Gauge symmetry of unimodular gravity in Hamiltonian formalism

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We work out the description of the gauge symmetry of unimodular gravity in the constrained Hamiltonian formalism. In particular, we demonstrate how the transversality conditions restricting the diffeomorphism parameters emerge from the algebra of the Hamiltonian constraints. The alternative form is long known as parametrizing the volume preserving diffeomorphisms by unrestricted two-forms instead of the transverse vector fields. This gauge symmetry is reducible. We work out the Hamiltonian description of this form of unimodular gravity (UG) gauge symmetry. Becchi-Rouet-Stora-Tyutin–Batalin-Fradkin-Vilkovisky (BFV-BRST) Hamiltonian formalism is constructed for both forms of the UG gauge symmetry. These two BRST complexes have a subtle inequivalence: Their BRST cohomology groups are not isomorphic. In particular, for the first complex, which is related to the restricted gauge parameters, the cosmological constant does not correspond to any nontrivial BRST cocycle, while for the alternative complex it does. In the wording of physics, this means Λ is a fixed parameter defined by the field asymptotics rather than the physical observable from the standpoint of the first complex. The second formalism views Λ as the observable with unrestricted initial data.

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I. INTRODUCTION

Unimodular gravity (UG) is the version of general relativity (GR) where the metrics are restricted by the unimodularity condition,

$$\det g_{\alpha\beta} = -1. \tag{1}$$

Given the restriction, the class of admissible gauge transformations reduces to the volume preserving diffeomorphisms,

$$\delta_\xi g_{\alpha\beta} = -\nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha, \qquad \partial_\alpha \xi^\alpha = 0. \tag{2}$$

The Lie brackets of the transverse vector fields are divergence-free, so the volume preserving diffeomorphisms (2) form a subgroup in the group of general coordinate transformations. This subgroup is singled out from the entire diffeomorphism group in a special way: The partial differential equation (PDE) is imposed restricting the gauge parameters ξ^α rather than the subset being explicitly picked out of the generators for the gauge subgroup. This is an example of the general phenomenon of unfree gauge symmetry [1], where the gauge variation of the action functional vanishes provided that the gauge parameters are subject to the PDE system. Among the other examples of unfree gauge symmetry, we can mention the spin-two Firtz-Pauli model [2,3] and some higher spin field theories [4–6]. The usual general theory of the systems with unconstrained gauge parameters cannot be directly applied to the case of unfree gauge symmetry as the PDEs restricting the gauge parameters, being essential constituents of the gauge algebra, have to be accounted for. The restrictions imposed on the gauge parameters result in modifications of the second Noether theorem and the Faddeev-Popov quantization rules [1]. Also the Batalin-Vilkovisky (BV) formalism has to be modified [7] to account for the distinctions of the unfree gauge symmetry from the case of unconstrained gauge parameters. The specifics of unfree gauge symmetry in the general constrained Hamiltonian formalism is worked out in the articles [8,9], including the modification of the Hamiltonian Becchi-Rouet-Stora-Tyutin–Batalin-Fradkin-Vilkovisky (BFV-BRST) formalism. A common feature for all the unfree gauge symmetries is that they admit an alternative formulation with unconstrained gauge parameters while the unrestricted gauge symmetry is reducible. This general fact is first noticed in [7], though for the specific models the reducible alternatives have been previously known. For the UG, the reducible gauge transformations can be parametrized by the two-form W,

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$$\delta_W g_{\alpha\beta} = -\frac{1}{2}\varepsilon^{\gamma\sigma\mu\nu}(\partial_\gamma g_{\alpha\beta}\partial_\sigma W_{\mu\nu} + g_{\alpha\gamma}\partial_\beta\partial_\sigma W_{\mu\nu} + g_{\gamma\beta}\partial_\alpha\partial_\sigma W_{\mu\nu}), \quad (3)$$

where $\varepsilon^{\gamma\sigma\mu\nu}$ is the Levi-Civita symbol. This parametrization of the volume preserving diffeomorphism is long known [10–12], and it is studied once and again mostly at the linearized level (see [13–15], and references therein).

The transformation (3) follows from (2) by substitution of the transverse vector ξ as the Hodge dual of the exact three-form $\xi = - * dW$. This substitution can be inequivalent to the original symmetry (2) if the manifold admits the third group of De Rham cohomology. Possible consequences of the inequivalence are discussed in the article [16]. Here, we do not elaborate on this issue.

Gauge symmetry (3) is reducible as it admits the sequence of gauge for gauge transformations:

$$\delta_\varphi W_{\alpha\beta} = \partial_\alpha\varphi_\beta - \partial_\beta\varphi_\alpha, \\ \delta_\psi\varphi_\alpha = \partial_\alpha\psi. \quad (4)$$

The higher spin analogs of this form of reducible gauge symmetry can be found in [6].

At the level of general constrained Hamiltonian formalism, the procedure of constructing the reducible alternative formulation with unconstrained gauge parameters has been worked out in the recent article [16] for the general unfree gauge symmetry. Proceeding from the two forms of gauge symmetry, two different BRST complexes can be associated with the same action functional. The first one is for the unfree gauge symmetry, and another one is for the reducible form of the symmetry. These two complexes are connected, though they are inequivalent, in general, in the sense that their BRST-cohomology groups are not necessarily isomorphic.[1]

The Hamiltonian formulation of GR has a long story. In particular, various ways have been discussed for decades of reproducing $4d$-diffeomorphism transformations of the Arnowitt-Deser-Misner (ADM) variables, including laps and shift functions, in Hamiltonian formalism. For pedagogical exposition of this subject, review, and references, see [17–19]. The recent discussion of the same issue for the Brans-Dicke theory can be found in [20]. Also for the UG, the constrained Hamiltonian formulation has been known for at least 30 years [11,12], and the topic has been extensively studied since then; for review and bibliography we refer to [21–23]. For the UG, unlike the GR, it is still not evident how the Hamiltonian constraints can reproduce the $4d$ gauge transformation (2) of the theory, including the transversality condition imposed on the gauge parameters. This puzzle is reported in the reviews (see, e.g., in [22]). Also the reducible form of the UG gauge symmetry (3), (4) has never been described in Hamiltonian formalism. Once the Hamiltonian description is lacking for the gauge symmetry (2) or (3), (4), the Hamiltonian BFV-BRST formalism is still unknown for UG. In this article, following the general procedure of Ref. [8], we work out the Hamiltonian description of the volume preserving diffeomorphisms (2). We also construct the Hamiltonian description for the alternative form of the gauge symmetry (3), (4) following the general recipe of Ref. [16]. Given the Hamiltonian form of the unfree gauge symmetry for the UG, we construct the Hamiltonian BFV-BRST complex for the UG following the prescription of the articles [8,16] for the general case of unfree gauge symmetry.

The paper is organized as follows. To make the article self-contained, in the next section we provide the general basics of describing the unfree gauge symmetry in Hamiltonian formalism. In Sec. III, proceeding from the general scheme of Sec. II, we provide the Hamiltonian description for the volume preserving diffeomorphisms and also the Hamiltonian analogs for the reducible gauge symmetry (3), (4). Section IV includes the BFV-BRST complexes for both forms of the gauge symmetry in UG. The last section includes concluding remarks.

II. UNFREE AND REDUCIBLE GAUGE SYMMETRY IN HAMILTONIAN FORMALISM

In this section, we briefly present the general scheme of deriving the unfree gauge symmetry for Hamiltonian constrained systems for the simplest case without tertiary and higher level constraints (the UG falls in this class of systems). For justification of the scheme, see in [8]. The Hamiltonian description of unfree gauge symmetry for the most general case with the tertiary and higher level constraints can be found in [9]. The method of finding the alternative Hamiltonian form of the unfree gauge symmetry with unrestricted reducible gauge parameters is worked out in the article [16]. It is briefly explained at the end of this section.

We begin with the action functional of constrained Hamiltonian system

$$S[q(t), p(t), \lambda(t)] = \int dt(p_i\dot{q}^i - H_T(q, p, \lambda)), \\ H_T(q, p, \lambda) = H(q, p) + \lambda^\alpha T_\alpha(q, p), \quad (5)$$

$\alpha = 1, \ldots, m$, where the time dependence is made explicit, while dependence on space points is implicit. Summation

[1]The subtle difference between the cohomology groups of the two complexes is related to the global conserved quantities, whose initial data are defined on the lower dimensional subset, not at the Cauchy surface. All the field theories with unfree gauge symmetry admit conserved quantities of this type (see in [16]). The simplest example of such a quantity is the cosmological constant of UG. For the first complex, these quantities turn out to be a coboundary, while for the alternative one, they are nontrivial cocycles (see in [16]). In this article, we notice this subtle distinction for the UG in the end of Sec. IV, though we do not elaborate on this fact in the present article.

over any condensed index includes integration over space. The action and the Hamiltonian H are supposed to be integrated over the space. All the constraints are assumed irreducible.

Let us assume the following involution relations of the Hamiltonian H, the primary constraints T_α, and the secondary ones τ_a:

$$\{T_\alpha, T_\beta\} = U^\gamma_{\alpha\beta}(\phi)T_\gamma, \tag{6}$$

$$\{T_\alpha, H\} = V^\beta_\alpha(\phi)T_\beta + V^a_\alpha(\phi)\tau_a, \tag{7}$$

$$\{\tau_a, H\} = V^\alpha_a(\phi)T_\alpha + V^b_a(\phi)\tau_b, \tag{8}$$

$$\{T_\alpha, \tau_a\} = U^\beta_{\alpha a}(\phi)T_\beta + U^b_{\alpha a}(\phi)\tau_b, \tag{9}$$

$$\{\tau_a, \tau_b\} = U^\alpha_{ab}(\phi)T_\alpha + U^c_{ab}(\phi)\tau_c, \tag{10}$$

where the uniform notation is introduced for the phase space variables $\phi = (q, p)$. These involution relations mean that conservation of the primary constraints T_α results in the secondary constraints τ_a while no tertiary ones arise. The complete set of constraints is of the first class. The subtlety that distinguishes the unfree-generated gauge symmetry from the case of symmetry with unconstrained gauge parameters lies in the structure of the coefficient $V^a_\alpha(\phi)$ in involution relation (7). This coefficient is supposed to be a nondegenerate differential operator in the sense that it has at most a finite dimensional kernel. The inverse does not exist to V in the class of differential operators, however. The simplest example of such V^a_α is a partial derivative. The kernel is one-dimensional (just any constant), while the inverse is not the differential operator. If the fields vanish at infinity, there is no kernel at all, while the operator ∂_i still does not admit a local inverse. In this case, the secondary constraints τ_a should vanish on shell for the sake of consistency of equations of motion, though relations $\tau_a \approx 0$ are *not differential consequences* of the primary constraints.

To represent the unfree gauge symmetry transformations in an economic way, it is convenient to introduce λ-dependent structure functions

$$\begin{aligned} W_1{}^\beta_\alpha(\phi,\lambda) &= V^\beta_\alpha - U^\beta_{\gamma\alpha}\lambda^\gamma, \\ W_2{}^\alpha_a(\phi,\lambda) &= V^\alpha_a - U^\alpha_{\beta a}\lambda^\beta, \\ \Gamma^b_a(\phi,\lambda) &= V^b_a - U^b_{\alpha a}\lambda^\alpha. \end{aligned} \tag{11}$$

In terms of these structure functions, the involution relations (6)–(9) read

$$\begin{aligned} \{T_\alpha(\phi), H_T(\phi,\lambda)\} = {}& W_1{}^\beta_\alpha(\phi,\lambda)T_\beta(\phi) \\ & + V^a_\alpha(\phi)\tau_a(\phi), \end{aligned} \tag{12}$$

$$\begin{aligned} \{\tau_a(\phi), H_T(\phi,\lambda)\} = {}& W_2{}^\alpha_a(\phi,\lambda)T_\alpha(\phi) \\ & + \Gamma^b_a(\phi,\lambda)\tau_b(\phi). \end{aligned} \tag{13}$$

In this form, the structure is evident of the Dirac-Bergmann algorithm for the system: Conservation of the primary constraints (12) leads to the secondary constraints τ_a as the coefficient V^a_α is nondegenerate even though it is not invertible in the class of differential operators. Conservation of the secondary constraints (13) does not lead to the tertiary ones nor does it define any Lagrange multiplier.

As the consequence of involution relations (12) and (13), the Hamiltonian action (5) is invariant [8] under the gauge transformations of phase space variables $\phi = (q, p)$ and Lagrange multipliers λ^α,

$$\delta_\epsilon\phi = \{\phi, T_\alpha\}\epsilon^\alpha + \{\phi, \tau_a\}\epsilon^a, \tag{14}$$

$$\delta_\epsilon\lambda^\alpha = \dot\epsilon^\alpha + W_1{}^\alpha_\beta(\phi,\lambda)\epsilon^\beta + W_2{}^\alpha_a(\phi,\lambda)\epsilon^a, \tag{15}$$

provided that the gauge parameters ϵ^α and ϵ^a are subject to the differential equations,

$$\dot\epsilon^a + \Gamma^a_b(\phi,\lambda)\epsilon^b + V^a_\alpha(\phi)\epsilon^\alpha = 0. \tag{16}$$

If the structure coefficient V^a_α admitted the inverse, being a differential operator, the gauge parameters ϵ^α could be expressed from (16) as combinations of gauge parameters ϵ^a and their derivatives. In this case, we would have the irreducible gauge symmetry with unconstrained gauge parameters ϵ^a. This symmetry would involve the second order time derivatives of the gauge parameters. Once the nondegenerate structure coefficient V^a_α does not admit any local inverse, the gauge symmetry (14), (15), (16) is unfree indeed. To demonstrate that the above transformation is a gauge symmetry only under the restrictions (16) imposed on the gauge parameters, let us compute the gauge variation (14) and (15) of the action (5),

$$\begin{aligned} \delta_\epsilon S \equiv \int dt \bigg(& (\dot\epsilon^a + \Gamma^a_b(\phi,\lambda)\epsilon^b + V^a_\alpha(\phi)\epsilon^\alpha)\tau_a \\ & - \frac{d}{dt}(T_\alpha\epsilon^\alpha + \tau_a\epsilon^a)\bigg). \end{aligned} \tag{17}$$

Once the secondary constraints τ_a are assumed to be irreducible, the variation integrand can reduce to the total derivative only under the condition that the coefficients vanish at τ_a. This leads one to impose restrictions (16) on the gauge parameters.

In the next section, the general relations (14)–(16) are specified for the UG reproducing the volume-preserving diffeomorphism in the Hamiltonian setup.

Any Hamiltonian action with unfree gauge symmetry admits the alternative reducible form of the gauge symmetry with unrestricted gauge parameters [16]. This reducible symmetry involves the higher order time derivatives of the gauge parameters. To arrive to this form of gauge symmetry, we introduce an alternative (overcomplete) set of the secondary constraints $\tilde{\tau}_\alpha$ that absorbs the structure coefficients V^a_α,

$$\tilde{\tau}_\alpha \equiv V^a_\alpha(\phi)\tau_a(\phi). \tag{18}$$

In general, the constraints above are reducible unlike the independent secondary constraints τ_a (7). In this article, we assume the simplest possible form of the reducibility conditions

$$Z_1{}^\alpha_A \tilde{\tau}_\alpha = 0, \qquad Z_2{}^A_{A_1} Z_1{}^\alpha_A = 0, \tag{19}$$

with field-independent null vectors $Z_1{}^\alpha_A$ and $Z_2{}^A_{A_1}$. The general case with off-shell nontrivial contributions to the reducibility relations can be found in the paper [16]. Making use of the reducible generating set of the secondary constraints, the involution relations (6)–(10) are reorganized as follows:

$$\{T_\alpha, T_\beta\} = U^\gamma_{\alpha\beta}(\phi)T_\gamma, \tag{20}$$

$$\{T_\alpha, H\} = V^\beta_\alpha(\phi)T_\beta + \tilde{\tau}_\alpha, \tag{21}$$

$$\{\tilde{\tau}_\alpha, H\} = \tilde{V}^\beta_\alpha(\phi)T_\beta + \tilde{V}'^\beta_\alpha(\phi)\tilde{\tau}_\beta, \tag{22}$$

$$\{T_\alpha, \tilde{\tau}_\beta\} = \tilde{\mathcal{U}}^\gamma_{\alpha\beta}(\phi)T_\gamma + \tilde{\mathcal{U}}'^\gamma_{\alpha\beta}(\phi)\tilde{\tau}_\gamma, \tag{23}$$

$$\{\tilde{\tau}_\alpha, \tilde{\tau}_\beta\} = \tilde{\mathfrak{U}}^\gamma_{\alpha\beta}(\phi)T_\gamma + \tilde{\mathfrak{U}}'^\gamma_{\alpha\beta}(\phi)\tilde{\tau}_\gamma. \tag{24}$$

We introduce λ-dependent structure functions, being analogs of (11) for the reducible set of constraints:

$$\tilde{W}_2{}^\beta_\alpha = \tilde{V}^\beta_\alpha - \tilde{\mathcal{U}}^\beta_{\gamma\alpha}\lambda^\gamma, \qquad \tilde{\Gamma}^\beta_\alpha = \tilde{V}'^\beta_\alpha - \tilde{\mathcal{U}}'^\beta_{\gamma\alpha}\lambda^\gamma. \tag{25}$$

In terms of these structure functions, the involution relations read

$$\{T_\alpha(\phi), H_T(\phi,\lambda)\} = W_1{}^\beta_\alpha(\phi,\lambda)T_\beta(\phi) + \tilde{\tau}_\alpha(\phi), \tag{26}$$

$$\{\tilde{\tau}_\alpha(\phi), H_T(\phi,\lambda)\} = \tilde{W}_2{}^\beta_\alpha(\phi,\lambda)T_\beta(\phi) + \tilde{\Gamma}^\beta_\alpha(\phi,\lambda)\tilde{\tau}_\beta(\phi). \tag{27}$$

The key distinction of relations (26) from (12) is that reducible secondary constraints $\tilde{\tau}_\alpha$ *are the differential consequences* of the primary constraints while the irreducible ones τ_a are not. It is the distinction that leads to the reducible gauge symmetry of the action (5) generated by T_α and $\tilde{\tau}_\alpha$ with the second time derivatives of unrestricted gauge parameters [16]. These gauge transformations read

$$\delta_\varepsilon\phi = \{\phi, T_\alpha\}(\dot{\varepsilon}^\alpha + \tilde{\Gamma}^\alpha_\beta\varepsilon^\beta - Z_1{}^\alpha_A\varepsilon^A) - \{\phi, \tilde{\tau}_\alpha\}\varepsilon^\alpha, \tag{28}$$

$$\delta_\varepsilon\lambda^\alpha = \left(\delta^\alpha_\beta\frac{d}{dt} + W_1{}^\alpha_\beta\right)(\dot{\varepsilon}^\beta + \tilde{\Gamma}^\beta_\gamma\varepsilon^\gamma - Z_1{}^\beta_A\varepsilon^A) - \tilde{W}_2{}^\alpha_\beta\varepsilon^\beta. \tag{29}$$

Given the involution relations of reducible constraints (19), (26), and (27), the above transformations leave the Hamiltonian action (5) invariant modulo a total derivative while no restrictions are imposed on the gauge parameters ε^α and ε^A.

Because of the reducibility of secondary constraints (19), gauge transformations (28) and (29) enjoy a gauge symmetry of their own. For the simplest case[2] of the constant null vectors, the gauge transformations of the original gauge parameters read

$$\delta_\omega\varepsilon^\alpha = Z_1{}^\alpha_A\omega^A, \qquad \delta_\omega\varepsilon^A = \dot{\omega}^A - Z_2{}^A_{A_1}\omega^{A_1}. \tag{30}$$

Because of (19), these symmetries are further reducible:

$$\delta_\eta\omega^A = Z_2{}^A_{A_1}\eta^{A_1}, \qquad \delta_\eta\omega^{A_1} = \dot{\eta}^{A_1}. \tag{31}$$

In the next section, proceeding from general relations (28)–(31) we find the Hamiltonian form of the reducible gauge transformations (3) and (4) for the UG.

III. UNIMODULAR GRAVITY

In this section, following the general scheme of the previous one, we construct the transverse diffeomorphism transformations in the Hamiltonian formalism of the UG. We also find the Hamiltonian form of the reducible description for the volume-preserving diffeomorphisms (3) and (4).

In this section, we use the ADM variables

$$g_{\alpha\beta} = \begin{pmatrix} N^2 + N_kN^k & N_j \\ N_i & \overset{*}{g}_{ij} \end{pmatrix},$$
$$g^{\alpha\beta} = \begin{pmatrix} N^{-2} & -N^jN^{-2} \\ -N^iN^{-2} & \overset{*}{g}{}^{ij} + N^iN^jN^{-2} \end{pmatrix}, \tag{32}$$

where Latin indices $i, j, k, \ldots$ run the values 1,2,3 and $N^i = \overset{*}{g}{}^{ij}N_j$. In these variables, the unimodularity condition $\det g_{\alpha\beta} = -1$ reads

$$N^2 = -\frac{1}{\overset{*}{g}}, \qquad \overset{*}{g} = \det \overset{*}{g}_{ij}. \tag{33}$$

[2]For the case of general reducibility, with off-shell nontrivial contributions, see [16].

This allows one to exclude from the set of variables the laps function N replacing it[3] by $(-\overset{*}{g})^{-1/2}$.

In terms of the ADM variables, with N excluded according to (33), the Einstein-Hilbert action[4] of UG is brought to the Hamiltonian form (5):

$$S = \int d^4x(\Pi^{ij}\partial_0\overset{*}{g}_{ij} - \mathcal{H}_T), \qquad \mathcal{H}_T = \mathcal{H}_0 + N^i T_i, \tag{34}$$

where the Hamiltonian reads

$$\mathcal{H}_0 = -\frac{1}{\overset{*}{g}}\mathcal{G}_{ijkl}\Pi^{ij}\Pi^{kl} + \overset{*}{R}. \tag{35}$$

Here, the usual definition is adopted for the De Witt metrics

$$\mathcal{G}_{ijkl} = \frac{1}{2}(\overset{*}{g}_{ik}\overset{*}{g}_{jl} + \overset{*}{g}_{il}\overset{*}{g}_{jk}) - \frac{1}{2}\overset{*}{g}_{ij}\overset{*}{g}_{kl}. \tag{36}$$

The inverse reads

$$\mathcal{G}^{ijkl} = \frac{1}{2}(\overset{*}{g}{}^{ik}\overset{*}{g}{}^{jl} + \overset{*}{g}{}^{il}\overset{*}{g}{}^{jk}) - \overset{*}{g}{}^{ij}\overset{*}{g}{}^{kl},$$
$$\mathcal{G}^{ijkl}\mathcal{G}_{klsm} = \frac{1}{2}(\delta^i_s\delta^j_m + \delta^i_m\delta^j_s). \tag{37}$$

The action includes the primary constraints

$$T_i = -2\overset{*}{g}_{ij}(\partial_k\Pi^{kj} + \overset{*}{\Gamma}{}^j_{kl}\Pi^{kl}), \tag{38}$$

with the shift functions N^i serving as the Lagrange multipliers. The Hamiltonian is a scalar, while the constraints (38) are covariant $3d$ vector densities.

The requirement of stability (12) of the primary constraints (38) reads

$$\left\{T_i, \int d^3x\,\mathcal{H}_T\right\} = T_j\partial_i N^j + \partial_j(T_i N^j) + \partial_i\left(\frac{1}{\overset{*}{g}}\mathcal{G}_{sm\,kl}\Pi^{sm}\Pi^{kl} - \overset{*}{R}\right) \approx 0. \tag{39}$$

The role of the structure coefficient V^a_α [cf. (7) and (12)] is played by the operator of the partial derivative. This operator does not admit the local inverse, while the one-dimensional kernel of ∂_i is formed by constants. So, we arrive at the single secondary constraint τ which is defined by the stability condition (39) modulo arbitrary additive constant,

$$\tau = -\frac{1}{\overset{*}{g}}\mathcal{G}_{ij\,kl}\Pi^{ij}\Pi^{kl} + \overset{*}{R} - \Lambda \approx 0, \qquad \Lambda = \text{const}. \tag{40}$$

The above constraint is the Hamiltonian counterpart of the relation $R = \Lambda$ being the well-known consequence of the UG field equations in Lagrangian formalism (see, e.g., [21]). The "integration" constant Λ is defined by the value of the metric and its derivatives at any single point of space, or by the asymptotics rather than by initial data at the entire Cauchy surface. For asymptotically flat space, $\Lambda = 0$, for example. Existence of the conserved quantities of this type is a common feature for all the systems with unfree gauge symmetry (see in [16]). The secondary constraint τ is $3d$ scalar, which differs from the Hamiltonian (35) by the "integration" constant Λ. In this sense, the constraint (40) means the off-shell conservation of energy density in the UG, given the metrics at any single point of the spacetime. This is in line with the recent discussion of the meaning of the Hamiltonian in the UG [23].

Stability condition for the secondary constraint τ does not lead to a tertiary constraint nor does it define any Lagrange multiplier [cf. (13)]

$$\left\{\tau, \int d^3x\,\mathcal{H}_T\right\} = -\partial_i((-\overset{*}{g})^{-1}\overset{*}{g}{}^{ij}T_j) + N^i\partial_i\tau. \tag{41}$$

The involution relations of all the constraints can be conveniently represented in terms of functionals being contractions with arbitrary test functions

$$T(\zeta) = \int d^3x\,T_i(x)\zeta^i(x),$$
$$\tau(\rho') = \int d^3x\,\tau(x)\rho'(x), \tag{42}$$

where $\xi^i(x)$ and $\zeta^i(x)$ are test vector fields and $\rho'(x)$ and $\sigma'(x)$ are the test functions being arbitrary scalar densities with weights 1 (letters with strokes). For these functionals, the involution relations (equal time P.B.) read

$$\{T(\xi), T(\zeta)\} = T([\xi,\zeta]), \tag{43}$$

$$\{T(\xi), \tau(\rho')\} = \tau([\xi,\rho']), \tag{44}$$

$$\{\tau(\rho'), \tau(\sigma')\} = T([\rho',\sigma']), \tag{45}$$

where

$$[\xi,\zeta]^i = \xi^j\partial_j\zeta^i - \partial_j\xi^i\zeta^j, \tag{46}$$

[3]Recently the modifications of the UG have been introduced that suggest to replace formula (33) by a more general relation between the lapse and the $3d$ metrics [24,25]. The techniques we propose for describing the gauge symmetry of the Hamiltonian formalism of UG are not particularly sensitive to the way of excluding N.

[4]We use the following definitions for the $3d$ Riemann tensor, Ricci tensor, and scalar curvature: $\overset{*}{R}{}^i{}_{jkl} = \partial_k\overset{*}{\Gamma}{}^i_{lj} - \partial_l\overset{*}{\Gamma}{}^i_{kj} + \overset{*}{\Gamma}{}^i_{ks}\overset{*}{\Gamma}{}^s_{lj} - \overset{*}{\Gamma}{}^i_{ls}\overset{*}{\Gamma}{}^s_{kj}$, $\overset{*}{R}_{ij} = \overset{*}{R}{}^s{}_{isj}$, and $\overset{*}{R} = \overset{*}{g}{}^{ij}\overset{*}{R}_{ij}$.

$$[\xi,\rho'] = \partial_i(\xi^i\rho'), \qquad [\rho',\xi] = -\partial_i(\xi^i\rho'), \tag{47}$$

$$[\rho',\sigma']^i = \frac{1}{\overset{*}{g}}\overset{*}{g}{}^{ij}(\rho'\partial_j\sigma' - \partial_j\rho'\sigma'). \tag{48}$$

Here $[\xi,\zeta]^i$ is a vector, $[\xi,\rho'], [\rho',\xi]$—scalar density with weight 1, $[\rho',\sigma']^i$—vector. The involution relations of the constraints and Hamiltonian read

$$\left\{T_i, \int d^3x\,\mathcal{H}_0\right\} = -\partial_i\tau, \tag{49}$$

$$\left\{\tau, \int d^3x\,\mathcal{H}_0\right\} = -\partial_i((-\overset{*}{g})^{-1}\overset{*}{g}{}^{ij}T_j). \tag{50}$$

Relations (39) and (41) present the explicit form of general involutional relations (12) and (13) in the case of UG. This allows one to identify the unfree gauge symmetry of the Hamiltonian UG action (34) making use of the general recipe (14)–(16). The gauge variations (14) and (15) for the case of UG read

$$\delta_\epsilon\overset{*}{g}_{ij} = \epsilon^k\partial_k\overset{*}{g}_{ij} + \overset{*}{g}_{ik}\partial_j\epsilon^k + \overset{*}{g}_{jk}\partial_i\epsilon^k - \frac{2}{\overset{*}{g}}\mathcal{G}_{ij\,kl}\Pi^{kl}\epsilon, \tag{51}$$

$$\begin{aligned}\delta_\epsilon\Pi^{ij} = {}& \partial_k(\Pi^{ij}\epsilon^k) - \Pi^{ik}\partial_k\epsilon^j - \Pi^{jk}\partial_k\epsilon^i \\ &+ \left(\frac{2}{\overset{*}{g}}\overset{*}{g}_{kl}\Pi^{ik}\Pi^{jl} - \frac{1}{\overset{*}{g}}\Pi\Pi^{ij} \right.\\ &\left. - \frac{1}{\overset{*}{g}}\overset{*}{g}{}^{ij}\mathcal{G}_{kl\,sm}\Pi^{kl}\Pi^{sm} + \overset{*}{R}{}^{ij}\right)\epsilon \\ &- \mathcal{G}^{ij\,kl}\sqrt{-\overset{*}{g}}(\partial_k\partial_l - \overset{*}{\Gamma}{}^s_{kl}\partial_s)\frac{\epsilon}{\sqrt{-\overset{*}{g}}},\end{aligned} \tag{52}$$

$$\delta_\epsilon N^i = \dot\epsilon^i + \epsilon^j\partial_j N^i - N^j\partial_j\epsilon^i - \frac{1}{\overset{*}{g}}\overset{*}{g}{}^{ij}\partial_j\epsilon, \tag{53}$$

where $\Pi = \overset{*}{g}_{ij}\Pi^{ij}$. The general equations (16) constraining the gauge parameters are specialized for UG as

$$\dot\epsilon + \partial_i(\epsilon^i - N^i\epsilon) = 0. \tag{54}$$

Relations (51)–(54), being deduced for the case of UG by the general procedure of the previous section, should represent the gauge symmetry of the Hamiltonian action (34). Let us verify this fact by directly varying the action given the variations of variables (51), (52), and (53),

$$\begin{aligned}\delta_\epsilon S &\equiv \int d^4x(\delta_\epsilon\Pi^{ij}\partial_0\overset{*}{g}_{ij} - \partial_0\Pi^{ij}\delta_\epsilon\overset{*}{g}_{ij} - \delta_\epsilon\mathcal{H}_T) \\ &\equiv \int d^4x((\dot\epsilon + \partial_i(\epsilon^i - N^i\epsilon))\tau \\ &\quad - \partial_0(T_i\epsilon^i + \tau\epsilon)).\end{aligned} \tag{55}$$

As one can see, the gauge variation would vanish off-shell modulo total time derivative provided that gauge parameters ϵ and ϵ^i are subject to Eq. (54).

Notice that the $4d$ diffeomorphism transformation parameters ξ^μ (2) are connected with their $(1+3)$-split Hamiltonian counterparts ϵ and ϵ^i involved in the transformations (51)–(54) by relations

$$\xi^0 = -\epsilon, \qquad \xi^i = -(\epsilon^i - N^i\epsilon). \tag{56}$$

This locally invertible change of gauge parameters brings Eq. (54) to the transversality condition $\partial_\mu\xi^\mu = 0$. Also notice that Hamiltonian gauge transformations (51) and (53) of $\overset{*}{g}_{ij}$ and N^i define the transformations of $4d$ unimodular metrics $g_{\mu\nu}$ as it is parametrized by the ADM variables (32) and (33). The transformations (51) and (53) involve, besides $\overset{*}{g}_{ij}$ and N^i, also the canonical momenta Π^{ij}. The momenta are defined by the Hamiltonian equations as functions of the ADM variables and the time derivatives of $3d$ metrics:

$$\begin{aligned}&\partial_0\overset{*}{g}_{ij} = \left\{\overset{*}{g}_{ij}, \int d^3x\mathcal{H}_T\right\} \Rightarrow \\ &\Pi^{ij} = -\frac{1}{2}\overset{*}{g}\mathcal{G}^{ijkl}\left(\partial_0\overset{*}{g}_{kl} - 2\partial_k N_l + 2\overset{*}{\Gamma}{}^s_{kl}N_s\right).\end{aligned} \tag{57}$$

As a result, upon the change of parameters $\epsilon,\epsilon^i \mapsto \xi^\mu$ [see (56)], the Hamiltonian gauge transformations (51) and (53) define the diffeomorphism of $4d$ unimodular metrics:

$$\delta_\epsilon g_{\mu\nu}(\overset{*}{g}_{ij},N^i)\Big|_{\substack{\epsilon\mapsto\epsilon(\xi)\\ \Pi\mapsto\Pi(\overset{*}{g}_{ij},N^i,\partial_0\overset{*}{g}_{ij})}} = -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu. \tag{58}$$

So, one can see that the Hamiltonian gauge transformations (51)–(54) represent indeed the $(1+3)$-split form of $4d$ volume-preserving diffeomorphism (2).

Now, let us deduce the Hamiltonian form of the reducible gauge symmetry (3) of the UG. The general scheme of replacing the unfree gauge symmetry by the alternative reducible counterpart with unrestricted gauge parameters is described in the end of previous section. It begins with absorbing the structure coefficient V^a_α of the involution relations (7) by the secondary constraint [see (18)]. For the UG, relations (7) read as (39) and (40) with V^a_α identified as $-\partial_i$. So, the reducible secondary constraints for the UG are defined as

$$\tilde{\tau}_i = -\partial_i \tau, \tag{59}$$

where τ is given by (40). Obviously, conservation of the primary constraints (38) identifies $\tilde{\tau}_i$ as the secondary ones

$$\left\{T_i, \int d^3x\, \mathcal{H}_T\right\} = T_j \partial_i N^j + \partial_j (T_i N^j) + \tilde{\tau}_i. \tag{60}$$

These constraints conserve

$$\left\{\tilde{\tau}_i, \int d^3x \mathcal{H}_T\right\} = \partial_i \partial_j ((-\overset{*}{g})^{-1} \overset{*}{g}{}^{jk} T_k) + \partial_i (\tilde{\tau}_j N^j). \tag{61}$$

The involution relations of all the constraints can be represented in terms of functionals

$$T(\zeta) = \int d^3x\, T_i(x) \zeta^i(x),$$
$$\tilde{\tau}(\rho') = \int d^3x\, \tilde{\tau}_i(x) \rho'^i(x), \tag{62}$$

where $\rho'^i(x)$ and $\sigma'^i(x)$ are the test functions being arbitrary vector densities with weights 1 and $\xi^i(x)$ and $\zeta^i(x)$ are test vector fields. For these functionals, the involution relations read

$$\{T(\xi), T(\zeta)\} = T([\xi, \zeta]), \tag{63}$$

$$\{T(\xi), \tilde{\tau}(\rho')\} = \tilde{\tau}([\xi, \rho']), \tag{64}$$

$$\{\tilde{\tau}(\rho'), \tilde{\tau}(\sigma')\} = T([\rho', \sigma']), \tag{65}$$

where the brackets denote the following bilinear skew-symmetric forms of the test functions:

$$[\xi, \zeta]^i = \xi^j \partial_j \zeta^i - \partial_j \xi^i \zeta^j, \tag{66}$$

$$[\xi, \rho']^i = \xi^i \partial_j \rho'^j, \qquad [\rho', \xi]^i = -\xi^i \partial_j \rho'^j, \tag{67}$$

$$[\rho', \sigma']^i = \frac{1}{\overset{*}{g}} \overset{*}{g}{}^{ij} (\partial_k \rho'^k \partial_j \partial_l \sigma'^l - \partial_j \partial_k \rho'^k \partial_l \sigma'^l). \tag{68}$$

Here $[\xi, \zeta]^i$ is the vector field, $[\xi, \rho'], [\rho', \xi]$ is the vector density of the weight 1, and $[\rho', \sigma']^i$ is the vector. The involution relations of the constraints and Hamiltonian read

$$\{T_i, \int d^3x \mathcal{H}_0\} = \tilde{\tau}_i, \tag{69}$$

$$\left\{\tilde{\tau}_i, \int d^3x \mathcal{H}_0\right\} = \partial_i \partial_j ((-\overset{*}{g})^{-1} \overset{*}{g}{}^{jk} T_k). \tag{70}$$

The secondary constraints (59) are reducible,

$$\varepsilon^{Aij} \partial_i \tilde{\tau}_j \equiv 0, \tag{71}$$

where ε^{Aij} is totaly antisymmetric, taking values $\{0, 1, -1\}$.

The role of the null vector $Z_{1A}^{\ \alpha}$ [see (19)] is played by the dualized De Rham differential. This null vector is reducible again. The role of the null vector $Z_{2A_1}^{\ A}$ [see (19)] is played by ∂_A. This reducibility of the UG Hamiltonian constraints corresponds to the general pattern described in Sec. II [cf. (19)].

Given the explicit form of irreducible primary constraints, reducible secondary ones, and the sequence of null vectors, the reducible gauge symmetry is constructed for the UG by the general recipe (28)–(31):

$$\begin{aligned}\delta_\varepsilon \overset{*}{g}_{ij} = {} & (\dot{\varepsilon}^k - N^k \partial_s \varepsilon^s - \varepsilon^{ksA} \partial_s \varepsilon_A) \partial_k \overset{*}{g}_{ij} \\ & + \overset{*}{g}_{ik} \partial_j (\dot{\varepsilon}^k - N^k \partial_s \varepsilon^s - \varepsilon^{ksA} \partial_s \varepsilon_A) \\ & + \overset{*}{g}_{jk} \partial_i (\dot{\varepsilon}^k - N^k \partial_s \varepsilon^s - \varepsilon^{ksA} \partial_s \varepsilon_A) \\ & + \frac{2}{\overset{*}{g}} \mathcal{G}_{ijkl} \Pi^{kl} \partial_s \varepsilon^s,\end{aligned} \tag{72}$$

$$\begin{aligned}\delta_\varepsilon \Pi^{ij} = {} & \partial_k (\Pi^{ij} (\dot{\varepsilon}^k - N^k \partial_s \varepsilon^s - \varepsilon^{ksA} \partial_s \varepsilon_A)) \\ & - \Pi^{ik} \partial_k (\dot{\varepsilon}^j - N^j \partial_s \varepsilon^s - \varepsilon^{jsA} \partial_s \varepsilon_A) \\ & - \Pi^{jk} \partial_k (\dot{\varepsilon}^i - N^i \partial_s \varepsilon^s - \varepsilon^{isA} \partial_s \varepsilon_A) \\ & - \left(\frac{2}{\overset{*}{g}} \overset{*}{g}_{kl} \Pi^{ik} \Pi^{jl} - \frac{1}{\overset{*}{g}} \Pi \Pi^{ij} \right. \\ & \left. - \frac{1}{\overset{*}{g}} \overset{*}{g}{}^{ij} \mathcal{G}_{kl\,nm} \Pi^{kl} \Pi^{nm} + \overset{*}{R}{}^{ij}\right) \partial_s \varepsilon^s \\ & + \mathcal{G}^{ijkl} \sqrt{-\overset{*}{g}} (\partial_k \partial_l - \overset{*}{\Gamma}{}^m_{kl} \partial_m) \frac{\partial_s \varepsilon^s}{\sqrt{-\overset{*}{g}}},\end{aligned} \tag{73}$$

$$\begin{aligned}\delta_\varepsilon N^i = {} & \frac{d}{dt} (\dot{\varepsilon}^i - N^i \partial_s \varepsilon^s - \varepsilon^{isA} \partial_s \varepsilon_A) \\ & + (\dot{\varepsilon}^j - N^j \partial_s \varepsilon^s - \varepsilon^{jsA} \partial_s \varepsilon_A) \partial_j N^i \\ & - N^j \partial_j (\dot{\varepsilon}^i - N^i \partial_s \varepsilon^s - \varepsilon^{isA} \partial_s \varepsilon_A) \\ & + \frac{1}{\overset{*}{g}} \overset{*}{g}{}^{ij} \partial_j \partial_s \varepsilon^s,\end{aligned} \tag{74}$$

$$\delta_\omega \varepsilon^i = \varepsilon^{ijA} \partial_j \omega_A, \qquad \delta_\omega \varepsilon_A = \dot{\omega}_A - \partial_A \omega, \tag{75}$$

$$\delta_\eta \omega_A = \partial_A \eta, \qquad \delta_\eta \omega = \dot{\eta}. \tag{76}$$

By construction, the reducible gauge variation of the ADM variables (72)–(74) with unrestricted gauge parameters ε^i and ε_A should leave the action (34) invariant modulo integral of a total divergence. This fact can be verified by explicitly varying the action:

$$\delta_\varepsilon S \equiv \int d^4x(\delta_\varepsilon \Pi^{ij}\partial_0 \overset{*}{g}_{ij} - \partial_0\Pi^{ij}\delta_\varepsilon \overset{*}{g}_{ij} - \delta_\varepsilon \mathcal{H}_T) \equiv -\int d^4x \varepsilon_A \varepsilon^{Aij}\partial_j \tilde{\tau}_i \quad (mod\, div). \tag{77}$$

Given reducibility of the secondary constraints (71), the action is invariant indeed.

Once the reducible gauge symmetry with the second order derivatives of unrestricted gauge parameters have been derived for the UG Hamiltonian action, the question appears about connection of these transformations with $4d$-covariant reducible symmetry (3), (4). To answer this question, we first establish correspondence between the gauge parameters of the Hamiltonian form of gauge symmetry $\varepsilon^i, \varepsilon_A, \omega_A, \omega, \eta$ [see (72), (74)–(76)] and their $4d$ counterparts [see (3), (4)]:

$$\varepsilon^i = \frac{1}{2}\varepsilon^{ijk}W_{jk}, \qquad \varepsilon_A = W_{0A}, \tag{78}$$

$$\omega_A = \varphi_A, \qquad \omega = \varphi_0, \qquad \eta = \psi. \tag{79}$$

Second, the Hamiltonian gauge transformations of the ADM variables involve the canonical momenta Π^{ij}, while the $4d$ covariant transformations do not. Replacing Π^{ij} by their on-shell expressions (57) in the reducible Hamiltonian gauge transformations, we recover explicitly covariant transformations (3), (4) upon the above identification of gauge parameters.

IV. HAMILTONIAN BFV-BRST FORMALISM OF THE UG

As we have demonstrated in the previous section, the constrained Hamiltonian formalism of UG admits two alternative formulations for gauge symmetry. The first one corresponds to the diffeomorphisms generated by transverse vector fields (2), while another one corresponds to the reducible gauge transformations (3) with the second order time derivatives of unrestricted gauge parameters being components of antisymmetric tensor. Existence of the alternative parametrizations of this type is a property of any system with unfree gauge symmetry [16]. Therefore, one can associate two different BRST complexes with the same action functional once it enjoys the unfree gauge symmetry. Given the Hamiltonian and constraints, these two complexes can be constructed along the usual lines of the BFV-BRST formalism in the *minimal sector* [26], while the specifics of unfree gauge symmetry reveals itself in the nonminimal sector [16] involved in the gauge fixing. For GR, the Hamiltonian BRST invariant action has been first constructed by Fradkin and Vilkovisky in the unpublished preprint [27]. The Fradkin-Vilkovisky action included the four ghost vertex in the gauge fixing terms due to the field dependence of the structure functions of involution relations. This work includes the BRST transformations, though it does not explicitly present the BRST charge for GR. The latter has been first explicitly presented in the article [28]. For the UG, the Hamiltonian BRST formalism has been unknown for the irreducible explicitly Λ-dependent set of constraints (38) and (40), and it is not known for the reducible Λ-independent generating set of constraints (38) and (59). In this section, we construct the Hamiltonian BFV-BRST formalism for the UG in both pictures (the first one with irreducible constraints generating the transverse diffeomorphisms, and the second one for the reducible constraints generating the second order reducible gauge symmetry) proceeding along the lines of the general method recently proposed in Refs. [8,16].

Let us begin with introducing the canonical set of ghosts for the irreducible set of constraints (38) and (40). The canonical ghost pairs of the minimal sector are assigned to all the constraints

$$\{C^i, \bar{P}_j\} = \delta^i_j, \qquad \mathrm{gh}\, C^i = -\mathrm{gh}\bar{P}_i = 1, \qquad \varepsilon(C^i) = \varepsilon(\bar{P}_i) = 1; \tag{80}$$

$$\{C, \bar{P}\} = 1, \qquad \mathrm{gh}C = -\mathrm{gh}\bar{P} = 1, \qquad \varepsilon(C) = \varepsilon(\bar{P}) = 1. \tag{81}$$

The original Hamiltonian action (34) involves the primary constraints T_i and corresponding Lagrange multipliers N^i. It does not explicitly involve the secondary constraint τ, nor does it contain any independent Lagrange multiplier for τ. This action enjoys the volume-preserving diffeomorphism (51)–(54).

Let us discuss now the gauge fixing for the UG in Hamiltonian BRST formalism. Once the four gauge parameters are constrained by one equation, to fix this gauge symmetry one has to impose three independent gauge conditions.[5] We are going to project out the three independent conditions from the Lorentz-covariant gauge of GR. We start with generalized de Donder–Fock conditions of GR [29]

$$D^\mu = \partial_\nu((-g)^\Delta g^{\mu\nu}) = 0, \qquad \Delta \in \mathbb{R}. \tag{82}$$

[5]Imposing three independent conditions we break explicit Lorentz invariance. To preserve the Lorentz symmetry explicitly one could consider four redundant gauge conditions. For the unfree gauge symmetry corresponding procedure of inclusion the redundant gauges are presented in Ref. [7]. In this article the general scheme is exemplified by the de Donder–Fock condition which is redundant for the linearized UG. For the nonlinear UG, the de Donder–Fock conditions are independent, so this gauge cannot be imposed. At the moment, no redundant Lorentz covariant gauge is known for the full nonlinear UG.

The Hamiltonian BFV-BRST formalism implies to impose the gauge conditions in the canonical form which is explicitly resolved with respect to the time derivatives of Lagrange multipliers. Let us express the gauge conditions (82) in terms of ADM variables (the special case $\Delta = 1$ is skipped) and resolve with respect to $\partial_0 N, \partial_0 N^i$:

$$\frac{D^0}{2(\Delta-1)N^{2\Delta-3}(-\overset{*}{g})^{\Delta}} = \partial_0 N - N^j\partial_j N - \frac{\Delta}{2(\Delta-1)}N^2(-\overset{*}{g})^{-\frac{1}{2}}\Pi + \frac{2\Delta-1}{2(\Delta-1)}N\partial_j N^j = 0, \tag{83}$$

$$-\frac{D^i + N^i D^0}{N^{2\Delta-2}(-\overset{*}{g})^{\Delta}} = \partial_0 N^i - N^2\partial_j \overset{*}{g}{}^{ji} - N^j\partial_j N^i - \Delta(-\overset{*}{g})^{-1}\overset{*}{g}{}^{ij}(2N\partial_j N(-\overset{*}{g}) + N^2\partial_j(-\overset{*}{g})) = 0. \tag{84}$$

For the UG, these conditions are simplified upon the account of the unimodularity condition $(-g) = N^2(-\overset{*}{g}) = 1$:

$$\frac{D^0}{2(\Delta-1)(-\overset{*}{g})^{\frac{3}{2}}} = \frac{1}{2(\Delta-1)(-\overset{*}{g})^{\frac{3}{2}}} \times(-\Pi + (-\overset{*}{g})\partial_j N^j) = 0, \tag{85}$$

$$-\frac{D^i + N^i D^0}{(-\overset{*}{g})} = \partial_0 N^i - (-\overset{*}{g})^{-1}\partial_j \overset{*}{g}{}^{ji} - N^j\partial_j N^i = 0. \tag{86}$$

One can see that the spatial projection (86) of the de Donder–Fock Lorentz invariant conditions (82) is an admissible gauge, while the time component (85) does not involve the time derivative of any Lagrange multiplier in the case of UG. So, we choose the independent gauge conditions

$$\partial_0 N^i - \chi^i = 0, \qquad \chi^i = (-\overset{*}{g})^{-1}\partial_j \overset{*}{g}{}^{ji} + N^j\partial_j N^i. \tag{87}$$

This gauge does not depend on momenta, unlike the complete de Donder–Fock conditions (83) in the GR. Gauge conditions (87) imply to introduce Lagrange multipliers π_i being canonically conjugate to N^i,

$$\{N^i, \pi_j\} = \delta^i_j, \qquad \mathrm{gh}N^i = -\mathrm{gh}\pi_i = 0, \qquad \varepsilon(N^i) = \varepsilon(\pi_i) = 0. \tag{88}$$

Given the canonical pairs of the above Lagrange multipliers, we introduce corresponding ghosts of the nonminimal sector,

$$\{P^i, \bar{C}_j\} = \delta^i_j, \qquad \mathrm{gh}P^i = -\mathrm{gh}\bar{C}_i = 1, \qquad \varepsilon(P^i) = \varepsilon(\bar{C}_i) = 1. \tag{89}$$

Since no gauge condition is imposed being paired with the "super-Hamiltonian" constraint τ (40), then the corresponding Lagrange multipliers and nonminimal sector ghosts are not introduced. Given the involution relations, Lagrange multipliers, and ghosts, the complete BRST charge reads

$$Q = \int d^3x(T_i C^i + \tau C - \bar{P}_i C^j\partial_j C^i - \bar{P}\partial_i(C^i C) + \bar{P}_i(-\overset{*}{g})^{-1}\overset{*}{g}{}^{ij}C\partial_j C + \pi_i P^i). \tag{90}$$

Notice that the cosmological constant Λ is explicitly involved in this charge through the secondary constraint τ defined by relation (40). This means the class of admissible fields is assumed restricted by the boundary conditions consistent with particular Λ.

Let us discuss the BRST invariant extension of the Hamiltonian. In the minimal ghost sector, given the involution relations (49) and (50), the BRST invariant Hamiltonian of the UG reads

$$\mathcal{H} = \int d^3x(\mathcal{H}_0(\phi) - \bar{P}\partial_i C^i - \bar{P}_i(-\overset{*}{g})^{-1}\overset{*}{g}{}^{ij}\partial_j C), \tag{91}$$

where $\mathcal{H}_0(\phi)$ is the original Hamiltonian (35). This Hamiltonian is BRST-exact modulo the additive constant Λ included in the secondary constraint (40),

$$\mathcal{H} = \int d^3x\Lambda + \left\{\int d^3x\bar{P}, Q\right\}. \tag{92}$$

At the level of BRST formalism, this corresponds to the earlier noticed fact that the Hamiltonian of UG (35) reduces on shell [given the secondary constraint (40)] to the constant defined by the field value at single point, or at asymptotics, rather than by the entire Cauchy surface. Since this complex treats Λ as the predefined parameter, it does not have the BRST cohomology element corresponding to energy.[6]

Now, let us construct the complete gauge fixed BRST-invariant Hamiltonian. Introduce the gauge fermion which includes independent gauges (87)

[6]This contrasts to the alternative BRST complex associated with the reducible form of the volume preserving diffeomorphisms. This issue is considered in the final part of this section.

$$\Psi = \int d^3x(\bar{C}_i\chi^i + \bar{P}_iN^i). \tag{93}$$

Given the gauge fermion Ψ and the BRST-invariant Hamiltonian of the minimal sector (91), the complete gauge-fixed BRST-invariant Hamiltonian reads

$$\begin{aligned} H_\Psi &= \mathcal{H} + \{Q,\Psi\} \\ &= \int d^3x\{\mathcal{H}_0(\phi) + T_iN^i + \pi_i\chi^i \\ &\quad - \bar{P}\partial_iC^i - \bar{P}_i(-\overset{*}{g})^{-1}\overset{*}{g}{}^{ij}\partial_jC + \bar{P}\partial_i(CN^i) + \bar{P}_i(\partial_jC^iN^j - C^j\partial_jN^i) \\ &\quad - \bar{C}_i(-\overset{*}{g})^{-1}(2\partial_j\overset{*}{g}{}^{ij}\overset{*}{\nabla}_kC^k + \partial_j(\overset{*}{\nabla}{}^jC^i + \overset{*}{\nabla}{}^iC^j)) \\ &\quad + \bar{C}_i(-\overset{*}{g})^{-1}(\partial_j\overset{*}{g}{}^{ij}(-\overset{*}{g})^{-1}\Pi C - 2\partial_j((-\overset{*}{g})^{-1}\Pi^{ij}C) + \partial_j(\overset{*}{g}{}^{ij}(-\overset{*}{g})^{-1}\Pi C)) \\ &\quad + \bar{C}_i(\partial_jN^iP^j + N^j\partial_jP^i) + \bar{P}_iP^i\}. \end{aligned} \tag{94}$$

Let us notice the two distinctions of the UG Hamiltonian H_Ψ from the long known GR counterpart [27]. First, the UG Hamiltonian includes the term with the original Hamiltonian $\mathcal{H}_0(\phi)$ (35) while in the GR the analogous term is absorbed by the super-Hamiltonian constraint multiplied by the lapse function N being an independent variable. In the UG, we do not have this variable, while the super-Hamiltonian term can be reduced to the constant Λ by shifting the gauge Fermion $\Psi \mapsto \Psi - \int d^3x\bar{P}$ [cf. (92)]. In the GR, the Π-squared contribution cannot be eliminated from the Hamiltonian H_Ψ by any admissible choice of the gauge fermion. The second distinction concerns the gauge fixing terms and related ghosts. In the UG, the relativistic gauge (87), being the spatial projection of the Lorentz invariant condition (82), does not result in the four ghost vertices unlike the GR counterpart [27].

To summarize this version of the Hamiltonian BFV-BRST formulation of the UG, we see that it corresponds to the fixed asymptotics of the fields and explicitly involves the corresponding cosmological constant as a predefined parameter. It differs, however, from the BFV formulation of GR by the content of the phase space, and by the structure of the BRST-invariant gauge-fixed Hamiltonian. This version of the BFV-BRST formalism suits well to the interpretation of the UG as the system with the fixed field asymptotics such that it corresponds to a certain predefined value of the cosmological constant. If the gravity could be quantized beyond the formal level (our consideration is formal) proceeding from this version of the BRST formalism, the cosmological constant would be involved in the quantum theory just as a numerical parameter whose value is fixed from the outset. This leaves no room for quantum transitions between the states with different values of Λ.

Let us construct now an alternative BFV-BRST formalism which proceeds from the reducible set of the secondary constraints (59). As we shall see, this formalism does not assume to fix the field asymptotics, nor does it explicitly involve the cosmological constant as the predefined parameter.

In the construction of the formalism, we follow the general scheme of the article [16] concerning the BFV-BRST formalism with reducible *secondary* constraints. This scheme differs from the classical BFV construction [26] by the nonminimal sector, including ghosts, Lagrange multipliers, and gauge conditions.

Given the irreducible primary constraints (38), and reducible secondary ones (59) and (71), we introduce the ghosts of the minimal sector,

$$\begin{aligned} \{C^i,\bar{P}_j\} &= \delta^i_j, \qquad \mathrm{gh}C^i = -\mathrm{gh}\bar{P}_i = 1, \\ \varepsilon(C^i) &= \varepsilon(\bar{P}_i) = 1; \end{aligned} \tag{95}$$

$$\begin{aligned} \{\mathcal{C}^i,\bar{\mathcal{P}}_j\} &= \delta^i_j, \qquad \mathrm{gh}\mathcal{C}^i = -\mathrm{gh}\bar{\mathcal{P}}_i = 1, \\ \varepsilon(\mathcal{C}^i) &= \varepsilon(\bar{\mathcal{P}}_i) = 1; \end{aligned} \tag{96}$$

$$\begin{aligned} \{\mathcal{C}_A,\bar{\mathcal{P}}^B\} &= \delta^B_A, \qquad \mathrm{gh}\mathcal{C}_A = -\mathrm{gh}\bar{\mathcal{P}}^A = 2, \\ \varepsilon(\mathcal{C}_A) &= \varepsilon(\bar{\mathcal{P}}^A) = 0; \end{aligned} \tag{97}$$

$$\begin{aligned} \{\mathcal{C},\bar{\mathcal{P}}\} &= 1, \qquad \mathrm{gh}\mathcal{C} = -\mathrm{gh}\bar{\mathcal{P}} = 3, \\ \varepsilon(\mathcal{C}) &= \varepsilon(\bar{\mathcal{P}}) = 1. \end{aligned} \tag{98}$$

The shift functions N^i, being the Lagrange multipliers to the *primary* constraints, are complemented by the conjugate momenta

$$\begin{aligned} \{N^i,\pi_j\} &= \delta^i_j, \qquad \mathrm{gh}N^i = -\mathrm{gh}\pi_i = 0, \\ \varepsilon(N^i) &= \varepsilon(\pi_i) = 0. \end{aligned} \tag{99}$$

These momenta are to serve as Lagrange multipliers to the *irreducible* gauge conditions (87).

The canonical conjugate pairs of ghosts are introduced for the irreducible gauge conditions

$$\{P^i, \bar{C}_j\} = \delta^i_j, \qquad \mathrm{gh}P^i = -\mathrm{gh}\bar{C}_i = 1, \\ \varepsilon(P^i) = \varepsilon(\bar{C}_i) = 1. \tag{100}$$

As far as the primary constraints are concerned, the nonminimal sector is constructed following the pattern of the irreducible gauge symmetry and the gauge fixing without redundancy.

Now, we turn to the nonminimal sector related to the reducible secondary constraints (59). Neither are Lagrange multipliers present in the theory for these constraints nor are gauge conditions imposed being paired with these constraints. So, the nonminimal sector of the secondary constraints does not include the canonical pairs of Lagrange multipliers, nor are introduced the ghosts related to the gauge conditions. Once the secondary constraints are reducible, they generate the redundant gauge symmetry (72)–(76) of the original variables. This leads to the ghosts for ghosts in the minimal sector, and this requires one to impose the gauge conditions on the ghosts related to the secondary constraints and their reducibilities. These gauge conditions, in their own turn, require corresponding extra ghosts. These are introduced following the pattern of redundant gauge conditions in the sector of the gauge of the ghosts, while the Lagrange multipliers and related ghosts are not introduced in this sector.

The nonminimal sector ghosts of the first reducibility of secondary constraints read

$$\{\mathcal{P}_A, \bar{\mathcal{C}}^B\} = \delta^B_A, \qquad \mathrm{gh}\mathcal{P}_A = -\mathrm{gh}\bar{\mathcal{C}}^A = 2, \\ \varepsilon(\mathcal{P}_A) = \varepsilon(\bar{\mathcal{C}}^A) = 0. \tag{101}$$

Also Lagrange multipliers are introduced to the first level reducibility and corresponding gauge conditions imposed on the original ghosts for reducible secondary constraints,

$$\{\lambda_A, \pi^B\} = \delta^B_A, \qquad \mathrm{gh}\lambda_A = -\mathrm{gh}\pi^A = 1, \\ \varepsilon(\lambda_A) = \varepsilon(\pi^A) = 1. \tag{102}$$

A similar set of the nonminimal sector ghosts and multipliers is introduced at the second reducibility level,

$$\{\mathcal{P}, \bar{\mathcal{C}}\} = 1, \qquad \mathrm{gh}\mathcal{P} = -\mathrm{gh}\bar{\mathcal{C}} = 3, \\ \varepsilon(\mathcal{P}) = \varepsilon(\bar{\mathcal{C}}) = 1; \tag{103}$$

$$\{\lambda, \pi\} = 1, \qquad \mathrm{gh}\lambda = -\mathrm{gh}\pi = 2, \\ \varepsilon(\lambda) = \varepsilon(\pi) = 0. \tag{104}$$

The second reducibility also requires one to introduce the extra ghosts

$$\{\lambda^{(1')}, \pi^{(1')}\} = 1, \qquad \mathrm{gh}\lambda^{(1')} = -\mathrm{gh}\pi^{(1')} = 1, \\ \varepsilon(\lambda^{(1')}) = \varepsilon(\pi^{(1')}) = 1; \tag{105}$$

$$\{\mathcal{P}^{(1')}, \bar{\mathcal{C}}^{(1')}\} = 1, \qquad \mathrm{gh}\mathcal{P}^{(1')} = -\mathrm{gh}\bar{\mathcal{C}}^{(1')} = 2, \\ \varepsilon(\mathcal{P}^{(1')}) = \varepsilon(\bar{\mathcal{C}}^{(1')}) = 0. \tag{106}$$

Given the ghost and Lagrange multiplier spectrum, constraints, and related null vectors, the complete BRST charge for the reducible gauge symmetry of UG reads

$$\begin{aligned} Q = \int d^3x \bigg(& T_i C^i + \tilde{\tau}_i \mathcal{C}^i + \bar{\mathcal{P}}_i \varepsilon^{ijA} \partial_j \mathcal{C}_A + \bar{\mathcal{P}}^A \partial_A \mathcal{C} \\ & - \bar{P}_i C^j \partial_j C^i - \bar{\mathcal{P}}_i C^i \partial_j \mathcal{C}^j + \bar{P}_i (-\overset{*}{g})^{-1} \overset{*}{g}{}^{ij} \partial_k \mathcal{C}^k \partial_j \partial_l \mathcal{C}^l \\ & + \frac{1}{2} \bar{\mathcal{P}}^A \varepsilon_{Aij} C^i C^j \partial_k \mathcal{C}^k + \frac{1}{6} \bar{\mathcal{P}} \varepsilon_{ijk} C^i C^j C^k \partial_l \mathcal{C}^l \\ & + \pi_i P^i + \pi^A \mathcal{P}_A + \pi \mathcal{P} + \pi^{(1')} \mathcal{P}^{(1')} \bigg), \end{aligned} \tag{107}$$

where ε_{ijk} is totally antisymmetric and takes values $\{0, 1, -1\}$, and $\varepsilon^{ijk} \varepsilon_{ksm} = -(\delta^i_s \delta^j_m - \delta^i_m \delta^j_s)$.

This BRST charge involves the ghost terms up to the fifth order even though the constraint algebra is closed [see involution relations (63)–(65)]. These higher order ghost terms are related to the off-shell disclosure of the algebra of reducible constraints: the null vectors are involved in the compatibility conditions for the structure functions of the involution relations.

In the minimal sector, the BRST invariant extension $\mathcal{H}$ of the original UG's Hamiltonian $\mathcal{H}_0$ reads

$$\begin{aligned} \mathcal{H} = \int d^3x \bigg(& \mathcal{H}_0(\phi) - \bar{\mathcal{P}}_i \mathcal{C}^i - \bar{P}_i (-\overset{*}{g})^{-1} \overset{*}{g}{}^{ij} \partial_j \partial_k \mathcal{C}^k \\ & + \frac{1}{2} \bar{\mathcal{P}}^A \varepsilon_{Aij} C^i C^j + \frac{1}{6} \bar{\mathcal{P}} \varepsilon_{ijk} C^i C^j C^k \bigg). \end{aligned} \tag{108}$$

Gauge fixing implies to involve the nonminimal sector through the gauge fermion Ψ,

$$H_\Psi = \mathcal{H} + \{Q, \Psi\}. \tag{109}$$

We suggest to choose the gauge Fermion in the following way:

$$\Psi = \int d^3x(\bar{C}_i\chi^i + \bar{P}_iN^i + (-\overset{*}{g})^{-1}\bar{\mathcal{C}}^A\varepsilon_{Aji}\overset{*}{\nabla}^j\mathcal{C}^i + \bar{\mathcal{P}}^A\lambda_A + (-\overset{*}{g})^{-1}\bar{\mathcal{C}}\overset{*}{\nabla}^A\mathcal{C}_A + \bar{\mathcal{P}}\lambda - (-\overset{*}{g})^{-1}\overset{*}{\nabla}^A\bar{\mathcal{C}}^{(1')}\lambda_A + \bar{\mathcal{C}}^A\overset{*}{\nabla}_A\lambda^{(1')} - \bar{\mathcal{C}}\mathcal{P}^{(1')}) \tag{110}$$

This choice breaks the spatial reparametrization invariance in the sector of original variables (as the de Donder–Fock conditions χ^i are involved), while it involves ghosts and ghost for ghosts in a reparametrization invariant way.

Given the gauge fermion, the gauge-fixed BRST invariant Hamiltonian of UG reads

$$\begin{aligned} H_\Psi = \int d^3x\Big\{ & \mathcal{H}_0(\phi) + T_iN^i + \pi_i\chi^i \\ & + (\varepsilon^{Aji}\overset{*}{\nabla}_j\bar{\mathcal{P}}_i - (-\overset{*}{g})^{-1}\overset{*}{\nabla}^A\pi^{(1')})\lambda_A - \overset{*}{\nabla}_A\bar{\mathcal{P}}^A\lambda \\ & + \pi^A((-\overset{*}{g})^{-1}\varepsilon_{Aji}\overset{*}{\nabla}^j\mathcal{C}^i + \overset{*}{\nabla}_A\lambda^{(1')}) + \pi((-\overset{*}{g})^{-1}\overset{*}{\nabla}^A\mathcal{C}_A - \mathcal{P}^{(1')}) \\ & - \bar{\mathcal{P}}_i(C^i - N^i\partial_k\mathcal{C}^k) - \bar{P}_i(-\overset{*}{g})^{-1}\overset{*}{g}^{ij}\partial_j\partial_k\mathcal{C}^k + \bar{P}_i(\partial_jC^iN^j - C^j\partial_jN^i) \\ & - \bar{C}_i(-\overset{*}{g})^{-1}(\partial_j\overset{*}{g}^{ij}X + \partial_jX^{ij}) + \bar{C}_i(\partial_jN^iP^j + N^j\partial_jP^i) \\ & + \bar{\mathcal{C}}^A((-\overset{*}{g})^{-1}(\delta_A^B\overset{*}{\Delta} - \overset{*}{\nabla}^B\overset{*}{\nabla}_A)\mathcal{C}_B + \overset{*}{\nabla}_A\mathcal{P}^{(1')}) + \bar{\mathcal{C}}(-\overset{*}{g})^{-1}\overset{*}{\Delta}\mathcal{C} \\ & + \bar{P}_iP^i + (\bar{\mathcal{P}}^A - (-\overset{*}{g})^{-1}\overset{*}{\nabla}^A\bar{\mathcal{C}}^{(1')})\mathcal{P}_A + \bar{\mathcal{P}}\mathcal{P} \\ & + \frac{1}{2}\bar{\mathcal{P}}^A\varepsilon_{Aij}(C^iC^j - 2C^iN^j\partial_k\mathcal{C}^k) - \bar{\mathcal{C}}^A(-\overset{*}{g})^{-1}\varepsilon_{Aji}\overset{*}{\nabla}^j(C^i\partial_k\mathcal{C}^k) \\ & + \bar{\mathcal{C}}^A(-\overset{*}{g})^{-1}\varepsilon_{Aji}\left(X\overset{*}{\nabla}^j\mathcal{C}^i + X^{jk}\overset{*}{\nabla}_k\mathcal{C}^i - \overset{*}{\nabla}^jX^i{}_k\mathcal{C}^k + \frac{1}{2}\overset{*}{\nabla}^jX\mathcal{C}^i\right) - \bar{\mathcal{C}}^A\overset{*}{\nabla}_AX\lambda^{(1')} \\ & - \left(\overset{*}{\nabla}^A\bar{\mathcal{C}}^{(1')}X + \overset{*}{\nabla}_i\bar{\mathcal{C}}^{(1')}X^{iA} + \frac{3}{2}\bar{\mathcal{C}}^{(1')}\overset{*}{\nabla}^AX\right)(-\overset{*}{g})^{-1}\lambda_A \\ & - \bar{\mathcal{C}}(-\overset{*}{g})^{-1}\left(X\overset{*}{\nabla}^A\mathcal{C}_A + \overset{*}{\nabla}_i(X^{iA}\mathcal{C}_A) - \frac{1}{2}\overset{*}{\nabla}^AX\mathcal{C}_A\right) \\ & + \frac{1}{6}\bar{\mathcal{P}}\varepsilon_{ijk}(C^iC^jC^k - 3C^iC^jN^k\partial_l\mathcal{C}^l) + \frac{1}{2}\bar{\mathcal{C}}(-\overset{*}{g})^{-1}\varepsilon_{Aij}\overset{*}{\nabla}^A(C^iC^j\partial_k\mathcal{C}^k)\Big\}, \end{aligned} \tag{111}$$

where $\overset{*}{\Delta} = \overset{*}{\nabla}_i\overset{*}{\nabla}^i$ and the following abbreviation is used:

$$X^{ij} = \overset{*}{\nabla}^iC^j + \overset{*}{\nabla}^jC^i + (-\overset{*}{g})^{-1}(2\Pi^{ij} - \overset{*}{g}^{ij}\Pi)\partial_k\mathcal{C}^k,$$
$$X = \overset{*}{g}_{ij}X^{ij} = 2\overset{*}{\nabla}_kC^k - (-\overset{*}{g})^{-1}\Pi\partial_k\mathcal{C}^k. \tag{112}$$

Covariant derivatives in (110), (111), and (112) are defined as for tensor densities of corresponding weight

$$\begin{aligned} p(\overset{*}{g}_{ij},\Pi^{ij}) &= p(C^i,\bar{P}_i) = p(\mathcal{C}_A,\bar{\mathcal{P}}^A) \\ &= p(\mathcal{C},\bar{\mathcal{P}}) = (0,1), \\ p(\mathcal{C}^i,\bar{\mathcal{P}}_i) &= (1,0), \\ p(N^i,\pi_i) &= p(P^i,\bar{C}_i) = p(\lambda_A,\pi^A) = p(\mathcal{P}_A,\bar{\mathcal{C}}^A) \\ &= p(\lambda,\pi) = p(\mathcal{P},\bar{\mathcal{C}}) = (-1,2), \\ p(\lambda^{(1')},\pi^{(1')}) &= p(\mathcal{P}^{(1')},\bar{\mathcal{C}}^{(1')}) = (-2,3), \\ p((-\overset{*}{g})^{\frac{1}{2}}) &= p(\varepsilon^{ijk}) = -p(\varepsilon_{ijk}) = 1. \end{aligned} \tag{113}$$

As we have seen in this section, the same classical Hamiltonian action of the UG (34) and (35) gives rise to two different Hamiltonian BRST formalisms. The first one involves the irreducible secondary constraints (40). Corresponding constraint algebra generates the unfree gauge symmetry with the gauge parameters obeying the transversality condition (see in Sec. III). The irreducible secondary constraints (40), and hence the BRST charge, explicitly involve the cosmological constant as the parameter predefined by the field asymptoptics. So, the cosmological constant is not a physical quantity in this setup as no BRST cocycle is associated with Λ. Another BRST complex, being based on the reducible set of secondary constraints corresponds to the reducible form of the volume preserving diffeomorphisms (see in Sec. III). The reducible constraints (59) do not explicitly involve Λ. Corresponding physical quantity τ is a cocycle of the local BRST complex generated by the BRST charge (107). So, this BRST complex captures the UG dynamics with various cosmological constants. If the gravity could be quantized beyond the formal level (we do not discuss here the notorious problem of renormalizing quantum gravity) proceeding from this BRST formulation, this would mean that the initial quantum state could be a mixture of various cosmological constants, and the quantum transitions are admissible between the states with different Λ's. The choice between these two inequivalent BRST-complexes depends on the setup of the physical problem. If the asymptotics of the fields is predefined (hence, Λ is fixed from the outset), the first option has to be chosen. If various asymptotics are admitted for the metrics, the second option is chosen where the cosmological constant is BRST cocycle, and hence it can enjoy dynamics at quantum level.

V. CONCLUSION

Let us briefly summarize the results and discuss further perspectives.

For the UG, we have found the Hamiltonian form of the volume preserving diffeomorphism transformations with the gauge parameters restricted by the transversality equations. We also find the Hamiltonian counterpart of the reducible form of UG gauge symmetry with the unrestricted gauge parameters (3) enjoying gauge symmetry of their own. Proceeding from these two alternative Hamiltonian descriptions of the volume preserving diffeomorphisms, we construct two alternative Hamiltonian BFV-BRST formalisms for the UG. These constructions are worked out along the lines of the recent article [16] which formulates the Hamiltonian description for a general system with unfree gauge symmetry. Let us mention some specifics of these BFV-BRST formalisms. The first of them, being related to the unfree form of the gauge symmetry, explicitly involves the cosmological constant Λ. This implies to fix the asymptotics of the fields, and the constant Λ is a fixed parameter from the viewpoint of this complex rather than an element of the cohomology group. Also it is interesting to note that this form of the UG Hamiltonian BFV-BRST formalism, being very close to the GR analog, admits such a projection of the de Donder–Fock condition that does not result in the four ghost vertex in the gauge fixed BRST invariant Hamiltonian, while for the GR this higher-order ghost vertex is inevitable for any known relativistic gauge fixing. The second form of the BFV-BRST Hamiltonian formalism corresponds to the reducible form of the UG gauge symmetry with unrestricted gauge parameters (3). In this picture, the BRST charge does not explicitly involve the cosmological constant, nor does the BRST invariant Hamiltonian. From the standpoint of the latter complex, the cosmological constant is a nontrivial element of the BRST cohomology group. This corresponds to the interpretation of Λ as the "global degree of freedom" while the former BRST complex treats the constant as a fixed parameter. In the BV formulation of the UG with the reducible parametrization of gauge symmetry (3), the cosmological constant is treated as the element of the local BRST cohomology group in the recent work [30].

Let us mention that any case of unfree gauge symmetry leads to the "global conserved quantities" [9,16,31]. Equally uniform for any unfree symmetry is the existence of an alternative form of gauge transformations with unrestricted gauge parameters enjoying the gauge symmetry of their own [16]. Various modifications of the UG are discussed in the literature (see [24,25,32,33] and references therein) where the cosmological constant, or even the Newtonian one, arises as the global conserved quantity. If one is going to treat these constants as the degrees of freedom, not as the parameters fixed by the predefined asymptotics of the fields, the reducible form of the gauge symmetry seems preferable.

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