Conditions for superradiant instability of the Kerr-Newman black holes

Yun Soo Myung[*]

Institute of Basic Sciences and Department of Computer Simulation, Inje University, Gimhae 50834, Korea and Asia Pacific Center for Theoretical Physics, Pohang 37673, Korea

(Received 5 May 2022; accepted 25 May 2022; published 8 June 2022)

We find two conditions for superradiant instability of Kerr-Newman black holes under a charged massive scalar perturbation by analyzing the asymptotic scalar potential and far-region wave function. Actually, they correspond to the condition for getting a trapping well. Also, we obtain the conditions for superradiant stability of Kerr-Newman black holes which states that there is no trapping well. The analysis is applied to Kerr black holes to find a condition for superradiant instability.

DOI: 10.1103/PhysRevD.105.124015

I. INTRODUCTION

If a light boson exists with proper mass, gravitational bound states are formed around rotating charged black holes. These bound states could continuously extract electromagnetic or rotational energy from black holes [1]. This is the superradiance phenomenon in black holes [2]. The existence of superradiant modes can be converted into an instability of the black hole background if a mirror mechanism to trap these modes is installed near the black hole. This is called superradiant instability. If a scalar has a mass μ, its mass would act as a reflecting mirror [3]. The superradiant instability of the Kerr black hole was found for $M\mu \gg 1$ [4], $M\mu \ll 1$ [5], and $M\mu \leq 0.5$ [6]. A scalar potential including a trapping well is essential for generating a quasibound state [1,4,7] whose wave function is peaked far outside the ergoregion. If there is no trapping well in the potential, it may correspond to superradiant stability with a bound state.

In a Kerr-Newman black hole (KNBH) with mass M, charge Q, and angular momentum J, the superradiant instability condition for a charged massive scalar with mass μ and charge q was first obtained as $qQ < \mu M$ which may be a condition for a trapping well [8]. However, $qQ < \mu M$ is not satisfied simultaneously if one imposes the superradiance condition ($\omega < \omega_c$ with $\omega_c = m\Omega_H + q\Phi_H$) and thus, it is a condition for bound states [9]. In other words, the condition of $qQ < \mu M$ corresponds to the Newton-Coulomb requirement for the gravitational force to exceed the electrostatic force. Also, it is noted that their effective potential $V_{\rm eff}$ is not a correct form. Scalar clouds with $\omega = \omega_c$ and $\omega < \mu$ were obtained in Refs. [10,11]. The absorption cross section was recently computed to give a negative one for corotating spherical waves [12].

On the other hand, there was an approach to analyzing superradiant instability based on the scalar potentials [13]. We stress that the appearance/disappearance of a trapping well is a decisive condition for superradiant instability/stability. Recently, it was shown that the superradiant stability of KNBHs under a charged massive scalar perturbation can be achieved if $qQ > \mu M$ and $r_-/r_+ \leq 1/3$ are satisfied [14], in addition to $\omega < \omega_c$ and the bound state condition ($\omega < \mu$). However, their potential based on the analysis is incorrect because $\Psi_{\ell m} = \sqrt{\Delta}R_{\ell m}$ is used and a tortoise coordinate defined by $dr_* = (r^2 + a^2)dr/\Delta$ is not used [15]. So, it suggests that $qQ > \mu M$ is not a condition for superradiant stability. The superradiant stability of a charged massive scalar based on a desirable potential was discussed in the KNBH background [16].

In this work, we wish to find two conditions for getting a trapping well of a KNBH under a charged massive scalar perturbation by analyzing the asymptotic scalar potential $V_{aaKN}(r)$ and far-region wave function $U[p, s; cr]$. They are given by $V'_{aaKN}(r) > 0 (M\mu^2 > qQ\omega)$ and $U'[p, s; cr] > 0$ ($p < 0$). In addition, the conditions for no trapping well are $V'_{aaKN}(r) < 0$ ($M\mu^2 < qQ\omega$) and $U'[p, s; cr] < 0$ ($p > 0$). We apply the same analysis to a Kerr black hole under a massive scalar propagation to find the condition for a trapping well.

II. POTENTIALS AROUND KNBHs

Let us introduce the Boyer-Lindquist coordinates to represent a KNBH with mass M, charge Q, and angular momentum J

$$ds^2_{\rm KN} = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - adt]^2 \quad (1)$$

[*]ysmyung@inje.ac.kr

with

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \quad \text{and} \quad a = \frac{J}{M}. \tag{2}$$

We choose the electromagnetic potential as

$$\bar{A}_\mu = \frac{Qr}{\rho^2}(-1, 0, 0, a\sin^2\theta). \tag{3}$$

The outer and inner horizons are obtained from $\Delta = (r - r_+)(r - r_-) = 0$ $(\bar{g}^{rr} = 0)$ as

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{4}$$

One describes a charged massive scalar perturbation Φ on the background of KNBHs by adapting the perturbed scalar equation

$$(\bar{\nabla}^\mu - iq\bar{A}^\mu)(\bar{\nabla}_\mu - iq\bar{A}_\mu)^*\Phi - \mu^2\Phi = 0. \tag{5}$$

Considering the static and axisymmetric background (1), it is plausible to separate the scalar perturbation into modes

$$\Phi(t, r, \theta, \phi) = \Sigma_{lm} e^{-i\omega t + im\phi} S_{lm}(\theta) R_{lm}(r), \tag{6}$$

where $S_{\ell m}(\theta)$ are the spheroidal harmonics with $-m \le \ell \le m$ and $R_{lm}(r)$ describes the radial part of the wave function. Substituting Eq. (6) into Eq. (5), we have the angular equation for $S_{lm}(\theta)$ and the Teukolsky equation as [10]

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta S_{\ell m}(\theta)) + \left[\lambda_{lm} + a^2(\mu^2 - \omega^2)\sin^2\theta - \frac{m^2}{\sin^2\theta}\right]S_{lm}(\theta) = 0, \tag{7}$$

$$\Delta\partial_r(\Delta\partial_r R_{\ell m}(r)) + U(r)R_{lm}(r) = 0, \tag{8}$$

where

$$U(r) = [\omega(r^2 + a^2) - am - qQr]^2 + \Delta[2am\omega - \mu^2(r^2 + a^2) - \lambda_{lm}]. \tag{9}$$

We note that Eq. (8) could be used directly for computing the absorption cross section, quasinormal modes of the scalar, and scalar clouds. At this stage, we introduce the tortoise coordinate r_* defined by $dr_* = \frac{r^2+a^2}{\Delta}dr$ to derive the Schrödinger-type equation. In this case, an interesting region of $r \in [r_+, \infty)$ could be mapped into the whole region of $r_* \in (-\infty, \infty)$. Then, the radial equation (8) takes the Schrödinger form when setting $\Psi_{lm} = \sqrt{a^2 + r^2}R_{lm}$

$$\frac{d^2\Psi_{lm}(r_*)}{dr_*^2} + [\omega^2 - V_{KN}(r)]\Psi_{lm}(r_*) = 0, \tag{10}$$

where the potential $V_{KN}(r)$ is found to be [11]

$$V_{KN}(r) = \omega^2 - \frac{3\Delta^2 r^2}{(a^2 + r^2)^4} + \frac{\Delta[\Delta + 2r(r - M)]}{(a^2 + r^2)^3} + \frac{\Delta\mu^2}{a^2 + r^2} - \left[\omega - \frac{am}{a^2 + r^2} - \frac{qQr}{a^2 + r^2}\right]^2 - \frac{\Delta}{(a^2 + r^2)^2}[2am\omega - \lambda_{lm}]. \tag{11}$$

Replacing λ_{lm} by $\tilde{\lambda}_{lm} + a^2(\omega^2 - \mu^2)$ with $\tilde{\lambda}_{lm} = l(l+1) + \cdots$, one finds a familiar angular equation

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta S_{\ell m}(\theta)) + \left[\tilde{\lambda}_{lm} + a^2(\omega^2 - \mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta}\right]S_{lm}(\theta) = 0. \tag{12}$$

Before we proceed, we would like to mention the superradiant scattering of the scalar of the KNBHs. We find two limits such that $V_{KN}(r \to \infty) \to \mu^2$ and $V_{KN}(r \to r_+) \to \omega^2 - (\omega - \omega_c)^2$. In this case, we have plane waves as scattering waves [2]

$$\Psi_{lm} \sim e^{-i\sqrt{\omega^2-\mu^2}r_*}(\leftarrow) + \mathcal{R}e^{+i\sqrt{\omega^2-\mu^2}r_*}(\rightarrow), \quad r_* \to +\infty(r \to \infty), \tag{13}$$

$$\Psi_{lm} \sim \mathcal{T}e^{-i(\omega-\omega_c)r_*}(\leftarrow), \qquad r_* \to -\infty(r \to r_+) \tag{14}$$

where $\mathcal{T}(\mathcal{R})$ are the transmission (reflection) amplitudes. The Wronskian $W(\Psi, \Psi^*)$ condition of $i\frac{d}{dr_*}W(\Psi, \Psi^*) = 0$ leads to

$$|\mathcal{R}|^2 = 1 - \frac{\omega - \omega_c}{\sqrt{\omega^2 - \mu^2}}|\mathcal{T}|^2, \tag{15}$$

which implies that outgoing waves with $\omega > \mu$ propagate to infinity and superradiant scattering occurs ($|\mathcal{R}|^2 > |\mathcal{I}|^2$) whenever the superradiance condition is satisfied

$$\omega < \omega_c. \tag{16}$$

Curiously, superradiance is associated with having a negative absorption cross section [12]. For a KNBH, the total absorption cross section becomes negative for corotating spherical waves at low frequencies.

Now, we wish to briefly describe superradiant instability. The two boundary conditions imply an exponentially decaying wave (bound state) away from the trapping well and a purely outgoing wave (superradiance) near the outer horizon under Eq. (16):

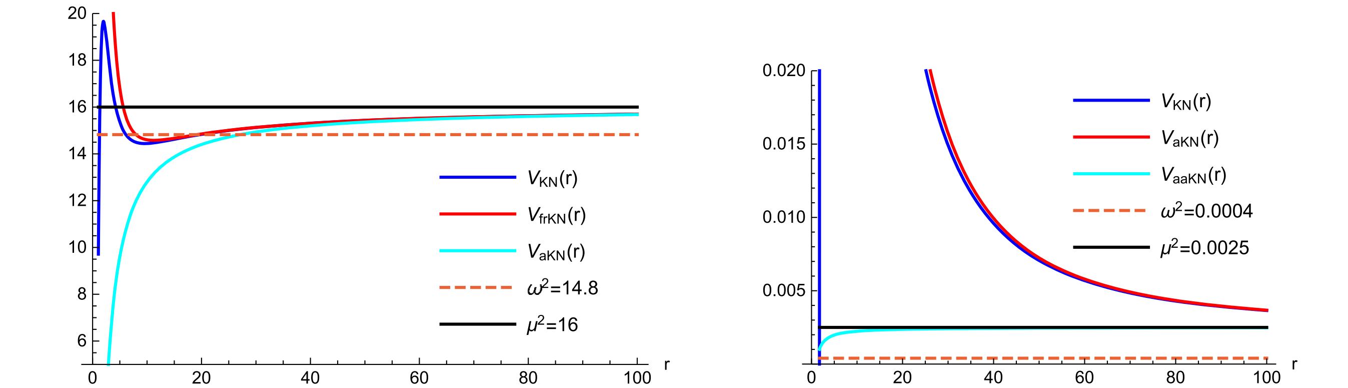

FIG. 1. Left: potential with trapping well $V_{KN}(r)$, its far-region $V_{aKN}(r)$, and asymptotic potential $V_{aaKN}(r)$ as functions of $r \in [r_+ = 1.06, 100]$ with $M = 1$, $Q = 0.01$, $\omega = 3.85$, $a = 0.998$, $m = 13$, $q = 0.2$, $\lambda_{lm} = 180$, and $\mu = 4$. $V_{KN}(r)$ has a trapping well located at $r = 9.61$. We check $\omega < \omega_c$ $(= 6.11)$ and $\omega < \mu$ as two conditions for superradiant instability. Right: potential without trapping well $V_{KN}(r)$ as a function of $r \in [r_+ = 1.741, 100]$ with $M = 1$, $Q = 0.6$, $\omega = 0.02$, $a = 0.3$, $m = 1$, $q = 0.1$, $\lambda_{lm} = 12$, and $\mu = 0.05$. We check the two conditions $\omega < \omega_c(= 0.13)$ and $\omega < \mu$.

$$\Psi \sim e^{-\sqrt{\mu^2-\omega^2}r}, \qquad r_* \to \infty (r \to \infty), \tag{17}$$

$$\Psi \sim e^{-i(\omega-\omega_c)r_*}, \qquad r_* \to -\infty (r \to r_+). \tag{18}$$

From Eq. (17), one needs the bound state condition to obtain an exponentially decaying mode

$$\omega < \mu. \tag{19}$$

Furthermore, superradiant instability/stability is determined by the shape of the potential, in addition to Eqs. (16) and (19). Importantly, a key condition for superradiant instability is to include a positive trapping well in the potential. If there is no trapping well in the potential, it implies superradiant stability.

To find out the condition of a trapping well, we have to consider the potential $V_{aKN}(r)$ obtained when expanding $V_{KN}(r)$ in the far region

$$V_{aKN}(r) = \mu^2 - \frac{2(M\mu^2 - qQ\omega)}{r} + \frac{\lambda_{lm} + Q^2(\mu^2 - q^2)}{r^2}. \tag{20}$$

Its first derivative is

$$V'_{aKN}(r) = \frac{2(M\mu^2 - qQ\omega)}{r^2} - \frac{2[\lambda_{lm} + Q^2(\mu^2 - q^2)]}{r^3}, \tag{21}$$

where the mass term μ^2 disappeared. We note that there are no restrictions on parameters in deriving $V_{aKN}(r)$. Here, one is tempted to say that the condition for a trapping well [no trapping well] is given by $V'_{aKN}(r) > 0$ [$V'_{aKN}(r) < 0$]. Nevertheless, it is difficult to find any analytic condition from $V'_{aKN}(r) > 0$ ($V'_{aKN}(r) < 0$). To this end, we consider the asymptotic potential $V_{aaKN}(r)$ obtained when expanding $V_{KN}(r)$ at $r = \infty$ and its first derivative

$$V_{aaKN}(r) = \mu^2 - \frac{2(M\mu^2 - qQ\omega)}{r},$$
$$V'_{aaKN}(r) = \frac{2(M\mu^2 - qQ\omega)}{r^2}. \tag{22}$$

Here, one requires $V'_{aaKN}(r) > 0$ $(M\mu^2 > qQ\omega)$ for a trapping well, whereas $V'_{aaKN}(r) > 0$ $(M\mu^2 < qQ\omega)$ is required for no trapping well. However, this is not a sufficient condition to get a trapping well. In the next section, we will find the other condition by analyzing the far-region scalar function. The other condition for a trapping well (no trapping well) will be given by $U'[p, s; cr] > 0$ $(U'[p, s; cr] < 0)$ where $U[p, s, cr]$ is the confluent hypergeometric function.

At this stage, we display two potentials: one (left panel of Fig. 1) includes a trapping well and the other (right panel of

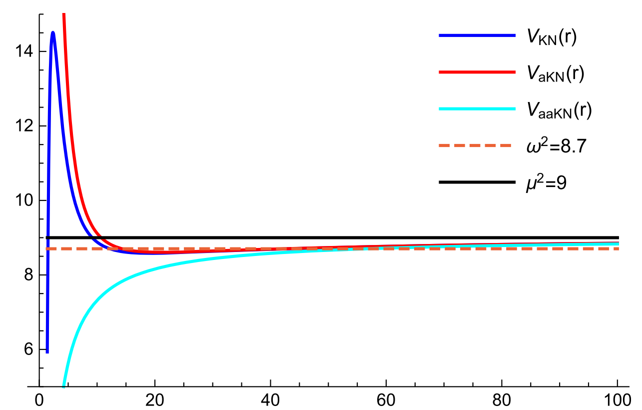

FIG. 2. Left: potential with trapping well $V_{KN}(r)$, its far-region $V_{aKN}(r)$, and asymptotic potential $V_{aaKN}(r)$ as functions of $r \in [r_+ = 1.39, 100]$ with $M = 1$, $Q = 0.01$, $\omega = 2.95$, $a = 0.9$, $m = 13$, $q = 20$, $\lambda_{lm} = 180$, and $\mu = 3$. Here, we have $q > \mu$, which shows a feature of KNBH. $V_{KN}(r)$ has a trapping well located at $r = 19.3$. We check that the two conditions $\omega < \omega_c(= 4.38)$ and $\omega < \mu$ are satisfied for superradiant instability.

Fig. 1) does not include a trapping well, but it matches up with $M\mu^2 > qQ\omega$. Figure 2 indicates a potential with a trapping well for $q > \mu$ and $M\mu^2 > qQ\omega$, which shows a feature of a charged massive scalar propagating around the KNBH. It is worth noting that all have $V'_{aaKN}(r) > 0$. One needs to find the other condition to have no trapping well.

Finally, we describe two conditions for superradiant instability and stability under a charged massive scalar propagating around the KNBHs:

(i) Superradiant instability → $\omega < \omega_c$ and $\omega < \mu$ with a positive trapping well.
(ii) Superradiant stability → $\omega < \omega_c$ and $\omega < \mu$ without a positive trapping well.

III. CONDITION FOR TRAPPING WELL

One needs to observe far-region scalar functions to distinguish between potential with trapping well and potential without trapping well. For this purpose, we wish to derive a scalar equation in the far region.

In the far region where one takes $r_* \sim r$, we obtain an equation from Eqs. (10) and (20) as

$$\left[\frac{d^2}{dr^2} + \omega^2 - V_{aKN}(r)\right]\Psi_{lm}(r) = 0. \tag{23}$$

The above equation could be rewritten as

$$\left[\frac{d^2}{dr^2} - A^2 + \frac{B}{r} - \frac{C}{r^2}\right]\Psi_{lm}(r) = 0, \tag{24}$$

where the three coefficients are given by

$$A = \sqrt{\mu^2 - \omega^2}, \qquad B = 2(M\mu^2 - qQ\omega),$$
$$C = \lambda_{lm} + Q^2(\mu^2 - q^2). \tag{25}$$

The solution is given exactly by the Whittaker function $W[p, s; cr]$ and the confluent hypergeometric function $U[p, s; cr]$ as

$$\Psi_{lm}(r) = c_1 W\left[\frac{B}{2A}, k; 2Ar\right] \tag{26}$$

$$= c_1 e^{-Ar}(2Ar)^{k+\frac{1}{2}} U\left[k + \frac{1}{2} - \frac{B}{2A}, 1 + 2k; 2Ar\right], \tag{27}$$

where

$$k = \frac{1}{2}\sqrt{1 + 4C}. \tag{28}$$

Here, we find a bound state $e^{-\sqrt{\mu^2-\omega^2}r}$ in Eq. (17). $M\mu^2 > qQ\omega$ ($M\mu^2 < qQ\omega$) corresponds to $B > 0$ ($B < 0$). In the asymptotic region, one has an asymptotic wave function

$$\Psi^A_{lm}(r) \sim e^{-\sqrt{\mu^2-\omega^2}r}\left(2\sqrt{\mu^2 - \omega^2}r\right)^{\frac{M\mu^2-qQ\omega}{\sqrt{\mu^2-\omega^2}}}, \tag{29}$$

which always leads to zero as $r \to \infty$ for $\sqrt{\mu^2 - \omega^2} > 0$.

Let us observe a radial mode $\Psi_{lm}(r)$ for a trapping well (see left panel of Fig. 1), implying superradiant instability. As is shown in the left panel of Fig. 3, Eq. (27) shows a quasibound state whose wave function is a peak located far from the outer horizon. A similar picture is found in the left panel of Fig. 4 which is based on Fig. 2 with $q > \mu$. This case indicates a feature of a charged massive scalar propagating around the KNBH.

In contrast, we consider a radial mode for a potential without a trapping well [shown in the right panel of Fig. 1 with $V'_{aaKN} > 0$ ($M\mu^2 > qQ\omega$)], implying superradiant stability. As is shown in Fig. 5, Eq. (27) shows an exponentially decaying mode. This case explains why one needs to find the other condition for no trapping well.

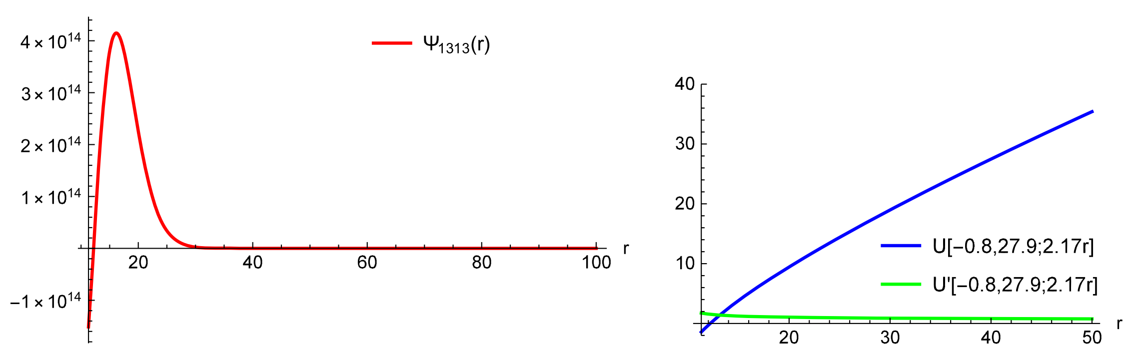

FIG. 3. Left: radial mode showing a quasibound state $\Psi_{1313}(r)$ as a function of $r \in [11.3, 100]$ with a trapping well. Right: the confluent hypergeometric function $U[-0.8, 27.8; 2.17r]$ is an increasing function of r and its derivative $U'[-0.8, 27.8; 2.17r]$ is positive. All parameters go together with the left panel of Fig. 1.

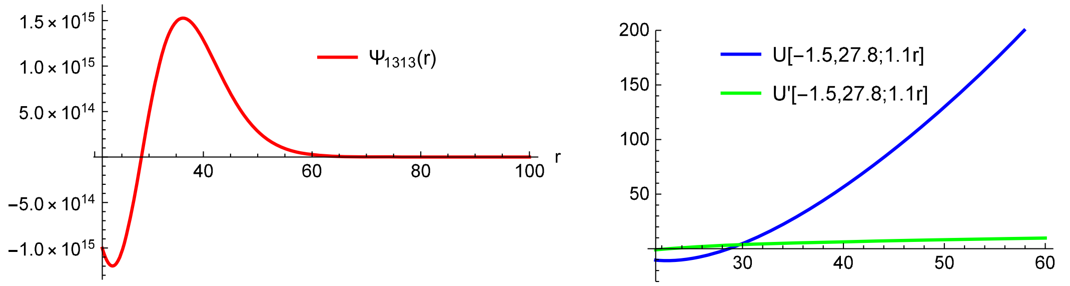

FIG. 4. Left: radial mode showing a quasibound state $\Psi_{1313}(r)$ as a function of $r \in [21.4, 100]$ with a trapping well. Right: the confluent hypergeometric function $U[-1.5, 27.8; 1.1r]$ is an increasing function of r and its derivative $U'[-1.5, 27.8; 1.1r]$ is positive. All parameters go together with Fig. 2.

At this stage, it is useful to introduce the asymptotic form of $U[p, s; cr]$ as [17]

$$U[p, s; cr \to \infty] \to (cr)^{-p}\left[1 - \frac{p(1+p-s)}{cr} + \mathcal{O}\left(\frac{1}{cr}\right)^2\right]. \tag{30}$$

Here, one observes an increasing function $U[p, s; cr]$ for a negative p (right panel of Fig. 3 and right panel of Fig. 4), while one finds a decreasing function for a positive p (Fig. 5). Furthermore, considering the first derivative of $U[p, s; cr]$ with respect to r ($c > 0$) as

$$U'[p, s; cr] = -pcU[1+p, 1+s; cr], \tag{31}$$

it implies that the condition for a trapping well is

$$U'[p, s; cr] > 0 \to p < 0, \tag{32}$$

whereas the condition for no trapping well is given by

$$U'[p, s; cr] < 0 \to p > 0. \tag{33}$$

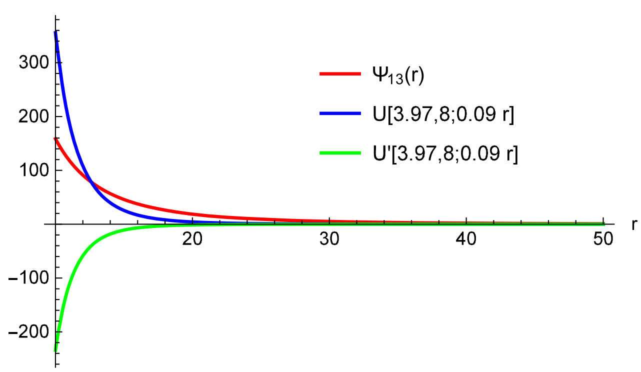

FIG. 5. Bound state function $\Psi_{13}(r)$ as $r \in [5, 100]$ without a trapping well. The confluent hypergeometric function $U[3.97, 8; 0.09r]$ represents a decreasing function of r and its derivative $U'[3.97, 8; 0.09r]$ is negative. All parameters go together with the right panel of Fig. 1.

Therefore, the quasibound state with a peak could be found when p is negative as

$$p < 0 \to \sqrt{1+4C} < \frac{B}{A} - 1 \to \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} > k + \frac{1}{2}, \tag{34}$$

which is the other condition for a trapping well, in addition to $M\mu^2 > qQ\omega$ $[V'_{aaKN}(r) > 0]$. On the other hand, the bound state could be found for a positive p as

$$p > 0 \to \sqrt{1+4C} > \frac{B}{A} - 1 \to \frac{M\mu^2 - qQ\omega}{\sqrt{\mu^2 - \omega^2}} < k + \frac{1}{2}, \tag{35}$$

which denotes the other condition for no trapping well, in addition to $M\mu^2 < qQ\omega$ $[(V'_{aaKN}(r) < 0)]$. We could not find a condition of $M\mu/qQ < 1$ for superradiant stability [14]. However, we obtain one condition from $V'_{aaKN}(r) < 0$ as

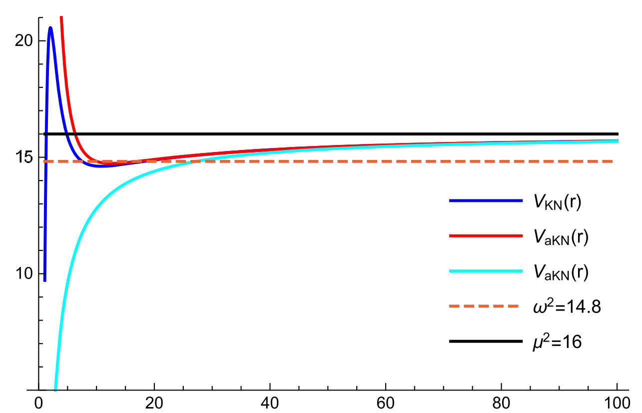

FIG. 6. Potential with a trapping well $V_{KN}(r)$, its far-region $V_{aKN}(r)$, and asymptotic potential $V_{aaKN}(r)$ as functions of $r \in [r_+ = 1.062, 100]$ with $M = 1$, $Q = 0.01$, $\omega = 3.85$, $a = 0.998$, $m = 13$, $q = 0.2$, λ_{lm} and $\mu = 4$. Here, we choose $\lambda_{lm} = 202.461$ to find $a = -5.9 \times 10^{-9} \sim 0$ in $U[p, s; cr]$. $V_{KN}(r)$ has a trapping well located at $r = 10.8$.

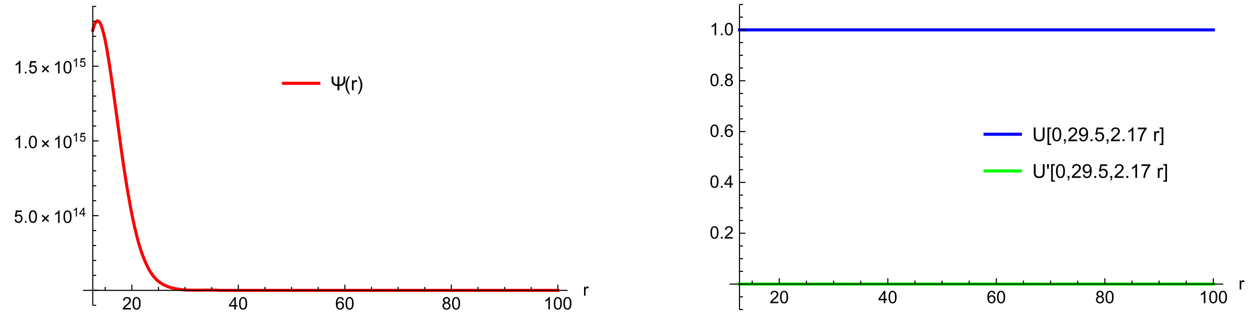

FIG. 7. Left: radial mode $\Psi(r)$ showing a half of a peak as a function of $r \in [12.66, 100]$ with a trapping well. Right: the confluent hypergeometric function $U[-5.9 \times 10^{-9} \simeq 0, 29.5; 2.17r]$ represents a constant and its derivative $U'[-5.9 \times 10^{-9} \simeq 0, 29.5; 2.17r]$ is nearly zero. All coefficients go together with Fig. 6.

$$\frac{M\mu}{qQ} < \frac{\omega}{\mu} < 1. \tag{36}$$

Finally, we consider the case of $p \simeq 0$ to denote a boundary state between quasibound and bound states. Here, we display a corresponding potential in Fig. 6 which seems to have a trapping well. The asymptotic wave function shows a half of a peak (left panel of Fig. 7) and thus, it does not represent a quasibound state completely. Its confluent hypergeometric function $U[p, 29.5; 2.17]$ with $p = -5.9 \times 10^{-9} \simeq 0$ is nearly a constant (right panel of Fig. 7), which implies $U'[p, 29.5; 2.17] \simeq 0$. This completes our classification: $p < 0$, $p = 0$, $p > 0$.

IV. CONDITION FOR TRAPPING WELL IN KERR BLACK HOLES

Now, we are in a position to study the superradiant instability/stability of Kerr black holes under a massive scalar propagation. We obtain this case when choosing $q = Q = 0$. Its far-region equation is given by Eq. (24) with $A = \sqrt{\mu^2 - \omega^2}$, $B = 2M\mu^2$, $C = l(l+1)$, and $k = l + 1/2$. Here, one should find the other condition for no trapping well because $B > 0$ $[V'_{aaK}(r) > 0]$ implies that a trapping well necessarily arises. In this case, the solution is given by [18]

$$\Psi^K_{lm}(r) = c_2 e^{-\sqrt{\mu^2-\omega^2}r}\left(2\sqrt{\mu^2-\omega^2}r\right)^{l+1} U \times \left[l + 1 - \frac{M\mu^2}{\sqrt{\mu^2-\omega^2}}, 2l+2; 2\sqrt{\mu^2-\omega^2}r\right]. \tag{37}$$

Here, the condition for a trapping well is obtained from the first argument ($p < 0$) in $U[p, s; cr]$ as

$$\frac{M\mu^2}{\sqrt{\mu^2-\omega^2}} > l + 1, \tag{38}$$

while the condition for no trapping well takes the form

$$\frac{M\mu^2}{\sqrt{\mu^2-\omega^2}} < l + 1. \tag{39}$$

Equation (38) is satisfied for the potential with a trapping well (similar to the left panel of Fig. 1) whose parameters are given by $M = 1$, $\omega = 3.85$, $a = 0.998$, $m = 13$, $l = 13$, $\mu = 4$, and $\omega_c = 6.101 > \omega$. We note that its confluent hypergeometric function $U[-0.61, 28, 2.19r]$ is an increasing function of r. Also, we have checked the stability condition (39) for the potential without a trapping well (similar to the right panel of Fig. 1) whose parameters are $M = 1$, $\omega = 0.02$, $a = 0.3$, $m = 1$, $l = 3$, $\mu = 0.05$, and $\omega_c = 0.076 > \omega$. As is expected, its confluent hypergeometric function $U[3.95, 8, 0.09r]$ is a decreasing function of r.

For a complex $\omega = 3.85 + 10^{-6}i$ ($|\omega_I| \ll \omega_R, \omega_R < m\Omega_H$) with the same parameters [18,19], one has the asymptotic solution

$$\Psi_{13,13} \sim (51422 - 2.35i)e^{(-1.09+3.5\times10^{-6}i)r} r^{14} U \times [-0.74 - 5 \times 10^{-5}i, 28, (2.17 - 7 \times 10^{-6}i)r] \tag{40}$$

whose real part takes the form of a peak as in the left panel of Fig. 3. Here, one may rewrite p in $U[p, s; cr]$ as

$$p = -n - \delta\nu, \tag{41}$$

where $n = 1$ and $\delta\nu = -0.26 + 5 \times 10^{-5}i(|\delta\nu| = 0.26 < 1)$. The latter complex number might represent a deviation from the hydrogen wave functions. A state of superradiant instability could not be approximately described by a small shift of hydrogen energy levels because $|\delta\nu| = 0.26$ is not a small shift. This may be so because we use $M\mu = 4 > 1$ (not ultralight bosons with $\omega \sim \mu \ll 1/M$).

V. DISCUSSIONS

First of all, we would like to mention the conditions for superradiant instability and stability. Superradiant instability can be achieved for $\omega < \omega_c$ and $\omega < \mu$ with a positive trapping well, whereas superradiant stability can be found for $\omega < \omega_c$ and $\omega < \mu$ without a positive trapping well. The presence of a trapping well is regarded as a decisive condition for superradiant instability. If there is no trapping well, it corresponds to superradiant stability.

In this work, we have first obtained two conditions for getting a trapping well of KNBHs under a charged massive scalar perturbation by analyzing the asymptotic scalar potential $[V_{aaKN}(r)]$ and far-region wave function $(U[p,s;cr])$. They are given by $V'_{aaKN}(r) > 0$ $(M\mu^2 > qQ\omega)$ and $U'[p,s;cr] > 0$ $(p < 0)$. Also, the two conditions for no trapping well are $V'_{aaKN}(r) < 0$ $(M\mu^2 < qQ\omega)$ and $U'[p,s;cr] < 0$ $(p > 0)$. From the former, we have derived one condition of Eq. (36) for superradiant stability.

We have carried out the same analysis for a Kerr black hole under a massive scalar propagation to find the conditions for superradiant instability and stability. The superradiant instability condition is given by $M\mu^2/\sqrt{\mu^2-\omega^2} > l+1$, while the stability condition is $M\mu^2/\sqrt{\mu^2-\omega^2} < l+1$ because $V'_{aaK}(r) > 0$ $(M\mu^2 > 0)$ is always satisfied.

ACKNOWLEDGMENTS

This work was supported by a grant from Inje University for the Research in 2021 (20210040).

[1] A. Arvanitaki and S. Dubovsky, Phys. Rev. D **83**, 044026 (2011).
[2] R. Brito, V. Cardoso, and P. Pani, Lect. Notes Phys. **906**, 1 (2015).
[3] T. Damour, N. Deruelle, and R. Ruffini, Lett. Nuovo Cimento **15**, 257 (1976).
[4] T. J. M. Zouros and D. M. Eardley, Ann. Phys. (N.Y.) **118**, 139 (1979).
[5] S. L. Detweiler, Phys. Rev. D **22**, 2323 (1980).
[6] S. R. Dolan, Phys. Rev. D **76**, 084001 (2007).
[7] R. A. Konoplya and A. Zhidenko, Rev. Mod. Phys. **83**, 793 (2011).
[8] H. Furuhashi and Y. Nambu, Prog. Theor. Phys. **112**, 983 (2004).
[9] C. A. R. Herdeiro, J. C. Degollado, and H. F. Rúnarsson, Phys. Rev. D **88**, 063003 (2013).
[10] S. Hod, Phys. Rev. D **90**, 024051 (2014).
[11] C. L. Benone, L. C. B. Crispino, C. Herdeiro, and E. Radu, Phys. Rev. D **90**, 104024 (2014).
[12] C. L. Benone and L. C. B. Crispino, Phys. Rev. D **99**, 044009 (2019).
[13] Y. Huang and D. J. Liu, Phys. Rev. D **94**, 064030 (2016).
[14] J. H. Xu, Z. H. Zheng, M. J. Luo, and J. H. Huang, Eur. Phys. J. C **81**, 402 (2021).
[15] S. Hod, Phys. Lett. B **708**, 320 (2012).
[16] Y. S. Myung, arXiv:2201.06706.
[17] Y. S. Myung, arXiv:2204.04789.
[18] R. Z. Guo, C. Yuan, and Q. G. Huang, Phys. Rev. D **105**, 064029 (2022).
[19] H. Kodama and H. Yoshino, Int. J. Mod. Phys. Conf. Ser. **07**, 84 (2012).