Consistent mass formulas for the four-dimensional dyonic NUT-charged spacetimes

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In our previous work [Phys. Rev. D **100**, 101501(R) (2019)], a novel idea that the Newman-Unti-Tamburino (NUT) charge can be thought of as a thermodynamical multihair has been advocated to describe perfectly the thermodynamical character of the generic four-dimensional Taub-NUT spacetimes. According to this scheme, the Komar mass (M), the gravitomagnetic charge (N), and/or the dual (magnetic) mass ($\tilde{M} = N$), together with a new secondary hair ($J_N = MN$), namely, a Kerr-like conserved angular momentum, enter into the standard forms of the first law and Bekenstein-Smarr mass formula. Distinguished from other recent attempts, our consistent thermodynamic differential and integral mass formulas are both obtainable from a meaningful Christodoulou-Ruffini-type squared-mass formula of almost all of the four-dimensional NUT-charged spacetimes. As an excellent consequence, the famous Bekenstein-Hawking one-quarter area-entropy relation can be naturally restored not only in the Lorentzian sector and but also in the Euclidian counterpart of the generic Taub-NUT-type spacetimes without imposing any constraint condition. However, only purely electric-charged cases in the four-dimensional Einstein-Maxwell gravity theory with a NUT charge have been addressed there. In this paper, we shall follow the simple, systematic way proposed in that article to further investigate the dyonic NUT-charged case. It is shown that the standard thermodynamic relations continue to hold true provided that no new secondary charge is added; however, the so-obtained electrostatic and magnetostatic potentials are not coincident with those computed via the standard method. To rectify this inconsistence, a simple strategy is provided by further introducing two additional secondary hairs, $Q_N = QN$ and $P_N = PN$, together with their thermodynamical conjugate potentials, so that the first law and Bekenstein-Smarr mass formula are still satisfied, where Q and P being the electric and magnetic charges, respectively.

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I. INTRODUCTION

In recent years, thermodynamics of the four-dimensional Lorentzian Taub-NUT spacetimes in the Einstein-Maxwell gravity theory have attracted a lot of attention [1–12]. In particular, in our previous work [13], we have advocated for a new idea that "the NUT charge is a thermodynamical multihair" and put forward a simple, systematic way to study the consistent thermodynamics of almost all of the four-dimensional NUT-charged spacetimes. The consistent first law and Bekenstein-Smarr mass formula of these NUT-charged spacetimes are deduced by first deriving a new meaningful Christodoulou-Ruffini-type squared-mass formula satisfied by the four-dimensional NUT-charged spacetimes with a new secondary hair; $J_N = MN$. In contrast, it should be mentioned that there is no analogous expression of the Christodoulou-Ruffini-type squared-mass formula [14,15] in all of the previous works [1–12]. As a fact that has already been demonstrated in Ref. [16], our new secondary hair $J_N = MN \equiv M_5$ exactly corresponds to the mass of the five-dimensional gravitational magnetic monopole, so at least from the five-dimensional point of view, it is very natural to consider it as a global conserved charge; then it is reasonable to include it to the first law and Bekenstein-Smarr mass formula. There are many reasons to support such an idea. For instance, it helps to explain the gyromagnetic ratio of Kerr-NUT-type spacetime [17], and the quantization condition for a gravitational monopole [18–20]. Additionally, it is proved in Ref. [21] that only by considering the secondary hair $J_N = MN$ as a independent charge, can the area (or entropy) products of the NUT-charged spacetimes be subject to the universal rules [22], and the mass be expressed as a sum of the surface energy, the rotational energy, and the electromagnetic energy [23].

According to the scheme advocated in our previous paper [13], the traditional elegant Bekenstein-Hawking one-quarter area-entropy relation can be naturally restored in the Lorentzian and Euclidian sectors of the generic NUT-

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charged spacetimes (and all of their extensions) in the four-dimensional Einstein-Maxwell gravity theory without imposing any constraint condition. Due to the fact that the NUT charge not only acts as a dual (magnetic) mass, but also simultaneously has the rotationlike and electromagnetic chargelike characters, we arrive at a new recognition that it must be a thermodynamical multihair. This viewpoint is in sharp contrast with all previous knowledge that it has merely one physical feature, or that it is purely a single-solution parameter, what is more, the physical meaning of the NUT parameter as a polyfacet can be completely uncovered in the thermodynamical sense.

The four-dimensional NUT-charged spacetimes studied in our previous work [13] are either static charged (including a nonzero negative cosmological constant) or rotating charged (with a vanishing cosmological constant) in the Einstein-Maxwell theory with a purely electric charge. Note that the purely magnetic-charged case can be identically treated via the electric-magnetic duality relation. However, that paper did not consider the case of the four-dimensional dyonic NUT-charged spacetimes, nor did it deal with the higher-dimensional case [24–30] and those four-dimensional NUT-charged spacetimes beyond the Einstein-Maxwell theory (such as Kaluza-Klein (K-K) theory [31–33], Einstein-Maxwell-Dilaton-Axion (EMDA) theory [34], and more general (gauged) STU supergravity theory [35–37]), all of which need to be studied promptly. In the present paper, we shall also focus on the thermodynamics of the four-dimensional Lorentzian dyonic NUT-charged spacetimes in the Einstein-Maxwell gravity theory.

The remaining part of this paper is organized as follows. In Sec. II, we begin with a brief introduction of some aspects of the four-dimensional Lorentzian dyonic Reissner-Nordström-NUT (RN-NUT) solution and then construct a new Christodoulou-Ruffini-like squared-mass formula, from which both the differential and integral mass formulas can be derived via a simple mathematical manipulation by only including the secondary hair $J_N = MN$, as did before in Ref. [13]. However, there exists a contradiction between the obtained electrostatic and he magnetostatic potentials with those computed by the standard method. We demonstrated that this inconsistency can be simply remedied by further introducing two new additional secondary hairs, $Q_N = QN$ and $P_N = PN$, together with their thermodynamical-conjugate potentials, where Q and P are the electric and magnetic charges, respectively, so that the standard thermodynamic relations can continue to hold true. In Sec. III, we turn to discuss the case of the dyonic RN-NUT-AdS$_4$ spacetime. We show that the dual (magnetic) mass must be further added to reproduce the familiar thermodynamical volume delivered in other literature. Then, in Sec. IV, we extend the above work to the case of the four-dimensional dyonic Kerr-Newman-NUT (KN-NUT) spacetime. In Sec. V, we discuss the impact of the secondary hair, J_N, on the mass formulas and present the reduced mass formulas. Finally, we present our conclusions in Sec. VI.

II. CONSISTENT MASS FORMULAS OF THE FOUR-DIMENSIONAL DYONIC RN-NUT SPACETIME

Let us start by summarizing some essential facts of the Lorentzian four-dimensional RN-NUT metric with both electric and magnetic charges in the Lorentz sector [38,39]. We adopt the following exquisite form of the line element in which the Misner strings [40] are symmetrically distributed along the polar axis,

$$ds^2 = -\frac{f(r)}{r^2+N^2}(dt + 2N\cos\theta d\phi)^2 + \frac{r^2+N^2}{f(r)}dr^2 + (r^2+N^2)(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $f(r) = r^2 - 2Mr - N^2 + Q^2 + P^2$, in which M, N, Q, and P are the mass, the NUT charge, and the electric and magnetic charges of the spacetime, respectively. In addition, the electromagnetic gauge potential one-form and its dual one-form are

$$\mathbf{A} = \frac{Qr - PN}{r^2+N^2}(dt + 2N\cos\theta d\phi) + P\cos\theta d\phi, \tag{2}$$

$$\tilde{\mathbf{A}} = \frac{Pr + QN}{r^2+N^2}(dt + 2N\cos\theta d\phi) - Q\cos\theta d\phi, \tag{3}$$

in which a gauge choice is made to let the temporal components of both potentials (2), (3) be zero at infinity, so that the corresponding electrostatic and magnetostatic potentials vanish at infinity. Alternatively, other often used expressions for them are given by [3,41]

$$\mathbb{A} = \frac{2QNr + P(r^2-N^2)}{2N(r^2+N^2)}(dt + 2N\cos\theta d\phi),$$

$$\tilde{\mathbb{A}} = \frac{2PNr - Q(r^2-N^2)}{2N(r^2+N^2)}(dt + 2N\cos\theta d\phi),$$

whose temporal components differ from ours by two constants, $P/(2N)$, and $-Q/(2N)$, respectively.

Traditionally, the spacetime (1) is termed as being asymptotically local flat. It has a lot of odd physical properties that are mainly due to the presence of the wire/line singularities at the polar axis ($\theta = 0, \pi$), which are often dubbed the Misner strings, are an analog of the Dirac string in electrodynamics. Misner [40] proposed to remove this kind of wire/line singularities (so as to ensure the regularity of the metric) by imposing a time periodical identification condition, $\beta = 8\pi n$. Then, the inevitable appearance of closed timelike curves subsequently led him [42] to claim that the NUT parameter was nonphysical and the Taub-NUT spacetime was "a counterexample to almost anything" in General Relativity. However in recent years, Clément *et al.* [43–45] demonstrated that it is not actually necessary to remove the Misner string by imposing a periodicity condition of the time coordinate. They illustrated that the Misner string singularities are far less problematic than previously thought, and argued that the Lorentzian Taub-NUT solutions without the Misner time periodicity condition are geodesically complete, and causality is not violated at all for geodesic observers, despite the existence of regions with closed timelike curves. An immediate consequence of their research is that the Lorentzian Taub-NUT spacetimes with the Misner strings may be physical in nature. This, in turn, invokes enthusiasm to explore other properties of these NUT-charged spacetimes.

In the following we will derive various mass formulas and discuss the consistent thermodynamics of the four-dimensional Lorentzian dyonic RN-NUT spacetime. As was done in Refs. [1,2,6,7,13,43,46–49], we will not impose the time-periodicity condition. Meanwhile, we shall also keep the Misner strings symmetrically present at the polar axes and only consider the conical singularities satisfying $f(r) = 0$; namely, the outer and inner horizons located at $r_h = r_\pm = M \pm \sqrt{M^2 + N^2 - Q^2 - P^2}$. Below, we will focus on the (exterior) event horizon—the discussions are also valid for the (interior) Cauchy horizon.

To begin with, let us recall some known quantities that can be evaluated via the standard method. First, the area and the surface gravity at the horizon are easily computed as

$$A_h = 4\pi(r_h^2 + N^2) = 4\pi\mathcal{A}_h, \quad \kappa = \frac{f'(r_h)}{2\mathcal{A}_h} = \frac{r_h - M}{r_h^2 + N^2}, \tag{4}$$

with a reduced horizon area, $\mathcal{A}_h$, being introduced [13,50] for conciseness,

$$\mathcal{A}_h = r_h^2 + N^2 = 2Mr_h + 2N^2 - Q^2 - P^2. \tag{5}$$

The electrostatic and magnetostatic potentials are gauge independent by virtue of the above specific gauge choice. They are simply given by

$$\Phi = \Phi_h = (\mathbf{A}_\mu \xi^\mu)|_{r=r_h} = \frac{Qr_h - PN}{r_h^2 + N^2},$$
$$\Psi = \Psi_h = (\tilde{\mathbf{A}}_\mu \xi^\mu)|_{r=r_h} = \frac{Pr_h + QN}{r_h^2 + N^2}, \tag{6}$$

where $\xi = \partial_t$ is a timelike Killing vector normal to the horizon.

As far as the calculation of the global conserved charges (M, N, Q, P) is concerned, the mass M can be computed via the Komar integral related to the timelike Killing vector ∂_t, while the electric and magnetic charges (Q, P) can be integrated by using the Gauss' law associated with the field strengths ($F = d\mathbf{A}$, $\tilde{F} = d\tilde{\mathbf{A}}$), respectively. The NUT charge N, however, has several different meanings, and so can be evaluated via different methods. If it appears as the dual or magnetic-type mass [20,51–53], then it can be determined via the dual Komar integral as $\tilde{M} = N$. On the other hand, if it acts as the gravitomagnetic charge, it can be calculated via the definition given in Ref. [54]. One cannot distinguish the dual or magnetic-type mass from the gravitomagnetic charge in the present case; however, we shall see below that they are significantly different from each other once a nonzero cosmological constant is included. In addition, the conserved charges (M, N, Q, P) as the primary hairs apparently appear in the leading order of the asymptotic expansions of the following components of the metric and the Abelian potentials at infinity,

$$\begin{aligned}
g_{tt} &\simeq -1 + \frac{2M}{r} + \mathcal{O}(r^{-2}),\\
g_{t\phi} &\simeq \left(-2N + \frac{4MN}{r}\right)\cos\theta + \mathcal{O}(r^{-2}),\\
\mathbf{A}_t &\simeq \frac{Q}{r} + \mathcal{O}(r^{-2}), \qquad \tilde{\mathbf{A}}_t \simeq \frac{P}{r} + \mathcal{O}(r^{-2}),\\
\mathbf{A}_\phi &\simeq \left(P + \frac{2QN}{r}\right)\cos\theta + \mathcal{O}(r^{-2}),\\
\tilde{\mathbf{A}}_\phi &\simeq \left(-Q + \frac{2PN}{r}\right)\cos\theta + \mathcal{O}(r^{-2}).
\end{aligned} \tag{7}$$

Note that our previously included secondary hair, $J_N = MN$, appears as the next-to-leading order of the asymptotic expansion of the metric component $g_{t\phi}$, and we can find that there are also two similar next-to-leading-order quantities, QN and PN, in the asymptotic expansions of two Abelian potentials' components, $\mathbf{A}_\phi$ and $\tilde{\mathbf{A}}_\phi$, indicating that as two secondary hairs, they might also play a key role in the mass formula.

A. Mass formulas with the secondary hair: $J_N = MN$ only

In order to derive the first law which is reasonable and consistent in both physical and mathematical sense, we adopt the method used in Refs. [13,50] to deduce a meaningful Christodoulou-Ruffini-type squared-mass formula. First, we rewrite the expression (5) of the reduced horizon area and get the following identity:

$$(\mathcal{A}_h - 2N^2 + Q^2 + P^2)^2 = 4M^2r_h^2 = 4M^2\mathcal{A}_h - 4M^2N^2. \tag{8}$$

Next, supposed that we only need to introduce the secondary hair: $J_N = MN$, as did in our previous work [13], then we can obtain a useful identity,

$$M^2 = \frac{1}{4\mathcal{A}_h}(\mathcal{A}_h - 2N^2 + P^2 + Q^2)^2 + \frac{J_N^2}{\mathcal{A}_h}, \qquad (9)$$

which is a Christodoulou-Ruffini-like squared-mass formula for the four-dimensional dyonic RN-NUT spacetime. We point out that Eq. (9) consistently reduces to the one obtained in the case of the four-dimensional RN-NUT spacetime [13] when the magnetic charge P is turned off.

Below, we will derive the differential and integral mass formulas for the dyonic RN-NUT spacetime, supposing that the primary hairs are the mass M, the NUT charge N, the electric and magnetic charges (Q, P) as well as the only one secondary hair, $J_N = MN$. Given that the secondary hair J_N can be viewed temporarily as a independent variable[1] at this moment, then the above squared-mass formula (9) can be viewed formally as a fundamental functional relation, $M = M(\mathcal{A}_h, N, J_N, Q, P)$. Differentiating it (multiplied by $4\mathcal{A}_h$) with respect to the thermodynamical variables $(\mathcal{A}_h, N, J_N, Q, P)$ yields their conjugate quantities, as was done in Refs. [50,55–60]. In doing so, we arrive at the differential and integral mass formulas, with the conjugate thermodynamic potentials given by the ordinary Maxwell relations.

Let us now demonstrate the above conclusion in more detail. Differentiating the squared-mass formula (9) with respect to the reduced horizon area $\mathcal{A}_h$ yields one half of the surface gravity

$$\kappa = 2\frac{\partial M}{\partial \mathcal{A}_h}\bigg|_{(N,J_N,Q,P)} = \frac{\mathcal{A}_h - 2N^2 + Q^2 + P^2 - 2M^2}{2M\mathcal{A}_h} = \frac{r_h - M}{r_h^2 + N^2}, \qquad (10)$$

which is entirely identical to the one given in Eq. (4). Similarly, by the differentiation of the squared-mass formula (9) with respect to the NUT charge N and the secondary hair J_N, one can obtain the conjugate gravitomagnetic potential ψ_h and the conjugate quasiangular momentum ω_h as

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A}_h,J_N,Q,P)} = \frac{-N(\mathcal{A}_h - 2N^2 + Q^2 + P^2)}{M\mathcal{A}_h} = \frac{-2Nr_h}{r_h^2 + N^2}, \qquad (11)$$

$$\omega_h = \frac{\partial M}{\partial J_N}\bigg|_{(\mathcal{A}_h,N,Q,P)} = \frac{J_N}{M\mathcal{A}_h} = \frac{N}{r_h^2 + N^2}. \qquad (12)$$

The electrostatic and magnetostatic potentials, which are conjugate to Q and P, respectively, can be computed as

$$\hat{\Phi} = \frac{\partial M}{\partial Q}\bigg|_{(\mathcal{A}_h,N,J_N,P)} = \frac{Q(\mathcal{A}_h - 2N^2 + Q^2 + P^2)}{2M\mathcal{A}_h} = \frac{Qr_h}{r_h^2 + N^2}, \qquad (13)$$

$$\hat{\Psi} = \frac{\partial M}{\partial P}\bigg|_{(\mathcal{A}_h,N,J_N,Q)} = \frac{P(\mathcal{A}_h - 2N^2 + Q^2 + P^2)}{2M\mathcal{A}_h} = \frac{Pr_h}{r_h^2 + N^2}, \qquad (14)$$

which coincide with their corresponding ones only in the purely electric- or purely magnetic-charged case [13]. In the present dyonic case, these two quantities are apparently different from those given in Eq. (6). Nevertheless, we can verify that both the differential and integral mass formulas are completely satisfied

$$dM = (\kappa/2)d\mathcal{A}_h + \omega_h dJ_N + \psi_h dN + \hat{\Phi}dQ + \hat{\Psi}dP, \qquad (15)$$

$$M = \kappa\mathcal{A}_h + 2\omega_h J_N + \psi_h N + \hat{\Phi}Q + \hat{\Psi}P, \qquad (16)$$

with respect to all the above thermodynamical conjugate pairs.

It is worth mentioning that the above differential and integral mass formulas (15)–(16) can not only naturally reduce to the purely electric- or purely magnetic-charged case when the magnetic or electric charge vanishes ($P = 0$ or $Q = 0$), but also smoothly recovers the dyonic RN black hole case when the NUT charge vanishes ($N = 0$). Comparing our new mass formulas presented in Eqs. (15)–(16) with the standard ones, it is strongly suggested that one should use the following familiar identities

$$T = \frac{\kappa}{2\pi} = \frac{r_h - M}{2\pi(r_h^2 + N^2)}, \qquad S = \frac{A}{4} = \pi(r_h^2 + N^2), \qquad (17)$$

which restores the famous Bekenstein-Hawking one-quarter area-entropy relation of the dyonic RN-NUT spacetime in a very comfortable way. It is worth noting that one should assign a geometric entropy to the dyonic RN-NUT spacetime, which is just one-quarter of its horizon area. In the above derivation, we do not require in advance that the relation (17) must hold in order to obtain a reasonable first law, but rather it is a very natural result from the above thermodynamic derivation.

It is remarkable that unlike Ref. [3], our differential and integral mass formulas (15)–(16) attain their traditional forms which relate the global conserved charges (M, Q, P, N, J_N) measured at the infinity, to those quantities $(T, S, \hat{\Phi}, \hat{\Psi}, \psi_h, \omega_h)$ evaluated at the horizon. In this sense, it is quite reasonable to infer that the entire set of four laws of the usual black hole thermodynamics is completely applicable to the dyonic RN-NUT spacetime. It is time to formally call

[1] However, one may think that it actually is not independent. A careful discussion about its impact on the mass formulas is presented in Sec. V.

the dyonic NUT-charged spacetimes real black holes, at least from the thermodynamic point of view.

B. Two new secondary hairs $Q_N = QN$ and $P_N = PN$

In the last subsection, we derived the differential and integral mass formulas of the four-dimensional dyonic RN-NUT spacetime via differentiating the squared-mass formula (9), but with a fly in the ointment as mentioned earlier; namely, the derived expressions for the conjugate electrostatic and magnetostatic potentials are inconsistent with those previously calculated by using the standard method. In order to get a consistent and reasonable first law and Bekenstein-Smarr mass formula, this inconsistency must be removed. Noting that the expressions $\psi_h = -2Nr_h/(r_h^2 + N^2)$ and $\omega_h = N/(r_h^2 + N^2)$ in the mass formulas (15)–(16) do not explicitly contain the electric and magnetic charges (Q, P), so we can leave them unchanged and replace only the electrostatic and magnetostatic potentials ($\hat{\Phi}$, $\hat{\Psi}$) by the standard ones (Φ, Ψ) given in Eq. (6). First, using $\hat{\Phi}Q + \hat{\Psi}P = \Phi Q + \Psi P$, the integral mass formula (16) can be rewritten as

$$M = 2TS + 2\omega_h J_N + \psi_h N + \Phi Q + \Psi P. \qquad (18)$$

Next, the first law (15) can be rewritten as

$$\begin{aligned} dM &= TdS + \omega_h dJ_N + \psi_h dN + \Phi dQ + \Psi dP \\ &\quad + \frac{N}{r_h^2 + N^2}(PdQ - QdP), \\ &= TdS + \omega_h dJ_N + \psi_h dN + \Phi dQ + \Psi dP \\ &\quad + \frac{Pd(QN) - Qd(PN)}{r_h^2 + N^2}, \\ &= TdS + \omega_h dJ_N + \psi_h dN + \Phi dQ + \Psi dP \\ &\quad + \Phi_N dQ_N + \Psi_N dP_N, \end{aligned} \qquad (19)$$

provided that one further introduces two new additional secondary hairs, $Q_N = QN$ and $P_N = PN$, together with their thermodynamic conjugate potentials,

$$\Phi_N = \frac{P}{r_h^2 + N^2}, \qquad \Psi_N = \frac{-Q}{r_h^2 + N^2}. \qquad (20)$$

Also, since $\Phi_N Q_N + \Psi_N P_N = 0$, so the Bekenstein-Smarr mass formula (18) can be further rewritten as

$$M = 2TS + 2\omega_h J_N + \psi_h N + \Phi Q + \Psi P + \Phi_N Q_N + \Psi_N P_N. \qquad (21)$$

From the first law (19), it is easy to see that there are five cases with no need to introduce the secondary hairs, $Q_N = QN$ and $P_N = PN$ as well as their conjugate potentials (Φ_N, Ψ_N): (i) purely electric-charged case ($P = 0$); (ii) purely magnetic-charged case ($Q = 0$); (iii) dyonic RN solution ($N = 0$); (iv) self-dual vector-potential case ($Q = P$); and (v) antiself-dual vector potential case ($Q = -P$).

The above identities (18)–(19) are the expected standard forms of our consistent first law and Bekenstein-Smarr mass formula for the dyonic RN-NUT spacetime, suggesting that the NUT charge should be treated as a thermodynamic multihair. The advantage of introducing the above secondary hairs is as follows: 1) it can smoothly recover the cases where the solution parameters take some special values in our previous work [13]; 2) it can retain some thermodynamic quantities calculated by the standard method; 3) all the expressions of the related thermodynamic quantities are very concise and much more simple than those appeared in other literature.

Finally, if the squared-mass (9) is viewed as a binomial of the reduced horizon area

$$\begin{aligned} &\mathcal{A}_h^2 + 2(P^2 + Q^2 - 2N^2 - 2M^2)\mathcal{A}_h \\ &\quad + 4M^2N^2 + (P^2 + Q^2 - 2N^2)^2 = 0, \end{aligned} \qquad (22)$$

then the area product of the inner and outer horizons,

$$\mathcal{A}_+\mathcal{A}_- = 4J_N^2 + (P^2 + Q^2 - 2N^2)^2, \qquad (23)$$

can only be quantized when $J_N = MN$ is quantized in a manner like the quantization of the angular momentum, and the charges (Q, P, N) also take discrete values.

III. EXTENSION TO THE DYONIC RN-NUT-AdS$_4$ SPACETIME

In this section, we would like to extend the above work to the Lorentzian dyonic RN-NUT-AdS$_4$ spacetime with a nonzero cosmological constant. The metric, the Abelian gauge potential and its dual, are still given by Eqs. (1)–(3), but now $f(r) = r^2 - 2Mr - N^2 + Q^2 + P^2 + g^2(r^4 + 6N^2r^2 - 3N^4)$, in which $g = 1/l$ is the gauge coupling constant.

First, we will determine the conserved charges (primary hairs) of the dyonic RN-NUT-AdS$_4$ solution. The electric and magnetic charges (Q, P) as well as the gravitomagnetic charge N can be computed just like the case without a cosmological constant. We will adopt the conformal completion method to calculate its electric mass M and dual (magnetic) mass [61], and show that the dual (magnetic) mass $\tilde{M}$ is now different from the NUT charge N. The conformal boundary metric of the dyonic RN-NUT-AdS$_4$ spacetime is given by

$$\begin{aligned} ds_\infty^2 = \lim_{r\to\infty}\frac{ds^2}{r^2} &= -g^2(dt + 2N\cos\theta d\phi)^2 \\ &\quad + d\theta^2 + \sin^2\theta d\phi^2, \end{aligned} \qquad (24)$$

with $g^{rr} = g^2 r^4$ being used to define a normal vector $N^r = gr^2$, then the conserved charge $\mathcal{Q}[\xi]$ associated with the Killing vector $\xi = \partial_t$ is given by

$$\mathcal{Q}[\xi] = \frac{1}{8\pi g^3}\int rN^\alpha N^\beta C^\mu{}_{\alpha\nu\beta}\xi^\nu dS_\mu, \tag{25}$$

where $C^\mu{}_{\alpha\nu\beta}$ is the Weyl conformal tensor and

$$dS_\mu = g\sin\theta d\theta\wedge d\phi \tag{26}$$

is the area element of the two-spherical cross section of the conformal boundary. The conformal (electric) mass $\mathcal{M}$ is easily evaluated as

$$\mathcal{M} = \mathcal{Q}[\xi^t] = M. \tag{27}$$

Similarly, in order to evaluate the dual conformal mass, we can define a dual conserved charge $\tilde{\mathcal{Q}}[\xi]$ via replacing the Weyl conformal tensor by its left-dual

$$\tilde{C}_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}C^{\alpha\beta}{}_{\rho\sigma}, \tag{28}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita antisymmetry tensor. Then, the dual (magnetic) mass is computed as

$$\tilde{M} = \tilde{\mathcal{Q}}[\xi^t] = N(1+4g^2N^2), \tag{29}$$

which is unequal to the NUT charge N.

Next, we want to calculate some thermodynamic quantities associated with the Killing horizons defined by $f(r_h) = 0$. The surface gravity at the horizon is given by

$$\kappa = \frac{f'(r_h)}{2\mathcal{A}_h} = \frac{r_h - M + 2g^2(r_h^2+3N^2)r_h}{\mathcal{A}_h}, \tag{30}$$

while the horizon area still reads $A_h = 4\pi(r_h^2+N^2) = 4\pi\mathcal{A}_h$, in which the reduced horizon area is now

$$\begin{aligned}\mathcal{A}_h &= 2Mr_h + 2N^2 - Q^2 - P^2 - g^2(r_h^4 + 6N^2r_h^2 - 3N^4)\\ &= 2Mr_h + 2N^2 - Q^2 - P^2\\ &\quad - g^2(\mathcal{A}_h^2 + 4N^2\mathcal{A}_h - 8N^4).\end{aligned} \tag{31}$$

The electrostatic and magnetostatic potentials are

$$\Phi = \frac{Qr_h - PN}{r_h^2+N^2}, \qquad \Psi = \frac{Pr_h + QN}{r_h^2+N^2}, \tag{32}$$

which are the same expressions as those given in Eq. (6), although the horizon location r_h now has a different expression.

A. Mass formulas with the secondary hair: $J_N = MN$ only

Now, we assume that only the secondary hair $J_N = MN$ is needed as before, and deduce a squared-mass formula. The reduced horizon area (31) can be written as

$$2Mr_h = (1+4g^2N^2)(\mathcal{A}_h - 2N^2) + Q^2 + P^2 + g^2\mathcal{A}_h^2. \tag{33}$$

After squaring this identity and then adding $4M^2N^2$ to its left-hand side and $4J_N$ to its right-hand side, we obtain

$$\begin{aligned}M^2 = \frac{1}{4\mathcal{A}_h}[&(1+4g^2N^2)(\mathcal{A}_h - 2N^2)\\ &+ Q^2 + P^2 + g^2\mathcal{A}_h^2]^2 + \frac{J_N^2}{\mathcal{A}_h},\end{aligned} \tag{34}$$

which is nothing but the squared-mass formula

$$\begin{aligned}M^2 = \frac{1}{\mathcal{A}_h}\bigg[&\left(1+\frac{32\pi}{3}\mathcal{P}N^2\right)(\mathcal{A}_h - 2N^2)\\ &+ Q^2 + P^2 + \frac{8\pi}{3}\mathcal{P}\mathcal{A}_h^2\bigg]^2 + \frac{J_N^2}{\mathcal{A}_h},\end{aligned} \tag{35}$$

after introducing the generalized pressure $\mathcal{P} = 3g^2/(8\pi)$ to replace the cosmological constant. We also point out that the squared-mass formula (35) consistently reduces to the one obtained in the four-dimensional RN-NUT-AdS spacetime case [13] when the magnetic charge P vanishes.

In the following, the differential and integral mass formulas for the dyonic RN-NUT-AdS$_4$ spacetime are derived by assuming that the whole set of thermodynamic quantities is the mass M, the NUT charge N, the electric and magnetic charges (Q, P), the generalized pressure $\mathcal{P}$, and the only one secondary hair $J_N = MN$ which will also be viewed as a independent variable.[2] In this way, the squared-mass formula (35) then can be viewed as a fundamental functional relation $M = M(\mathcal{A}_h, N, J_N, Q, P, \mathcal{P})$ of its thermodynamical variables.

Applying a similar procedure as manipulated in the last section, i.e., performing the partial derivative of the above squared-mass formula (35) (multiplied by $4\mathcal{A}_h$) with respect to one of the thermodynamical quantities ($\mathcal{A}_h$, N, J_N, Q, P, $\mathcal{P}$) and simultaneously fixing the remaining ones, respectively, leads to its corresponding conjugate quantities. First, differentiating the squared-mass formula (35) with respect to the reduced horizon area $\mathcal{A}_h$ yields one half of the surface gravity,

[2]A detailed discussion about the impact of $J_N = MN$ on the mass formulas is presented in Sec. V.

$$\kappa = 2\frac{\partial M}{\partial \mathcal{A}_h}\bigg|_{(N,J_N,Q,P,\mathcal{P})} = \frac{r_h - M + 2g^2(r_h^2 + 3N^2)r_h}{r_h^2 + N^2}, \quad (36)$$

which coincides with the one given in Eq. (30). Next, the potential ψ_h and the quasiangular momentum ω_h, which are conjugate to N and J_N, respectively, are given by

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A}_h,J_N,Q,P,\mathcal{P})} = 2Nr_h\frac{-1 + 2g^2(r_h^2 - 3N^2)}{r_h^2 + N^2}, \quad (37)$$

$$\omega_h = \frac{\partial M}{\partial J_N}\bigg|_{(\mathcal{A}_h,N,Q,P,\mathcal{P})} = \frac{N}{r_h^2 + N^2}. \quad (38)$$

By differentiating the squared-mass formula (35) with respect to the electric and magnetic charges (Q, P), respectively, one can get the conjugate electrostatic and magneto-static potentials as follows:

$$\hat{\Phi} = \frac{\partial M}{\partial Q}\bigg|_{(\mathcal{A}_h,N,J_N,P,\mathcal{P})} = \frac{Qr_h}{r_h^2 + N^2}, \quad (39)$$

$$\hat{\Psi} = \frac{\partial M}{\partial P}\bigg|_{(\mathcal{A}_h,N,J_N,Q,\mathcal{P})} = \frac{Pr_h}{r_h^2 + N^2}. \quad (40)$$

These two quantities are also different from those given in Eq. (32). Finally, via the differentiation of the squared-mass formula (35) with respect to the pressure $\mathcal{P}$, one can obtain a conjugate thermodynamical volume,

$$\mathcal{V} = \frac{\partial M}{\partial \mathcal{P}}\bigg|_{(\mathcal{A}_h,N,J_N,Q,P)} = \frac{4\pi r_h(r_h^4 + 6N^2r_h^2 - 3N^4)}{3(r_h^2 + N^2)}. \quad (41)$$

Using all of the above thermodynamical conjugate pairs, we can easily check that both differential and integral mass formulas are completely obeyed,

$$dM = (\kappa/2)d\mathcal{A}_h + \omega_h dJ_N + \psi_h dN + \mathcal{V}d\mathcal{P} + \hat{\Phi}dQ + \hat{\Psi}dP, \quad (42)$$

$$M = \kappa\mathcal{A}_h + 2\omega_h J_N + \psi_h N - 2\mathcal{V}\mathcal{P} + \hat{\Phi}Q + \hat{\Psi}P. \quad (43)$$

It is natural to recognize

$$S = \frac{A_h}{4} = \pi\mathcal{A}_h, \qquad T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi\mathcal{A}_h}, \quad (44)$$

so that the solution behaves like a genuine black hole without violating the beautiful one-quarter area/entropy law. In sharp contrast with Ref. [3], we do not require in advance that the first law should be obeyed so as to obtain the consistent thermodynamical relations—rather it is just a very natural byproduct of the pure algebraic deduction.

B. Consistent mass formulas

One may notice that there are two shortcomings of our work done in the last subsection. The first one is that the obtained electrostatic and magnetostatic potentials do not coincide with those computed via the standard method, and the second one is that our derived conjugate thermodynamical volume $\mathcal{V}$ is not equal to the familiar one, $\tilde{\mathcal{V}} = 4\pi r_h(r_h^2 + 3N^2)/3$, which appeared in other literature [1–4,62]. Below, we will resolve these two inconsistences one by one.

To settle the first contradiction, like the case without a cosmological constant, we just need to further introduce two new additional secondary hairs, $Q_N = QN$ and $P_N = PN$, together with their thermodynamic conjugate electrostatic and magnetostatic potentials

$$\Phi_N = \frac{P}{r_h^2 + N^2}, \qquad \Psi_N = \frac{-Q}{r_h^2 + N^2}, \quad (45)$$

to get the standard forms of the Bekenstein-Smarr formula and the first law as follows:

$$M = 2TS + 2\omega_h J_N + \psi_h N + \Phi Q + \Psi P - 2\mathcal{V}\mathcal{P}, \quad (46)$$

$$\begin{aligned} dM &= TdS + \omega_h dJ_N + \psi_h dN + \Phi dQ + \Psi dP \\ &\quad + \mathcal{V}d\mathcal{P} + \frac{Pd(QN) - Qd(PN)}{r_h^2 + N^2}, \\ &= TdS + \omega_h dJ_N + \psi_h dN + \Phi dQ + \Psi dP \\ &\quad + \mathcal{V}d\mathcal{P} + \Phi_N dQ_N + \Psi_N dP_N, \end{aligned} \quad (47)$$

which are our consistent and reasonable thermodynamical first law and Bekenstein-Smarr mass formulas for the dyonic RN-NUT-AdS$_4$ spacetime. The first law (47) indicates that there are three classes of special cases without introducing the secondary hairs, $Q_N = QN$ and $P_N = PN$, as well as their conjugate potentials (Φ_N, Ψ_N): (i) purely electric-charged ($P = 0$) or purely magnetic-charged ($Q = 0$) case; (ii) NUT-less dyonic solution ($N = 0$); (iii) self-dual or antiself-dual vector-potential case ($|Q| = |P|$).

On the basis of this modification, we are now ready to remove the second conflict via replacing the derived conjugate thermodynamical volume $\mathcal{V}$ by $\tilde{\mathcal{V}} = 4\pi r_h(r_h^2 + 3N^2)/3$, and further introducing the dual (magnetic) mass $\tilde{M} = N(1 + 4g^2N^2)$ into the above differential and integral mass formulas (46)–(47). Now we get the following consistent mass formulas

$$\begin{aligned} dM &= TdS + \omega_h dJ_N + \widetilde{\psi_h}dN + \zeta d\tilde{M} + \Phi dQ + \Psi dP \\ &\quad + \Phi_N dQ_N + \Psi_N dP_N + \tilde{\mathcal{V}}d\mathcal{P}, \\ M &= 2TS + 2\omega_h J_N + \widetilde{\psi_h}N + \zeta\tilde{M} + \Phi Q + \Psi dP - 2\tilde{\mathcal{V}}\mathcal{P}, \end{aligned}$$

in which two new conjugate potentials are given by

$$\widetilde{\psi_h} = -\frac{2Nr_h}{\mathcal{A}_h} - (1-4g^2N^2)\zeta, \qquad \zeta = \frac{r_h(r_h^2-3N^2)}{4N\mathcal{A}_h}.$$

It is of little possibility to reproduce the thermodynamical volume $\tilde{\mathcal{V}}$ without the inclusion of the dual mass $\tilde{M}$.

It should be pointed out that unlike the formalism advocated in other papers [3,8] where there are electric-type, magnetic-type, mixed-type, and many other versions of the 'consistent' first law in which the thermodynamic mass also remains unmodified, here our consistent mass formulas are unique. In contrast, Awad *et al.* [12] proposed modifying the thermodynamic mass which includes the contribution from the Misner string so that the first law retains its usual form without introducing new thermodynamical conjugate pairs, although they used a four-dimensional planar NUT-charged spacetime as a special example. According to this fashion, it is shown in Appendix A that there are infinitely many consistent mass formulas for the dyonic RN-NUT-AdS$_4$ spacetime.

IV. CONSISTENT MASS FORMULAS OF THE DYONIC KN-NUT SPACETIME

Finally, we will show that the general rotating Lorentzian dyonic NUT-charged case without a cosmological constant can be treated completely in the same pattern as did in the last two sections. The line element of the dyonic KN-NUT spacetime with the Misner strings symmetrically distributed along the rotation axis, the electromagnetic one-form and its dual one-form are

$$ds^2 = -\frac{\Delta(r)}{\Sigma}X^2 + \frac{\Sigma}{\Delta(r)}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}Y^2, \tag{48}$$

$$\begin{aligned}\mathbf{A} &= \frac{Qr-PN}{\Sigma}X - \frac{P\cos\theta}{\Sigma}Y \\ &= \frac{Qr-P(N+a\cos\theta)}{\Sigma}X + P\cos\theta d\phi,\end{aligned} \tag{49}$$

$$\begin{aligned}\tilde{\mathbf{A}} &= \frac{Pr+QN}{\Sigma}X + \frac{Q\cos\theta}{\Sigma}Y \\ &= \frac{Pr+Q(N+a\cos\theta)}{\Sigma}X - Q\cos\theta d\phi,\end{aligned} \tag{50}$$

where $\Sigma = r^2 + (N + a\cos\theta)^2$, and

$$\begin{aligned}\Delta(r) &= r^2 + a^2 - 2Mr - N^2 + Q^2 + P^2, \\ X &= dt + (2N\cos\theta - a\sin^2\theta)d\phi, \\ Y &= adt - (r^2 + a^2 + N^2)d\phi.\end{aligned}$$

The global conserved charges for this spacetime are the Komar mass M, the angular momentum $J = Ma$, the electric and magnetic charges (Q, P), and the gravitomagnetic charge or dual (magnetic) mass (both of which are identical to the NUT charge N). These conserved charges display obviously in the leading order of the following asymptotic expansions of the metric components and the Abelian potentials at infinity,

$$\begin{aligned}g_{tt} &\simeq -1 + \frac{2M}{r} + \mathcal{O}(r^{-2}), \\ \mathbf{A}_t &\simeq \frac{Q}{r} + \mathcal{O}(r^{-2}), \qquad \tilde{\mathbf{A}}_t \simeq \frac{P}{r} + \mathcal{O}(r^{-2}), \\ g_{t\phi} &\simeq \left(-2N + \frac{4MN}{r}\right)\cos\theta - \frac{2Ma}{r}\sin^2\theta + \mathcal{O}(r^{-2}), \\ \mathbf{A}_\phi &\simeq \left(P + \frac{2QN}{r}\right)\cos\theta - \frac{Qa}{r}\sin^2\theta + \mathcal{O}(r^{-2}), \\ \tilde{\mathbf{A}}_\phi &\simeq \left(-Q + \frac{2PN}{r}\right)\cos\theta - \frac{Pa}{r}\sin^2\theta + \mathcal{O}(r^{-2}).\end{aligned} \tag{51}$$

Besides these primary hairs, there are also the secondary hairs, (MN, QN, PN), electric-dipole, and magnetic-dipole moments (Qa, Pa) appearing in the next-to-leading order of the above asymptotic expansions.

The event and Cauchy horizons are determined by $\Delta(r_h) = 0$, which gives $r_h = r_\pm = M \pm \sqrt{M^2 + N^2 - Q^2 - P^2 - a^2}$. The event horizon area is $A_h = 4\pi\mathcal{A}_h$, with the reduced horizon area now being $\mathcal{A}_h = r_h^2 + a^2 + N^2 = 2Mr_h + 2N^2 - Q^2 - P^2$.

At the horizon, the surface gravity and the angular velocity can be evaluated via the standard method as

$$\kappa = \frac{\Delta'(r_h)}{2\mathcal{A}_h} = \frac{r_h - M}{\mathcal{A}_h}, \qquad \Omega = \frac{-g_{t\phi}}{g_{\phi\phi}}\bigg|_{r=r_h} = \frac{a}{\mathcal{A}_h}. \tag{52}$$

The electrostatic and magnetostatic potentials simply identify with those at the horizons and read

$$\begin{aligned}\Phi &= \Phi_h = (\mathbf{A}_\mu\xi^\mu)|_{r=r_h} = \frac{Qr_h - PN}{\mathcal{A}_h}, \\ \Psi &= \Psi_h = (\tilde{\mathbf{A}}_\mu\xi^\mu)|_{r=r_h} = \frac{Pr_h + QN}{\mathcal{A}_h},\end{aligned} \tag{53}$$

where $\xi = \partial_t + \Omega\partial_\phi$ is the corotating Killing vector normal to the horizon.

A. Mass formulas with the secondary hair: $J_N = MN$ only

Adopting the same procedure as we did in the last two sections and supposing that only one secondary hair, $J_N = MN$, is needed to be included as before, we square the following identity $2Mr_h = \mathcal{A}_h - 2N^2 + Q^2 + P^2$. Then after adding $4M^2(a^2 + N^2)$ to its left-hand side and $4J^2 + 4J_N^2$ to its right-hand side, followed by dividing both sides with $4M^2$, we can obtain a squared-mass formula,

$$M^2 = \frac{1}{4\mathcal{A}_h}(\mathcal{A}_h - 2N^2 + Q^2 + P^2)^2 + \frac{J_N^2 + J^2}{\mathcal{A}_h}, \quad (54)$$

which consistently reduces to the one obtained in the four-dimensional KN-NUT spacetime case [13] when the magnetic charge P is turned off.

Incidentally, if the squared-mass (54) is rewritten as a binomial of the reduced horizon area,

$$\mathcal{A}_h^2 + 2(P^2 + Q^2 - 2N^2 - 2M^2)\mathcal{A}_h + 4J^2 + 4M^2N^2 + (P^2 + Q^2 - 2N^2)^2 = 0, \quad (55)$$

then the area product of the inner and outer horizons,

$$\mathcal{A}_+\mathcal{A}_- = 4J^2 + 4J_N^2 + (P^2 + Q^2 - 2N^2)^2 \quad (56)$$

can be quantized only when $J_N = MN$ is quantized in a manner just as the angular momentum $J = m\hbar$ is quantized, and the charges (Q, P, N) take discrete values.

Suppose for a moment that the secondary hair, $J_N = MN$, is an independent conserved charge (namely, it can be treated as a independent thermodynamical variable)[3] then Eq. (54) formally represents a fundamental functional relation $M = M(\mathcal{A}_h, J, J_N, N, Q, P)$ with the whole set of the extensive variables being the NUT charge N, the electric and magnetic charges (Q, P), the angular momentum J, the secondary hair J_N, and $\mathcal{A}_h$ as the intense quantity of the dyonic KN-NUT spacetime. Then, differentiating the above squared-mass formula (54) with respect to one variable of the whole set of the thermodynamical quantities ($\mathcal{A}_h$, J, J_N, N, Q, P) and simultaneously fixing the remaining ones, respectively, gives rise to its corresponding conjugate quantities. Subsequently, one can derive the differential and integral mass formulas with the conjugate thermodynamical potentials reproduced by the ordinary Maxwell relations.

The conjugate quantity of the reduced horizon area $\mathcal{A}_h$ is one half of the surface gravity,

$$\kappa = 2\frac{\partial M}{\partial \mathcal{A}_h}\bigg|_{(J,J_N,N,Q,P)} = \frac{r_h - M}{\mathcal{A}_h}. \quad (57)$$

The angular velocity, which is conjugate to J, is given by

$$\Omega = \frac{\partial M}{\partial J}\bigg|_{(\mathcal{A}_h,J_N,N,Q,P)} = \frac{a}{\mathcal{A}_h}. \quad (58)$$

These two conjugate quantities are entirely identical to those given in Eq. (52). Differentiating the squared-mass formula (54) with respect to the NUT charge N and the secondary hair J_N, one can get the conjugate gravitomagnetic potential,

$$\psi_h = \frac{\partial M}{\partial N}\bigg|_{(\mathcal{A}_h,J,J_N,Q,P)} = -\frac{2Nr_h}{\mathcal{A}_h}, \quad (59)$$

and a conjugate quasiangular momentum,

$$\omega_h = \frac{\partial M}{\partial J_N}\bigg|_{(\mathcal{A}_h,J,N,Q,P)} = \frac{N}{\mathcal{A}_h}. \quad (60)$$

Differentiating the squared-mass formula (54) with respect to the electric and magnetic charges (Q, P), respectively, yields the conjugate electrostatic and magnetostatic potentials,

$$\hat{\Phi} = \frac{\partial M}{\partial Q}\bigg|_{(\mathcal{A}_h,J,J_N,N,P)} = \frac{Qr_h}{\mathcal{A}_h}, \quad (61)$$

$$\hat{\Psi} = \frac{\partial M}{\partial P}\bigg|_{(\mathcal{A}_h,J,J_N,N,Q)} = \frac{Pr_h}{\mathcal{A}_h}, \quad (62)$$

which are different from those given in Eq. (53).

One can also easily demonstrate both the differential and integral mass formulas are completely fulfilled

$$dM = (\kappa/2)d\mathcal{A}_h + \Omega dJ + \omega_h dJ_N + \psi_h dN + \hat{\Phi}dQ + \hat{\Psi}dP, \quad (63)$$

$$M = \kappa\mathcal{A}_h + 2\Omega J + 2\omega_h J_N + \psi_h N + \hat{\Phi}Q + \hat{\Psi}P, \quad (64)$$

after using all the above thermodynamical conjugate pairs.

The consistency of the above mass formulas (63)–(64) suggests that one should restore the well-known Bekenstein-Hawking area/entropy relation and Hawking temperature

$$S = \frac{A_h}{4} = \pi\mathcal{A}_h, \qquad T = \frac{\kappa}{2\pi} = \frac{r_h - M}{2\pi\mathcal{A}_h}, \quad (65)$$

which means that the whole class of the four-dimensional dyonic NUT-charged spacetimes should be viewed as generic black holes.

B. Two new secondary hairs $Q_N = QN$ and $P_N = PN$

In this subsection, we will show that a consistent and reasonable first law and Bekenstein-Smarr mass formula of the dyonic KN-NUT spacetime can still be obtained via introducing two new additional secondary hairs, $Q_N = QN$ and $P_N = PN$, together with their thermodynamic conjugate potentials,

$$\Phi_N = \frac{P}{\mathcal{A}_h}, \qquad \Psi_N = \frac{-Q}{\mathcal{A}_h}. \quad (66)$$

By the replacement of the electrostatic and magnetostatic potentials, it is not difficult to see that the integral mass formula becomes

[3]We will discuss the impact of $J_N = MN$ in Sec. V.

$$M = 2TS + 2\Omega J + 2\omega_h J_N + \psi_h N + \Phi Q + \Psi P, \quad (67)$$

while the differential mass formula is rewritten as follows:

$$\begin{aligned} dM &= TdS + \Omega dJ + \omega_h dJ_N + \psi_h dN + \Phi dQ \\ &\quad + \Psi dP + \frac{Pd(QN) - Qd(PN)}{\mathcal{A}_h}, \\ &= TdS + \Omega dJ + \omega_h dJ_N + \psi_h dN + \Phi dQ \\ &\quad + \Psi dP + \Phi_N dQ_N + \Psi_N dP_N. \end{aligned} \quad (68)$$

The first law (68) implies that there are three kind of special cases with no need of introducing the secondary hairs, $Q_N = QN$ and $P_N = PN$, as well as their conjugate potentials (Φ_N, Ψ_N): (i) purely electric-charged ($P = 0$) or purely magnetic-charged ($Q = 0$) case; (ii) dyonic KN solution ($N = 0$); (iii) self-dual or antiself-dual vector potential case ($Q = \pm P$).

Both Eqs. (67) and (68) are expressed in the standard forms, which are our expected consistent thermodynamical first law and Bekenstein-Smarr mass formula for the four-dimensional dyonic KN-NUT spacetime. They are not only simple, but also unique, unlike the work [5] which declared that there are several different versions for them.

V. REDUCED MASS FORMULAS

In the Secs. II A, III A, and IV A, the secondary hair, $J_N = MN$, has been viewed as a independent thermodynamic variable, its impact on the thermodynamical relations has been ignored. In this section, we investigate this issue and derive the corresponding reduced mass formulas of the dyonic RN-NUT, dyonic RN-NUT-AdS$_4$ and dyonic KN-NUT spacetimes, respectively. This is somewhat analogous to those about the chirality condition, $J = Ml$, (l is the cosmological radius) of the superentropic Kerr-Newman-AdS$_4$, ultraspinning Kerr-Sen-AdS$_4$ and ultraspinning dyonic Kerr-Sen-AdS$_4$ black holes [55–57].

Now, considering $J_N = MN$ as a redundant variable and taking into account its differentiation $dJ_N = MdN + NdM$, followed by eliminating them from the differential and integral mass formulas with the help of $N = J_N/M$, then the first law and Bekenstein-Smarr mass formula boil down to their nonstandard forms, which are listed below for the spacetimes considered before.

(a) Dyonic RN-NUT spacetime

$$\begin{aligned} (1 - \omega_h N)dM &= (\kappa/2)d\mathcal{A}_h + \bar{\psi}_h dN + \hat{\Phi} dQ + \hat{\Psi} dP, \\ (1 - \omega_h N)M &= \kappa\mathcal{A}_h + \bar{\psi}_h N + \hat{\Phi} Q + \hat{\Psi} P; \end{aligned}$$

*(b) Dyonic RN-NUT-*AdS$_4$ *spacetime*

$$\begin{aligned} (1 - \omega_h N)dM &= (\kappa/2)d\mathcal{A}_h + \bar{\psi}_h dN + \hat{\Phi} dQ + \hat{\Psi} dP + \mathcal{V} d\mathcal{P}, \\ (1 - \omega_h N)M &= \kappa\mathcal{A}_h + \bar{\psi}_h N + \hat{\Phi} Q + \hat{\Psi} P - 2\mathcal{V}\mathcal{P}; \end{aligned}$$

(c) Dyonic KN-NUT spacetime

$$\begin{aligned} (1 - \omega_h N)dM &= (\kappa/2)d\mathcal{A}_h + \Omega dJ + \bar{\psi}_h dN \\ &\quad + \hat{\Phi} dQ + \hat{\Psi} dP, \\ (1 - \omega_h N)M &= \kappa\mathcal{A}_h + 2\Omega J + \bar{\psi}_h N + \hat{\Phi} Q + \hat{\Psi} P, \end{aligned}$$

where $\bar{\psi}_h = \psi_h + \omega_h M$ in each case.

It is easy to see that all of the thermodynamic quantities in these reduced mass formulas cannot constitute the ordinary canonical conjugate pairs due to the presence of a factor $(1 - \omega_h N)$ in front of dM and M. By the way, we mention that the above nonstandard mass formulas partially appeared in some papers [63–66].

VI. CONCLUDING REMARKS

In this paper, we have extended our previous work [13] to the more general four-dimensional dyonic NUT-charged cases, and followed a simple, systematic way to naturally derive the thermodynamical first law and Bekenstein-Smarr mass formula via differentiating the Christodoulou-Ruffini-like squared-mass formula with respect to its thermodynamic variables. If only a secondary hair, $J_N = MN$, is included (as we did before) then the obtained thermodynamical conjugate pairs fulfill the standard forms of the differential and integral mass formulas, except that the derived electrostatic and magneto-static potentials are not equal to those calculated by the standard method. Then, we demonstrated that this contradiction can be rectified via further introducing two new additional secondary hairs, $Q_N = QN$ and $P_N = PN$, together with their thermodynamical conjugate potentials (Φ_N, Ψ_N). We have determined some special cases with no need to include them, of which the $Q^2 = P^2$ case with the self-dual and antiself-dual Abelian vector potentials is possible to be particularly interesting. After that, the impact of the secondary hair, $J_N = MN$, on the thermodynamics and the reduced mass formulas are discussed.

Our work demonstrated that, not only can the beautiful Bekenstein-Hawking one-quarter area-entropy relation be naturally restored, but also that the four laws of the usual black hole thermodynamics are completely applicable to the dyonic Taub-NUT-type spacetimes. We are believed that the strategy proposed in our papers has provided the best and simplest scheme to formulate the consistent thermodynamical relations for the NUT-charged spacetimes. A related issue to investigate is the consistent thermodynamics of the four-dimensional NUT-charged spacetimes in the K-K theory [31–33] and the EMDA theory [34]. We hope to report the related progress soon.

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APPENDIX MANY OTHER "CONSISTENT" MASS FORMULAS

Currently, there exist three different fashions to formulate the consistent first law of the four-dimensional NUT-charged spacetimes: (I) Keeping the thermodynamic mass unmodified and introducing new global-like charges (secondary hairs) and their conjugate potentials [13]; (II) Retaining the thermodynamic mass unchanged and introducing new nonglobal Misner charge and its conjugate variable [1–5,8]; (III) Only modifying the thermodynamic mass by including the contribution from the Misner string [12]. In our formalism [13], the consistent mass formulas are unique, and every expressions for the thermodynamical quantities are very simple and concise. In contrast, not only can the consistent first law of the NUTty dyonic spacetimes have ironically different possibilities to be formulated as the electric-type, magnetic-type, mixed-type versions [3,5], and even many others [8], but also the expressions of some related thermodynamical quantities are very complicated. Below, we will show that there are infinitely many consistent mass formulas for the dyonic RN-NUT-AdS$_4$ spacetime if the thermodynamic mass is modified *a la* the mode proposed by Awad *et al.* [12].

1. Infinitely many 'consistent' mass formulas for dyonic RN-NUT-AdS$_4$ spacetime

In the spirit of Ref. [12] we consider modifying the thermodynamic mass which receives the contribution from the Misner string so that the first law retains its usual form without introducing new thermodynamical conjugate pairs. Then we can find that there are infinitely many consistent mass formulas for the dyonic RN-NUT-AdS$_4$ spacetime as follows:

$$\tilde{M} = 2TS + \Phi\tilde{Q} + \Psi\tilde{P} - 2\tilde{\mathcal{V}}\mathcal{P} + N\chi, \tag{A1}$$

$$d\tilde{M} = TdS + \Phi d\tilde{Q} + \Psi d\tilde{P} + \tilde{\mathcal{V}}d\mathcal{P} + \chi dN, \tag{A2}$$

where the expressions of (T, S, Φ, Ψ) are given in Sec. III, the pressure is $\mathcal{P} = 3g^2/(8\pi)$, and

$$\tilde{M} = M - N\chi, \qquad \tilde{\mathcal{V}} = \frac{4\pi}{3}r_h(r_h^2 + 3N^2),$$
$$\tilde{Q} = Q + (w-1)N\Psi - 2w_1N\Phi,$$
$$\tilde{P} = P + (w+1)N\Phi - 2w_2N\Psi,$$

together with a new thermodynamic potential conjugate to the NUT charge N,

$$\begin{aligned}\chi &= N\frac{M - r_h + g^2r_h(r_h^2 - 3N^2)}{r_h^2 + N^2} \\ &\quad - w\Phi\Psi + w_1\Phi^2 + w_2\Psi^2 \\ &= \frac{-N}{2r_h}[1 - 3g^2(r_h^2 - N^2)] + \frac{N(Q^2 + P^2)}{2r_h(r_h^2 + N^2)} \\ &\quad - w\Phi\Psi + w_1\Phi^2 + w_2\Psi^2,\end{aligned}$$

in which w, w_1, and w_2 are three arbitrary constants.

Some special cases may be very interesting. Let $w_1 = w_2 = 0$, then $\tilde{Q} = Q$ and $\tilde{P} = P + 2N\Phi \equiv P_h$ when $w = 1$, whilst $\tilde{Q} = Q - 2N\Psi \equiv Q_h$ and $\tilde{P} = P$ when $w = -1$. A very simple case is to set $w = w_1 = w_2 = 0$, then χ is proportional to the value of the NUT (twist) potential measured at the horizon relative to the infinity. Since w, w_1, and w_2 can take arbitrary values, then a natural consequence is that there are many different versions for the Lorentzian dyonic RN-NUT-AdS$_4$ spacetime.

2. Other type of mass formulas for Kerr-NUT spacetime

Extend to the rotating Kerr-NUT case, other consistent mass formulas are given below,

$$\tilde{M} = 2TS + 2\Omega\tilde{J} + N\chi, \tag{A3}$$

$$d\tilde{M} = TdS + \Omega d\tilde{J} + \chi dN, \tag{A4}$$

where

$$T = \frac{r_h - M}{2\pi\mathcal{A}_h}, \qquad \Omega = \frac{a}{\mathcal{A}_h}, \qquad \chi = \frac{-N}{2r_h},$$
$$\tilde{M} = M - N\chi, \qquad \tilde{J} = (M - 2N\chi)a, \qquad S = \pi\mathcal{A}_h.$$

in which $\mathcal{A}_h = r_h^2 + a^2 + N^2 = 2Mr_h + 2N^2$.

Clearly, the above two appendixes show that there are many different versions of the consistent first law and Bekenstein-Smarr mass formula in the Lorentzian Taub-NUT-type spacetimes by modifying the thermodynamic mass without introducing a new thermodynamical conjugate pair. Thus, a natural puzzle is which version of the 'consistent' first law is the most appropriate one?

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