First-order thermodynamics of Horndeski gravity

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We extend to the Horndeski realm the irreversible thermodynamics description of gravity previously studied in "first generation" scalar-tensor theories. We identify a subclass of Horndeski theories as an out-of–equilibrium state, while general relativity corresponds to an equilibrium state. In this context, we identify an effective heat current, "temperature of gravity," and shear viscosity in the space of theories. The identification is accomplished by recasting the field equations as effective Einstein equations with an effective dissipative fluid, with Einstein gravity as the equilibrium state, following Eckart's first-order thermodynamics.

I. INTRODUCTION

A connection between gravity and thermodynamics has been drawn with the discovery of black hole and horizon thermodynamics and has been augmented by the suggestion that gravity may not be fundamental after all, but rather that it emerges from more fundamental constituents in the same way that a fluid emerges from its atoms or molecules. An indication that this idea may not be too far fetched is the derivation of the Einstein field equation of general relativity (GR) by Jacobson as an equation of state, using thermodynamical considerations [1]. This view of gravity has profound implications for quantum gravity as well, since it implies that quantizing the Einstein equation is rather meaningless—the analogy would be that it amounts to quantizing the macroscopic ideal gas equation of state. Such a quantization could, at most, find phonons but certainly not results as fundamental as the eigenfunctions and energy spectrum of the Hamiltonian of the hydrogen atom. If they exist, the "atoms of spacetime" would not be directly related to the Einstein equation.

One then wonders what role extensions of GR may play in the broader context of a gravity-thermodynamics connection. A bold idea was advanced in the second seminal paper [2], in which the field equation of metric $f(R)$ gravity was also obtained with thermodynamical considerations. According to [2], this fourth order modification of GR corresponds to a nonequilibrium state in a "thermodynamics of gravitational theories." Although the latter is not fully developed, a "bulk viscosity of spacetime" is introduced to

drive a dissipation process of gravity towards a thermodynamic equilibrium state, which is Einstein's gravity as previously indicated in [1]. In other words, GR would be an equilibrium state while $f(R)$ gravity is an excited (or nonequilibrium) state. By extension, any gravity theory containing extra dynamical degrees of freedom with respect to the two massless spin-two modes of GR should correspond to an "excited state". Indeed, referring to extra dynamical degrees of freedom makes the idea of GR as a "ground state" appear very reasonable.

Extensions of gravity are well-motivated from the theoretical point of view. It is well known that virtually any attempt to introduce quantum corrections to GR involves extending it by introducing higher order corrections or extra degrees of freedom, and one expects extensions to GR to appear as soon as one moves away from the lowest energy regime. For example, one-loop renormalization requires the introduction of terms quadratic in the curvature and, to date, the most successful scenario of early universe inflation [3,4], i.e., Starobinsky inflation [5], is based on R^2 corrections to the Einstein-Hilbert action. The low-energy limit of the bosonic string, the most rudimental string theory known, does not reproduce GR but gives instead an $\omega = -1$ Brans-Dicke theory (where ω is the Brans-Dicke coupling constant) [6,7].

Today, the main motivation for extended gravity theories is no doubt given by cosmology [8]. The 1998 discovery of the accelerated expansion of the universe made with type Ia supernovae calls for an explanation. The standard Λ-cold dark matter (ΛCDM) cosmological model based on GR requires the introduction of either an incredibly fine-tuned cosmological constant Λ, or of a completely *ad hoc* dark energy (see [9] for a discussion). An alternative consists of extending gravity at large cosmological scales without

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introducing dark energy [10,11]. Many approaches to extended gravity theories motivated by cosmology have been researched, with $f(R)$ gravity probably being the most popular (see [12–14] for reviews). Gravity is tested rather poorly at many scales and in certain regimes [15,16], which leaves ample room for extended gravity theories. In view of the above, it is only natural to contemplate the role of extensions when investigating the connection between gravity and thermodynamics.

The seminal papers [1,2] have been very influential. However, in spite of a large literature, no light has been shed on the equations ruling the dissipation of gravity towards the GR equilibrium state and the order parameter indicating how close the nonequilibrium is to the GR equilibrium state has not been identified. Overall, not much progress has been made in this direction since the works [1,2]. We ought to mention, however, an important result: Ref. [17] identified the essential role played by shear viscosity, while eliminating bulk viscosity from the thermodynamics of spacetime picture.

Perhaps the lack of progress is due to the fact that the ideas advanced in Refs. [1,2] are so fundamental that they require at least some more advanced knowledge of the basic ingredients in order to be developed. In previous papers [18,19], we proposed a more modest approach in a very different context, but in the same spirit. First, within the wide spectrum of gravity theories, we identified the scalar-tensor class generalizing Brans-Dicke gravity [20–24] as the most suitable candidate for our new approach. This class contains $f(R)$ gravity as a subclass [12–14]. Brans-Dicke gravity [20] and its scalar-tensor generalizations [21–24] are minimal extensions of GR since they contain only a scalar degree of freedom ϕ in addition to the usual two massless spin two modes of GR. The description of scalar-tensor gravity as an excited state of GR does not require fundamental assumptions such as those of Refs. [1,2] in spacetime thermodynamics (for instance, the notions of local causal horizon and of local Rindler frame to take advantage of a local Unruh effect). In fact, the structure of a dissipative fluid containing GR as its limit is already contained in the field equations of scalar-tensor gravity. More precisely, the contribution of the scalar degree of freedom ϕ to the field equations has the structure of the energy-momentum tensor of an effective relativistic dissipative fluid [25,26]. This fact can be derived in a straightforward manner by rewriting the field equations and is not an independent assumption.

By taking seriously the dissipative fluid nature of the effective stress-energy tensor of ϕ, one wonders what the minimal theory of relativistic dissipative fluids has to say. With this question in mind, we have applied Eckart's first-order thermodynamics [27–29] (which is noncausal and plagued by instabilities, but is nevertheless the most widely applied formalism to describe dissipation in GR) to the effective ϕ-fluid. Explicit expressions for the

corresponding thermodynamical quantities were obtained, including the heat current density, the sought-for "temperature of extended gravity," and shear viscosity.

In the last decade, "first generation" scalar-tensor gravity has been generalized by rediscovering and reformulating [30–32] the old Horndeski gravity [33], which still contains only an extra scalar degree of freedom obeying second order equations of motion but is much more general. These efforts were followed by the discovery that higher order Lagrangians still admit second order equations of motion and are healthy with respect to the Ostrogradski instability when subjected to a degeneracy condition, resulting in the so-called degenerate higher order scalar-tensor ("DHOST") theories [34,35]. These generalizations, developed in [36–41], introduce a large number of terms in the gravitational sector of the action and have generated a rich literature (see Refs. [42,43] for reviews). DHOST theories are restricted by theoretical constraints avoiding graviton decay into scalar field perturbations [44] and, above all, by the recent multimessenger observation of simultaneous gravitational waves and γ-rays in the GW170817/ GRB170817 event [45,46]. The latter sets severe constraints on the space of DHOST and Horndeski theories from the observed upper limits on the difference between the propagation speeds of gravitational and electromagnetic waves [47]. Although these constraints are strong at the present time, there is still some room for evading them in the early Universe, which seems to be possible at least in some theories with nonminimal derivative couplings of physical interest (see the discussion in Sec. VII).

Motivated by the explosion of interest in Horndeski gravity, we apply the effective fluid formalism which was successful in "first generation" scalar-tensor gravity to this more general scenario. In particular, we rewrite the field equations as effective Einstein equations containing an effective dissipative fluid in their right-hand side (at this stage, we complete and correct the previous work [48] which computes some of the effective fluid quantities for a subclass of Horndeski theories but does not work out the thermodynamics of the corresponding dissipative fluid). Then, we proceed to identify the corresponding heat flux density, anisotropic stresses, "temperature of gravity," and shear viscosity whenever possible.

As a result, the effective thermodynamics of scalar-tensor gravity does not apply to the most general Horndeski theory because some terms in the field equations explicitly break the thermodynamic analogy by spoiling the constitutive equations of Eckart's theory for the effective ϕ-fluid, contrary to the first generation scalar-tensor theories examined in [18,19]. However, this situation returns to "normal" when the most general Horndeski action is restricted by eliminating the terms violating the equality between the propagation speeds of gravitational and electromagnetic waves. In this sense, the thermodynamics of gravity seems to indicate the way to the physical constraints on Horndeski gravity. Notably, the equivalent

thermodynamical description breaks down for those operators which contain derivative nonminimal couplings and nonlinear contributions in the connection. Those are exactly the operators which do not permit a local field redefinition into the Einstein frame and give rise to intrinsic modifications of the helicity-2 field. It is intriguing that Eckart's first-order thermodynamical description is tightly related to these intrinsic changes of the helicity-2 sector. However, since the failure of rewriting the theory in the Einstein frame is directly related to the failure of separating the "gravity effects" from the "effective matter fluid" it is not a surprise after all.

We follow the notation of Ref. [49]: the metric signature is $(-+++)$ and units are used in which the speed of light c and Newton's constant G are unity.

II. KINEMATIC QUANTITIES FOR THE EFFECTIVE FLUID

Assuming that the scalar field gradient is timelike, $\nabla_a \phi \nabla^a \phi < 0$, and using the notation,

$$X \equiv -\frac{1}{2}\nabla_a\phi\nabla^a\phi > 0, \qquad (2.1)$$

it is natural to define the 4-velocity of the effective fluid associated with the scalar field ϕ as

$$u^a = \frac{\nabla^a\phi}{\sqrt{2X}}, \qquad (2.2)$$

which satisfies $u^a u_a = -1$. Identifying the field u^a identifies uniquely a frame comoving with the effective fluid and a $3+1$ splitting of spacetime in the time direction of these observers with 4-tangent u^a and their 3-space. The Riemannian metric on the 3-space orthogonal to u^a (i.e., the 3-space of the observers comoving with the effective fluid) is

$$h_{ab} = g_{ab} + u_a u_b = g_{ab} + \frac{\nabla_a\phi\nabla_b\phi}{2X}, \qquad (2.3)$$

while $h^a{}_b$ is the projection operator onto this 3-space. The 4-velocity gradient is

$$\nabla_a u_b = \frac{1}{\sqrt{2X}}\left(\nabla_a\nabla_b\phi - \frac{\nabla_a X\nabla_b\phi}{2X}\right). \qquad (2.4)$$

Using the fact that

$$\nabla_a X = -\frac{1}{2}\nabla_a(\nabla_e\phi\nabla^e\phi) = -\nabla_e\phi\nabla_a\nabla^e\phi, \qquad (2.5)$$

this 4-velocity gradient is written as

$$\nabla_a u_b = \frac{1}{\sqrt{-\nabla_e\phi\nabla^e\phi}}\left(\nabla_a\nabla_b\phi + \frac{\nabla_b\phi\nabla_c\phi\nabla_a\nabla^c\phi}{-\nabla_e\phi\nabla^e\phi}\right), \qquad (2.6)$$

that coincides with the corresponding expressions appearing in Refs. [18,26].

The effective fluid has 4-acceleration,

$$\dot{u}^a \equiv u^c\nabla_c u^a = \frac{\nabla^c\phi}{\sqrt{2X}}\frac{1}{\sqrt{2X}}\left(\nabla_c\nabla^a\phi - \frac{\nabla_c X\nabla^a\phi}{2X}\right)$$

$$= -\frac{1}{2X}\left(\nabla^a X + \frac{\nabla X\cdot\nabla\phi}{2X}\nabla^a\phi\right), \qquad (2.7)$$

where $\nabla X\cdot\nabla\phi \equiv g^{ab}\nabla_a X\nabla_b\phi$.

Using $h_a{}^b = \delta_a{}^b + u_a u^b$ and $u^a\nabla_b u_a = 0$, the (double) spatial projection of the velocity gradient is

$$V_{ab} \equiv h_a{}^c h_b{}^d\nabla_d u_c = \nabla_b u_a + \dot{u}_a u_b$$

$$= \frac{1}{\sqrt{2X}}\left(\nabla_b\nabla_a\phi - \frac{\nabla_b X\nabla_a\phi}{2X}\right)$$

$$- \frac{\nabla_b\phi}{\sqrt{2X}}\frac{1}{2X}\left(\nabla_a X + \frac{\nabla X\cdot\nabla\phi}{2X}\nabla_a\phi\right)$$

$$= \frac{1}{\sqrt{2X}}\left(\nabla_a\nabla_b\phi - \frac{\nabla_{(a}X\nabla_{b)}\phi}{X} - \frac{\nabla X\cdot\nabla\phi}{4X^2}\nabla_a\phi\nabla_b\phi\right). \qquad (2.8)$$

This tensor is symmetric, $V_{ab} = V_{(ab)}$, and its antisymmetric part vanishes identically,

$$\omega_{ab} \equiv V_{[ab]} = 0, \qquad (2.9)$$

so that the 4-velocity u^a of the effective ϕ-fluid is irrotational[1] and hypersurface-orthogonal, the spacetime line element $ds^2 = g_{ab}dx^a dx^b$ becomes diagonal in adapted coordinates $\{x^a\}$, and a foliation of three-dimensional hypersurfaces with Riemannian metric h_{ab} always exists [49]. The expansion scalar of the effective fluid is

$$\theta = \nabla_a u^a = \frac{1}{\sqrt{2X}}\left(\Box\phi - \frac{\nabla X\cdot\nabla\phi}{2X}\right)$$

$$= \frac{1}{\sqrt{-\nabla_c\phi\nabla^c\phi}}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla_e\phi\nabla^e\phi}\right), \qquad (2.10)$$

while the trace-free shear tensor reads

$$\sigma_{ab} \equiv V_{(ab)} - \frac{\theta}{3}h_{ab} = V_{ab} - \frac{\theta}{3}h_{ab}$$

$$= \frac{1}{\sqrt{2X}}\left[\nabla_a\nabla_b\phi - \frac{\nabla_{(a}X\nabla_{b)}\phi}{X} - \frac{\nabla X\cdot\nabla\phi}{4X^2}\nabla_a\phi\nabla_b\phi\right.$$

$$\left. - \frac{h_{ab}}{3}\left(\Box\phi - \frac{\nabla X\cdot\nabla\phi}{2X}\right)\right]. \qquad (2.11)$$

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[1]This feature was missed in Ref. [48].

These kinematic quantities for the effective ϕ-fluid coincide with those reported in the previous Refs. [18,26] for the "old" scalar-tensor gravities of [20–24]. This fact is expected because shear, expansion and vorticity are purely kinematic quantities, and cannot depend on the particular theory (i.e., on the action or the field equations), provided that only a scalar degree of freedom ϕ is added to the ordinary spin 2 massless polarizations of the metric tensor in Einstein gravity, as done in Brans-Dicke or in Horndeski gravity.

III. IMPERFECT FLUID DESCRIPTION OF HORNDESKI GRAVITY

The most general Lagrangian of Horndeski gravity reads

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \qquad (3.1)$$

where the individual interactions are given by

$$\begin{aligned}
\mathcal{L}_2 &= G_2, \\
\mathcal{L}_3 &= -G_3\Box\phi, \\
\mathcal{L}_4 &= G_4 R + G_{4X}[(\Box\phi)^2 - (\nabla_a\nabla_b\phi)^2], \\
\mathcal{L}_5 &= G_5 G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi(\nabla_a\nabla_b\phi)^2 \\
&\quad + 2(\nabla_a\nabla_b\phi)^3],
\end{aligned} \qquad (3.2)$$

where ϕ is the scalar degree of freedom, ∇_a is the covariant derivative of the metric g_{ab} (which has determinant g), and $\Box \equiv g^{ab}\nabla_a\nabla_b$ is d'Alembert's operator, G_{ab} denotes the Einstein tensor, while $G_i(\phi, X)$ $(i = 2, 3, 4, 5)$ are arbitrary functions of the scalar field ϕ and of the canonical kinetic term X. According to standard notation, $G_{i\phi} \equiv \partial G_i/\partial\phi$ and $G_{iX} \equiv \partial G_i/\partial X$.

This Lagrangian represents the most general scalar-tensor theory with second order equations of motion. The local and second order nature of the field equations then implies that the theory naturally avoids Ostrogradsky instabilities.

We consider the subclass of Horndeski gravity that implies a luminal propagation of gravitational waves, in which the interactions are restricted to

$$\bar{\mathcal{L}} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R, \qquad (3.3)$$

that is, we set $G_{4X} = 0$, $G_5 = 0$. This last Lagrangian will be the main focus in the following.

Performing its variation,

$$\delta(\sqrt{-g}\bar{\mathcal{L}}) = \sqrt{-g}\sum_{i=2}^{4}\mathcal{G}_{ab}^{(i)}\delta g^{ab} + \sum_{i=2}^{4}(\cdots)\delta\phi$$
$$+ \text{total derivative}, \qquad (3.4)$$

and specializing the results of Ref. [50] to our specific situation, we obtain

$$\mathcal{G}_{ab}^{(2)} = -\frac{1}{2}G_{2X}\nabla_a\phi\nabla_b\phi - \frac{1}{2}G_2 g_{ab}, \qquad (3.5)$$

$$\mathcal{G}_{ab}^{(3)} = \frac{1}{2}G_{3X}\Box\phi\nabla_a\phi\nabla_b\phi + \nabla_{(a}G_3\nabla_{b)}\phi - \frac{1}{2}g_{ab}\nabla_c G_3\nabla^c\phi, \qquad (3.6)$$

$$\mathcal{G}_{ab}^{(4)} = G_4 G_{ab} + g_{ab}(G_{4\phi}\Box\phi - 2XG_{4\phi\phi}) - G_{4\phi}\nabla_a\nabla_b\phi - G_{4\phi\phi}\nabla_a\phi\nabla_b\phi. \qquad (3.7)$$

Recalling now that $\nabla_a G_3 = G_{3\phi}\nabla_a\phi + G_{3X}\nabla_a X$, one can write

$$\begin{aligned}
\nabla_{(a}G_3\nabla_{b)}\phi &= G_{3\phi}\nabla_{(a}\phi\nabla_{b)}\phi + G_{3X}\nabla_{(a}X\nabla_{b)}\phi \\
&= G_{3\phi}\nabla_a\phi\nabla_b\phi + G_{3X}\nabla_{(a}X\nabla_{b)}\phi
\end{aligned} \qquad (3.8)$$

and

$$\begin{aligned}
\nabla_c G_3\nabla^c\phi &= G_{3\phi}\nabla_c\phi\nabla^c\phi + G_{3X}\nabla_c X\nabla^c\phi \\
&= -2XG_{3\phi} + G_{3X}\nabla_c X\nabla^c\phi;
\end{aligned} \qquad (3.9)$$

then we have that

$$\begin{aligned}
\mathcal{G}_{ab}^{(3)} &= \frac{1}{2}(2G_{3\phi} + G_{3X}\Box\phi)\nabla_a\phi\nabla_b\phi + G_{3X}\nabla_{(a}X\nabla_{b)}\phi \\
&\quad - \frac{1}{2}g_{ab}(-2XG_{3\phi} + G_{3X}\nabla X \cdot \nabla\phi).
\end{aligned} \qquad (3.10)$$

For future convenience we also rewrite $\mathcal{G}_{ab}^{(4)}$ as

$$\begin{aligned}
\mathcal{G}_{ab}^{(4)} &= G_4 G_{ab} + G_{4\phi}(g_{ab}\Box\phi - \nabla_a\nabla_b\phi) \\
&\quad - G_{4\phi\phi}(2Xg_{ab} + \nabla_a\phi\nabla_b\phi).
\end{aligned} \qquad (3.11)$$

We can now compute the effective stress-energy tensor of the ϕ-fluid, which is defined by writing the field equations of the specific subclass of Horndeski theories as the effective Einstein equations,

$$G_{ab} = \frac{1}{G_4}T_{ab}^{(m)} + T_{ab}^{(\text{eff})}, \qquad (3.12)$$

where the effective stress-energy tensor is

$$T_{ab}^{(\text{eff})} = T_{ab}^{(2)} + T_{ab}^{(3)} + T_{ab}^{(4)} \qquad (3.13)$$

with

$$T_{ab}^{(2)} = -\frac{\mathcal{G}_{ab}^{(2)}}{G_4} = \frac{1}{2G_4}(G_{2X}\nabla_a\phi\nabla_b\phi + G_2 g_{ab}), \qquad (3.14)$$

$$T_{ab}^{(3)} = -\frac{\mathcal{G}_{ab}^{(3)}}{G_4} = \frac{1}{2G_4}[g_{ab}(G_{3X}\nabla X \cdot \nabla\phi - 2XG_{3\phi})$$
$$- 2G_{3X}\nabla_{(a}X\nabla_{b)}\phi - (2G_{3\phi} + G_{3X}\Box\phi)\nabla_a\phi\nabla_b\phi], \quad (3.15)$$

$$T_{ab}^{(4)} = -\frac{\mathcal{G}_{ab}^{(4)} - G_4 G_{ab}}{G_4} = \frac{G_{4\phi}}{G_4}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)$$
$$+ \frac{G_{4\phi\phi}}{G_4}(2Xg_{ab} + \nabla_a\phi\nabla_b\phi). \quad (3.16)$$

The effective tensor given by Eqs. (3.13)–(3.16) has the form of an imperfect fluid stress-energy tensor,

$$T_{ab} = \rho u_a u_b + q_a u_b + q_b u_a + \Pi_{ab}, \quad (3.17)$$

where

$$\Pi_{ab} = T_{cd}h_a{}^c h_b{}^d = Ph_{ab} + \pi_{ab} \quad (3.18)$$

is the effective stress tensor[2] containing the isotropic pressure,

$$P = \frac{1}{3}g^{ab}\Pi_{ab} = \frac{1}{3}h^{ab}T_{ab}, \quad (3.19)$$

the anisotropic stresses,

$$\pi_{ab} = \Pi_{ab} - Ph_{ab}, \quad (3.20)$$

the effective energy density,

$$\rho = T_{ab}u^a u^b, \quad (3.21)$$

and the effective heat flux density,

$$q_a = -T_{cd}u^c h_a{}^d. \quad (3.22)$$

We compute these quantities separately for each contribution $T_{ab}^{(2,3,4)}$ to the effective energy-momentum tensor $T_{ab}^{(\text{eff})}$. Let us begin with the $T_{ab}^{(2)}$ contribution. Recalling that $\nabla_a\phi = \sqrt{2X}u_a$ and using the identities,

$$h_{ab}u^b = h_{ab}u^a = 0, \quad (3.23)$$

$$g_{ae}h^{eb} = h_a{}^b, \qquad h^a{}_e h^e{}_b = h^a{}_b, \quad (3.24)$$

$$g_{ab}h^{ab} = h^a{}_a = 3, \quad (3.25)$$

[2]The stress tensor Π_{ab}, the anisotropic stresses π_{ab}, and the heat current density are purely spatial tensors, $\Pi_{ab}u^a = \Pi_{ab}u^b = \pi_{ab}u^a = \pi_{ab}u^b = 0$, $q_a u^a = 0$.

one finds

$$T_{ab}^{(2)} = \frac{1}{2G_4}(2G_{2X}Xu_a u_b + G_2 g_{ab}), \quad (3.26)$$

and the effective fluid quantities for this part of the effective stress-energy tensor of the ϕ-fluid are

$$\rho^{(2)} = T_{ab}^{(2)}u^a u^b = \frac{1}{2G_4}(G_{2X}2Xu_a u_b + G_2 g_{ab})u^a u^b$$
$$= \frac{1}{2G_4}(2XG_{2X} - G_2), \quad (3.27)$$

$$q_a^{(2)} = -T_{cd}^{(2)}u^c h_a{}^d = -\frac{1}{2G_4}(G_{2X}2Xu_c u_d + G_2 g_{cd})u^c h_a{}^d$$
$$= -\frac{1}{2G_4}(-2XG_{2X}u_d + G_2 u_d)h_a{}^d = 0, \quad (3.28)$$

$$\Pi_{ab}^{(2)} = T_{cd}^{(2)}h_a{}^c h_b{}^d = \frac{1}{2G_4}(G_{2X}2Xu_c u_d + G_2 g_{cd})h_a{}^c h_b{}^d$$
$$= \frac{G_2}{2G_4}h_{ab}, \quad (3.29)$$

$$P^{(2)} = \frac{1}{3}g^{ab}\Pi_{ab}^{(2)} = \frac{1}{3}g^{ab}\frac{G_2}{2G_4}h_{ab} = \frac{1}{3}h^a{}_a\frac{G_2}{2G_4} = \frac{G_2}{2G_4}, \quad (3.30)$$

$$\pi_{ab}^{(2)} = \Pi_{ab}^{(2)} - P^{(2)}h_{ab} = 0. \quad (3.31)$$

Continuing with the effective fluid quantities associated with the contribution $T_{ab}^{(3)}$ to $T_{ab}^{(\text{eff})}$, one finds

$$T_{ab}^{(3)} = \frac{1}{2G_4}[g_{ab}(G_{3X}\nabla X \cdot \nabla\phi - 2XG_{3\phi})$$
$$- 2\sqrt{2X}G_{3X}\nabla_{(a}Xu_{b)} - 2X(2G_{3\phi} + G_{3X}\Box\phi)u_a u_b] \quad (3.32)$$

and

$$\rho^{(3)} = T_{ab}^{(3)}u^a u^b$$
$$= \frac{1}{2G_4}[-(G_{3X}\nabla X \cdot \nabla\phi - 2XG_{3\phi}) + 2G_{3X}\sqrt{2X}u^a\nabla_a X$$
$$- 2X(2G_{3\phi} + G_{3X}\Box\phi)]$$
$$= -\frac{1}{2G_4}(-G_{3X}\nabla X \cdot \nabla\phi + 2XG_{3\phi} + 2XG_{3X}\Box\phi). \quad (3.33)$$

To compute $q_a^{(3)}$ one makes use of the facts that

$$g_{cd}u^c h_a{}^d = u_d h_a{}^d = 0, \quad (3.34)$$

$$u^c h_a{}^d \nabla_{(a} X u_{b)} = \frac{1}{2} u^c h_a{}^d (\nabla_a X u_b + \nabla_b X u_a) = -\frac{1}{2} h_{ad} \nabla^d X, \tag{3.35}$$

$$u_c u_d u^c h_a{}^d = 0, \tag{3.36}$$

that yield

$$q_a^{(3)} = -T_{cd}^{(3)} u^c h_a{}^d = -\frac{G_{3X}}{2G_4} \sqrt{2X} \left(\nabla_a X + \frac{\nabla X \cdot \nabla \phi}{2X} \nabla_a \phi \right). \tag{3.37}$$

Now, to compute $\Pi_{ab}^{(3)}$ one uses

$$h_a{}^c h_b{}^d g_{cd} = h_{ab}, \tag{3.38}$$

$$h_a{}^c h_b{}^d \nabla_{(a} X u_{b)} = \frac{1}{2} h_a{}^c h_b{}^d (\nabla_a X u_b + \nabla_b X u_a) = 0, \tag{3.39}$$

$$h_a{}^c h_b{}^d u_c u_d = 0, \tag{3.40}$$

which lead to

$$\Pi_{ab}^{(3)} = \frac{h_{ab}}{2G_4} (G_{3X} \nabla X \cdot \nabla \phi - 2X G_{3\phi}), \tag{3.41}$$

$$P^{(3)} = \frac{1}{3} g^{ab} \Pi_{ab}^{(3)} = \frac{1}{2G_4} (G_{3X} \nabla X \cdot \nabla \phi - 2X G_{3\phi}), \tag{3.42}$$

and

$$\pi_{ab}^{(3)} = \Pi_{ab}^{(3)} - P^{(3)} h_{ab} = 0. \tag{3.43}$$

Finally, the $T_{ab}^{(4)}$ contribution,

$$T_{ab}^{(4)} = \frac{G_{4\phi}}{G_4} (\nabla_a \nabla_b \phi - g_{ab} \Box \phi) + 2X \frac{G_{4\phi\phi}}{G_4} (g_{ab} + u_a u_b) \tag{3.44}$$

is calculated using the intermediate result reported in Appendix A, obtaining

$$\rho^{(4)} = T_{ab}^{(4)} u^a u^b = u^a u^b \left[\frac{G_{4\phi}}{G_4} (\nabla_a \nabla_b \phi - g_{ab} \Box \phi) \right.$$
$$\left. + 2X \frac{G_{4\phi\phi}}{G_4} (g_{ab} + u_a u_b) \right]$$
$$= \frac{G_{4\phi}}{G_4} \left(\Box \phi - \frac{\nabla X \cdot \nabla \phi}{2X} \right), \tag{3.45}$$

$$q_a^{(4)} = -T_{cd}^{(4)} u^c h_a{}^d$$
$$= -\frac{G_{4\phi}}{G_4} u^c h_a{}^d \nabla_c \nabla_d \phi$$
$$= -\frac{G_{4\phi}}{G_4} \frac{\nabla^c \phi}{\sqrt{2X}} \left(\nabla_c \nabla_a \phi + \frac{\nabla_a \phi \nabla^d \phi \nabla_c \nabla_d \phi}{2X} \right)$$
$$= \frac{G_{4\phi}}{G_4} \frac{1}{\sqrt{2X}} \left(\nabla_a X + \frac{\nabla \phi \cdot \nabla X}{2X} \nabla_a \phi \right) \tag{3.46}$$

and

$$\Pi_{ab}^{(4)} = T_{cd}^{(4)} h_a{}^c h_b{}^d$$
$$= \left[\frac{G_{4\phi}}{G_4} (\nabla_c \nabla_d \phi - g_{cd} \Box \phi) \right.$$
$$\left. + 2X \frac{G_{4\phi\phi}}{G_4} g_{cd} + (\propto u_c u_d) \right] h_a{}^c h_b{}^d$$
$$= \frac{G_{4\phi}}{G_4} h_a{}^c h_b{}^d \nabla_c \nabla_d \phi + h_{ab} \left(2X \frac{G_{4\phi\phi}}{G_4} - \frac{G_{4\phi}}{G_4} \Box \phi \right), \tag{3.47}$$

or

$$\Pi_{ab}^{(4)} = \frac{G_{4\phi}}{G_4} \left(\nabla_a \nabla_b \phi - \frac{\nabla_{(a} X \nabla_{b)} \phi}{X} - \frac{\nabla X \cdot \nabla \phi}{4X^2} \nabla_a \phi \nabla_b \phi \right)$$
$$+ h_{ab} \left(2X \frac{G_{4\phi\phi}}{G_4} - \frac{G_{4\phi}}{G_4} \Box \phi \right), \tag{3.48}$$

while

$$P^{(4)} = \frac{1}{3} g^{ab} \Pi_{ab}^{(4)}$$
$$= -\frac{G_{4\phi}}{3G_4} \left(2\Box \phi + \frac{\nabla X \cdot \nabla \phi}{2X} \right) + 2X \frac{G_{4\phi\phi}}{G_4}, \tag{3.49}$$

$$\pi_{ab}^{(4)} = \Pi_{ab}^{(4)} - P^{(4)} h_{ab}$$
$$= -h_{ab} \left[-\frac{G_{4\phi}}{3G_4} \left(2\Box \phi + \frac{\nabla X \cdot \nabla \phi}{2X} \right) + 2X \frac{G_{4\phi\phi}}{G_4} \right]$$
$$= \frac{G_{4\phi}}{G_4} \left[\nabla_a \nabla_b \phi - \frac{\nabla_{(a} X \nabla_{b)} \phi}{X} - \frac{\nabla X \cdot \nabla \phi}{4X^2} \nabla_a \phi \nabla_b \phi \right.$$
$$\left. + \frac{h_{ab}}{3} \left(\frac{\nabla X \cdot \nabla \phi}{2X} - \Box \phi \right) \right]. \tag{3.50}$$

To summarize, the field equations of the chosen subclass of Horndeski theories of gravity have been rewritten in the form of effective Einstein equations by moving the Horndeski terms to their right-hand side and leaving the Einstein tensor in the left-hand side. It is a fact that the right-hand side of the field equations, recast in this form, assumes the form of the stress-energy tensor of a

dissipative effective fluid. Thus far, only a manipulation of the field equations has been performed and no extra assumption has been made. The results presented in this section confirm the ones discussed in [48].

IV. THERMODYNAMIC ANALOGY FOR THE EFFECTIVE DISSIPATIVE ϕ-FLUID

We are now ready to examine the consequences of writing the Horndeski field equations (for the class of Horndeski theories considered) in the form of effective Einstein equations with an effective dissipative fluid. Although this reduction has been performed many times in the literature in various special contexts [including Brans-Dicke or $f(R)$ gravity, nonminimally coupled scalar fields, Friedmann-Lemaître-Robertson-Walker metrics or cosmological perturbations in extended gravity], the physical interpretation of the effective dissipative fluid and of its thermodynamics is usually not attempted. We began looking for this physical interpretation, for simple Brans-Dicke-like and $f(R)$ gravity, in [18,19]. To this end, we adopt the most basic elements of Eckart's theory of gravity [27]. While it is well known that this theory is riddled with causality violation and instabilities, it is nevertheless the model of dissipative fluid most frequently used in relativity [28,29]. We assume the constitutive equations of Eckart's theory: these are phenomenological equations that could be assumed in a variety of theories of dissipation and constitute minimal assumptions on the physics of a (real or effective) dissipative fluid [27–29].

The three constitutive equations ([27], see also [29]), relate the viscous pressure $P_{\rm vis}$ with the fluid expansion θ, the heat current density q^a with the temperature $\mathcal{T}$, and the anisotropic stresses π_{ab} with the shear tensor σ_{ab},

$$P_{\rm vis} = -\zeta\theta, \tag{4.1}$$

$$q_a = -\mathcal{K}(h_{ab}\nabla^b\mathcal{T} + \mathcal{T}\dot{u}_a), \tag{4.2}$$

$$\pi_{ab} = -2\eta\sigma_{ab}, \tag{4.3}$$

where ζ, $\mathcal{K}$, and η are the thermal conductivity, bulk viscosity, and shear viscosity, respectively.

Let us begin with the phenomenological extension of Fourier's law relating heat flux density and temperature. The calculations of the previous section provide the effective heat flux density,

$$q_a^{\rm (eff)} = q_a^{(3)} + q_a^{(4)} = \frac{G_{4\phi} - XG_{3X}}{G_4\sqrt{2X}}\left(\nabla_a X + \frac{\nabla\phi \cdot \nabla X}{2X}\nabla_a\phi\right), \tag{4.4}$$

in the subclass of Horndeski theories. One infers from Eq. (2.7) that

$$q_a^{\rm (eff)} = -\frac{\sqrt{2X}(G_{4\phi} - XG_{3X})}{G_4}\dot{u}_a. \tag{4.5}$$

Turning to Eq. (4.2), it turns out [18,19,26] that for "old" scalar-tensor gravity, the spatial gradient $h_{ab}\nabla^b\mathcal{T}$ vanishes in the comoving frame,[3] leaving only the inertial term in the heat flux density,

$$q_a = -\mathcal{K}\mathcal{T}\dot{u}_a. \tag{4.6}$$

Comparing Eqs. (4.4) and (4.6), one can make the identifications,

$$\mathcal{K}h_{ab}\nabla^b\mathcal{T} = 0, \tag{4.7}$$

and

$$\mathcal{K}\mathcal{T} \equiv \frac{\sqrt{2X}(G_{4\phi} - XG_{3X})}{G_4}, \tag{4.8}$$

where $\mathcal{K}$ and $\mathcal{T}$ denote, respectively, the thermal conductivity and effective temperature of the ϕ-fluid for the subclass of Horndeski gravity considered. Here $\mathcal{T}$ is the "temperature of gravity," which reduces to the quantity already identified in "old" (i.e., Brans-Dicke-like) scalar-tensor theories in Refs. [18,19,26].

To continue on the lines of [18,19], we identify a shear viscosity for the effective ϕ-fluid. The latter has anisotropic stress tensor,

$$\pi_{ab}^{\rm (eff)} = \pi_{ab}^{(4)} = \frac{G_{4\phi}}{G_4}\left[\nabla_a\nabla_b\phi - \frac{\nabla_{(a}X\nabla_{b)}\phi}{X} - \frac{\nabla X \cdot \nabla\phi}{4X^2}\nabla_a\phi\nabla_b\phi - \frac{h_{ab}}{3}\left(\Box\phi - \frac{\nabla X \cdot \nabla\phi}{2X}\right)\right], \tag{4.9}$$

and, from Eq. (2.11), one infers that

$$\pi_{ab}^{\rm (eff)} = \frac{G_{4\phi}\sqrt{2X}}{G_4}\sigma_{ab}. \tag{4.10}$$

We now assume the second constitutive equation of Eckart's theory relating the anisotropic stresses π_{ab} with the shear tensor σ_{ab} in a dissipative fluid [27],

$$\pi_{ab} = -2\eta\sigma_{ab}, \tag{4.11}$$

where η is the shear viscosity. Comparing Eqs. (4.4) and (4.6), one is naturally led to identify

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[3]The spatial temperature gradient of a fluid does not always vanish in the frame comoving with it: for example, in a static fluid in thermal equilibrium in a static gravitational field, the temperature obeys the Tolman condition $\mathcal{T}\sqrt{-g_{00}} = {\rm const.}$ [51].

$$\eta = -\frac{\sqrt{X} G_{4\phi}}{\sqrt{2} G_4}, \qquad (4.12)$$

with the shear viscosity of the effective ϕ-fluid, where it is $G_4 > 0$ to guarantee a positive gravitational coupling of gravity to matter. Since one can always redefine the scalar field ϕ according to $\psi = G_4(\phi)$ (this relation is invertible whenever $G_{4\phi} \neq 0$), the shear viscosity becomes $\eta = -\frac{\sqrt{X}\psi_{,\phi}}{\sqrt{2}\psi}$ and is positive whenever $G_{4\phi} < 0$ and negative otherwise, for example in Brans-Dicke theory where $G_4(\phi) = \phi$ [20]. Negative viscosities can occur in fluid mechanics, atmospheric physics, ocean currents, liquid crystals, *etc.* Typically, they are related with turbulence and occur in nonisolated systems which receive energy from the outside (see, e.g., Refs. [52]). Indeed, the effective ϕ-fluid is not isolated since the scalar ϕ couples explicitly to gravity through the term $G_4 R$ in the Horndeski Lagrangian.

The structure of $T_{ab}^{(\mathrm{eff})}$ (in the form that we have chosen) does not allow for a viscous pressure; hence the bulk viscosity vanishes, $\zeta = 0$. It is interesting that, in the very different context of spacetime thermodynamics in $f(\mathcal{R})$ gravity, bulk viscosity is absent and shear viscosity is important [this fact was emphasized in Ref. [17] and corrects the previous interpretation of [2] of the thermodynamics of spacetime in $f(R)$ gravity].

A general interpretation of $\mathcal{K}$ and $\mathcal{T}$ emerges from the thermodynamic analogy. If one chooses

$$\mathcal{K} \equiv \sqrt{2X}(G_{4\phi} - X G_{3X}) \qquad (4.13)$$

and

$$\mathcal{T} \equiv \frac{1}{G_4}, \qquad (4.14)$$

then $\mathcal{T}$ automatically satisfies $h_{ab}\nabla^b \mathcal{T} = 0$. Indeed, $\mathcal{T} = \mathcal{T}(\phi)$ since $G_4 = G_4(\phi)$, thus $\nabla_a \mathcal{T} \propto \nabla_a \phi$. Furthermore, it must be $G_4 > 0$ to guarantee a positive coupling strength of gravity to matter, as is clear from Eq. (3.12), and the "temperature" of gravity $\mathcal{T}$ is non-negative. This fact was not guaranteed *a priori*. GR corresponds to $\phi = \mathrm{const.}$ and, therefore, to a unit value of the temperature (if coupling with matter is considered) and vanishing thermal conductivity in the spectrum of the specific subclass of Horndeski theories. This fact embodies the idea that GR is a "state of equilibrium" in a wider space of theories of gravity and any extension of gravity corresponds to a deviation from equilibrium, which is rather natural if extra degrees of freedom (in this case the scalar ϕ) are excited.

One can describe the approach to the GR equilibrium, or the departure from it, by differentiating Eq. (4.8) with respect to the proper time τ along the worldlines of the effective fluid,

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} \equiv u^c \nabla_c(\mathcal{K}\mathcal{T}) = \frac{\nabla^c \phi}{\sqrt{2X}} \nabla_c \left[\frac{\sqrt{2X}(G_{4\phi} - X G_{3X})}{G_4}\right]$$
$$= \frac{\nabla X \cdot \nabla \phi}{(2X)^{3/2}} \mathcal{K}\mathcal{T} + \nabla^c \phi \nabla_c \left(\frac{G_{4\phi} - X G_{3X}}{G_4}\right). \quad (4.15)$$

The use of the expression (2.10) of the expansion θ to eliminate $\nabla\phi \cdot \nabla X$ yields

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} = \left(\frac{\Box\phi}{\sqrt{2X}} - \theta\right)\mathcal{K}\mathcal{T} + \nabla^c \phi \nabla_c \left(\frac{G_{4\phi} - X G_{3X}}{G_4}\right). \qquad (4.16)$$

The second piece in the latter equation can be expanded as

$$\nabla^c \phi \nabla_c \left(\frac{G_{4\phi} - X G_{3X}}{G_4}\right) = -\frac{2X}{G_4^2}[G_4 G_{4\phi\phi} - X G_4 G_{3X\phi}$$
$$- G_{4\phi}(G_{4\phi} - X G_{3X})]$$
$$- \frac{(2X)^{3/2}}{G_4}\left(\frac{\Box\phi}{\sqrt{2X}} - \theta\right)$$
$$\times (G_{3X} + X G_{3XX}), \qquad (4.17)$$

where we have used again Eq. (2.10) to eliminate $\nabla\phi \cdot \nabla X$ and the definition of X.

Therefore, for the viable class of Horndeski theories one obtains

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} = \left(\frac{\Box\phi}{\sqrt{2X}} - \theta\right)\left[\mathcal{K}\mathcal{T} - \frac{(2X)^{3/2}}{G_4}(G_{3X} + X G_{3XX})\right]$$
$$- \frac{2X}{G_4^2}[G_4 G_{4\phi\phi} - X G_4 G_{3X\phi} - G_{4\phi}(G_{4\phi} - X G_{3X})]. \qquad (4.18)$$

The function G_2 does not enter this evolution equation, although it still plays a role by determining the dynamics of the scalar field ϕ and of its gradient, from which $\mathcal{K}\mathcal{T}$ depends explicitly.

The physical interpretation of Eq. (4.18) describing the approach to the GR equilibrium or, possibly, the departure from it, is made difficult by the many free functions appearing in it. However, one can extract some physics at least in simple cases. Consider, for illustration, vacuum Brans-Dicke theory with constant Brans–Dicke parameter ω and massless Brans–Dicke scalar ϕ, which is reproduced by the choice,

$$G_4 = \phi, \qquad G_3 = 0. \qquad (4.19)$$

In this case, Eq. (4.8) reduces to $\mathcal{K}\mathcal{T} = \sqrt{2X}/\phi$ and the equation of motion for the Brans-Dicke scalar is simply $\Box\phi = 0$. Then, Eq. (4.18) becomes [18,19]

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} = -\theta\mathcal{K}\mathcal{T} + (\mathcal{K}\mathcal{T})^2. \qquad (4.20)$$

When the effective ϕ-fluid is focused, i.e., $\theta < 0$, we have

$$\frac{d(\mathcal{K}\mathcal{T})}{d\tau} > (\mathcal{K}\mathcal{T})^2, \qquad (4.21)$$

and $\mathcal{K}\mathcal{T}$ diverges away from the GR equilibrium state very fast [if θ was constant, there would be a pole in $(\mathcal{K}\mathcal{T})(\tau)$ because of the superexponential growth]. In regions where the effective fluid gets concentrated (notably, near spacetime singularities) the deviations of Horndeski gravity from GR become extreme.

If instead $\theta > 0$, it is possible that the term $-\theta\mathcal{K}\mathcal{T}$ in the right-hand side of Eq. (4.18) dominate, in which case the solution $\mathcal{K}\mathcal{T}$ of (4.18) tends to zero and gravity approaches the GR equilibrium state ("expansion cools gravity"). This is not the only *a priori* possibility, though: if $\mathcal{K}\mathcal{T}$ is large, the positive term $(\mathcal{K}\mathcal{T})^2$ dominates the right-hand side of (4.18) and makes the solution $\mathcal{K}\mathcal{T}$ move away from the GR equilibrium. Approaching GR does not necessarily happen all the time.

Lastly, it is worth noting that if one expands the general Horndeski action on a spatially flat, homogeneous, and isotropic background to second order in the linear perturbations, then the corresponding dynamics is controlled by four functions of time (see e.g. [53,54], and references therein). Among these functions one finds the effective Planck mass $M_\star^2$ and the braiding $\alpha_{\rm B}$, quantifying the strength of the kinetic mixing between scalar and tensor perturbations, that for the case of the specific subclass of Horndeski theories simply read

$$M_\star^2 = 2G_4 \quad \text{and} \quad \alpha_{\rm B} = \frac{2\dot\phi}{HM_\star^2}(XG_{3X} - G_{4\phi}), \quad (4.22)$$

where ϕ, G_3, G_4, and their derivatives are evaluated on the background configuration and with H denoting the Hubble parameter. A comparison between (4.13), (4.14), and (4.22) shows that $\mathcal{T} \propto 1/M_\star^2$ and $\mathcal{K} \propto -\alpha_{\rm B}$ in this specific realization of the analogy. This suggests a deeper physical significance behind the choice of (4.13) and (4.14), beyond their simplicity, among the broader class of solutions of the system (4.7)–(4.8). However, the derivation of the functions $M_\star^2$ and $\alpha_{\rm B}$ is performed on cosmological backgrounds, which is a context that still escapes the formalism discussed here [18,19]. Hence, further consideration of this matter is postponed to future investigations.

V. MORE GENERAL HORNDESKI THEORIES

We now move on to more general Horndeski theories of gravity. Consider a Horndeski model such that

$$\delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}\mathcal{G}_{ab}\delta g^{ab} + (\cdots)\delta\phi + \text{total derivative}, \qquad (5.1)$$

with

$$\mathcal{G}_{ab} \supset \xi(\phi, X)R_{acbd}\nabla^c\phi\nabla^d\phi, \qquad (5.2)$$

which is a common feature of theories beyond the subclass we considered so far (see Ref. [50] for the corresponding field equations). As it becomes clear, they contain nonminimal derivative couplings. This choice implies

$$T_{ab} \supset \zeta(\phi, X)R_{acbd}\nabla^c\phi\nabla^d\phi, \qquad (5.3)$$

where $\zeta(\phi, X)$ is a function proportional to $\xi(\phi, X)$; from this one concludes that

$$\Pi_{ab} = T_{cd}h_a{}^c h_b{}^d \supset \zeta(\phi, X)h_a{}^c h_b{}^d R_{cedf}\nabla^e\phi\nabla^f\phi$$
$$= \zeta(\phi, X)R_{aebf}\nabla^e\phi\nabla^f\phi, \qquad (5.4)$$

taking advantage of the symmetries of the Riemann tensor, which lead to $R_{cedf}\nabla^e\phi\nabla^f\phi\nabla^c\phi\nabla^d\phi = 0$. Taking the trace of the stress tensor yields

$$P = \frac{1}{3}g^{ab}\Pi_{ab} \supset \frac{\zeta(\phi, X)}{3}g^{ab}R_{aebf}\nabla^e\phi\nabla^f\phi$$
$$= \frac{\zeta(\phi, X)}{3}R_{ef}\nabla^e\phi\nabla^f\phi, \qquad (5.5)$$

and then we have

$$\pi_{ab} = \Pi_{ab} - Ph_{ab} \supset \zeta(\phi, X)R_{aebf}\nabla^e\phi\nabla^f\phi$$
$$- \frac{\zeta(\phi, X)}{3}h_{ab}R_{ef}\nabla^e\phi\nabla^f\phi. \qquad (5.6)$$

While the term containing the Ricci tensor can, in principle, cancel out with similar terms coming from the field equations and contained in the effective energy-momentum tensor, the contribution to the right-hand side depending on the Riemann tensor cannot be traced away.[4] This Riemann tensor term breaks the proportionality between σ_{ab} (which is a kinematic quantity and, therefore, does not depend on the specific model) and π_{ab}. As a consequence, the three constitutive equations of Eckart's theory [27] no longer hold for the effective fluid. Of course, this proportionality could be broken by other terms coming from the variation of $\mathcal{L}_4$ and $\mathcal{L}_5$, though the feature discussed here involves a property that seem to be shared by the vast majority of models beyond the specific luminal Horndeski class.

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[4]Although it is not *a priori* unconceivable that this contribution is removed by imposing some relation between G_4 and G_5, the latter would be extremely fine-tuned and would give rise to a completely artificial theory.

As an example, let us discuss the Horndeski theory, not belonging to the restricted class (3.3), with Lagrangian density,

$$\mathcal{L} = G_4(X)R = XR,$$ (5.7)

for which the effective energy-momentum tensor of the ϕ-fluid is

$$T_{ab}^{(\text{eff})} = \frac{1}{X}\left\{\frac{R}{2}\nabla_a\phi\nabla_b\phi + \Box\phi\nabla_a\nabla_b\phi - \nabla_a\nabla_e\phi\nabla^e\nabla_b\phi - \frac{1}{2}g_{ab}[(\Box\phi)^2 - (\nabla\nabla\phi)^2] - 2R_{e(a}\nabla_{b)}\phi\nabla^e\phi \right.$$
$$\left. + g_{ab}R^{ef}\nabla_e\phi\nabla_f\phi - R_{aebf}\nabla^e\phi\nabla^f\phi\right\},$$ (5.8)

where $(\nabla\nabla\phi)^2 \equiv \nabla^c\nabla^d\phi\nabla_c\nabla_d\phi$. The effective stress tensor is then

$$\Pi_{ab}^{(\text{eff})} = T_{cd}^{(\text{eff})}h_a{}^ch_b{}^d = \frac{1}{X}\left\{\Box\phi\nabla_c\nabla_d\phi - \nabla_c\nabla_e\phi\nabla^e\nabla_d\phi - \frac{1}{2}g_{cd}[(\Box\phi)^2 - (\nabla\nabla\phi)^2]\right.$$
$$\left. + g_{cd}R^{ef}\nabla_e\phi\nabla_f\phi - R_{cedf}\nabla^e\phi\nabla^f\phi\right\}h_a{}^ch_b{}^d.$$ (5.9)

The terms $h_a{}^ch_b{}^d\Box\phi\nabla_c\nabla_d\phi$ and $h_a{}^ch_b{}^d\nabla^e\nabla_c\phi\nabla_e\nabla_d\phi$ appearing in this expression are computed in Appendix B; substituting them into Eq. (5.9), one obtains

$$\Pi_{ab}^{(\text{eff})} = \frac{1}{X}\left\{\Box\phi\left[\nabla_a\nabla_b\phi - \frac{1}{X}\nabla_{(a}X\nabla_{b)}\phi - \frac{1}{4X^2}\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c X\right] - \nabla_a\nabla_e\phi\nabla^e\nabla_b\phi + \frac{1}{X}\nabla_{(a}\phi\nabla_{b)}\nabla_e\phi\nabla^e X\right.$$
$$\left. - \nabla_a\phi\nabla_b\phi\frac{\nabla_e X\nabla^e X}{4X^2} - \frac{1}{2}h_{ab}[(\Box\phi)^2 - (\nabla\nabla\phi)^2] + h_{ab}R^{ef}\nabla_e\phi\nabla_f\phi - R_{cdef}h_a{}^ch_b{}^d\nabla^e\phi\nabla^f\phi\right\}.$$ (5.10)

From this one computes the effective fluid pressure,

$$P^{(\text{eff})} = \frac{1}{3}g^{ab}\Pi_{ab}^{(\text{eff})} = \frac{1}{3X}\left[-\frac{(\Box\phi)^2}{2} + \frac{3}{2}(\nabla\nabla\phi)^2 - \frac{\Box\phi}{2X}\nabla^c\phi\nabla_c X - \frac{\nabla^c X\nabla_c X}{2X} + 3R_{ef}\nabla^e\phi\nabla^f\phi - \nabla^a\nabla^e\phi\nabla_a\nabla_e\phi\right].$$ (5.11)

The anisotropic stresses are then obtained as

$$\pi_{ab}^{(\text{eff})} \equiv \Pi_{ab}^{(\text{eff})} - P^{(\text{eff})}h_{ab}$$
$$= \frac{1}{X}\left\{\Box\phi\left[\nabla_a\nabla_b\phi - \frac{\nabla_{(a}\phi\nabla_{b)}X}{X} - \frac{\nabla^c\phi\nabla_c X}{4X^2}\nabla_a\phi\nabla_b\phi + \frac{\nabla^c\phi\nabla_c X}{6X}h_{ab}\right]\right.$$
$$\left. - \nabla_a\nabla_e\phi\nabla^e\nabla_b\phi + \frac{\nabla_{(a}\phi\nabla_{b)}\nabla_e\phi\nabla^e X}{X} - \nabla_a\phi\nabla_b\phi\frac{\nabla^e X\nabla_e X}{4X^2} + \frac{\nabla^e X\nabla_e X}{6X}h_{ab} - \frac{(\Box\phi)^2}{6}h_{ab}\right.$$
$$\left. - R_{cdef}h_a{}^ch_b{}^d\nabla^e\phi\nabla^f\phi + \frac{1}{3}(\nabla^a\nabla^e\phi\nabla_a\nabla_e\phi)h_{ab}\right\}.$$ (5.12)

This quantity is definitely not proportional to the shear $\sigma_{ab}^{(\text{eff})}$, thus breaking the analogy with Eckart's constitutive relation (4.3). Similarly, one computes the effective heat current density,

$$q_a^{(\text{eff})} = -\frac{1}{X}[\Box\phi\nabla_c\nabla_d\phi - \nabla_c\nabla_e\phi\nabla^e\nabla_d\phi$$
$$- R_{cedf}\nabla^e\phi\nabla^f\phi]u^ch_a{}^d,$$ (5.13)

which cannot be reduced to the Eckart constitutive relation (4.2). It is exactly the presence of such derivative non-minimal couplings that breaks down the thermodynamical description. They are exactly those operators which quite generically forbids a dual description in the Einstein frame due to intrinsic changes in the gravity sector. In terms of a nonlocal field redefinition one could write the dual description in the Einstein frame where the effective fluid and its stress energy tensor would become nonlocal. However, in terms of a local description the separation in terms of

"gravity" and "matter fluid" fails and therefore also the application of Eckart's formalism.

VI. APPLICATION TO COSMOLOGICAL BLACK HOLES

Let us consider an application to cosmological black holes. The *LIGO* observations of black holes set constraints on primordial black holes in the Universe and almost eliminate the possibility that they explain a fraction of dark matter [55], but research on primordial black holes has seen a recent resurgence of interest following the *LIGO* detection of black holes in a mass range forbidden by stellar evolution [56]. The possibility of relaxing the *LIGO* bounds has been explored. In particular, it has been pointed out that it is probably incorrect to use the Schwarzschild geometry to describe a primordial black hole embedded in the early Universe. In fact, the latter varies on a scale not vastly separated from the local dynamical scale of the black hole (as is instead the case in the present Universe, where the Schwarzschild and the Hubble radii are completely separated). As a result, a dynamical black hole model was proposed to replace the Schwarzschild metric in this context [57,58] (see [59,60] for alternative uses of dynamical models of cosmological black holes). Although this model has been heavily criticized [61,62], a debate continues [63–65] but the need for dynamical models of black holes embedded in early FLRW universes seems clear. For this purpose, one can use McVittie [66] and generalized McVittie [67–69] metrics instead, and we expect to see more models of this kind in the future.

In the following we discuss what the thermodynamic formalism proposed in this work can say about this class of primordial black hole models in the early Universe, where deviations of gravity from GR are less constrained than today. As an example, consider the viable Horndeski theory described by the action [70],

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi} - 3H_0^2 + 6H_0\sqrt{2X} - 6X \right. $$
$$\left. + \ln\left(\frac{X}{\Lambda^4}\right)\Box\phi \right], \tag{6.1}$$

where $H_0 > 0$ and Λ are constants (the specific values of the coefficients in this action result from particular choices of more general functions made in Ref. [70]). This action was originally named "extended cuscuton" because it generalizes the original cuscuton model of [71–73] (see also [74–88]), but is now called "cuscuta-Galileon" [89,90]. A new name is appropriate because it distinguishes the original cuscuton model (and other models now called "extended cuscuton" in the literature) in which the scalar field ϕ does not propagate, from other models (such as the "cuscuta-Galileon" [89,90]) in which ϕ is instead a propagating degree of freedom [89,90].

In the standard Horndeski notation adopted here, it is

$$G_4 = \frac{1}{16\pi}, \tag{6.2}$$

$$G_2 = -3H_0^2 + 6H_0\sqrt{2X} - 6X, \tag{6.3}$$

$$G_3 = -\ln\left(\frac{X}{\Lambda^4}\right), \tag{6.4}$$

$$G_5 = 0. \tag{6.5}$$

The theory (6.1) admits the generalized McVittie solution [70] given, in isotropic coordinates, by [cf. Eqs. (31)–(33) of [70]]

$$ds^2 = -\left[\frac{1 - \frac{m(t)}{2a(t)r}}{1 + \frac{m(t)}{2a(t)r}}\right]^2 dt^2 + a^2(t)\left[1 + \frac{m(t)}{2a(t)r}\right]^4$$
$$\times (dr^2 + r^2 d\Omega_{(2)}^2), \tag{6.6}$$

$$H(t) = H_0 \coth(H_0 t), \tag{6.7}$$

$$m(t) = m_0 e^{-H_0 t} \sinh(H_0 t), \tag{6.8}$$

$$\phi(t) = \phi_0 + H_0 t - \ln \sinh(H_0 t), \tag{6.9}$$

where $m_0 > 0$ and ϕ_0 are constants and $d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere. The line element (6.6) describes a black hole embedded in a FLRW universe with a scale factor,

$$a(t) = a_0 \sinh(H_0 t) \tag{6.10}$$

(with constant $a_0 > 0$) and with time-dependent mass function $m(t)$, with a radial energy flux onto this central object [66–69]. The black hole nature of this central object is shown explicitly in [70]; see [67–70,91] for discussions of McVittie and generalized McVittie metrics. What is relevant here is that, in the absence of matter (or in situations in which the gravitational scalar ϕ dominates over matter), the action (6.1) admits the above solution, which could be taken as a model of a primordial black hole (arguably, a better model than the nonrotating Thakurta proposal of [57,58] criticized in [61,62]).

According to our prescription, the cuscuta-Galileon theory (6.1) has

$$\mathcal{K}T = \sqrt{2X}\frac{(G_{4\phi} - XG_{3X})}{G_4} = 16\pi\sqrt{2X}. \tag{6.11}$$

Let us evaluate this quantity on the specific cosmological black hole solution (6.6)–(6.9): one easily obtains

$$2X = -g^{00}\dot{\phi}^2 = \left[\frac{1 + \frac{m(t)}{2a(t)r}}{1 - \frac{m(t)}{2a(t)r}}\right]^2 \frac{H_0^2 e^{-2H_0 t}}{\sinh^2(H_0 t)} \qquad (6.12)$$

and

$$\mathcal{K}T = \frac{16\pi H_0}{e^{H_0 t} \sinh(H_0 t)} \left[\frac{1 + \frac{m(t)}{2a(t)r}}{1 - \frac{m(t)}{2a(t)r}}\right]. \qquad (6.13)$$

At late times, this black hole disappears and this solution reduces to a de Sitter spacetime of GR with constant scalar field and positive cosmological constant. In fact, as $t \to +\infty$, the scale factor (6.10) converges to the de Sitter exponential $a_\infty(t) = a_0 e^{H_0 t}$, $m(t)/a(t) = (m_0/a_0)e^{-H_0 t} \to 0$,

$$ds^2 \to ds_\infty^2 = -dt^2 + a_\infty^2(t)(dr^2 + r^2 d\Omega_{(2)}^2), \qquad (6.14)$$

and $\mathcal{K}T \to 0$.

The solution relaxes to the GR solution and the action (6.1), evaluated on this particular solution, reduces to that of GR with a cosmological constant,

$$S \to S_\infty = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi} - 3H_0^2\right). \qquad (6.15)$$

This example shows how a solution of Horndeski gravity can approach a GR state of equilibrium solution. In principle, one can consider an entire theory of gravity as a state of equilibrium (as done here for GR), or one can consider specific solutions of a theory of gravity and see whether it approaches a GR solution. In this case, it is not possible to make statements about the entire theory, but only about specific solutions of the modified gravity with extra degrees of freedom excited, and about "states of equilibrium solutions," which is what we do in this section. The use of McVittie and generalized McVittie solutions as models of primordial black holes in Horndeski gravity will be explored in future publications.

VII. DISCUSSION AND CONCLUSIONS

For first generation scalar-tensor theories, the description in terms of an effective ϕ-fluid which is dissipative leads, through Eckart's first order thermodynamics [27], to a formalism of "thermodynamics of gravity" in which GR is seen as the state of equilibrium and scalar-tensor gravity, with its extra scalar degree of freedom ϕ, as a nonequilibrium state. In many situations, the dissipation leads to an approach to GR [18,19]. It is natural to apply a similar formalism to Horndeski theories of gravity, which generalize "old" scalar-tensor gravity [20–24] and which have seen an explosion of activity during the past decade.

The first step consists of extracting an effective dissipative fluid of the scalar degree of freedom ϕ from the field equations. This step was started in Ref. [48], which we generalize introducing minor corrections. In particular, the kinematic quantities u^a, $\dot{u}^a$, θ, ω_{ab}, V_{ab}, and σ_{ab} are identical to the previous derivations in[5] Refs. [18,26], since they do not depend on the field equations. We emphasize that the energy-momentum tensor of the effective ϕ-fluid obtained does not satisfy any energy condition, nor it is expected to: it is built out of gravitational terms and does not arise from a kinetic theory. In spite of this shortcoming from a fluid-mechanicist's point of view, its interpretation as a dissipative fluid à la Eckart can provide an intriguing view of GR as the (constant) unit temperature state, with vanishing thermal conductivity, and of Horndeski gravity as a nonequilibrium state. This view is independent of Jacobson's thermodynamics of spacetime [1,2], but it echoes two of its main ideas.

The next step consists of applying Eckart's thermodynamics to this effective fluid. In comparison with Jacobson's thermodynamics of spacetime [1,2], the effective fluid approach and Eckart's description are minimalist in their assumptions.

The calculations are a bit tedious, according to how many terms are allowed in the Horndeski action. It turns out that our approach does not work for the most general Horndeski theory: even though one can define the effective fluid, including its heat current density and anisotropic stresses, they do not satisfy the constitutive relations (4.1)–(4.3) linking them with the viscous pressure, shear, fluid four-acceleration, and temperature gradient in Eckart's formalism (or in any thermodynamical theory in which reasonable constitutive relations are needed). However, when the terms violating the constraints on the speed of gravitational waves are dropped from the Horndeski action, the "temperature of gravity" formalism makes sense again, the temperature $\mathcal{T}$ is positive-definite, and GR corresponds, for instance, to $\mathcal{T} = 1$ and $\mathcal{K} = 0$. This interpretation is not the only possible one since $\mathcal{T}$ and $\mathcal{K}$ are ultimately defined by the system of equations given by (4.7) and (4.8). One could take this result to say that there could be a relation between physical constraints such as those usually imposed on Horndeski gravity (related to stability and the propagation of gravitational waves) and the validity of the thermodynamic analogy relating kinematic quantities and the components of the ϕ-fluid through Eckart's constitutive equations. The effective temperature of gravity formalism is still under development and far-fetching conclusions are premature; however this result is rather suggestive.

The verdict is not out on Horndeski theories of gravity with nonminimal derivative couplings. They do predict the speed of gravitational waves to be different from the speed of light [92–94], and this behavior is strongly constrained by the GW170817/GRB170817A event [45–47]. However, it is important to note that the constraint really applies to

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[5]The previous Ref. [25] did not provide the kinematical quantities.

certain solutions of the theory and not to the entire theory. It has been shown in Ref. [95] that the theory admits self-accelerating cosmologies that are stable in the future and with tensor perturbations propagating with speed very close to the speed of light. More precisely, the difference between the two speeds scales like a^{-3}; therefore, it can agree with high precision with the GW170817/GRB170817A limit today [95]. Since there are no observational data on the speed of gravitational waves at redshifts $z > 0.3$, it is suggested in [95] that this physics can be extrapolated to the early Universe. Therefore, Horndeski theories with nonminimal derivative couplings are not ruled out observationally. These considerations point to the fact that the failure of the scalar-tensor thermodynamical formalism to apply to Horndeski theories with $G_{4X} = G_5 = 0$ is probably unrelated with the discrepancy between light cones and "gravitational wave cones" and is related more with the particular way in which the dynamical variables enter the field equations. We are unable to offer more insight at the moment, but this point deserves to be understood better in the future.

Although intriguing, the approach followed here suffers from the limitations intrinsic to Eckart's first-order thermodynamics [27] (or better, of its constitutive equations, which is all that was used here). An attempt to generalize the present work to causal (second-order) thermodynamics will be presented elsewhere. Similarly, one can classify different extensions of GR based on their "thermodynamical running" to the GR fixed point, specifically based on the presence or absence of hairy solutions away from GR. Such an application to vector-tensor theories [96] is studied elsewhere.

ACKNOWLEDGMENTS

APPENDIX A: USEFUL RELATIONS

The following relations are useful to compute the various contributions to the stress-energy tensor of the effective dissipative fluid associated with the scalar field ϕ,

$$\nabla\phi \cdot \nabla X = -\nabla^a\phi(\nabla_c\phi\nabla_a\nabla^c\phi) = -\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi, \tag{A1}$$

$$h_b{}^d\nabla_c\nabla_d\phi = \left(\delta_b{}^d + \frac{\nabla_b\phi\nabla^d\phi}{2X}\right)\nabla_c\nabla_d\phi = \nabla_c\nabla_b\phi + \frac{\nabla_b\phi\nabla^d\phi\nabla_c\nabla_d\phi}{2X}, \tag{A2}$$

and

$$
\begin{aligned}
h_a{}^c h_b{}^d\nabla_c\nabla_d\phi &= \left(\delta_a{}^c + \frac{\nabla_a\phi\nabla^c\phi}{2X}\right)\left(\nabla_c\nabla_b\phi + \frac{\nabla_b\phi\nabla^d\phi\nabla_c\nabla_d\phi}{2X}\right) \\
&= \nabla_a\nabla_b\phi + \frac{\nabla_b\phi\nabla^d\phi\nabla_a\nabla_d\phi}{2X} + \frac{\nabla_a\phi\nabla^c\phi\nabla_c\nabla_b\phi}{2X} + \frac{\nabla_a\phi\nabla^c\phi\nabla_b\phi\nabla^d\phi\nabla_c\nabla_d\phi}{4X^2} \\
&= \nabla_a\nabla_b\phi + \frac{\nabla_b\phi\nabla^d\phi\nabla_a\nabla_d\phi}{2X} + \frac{\nabla_a\phi\nabla^c\phi\nabla_b\nabla_c\phi}{2X} + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d\phi}{4X^2} \\
&= \nabla_a\nabla_b\phi - \frac{\nabla_{(a}X\nabla_{b)}\phi}{X} - \frac{\nabla X \cdot \nabla\phi}{4X^2}\nabla_a\phi\nabla_b\phi.
\end{aligned}
\tag{A3}
$$

APPENDIX B: COMPUTATION OF $h_a{}^c h_b{}^d \Box\phi\nabla_c\nabla_d\phi$ AND $h_a{}^c h_b{}^d \nabla^e\nabla_c\phi\nabla_e\nabla_d\phi$

Here we compute two terms needed for the evaluation of the effective stress tensor (5.9). We have

$$
\begin{aligned}
h_a{}^c h_b{}^d \Box\phi\nabla_c\nabla_d\phi &= \Box\phi\left(\delta_a^c + \frac{\nabla_a\phi\nabla^c\phi}{2X}\right)\left(\delta_b^d + \frac{\nabla_b\phi\nabla^d\phi}{2X}\right)\nabla_c\nabla_d\phi \\
&= \Box\phi\left[\nabla_a\nabla_b\phi - \frac{1}{2X}(\nabla_a\phi\nabla_b X + \nabla_b\phi\nabla_a X) - \frac{1}{4X^2}\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\phi\right].
\end{aligned}
\tag{B1}
$$

We then need

$$
\begin{aligned}
h_a{}^c h_b{}^d \nabla^e \nabla_c \phi \nabla_e \nabla_d \phi &= (\delta_a^c + \frac{\nabla_a \phi \nabla^c \phi}{2X})(\delta_b^d + \frac{\nabla_b \phi \nabla^d \phi}{2X}) \nabla_e \nabla_c \phi \nabla^e \nabla_d \phi \\
&= \nabla_e \nabla_a \phi \nabla^e \nabla_b \phi + \frac{1}{2X}[(\nabla_e \nabla_a \phi \nabla^e \nabla_d \phi)\nabla_b \phi \nabla^d \phi + (\nabla_e \nabla_c \phi \nabla^e \nabla_b \phi)\nabla_a \phi \nabla^c \phi] \\
&\quad + \frac{1}{4X^2}\nabla_a \phi \nabla_b \phi(\nabla^e \nabla_c \phi \nabla_e \nabla_d \phi)\nabla^c \phi \nabla^d \phi \\
&= \nabla_e \nabla_a \phi \nabla^e \nabla_b \phi - \frac{1}{2X}[\nabla_e \nabla_a \phi \nabla^e X \nabla_b \phi + \nabla^e \nabla_b \phi \nabla_e X \nabla_a \phi] + \frac{\nabla_e X \nabla^e X \nabla_a \phi \nabla_b \phi}{4X^2}. \quad \text{(B2)}
\end{aligned}
$$

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