of the emitted particles form the axes in the space. The shaded region is the microstate hyperplane.

$$\mathcal{S}_{n-1} = \sqrt{n}\,\frac{S_{\mathrm{BH}}^{n-1}}{(n-1)!}, \tag{22}$$

and the area of the unit cell on the microstate hyperplane is (see Appendix A)

$$\mathcal{A}_{n-1} = \sqrt{n}. \tag{23}$$

Therefore,

$$\Omega_n = \frac{\sqrt{n}\,\frac{S_{\mathrm{BH}}^{n-1}}{(n-1)!}}{\sqrt{n}} = \frac{S_{\mathrm{BH}}^{n-1}}{(n-1)!} = \frac{(4\pi G M^2)^{n-1}}{(n-1)!}. \tag{24}$$

This gives us the number of different possible ways in which the black hole can be evaporated by the emission of n particles. Now, the total number of ways in which the black hole can evaporate is

$$\Omega_{\mathrm{total}} = \sum_{n=1}^{\infty} \Omega_n = \sum_{n=1}^{\infty} \frac{(4\pi G M^2)^{n-1}}{(n-1)!} = e^{4\pi G M^2}. \tag{25}$$

Therefore, the probability that the black hole is completely evaporated by the emission of n particles is given by

$$q_n = \frac{\Omega_n}{\Omega_{\mathrm{total}}} = \frac{(4\pi G M^2)^{n-1}}{(n-1)!} e^{-4\pi G M^2}. \tag{26}$$

It is easy to verify that the probabilities add up to unity:

$$\sum_{n=1}^{\infty} q_n = e^{-4\pi G M^2} \sum_{n=1}^{\infty} \frac{(4\pi G M^2)^{n-1}}{(n-1)!} = e^{-4\pi G M^2} \times e^{4\pi G M^2} = 1. \tag{27}$$

That is, the probabilities q_n are normalized.

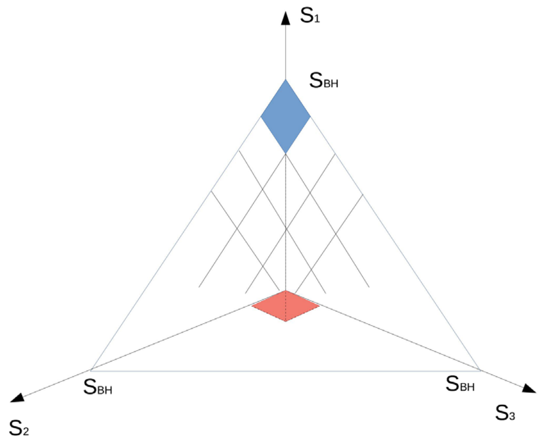

FIG. 2. Unit cell on the microstate hyperplane is shown by the blue shaded region. It projects a unit square (two-dimensional cube) on the space spanned by S_2 and S_3 (red shaded region). The area of the unit cell is $\mathcal{A}_2 = \sqrt{3}$.

B. Displacement relation

If we plot the probabilities q_n as a function of the numbers of emitted particles n from Eq. (26), we get the probability distribution for the black hole to be evaporated completely by the emission of different possible numbers of particles. The probability distribution is parametrized by the entropy of the black hole ($S_{\rm BH}$).

From Fig. 3, we see that for a black hole of given mass, there exists a certain number of particles, $n_{\rm max}$, for which the probability distribution peaks. That is, the black hole is most likely to evaporate completely by the emission of $n_{\rm max}$ particles. We further see that, as the mass, and hence the entropy, of the black hole increases, the peak of the distribution decreases and shifts towards higher values of $n_{\rm max}$. For a black hole of given mass, $n_{\rm max}$ has the nearest integer value between $S_{\rm BH}$ and $S_{\rm BH}+1$ (see Appendix B):

$$S_{\rm BH} \le n_{\rm max} \le S_{\rm BH} + 1, \tag{28}$$

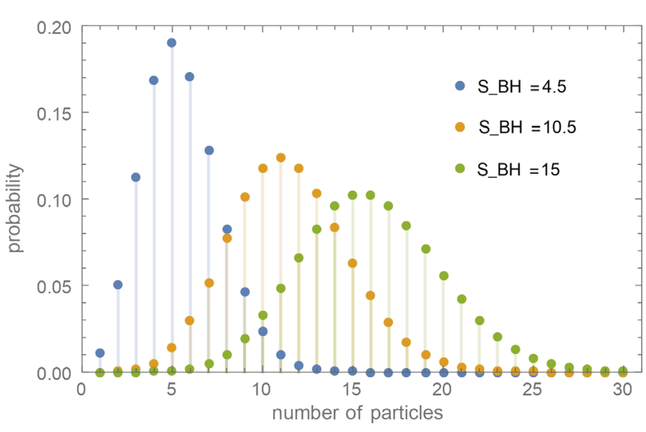

FIG. 3. Probability distribution for black holes of different masses to be evaporated by the emission of different possible numbers of particles.

$$\text{i.e,} \qquad n_{\rm max} \approx S_{\rm BH} = 4\pi G M^2 = \frac{1}{16\pi G T_{\rm BH}^2}, \tag{29}$$

where $T_{\rm BH}$ is the Hawking temperature of the initial black hole. In other words,

$$n_{\rm max} T_{\rm BH}^2 = \frac{1}{16\pi G} = \text{constant}. \tag{30}$$

Equation (30) relates the most probable number of emitted particles with the temperature of the initial black hole. This relation resembles Wien's displacement law for blackbody radiation. According to Wien's displacement law, the wavelength of blackbody radiation for which the spectral energy density is maximum, $\lambda_{\rm max}$, is inversely proportional to the temperature of the blackbody,

$$\lambda_{\rm max} T = 2.898 \times 10^{-3}\ \text{mK} \quad (\text{not in natural units}) \\ = \text{constant}. \tag{31}$$

C. Entropy of the system

We have seen how the conservation of entropy by the nonthermal emission of particles from black holes plays an important role in determining the possible number of ways in which a black hole can evaporate completely by emitting a given number of particles. Before concluding, let us delve a little more into the entropy of the radiation system, or the system of emitted particles after complete evaporation of the black hole. In information theory, the entropy of a system is considered as a "measure of uncertainty" of the system. If a system can exist in multiple possible states, then there is an uncertainty about the state in which the system is. For each possible state, there is a probability which measures the likeliness of the system to exist in that state. Given a probability distribution for the system, we can quantify the uncertainty in terms of Shannon entropy as

$$S = -\sum_i p_i \ln p_i, \tag{32}$$

where p_i is the probability of the ith state of the system, and the sum is over all possible states.

In our case, the system of particles obtained after the complete evaporation of the black hole has twofold uncertainties: (i) how many particles have been emitted is not predetermined, and (ii) if it is given that the number of emitted particles is known, the particular microstate in which the black hole evaporated is not known. Both of these uncertainties are there in the total entropy of the radiation system. This is illustrated below.

The probability that the black hole will emit n particles before complete evaporation is given by Eq. (26). So, the first uncertainty is represented by a form of entropy denoted by $S(\text{number})$ as

$$
\begin{aligned}
S(\text{number}) &= -\sum_{n=1}^{\infty} q_n \ln q_n \\
&= \sum_{n=1}^{\infty} \frac{\Omega_n}{\Omega_{\text{total}}} \ln \Omega_{\text{total}} - \sum_{n=1}^{\infty} \frac{\Omega_n}{\Omega_{\text{total}}} \ln \Omega_n \\
&= \frac{\ln \Omega_{\text{total}}}{\Omega_{\text{total}}} \sum_{n=1}^{\infty} \Omega_n - \frac{1}{\Omega_{\text{total}}} \sum_{n=1}^{\infty} \Omega_n \ln \Omega_n \\
&= \ln \Omega_{\text{total}} - \frac{\sum_{n=1}^{\infty} \Omega_n \ln \Omega_n}{\sum_{n=1}^{\infty} \Omega_n}. \qquad (33)
\end{aligned}
$$

Now, consider that the black hole is evaporated by the emission of n particles. The first particle is emitted with energy E_1, the second particle with energy E_2, and the nth particle with energy E_n. The joint probability for this sequence of emission is

$$
\begin{aligned}
P &= \Gamma(E_1; M) \times \Gamma(E_2; M - E_1) \times \cdots \times \Gamma\left(E_n; M - \sum_{j=1}^{n-1} E_j\right) \\
&= \exp(-4\pi G M^2). \qquad (34)
\end{aligned}
$$

It is readily seen that this probability holds true for any value of n, and for any partition of the total energy M among the emitted particles. That is to say, all Ω_{total} ways of black hole evaporation occur with this same probability. Since all Ω_{total} possibilities occur with equal probability, it easily follows that

$$
P = \frac{1}{\Omega_{\text{total}}} = e^{-4\pi G M^2}, \qquad (35)
$$

reconfirming Eq. (25). Again, the probability of occurrence of one of the Ω_{total} possibilities of black hole evaporation can also be written as

$$
\begin{aligned}
P &= \text{Probability that the black hole has emitted} \\
&\quad n \text{ particles} \times \text{Probability of occurrence of} \\
&\quad \text{one among } \Omega_n \text{ microstates} \\
&= q_n \times P_\alpha(\text{microstate}|\text{number}), \qquad (36)
\end{aligned}
$$

where $P_\alpha(\text{microstate}|\text{number}), \alpha = 1, 2, \ldots, \Omega_n$ is the conditional probability of the occurrence of the αth microstate among the Ω_n possible microstates provided that the black hole has been evaporated by the emission of n particles [35]. Therefore,

$$
\begin{aligned}
\frac{1}{\Omega_{\text{total}}} &= \frac{\Omega_n}{\Omega_{\text{total}}} \times P_\alpha(\text{microstate}|\text{number}), \\
P_\alpha(\text{microstate}|\text{number}) &= \frac{1}{\Omega_n}. \qquad (37)
\end{aligned}
$$

The second uncertainty about the specific microstate, given the knowledge of the number of particles emitted, is represented by another form of entropy, denoted by $S(\text{microstate}|\text{number})$. For the emission of n particles, this entropy is expressed as

$$
\begin{aligned}
S_n(\text{microstate}|\text{number}) &= -\sum_{\alpha=1}^{\Omega_n} P_\alpha(\text{microstate}|\text{number}) \\
&\quad \times \ln P_\alpha(\text{microstate}|\text{number}) \\
&= \sum_{\alpha=1}^{\Omega_n} \frac{1}{\Omega_n} \ln \Omega_n \\
&= \ln \Omega_n. \qquad (38)
\end{aligned}
$$

Finally, the total uncertainty about the particular way among the Ω_{total} possibilities in which the black hole has evaporated encompasses both of the uncertainties that we discussed earlier. This total uncertainty is represented by the total entropy of the system as

$$
S_{\text{rad}} = -\sum_{k=1}^{\Omega_{\text{total}}} \frac{1}{\Omega_{\text{total}}} \ln \frac{1}{\Omega_{\text{total}}} = \ln \Omega_{\text{total}} = 4\pi G M^2 = S_{\text{BH}}. \qquad (39)
$$

So,

$$
\begin{aligned}
S(\text{number}) &= S_{\text{BH}} - \frac{\sum_{n=1}^{\infty} \Omega_n S_n(\text{microstate}|\text{number})}{\sum_{n=1}^{\infty} \Omega_n}; \\
S_{\text{BH}} &= S(\text{number}) + S_{\text{avg}}(\text{microstate}|\text{number}), \\
&= S_{\text{rad}}, \qquad (40)
\end{aligned}
$$

where $S_{\text{avg}}(\text{microstate}|\text{number})$ is the conditional entropy $S(\text{microstate}|\text{number})$ averaged over all possible numbers of emitted particles. So, we see that the total entropy of the system ($S_{\text{rad}} = S_{\text{BH}}$) contains two parts:

(i) Entropy due to uncertainty in the number of particles emitted ($S(\text{number})$).

(ii) Average entropy due to uncertainty in the microstates of a given number of emitted particles ($S_{\text{avg}}(\text{microstate}|\text{number})$).

IV. SUMMARY AND CONCLUSIONS

We have revisited the process of nonthermal radiation from black holes. The nonthermal correction to Hawking's original calculations of black hole radiation comes from taking into account the backreaction of the emitted particles on the black hole spacetime. Entropy is conserved during the evaporation process of black holes due to nonthermal radiation, irrespective of the number of particles emitted. In this work, we have tried to answer the following question: What is the probability that a black hole of some given mass emits a certain number of particles before being completely evaporated? Conservation of entropy during the evaporation process plays a crucial role in determining the probability. We have found that a black hole of mass M evaporates completely by emitting n particles in $\Omega_n = \frac{(4\pi GM^2)^{n-1}}{(n-1)!}$ different possible ways. These different possible ways correspond to different possible partitionings of the Bekenstein-Hawking entropy of the initial black hole among the emitted particles. If we let the number of emitted particles be arbitrary, then there are in total $\Omega_{\text{total}} = e^{-4\pi GM^2}$ ways in which the black hole can evaporate completely. From these, we find the probability of the emission of n particles before complete black hole evaporation to be $q_n = \frac{(4\pi GM^2)^{n-1}}{(n-1)!} e^{4\pi GM^2}$.

From the probability distribution obtained for different numbers of emitted particles, we find that for black holes of mass M, there is a *most probable number* of emitted particles, $n_{\max} = 4\pi GM^2$. That is, the black hole is most likely to evaporate completely by emitting $n_{\max}$ particles. This implies that, for more massive black holes, a larger number of particles is expected to be emitted before their complete evaporation. We have expressed this conclusion in the form of a displacement relation between the most probable number of particles emitted and the temperature of the initial black hole, $n_{\max} T_{\text{BH}}^2 = \frac{1}{16\pi G} = \text{constant}$. This displacement relation resembles Wien's displacement law for blackbody radiation.

Finally, we have examined the entropy of the system of radiated particles obtained after the complete evaporation of a black hole from a different perspective. As mentioned earlier, this entropy matches the entropy of the initial black hole. When the entropy is interpreted as a measure of uncertainty (or, equivalently, hidden information), then we see that the total entropy of the system contains two parts. One part contains the information about the number of the particles in the system (the number of particles emitted before the complete evaporation of the black hole). The other part contains the information about the particular way in which the black hole has emitted the given number of particles. This interpretation may give a new meaning to black hole entropy.

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APPENDIX A: SURFACE AREA OF THE MICROSTATE HYPERPLANE AND AREA OF A UNIT CELL IN THE MICROSTATE HYPERPLANE

1. Surface area of the microstate hyperplane

In this section, we will calculate the surface area of an $(n-1)$-dimensional hyperplane given by the equation

$$\sum_{i=1}^{n} x_i = C \tag{A1}$$

in the positive sector of the coordinates ($x_i \geq 0$, $\forall\, i = 1, 2, \ldots, n$).

But first, let us calculate the volume of the region bounded by this plane and the planes $x_i = 0$, $\forall\, i = 1, 2, \ldots, n$. The n-dimensional volume is given by

$$\begin{aligned}\mathcal{V}_n &= \int_{x_1=0}^{C} \int_{x_2=0}^{C-x_1} \cdots \int_{x_n=0}^{C-\sum_{i=1}^{n-1} x_i} \prod_{i=1}^{n} dx_i \\ &= \int_{x_1=0}^{C} \cdots \int_{x_{n-1}=0}^{C-\sum_{i=1}^{n-2} x_i} \left(C - \sum_{i=1}^{n-1} x_i\right) \prod_{i=1}^{n-1} dx_i.\end{aligned}$$

Let

$$C - \sum_{i=1}^{n-2} x_i = \alpha. \tag{A2}$$

Therefore,

$$\begin{aligned}\mathcal{V}_n &= \int_{x_1=0}^{C} \cdots \int_{x_{n-1}=0}^{\alpha} (\alpha - x_{n-1}) \prod_{i=1}^{n-1} dx_i \\ &= \frac{1}{2} \int_{x_1=0}^{C} \cdots \int_{x_{n-2}=0}^{C-\sum_{i=1}^{n-3} x_i} \left(C - \sum_{i=1}^{n-2} x_i\right)^2 \prod_{i=1}^{n-2} dx_i.\end{aligned}$$

Again, let

$$C - \sum_{i=1}^{n-3} x_i = \beta. \tag{A3}$$

Therefore,

$$\begin{aligned}\mathcal{V}_n &= \int_{x_1=0}^{C} \cdots \int_{x_{n-2}=0}^{\beta} (\beta - x_{n-2})^2 \prod_{i=1}^{n-2} dx_i \\ &= \frac{1}{2 \times 3} \int_{x_1=0}^{C} \cdots \int_{x_{n-3}=0}^{C-\sum_{i=1}^{n-4} x_i} \left(C - \sum_{i=1}^{n-3} x_i\right)^3 \prod_{i=1}^{n-3} dx_i.\end{aligned}$$

Let

$$C - \sum_{i=1}^{n-4} x_i = \delta. \tag{A4}$$

Therefore,

$$\begin{aligned}\mathcal{V}_n &= \int_{x_1=0}^{C} \cdots \int_{x_{n-3}=0}^{\delta} (\delta - x_{n-3})^3 \prod_{i=1}^{n-3} dx_i \\ &= \frac{1}{4 \times 3!} \int_{x_1=0}^{C} \cdots \int_{x_{n-4}=0}^{C-\sum_{i=1}^{n-5} x_i} \left(C - \sum_{i=1}^{n-4} x_i\right)^4 \prod_{i=1}^{n-4} dx_i.\end{aligned}$$

Proceeding in similar way, we get

$$\mathcal{V}_n = \frac{1}{(n-2)!}\int_{x_1=0}^{C}\int_{x_2=0}^{C-x_1}(C-x_1-x_2)^{n-2}dx_2dx_1$$
$$= \frac{1}{(n-1)!}\int_{x_1=0}^{C}(C-x_1)^{n-1}dx_1$$
$$= \frac{1}{n!}C^n. \tag{A5}$$

Now, we turn back to calculate the surface area of the hyperplane represented by Eq. (A1). In general, the surface area of any $(n-1)$-dimensional surface embedded in n dimensions, represented by

$$x_n = \phi(x_1, x_2, \ldots, x_{n-1}), \tag{A6}$$

can be written as

$$\mathcal{S}_{n-1} = \int_{\mathcal{D}}\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial\phi}{\partial x_i}\right)^2}\prod_{i=1}^{n-1}dx_i, \tag{A7}$$

where $\mathcal{D}$ is the projection of the hyperplane on the space spanned by $(x_1, x_2, \ldots, x_{n-1})$.

For our case, the equation of the hyperplane can be written as

$$x_n = \phi = C - \sum_{i=1}^{n-1}x_i. \tag{A8}$$

So,

$$\frac{\partial\phi}{\partial x_i} = -1, \quad \forall \ \ i = 1, 2, \ldots, n-1.$$

Therefore,

$$\mathcal{S}_{n-1} = \int_{\mathcal{D}}\sqrt{n}\prod_{i=1}^{n-1}dx_i. \tag{A9}$$

Here, the projected region $\mathcal{D}$ on the space spanned by $(x_1, x_2, \ldots, x_{n-1})$ is bounded by $(n-2)$-dimensional surfaces represented by the equations $x_i = 0$, $\forall\, i = 1, 2, \ldots, n-1$ (since we are only concerned about the positive sector of the space), and $C - \sum_{i=1}^{n-1} x_i = 0$. So, the term $\int_{\mathcal{D}}\prod_{i=1}^{n-1} dx_i$ is simply the volume of the $(n-1)$-dimensional space bounded by the said $(n-2)$-dimensional surfaces. Now, the surface area can be written as

$$\mathcal{S}_{n-1} = \sqrt{n}\int_{x_1=0}^{C}\int_{x_2=0}^{C-x_1}\cdots\int_{x_{n-1}=0}^{C-\sum_{i=1}^{n-2}x_i}\prod_{i=1}^{n-1}dx_i$$
$$= \sqrt{n}\mathcal{V}_{n-1} = \sqrt{n}\frac{C^{n-1}}{(n-1)!}. \tag{A10}$$

This gives the surface area of the hyperplane. The area of the microstate plane, whose equation is given by $\sum_{i=1}^{n} S_i = S_{\mathrm{BH}}$, can be calculated using this formula in a straightforward way.

2. Area of the unit cell

The area of a unit cell in the microstate hyperplane can be calculated using Eq. (A9), where the region $\mathcal{D}$ is the projection of the unit cell on the $(n-1)$-dimensional space spanned by $(S_1, S_2, \ldots, S_{n-1})$. The projected region $\mathcal{D}$ forms a $(n-1)$-dimensional unit cube, then the area of the unit cell in the microstate hyperplane is given by

$$\mathcal{A}_{n-1} = \sqrt{n}. \tag{A11}$$

The unit cell in the microstate hyperplane and the projected region $\mathcal{D}$ are shown in Fig. 2 for the case when the black hole is evaporated by three particles.

APPENDIX B: VALUE OF THE MOST PROBABLE NUMBER OF EMITTED PARTICLES ($n_{\max}$) FROM THE DISTRIBUTION FUNCTION

In this section, we will evaluate the most probable number of emitted particles ($n_{\max}$) analytically from the probability distribution function. We have the probability distribution function as

$$q_n = e^{-S_{\mathrm{BH}}}\frac{S_{\mathrm{BH}}^{n-1}}{(n-1)!}. \tag{B1}$$

For the most probable number of emitted particles,

$$q_{n_{\max}} = \text{maximum}.$$

Now, q_n is a function of discrete variable. To obtain the maxima for this function, the following two conditions have to be satisfied simultaneously:

$$q_{(n_{\max}+1)} - q_{(n_{\max})} \le 0,$$
$$q_{(n_{\max})} - q_{(n_{\max}-1)} \ge 0. \tag{B2}$$

Therefore,

$$e^{-S_{\mathrm{BH}}}\left[\frac{S_{\mathrm{BH}}^{(n_{\max})}}{(n_{\max})!} - \frac{S_{\mathrm{BH}}^{(n_{\max}-1)}}{(n_{\max}-1)!}\right] \le 0$$
$$\frac{S_{\mathrm{BH}}^{(n_{\max})}}{(n_{\max})!} - \frac{n_{\max}S_{\mathrm{BH}}^{(n_{\max}-1)}}{(n_{\max})!} \le 0$$
$$S_{\mathrm{BH}} \le n_{\max}, \tag{B3}$$

and

$$e^{-S_{BH}}\left[\frac{S_{BH}^{(n_{max}-1)}}{(n_{max}-1)!}-\frac{S_{BH}^{(n_{max}-2)}}{(n_{max}-2)!}\right]\geq 0$$
$$\frac{S_{BH}^{(n_{max}-1)}}{(n_{max}-1)!}-\frac{(n_{max}-1)S_{BH}^{(n_{max}-2)}}{(n_{max}-1)!}\geq 0$$
$$S_{BH}+1\geq n_{max}. \quad \text{(B4)}$$

So, for the given probability distribution, n_{max} has the integer value in the range

$$S_{BH}\leq n_{max}\leq S_{BH}+1. \quad \text{(B5)}$$

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