

S-wave pion condensation in symmetric nuclear matterD. N. Voskresensky *BLTP, Joint Institute for Nuclear Research, RU-141980 Dubna, Russia
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S-wave pion-nucleon interactions in the linear sigma model, and in Manohar-Georgi and Gasser-Sainio-Svarc models with finite number of terms in Lagrangians, as well as in a general phenomenological approach are reviewed. Subtleties associated with the current algebra theorems and field redefinitions are discussed. In the first and third models most likely the *s*-wave pion condensation in the isospin-symmetric matter does not occur at least up to high densities, whereas within the second model it may appear already at moderate densities. In the phenomenological approach two parametrizations of the *s*-wave pion-nucleon scattering amplitude and the pion polarization operator used in the literature are considered. The first parametrization employs the off-mass-shell amplitude and allows to fulfil the current algebra theorems. Using it the *s*-wave pion polarization operator in the isospin-symmetric matter is reconstructed within the gas approximation. With this pion polarization operator the *s*-wave pion condensation in the isospin-symmetric matter does not occur at least up to high densities. Second parametrization uses the on-mass-shell pion-nucleon scattering amplitude and does not satisfy the Adler and Weinberg conditions. With such a parametrization most likely the *s*-wave pion condensation in the isospin-symmetric matter may occur already at the nucleon density $n \simeq (1.4 - 2.5)n_0$, where n_0 is the density of the atomic nucleus, that should result in observable effects. Both parametrizations allow us to successfully describe the pion atom data.

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Intensive study of pion degrees of freedom in nuclear matter started in 1970s [1–13], cf. reviews [14–18] and further references therein. Frequency-momentum (ω, \vec{q}) dependent pion polarization operator and pion spectra were constructed for nuclear matter with arbitrary ratio of neutrons and protons, N/Z , and a possibility of the *p*-wave pion condensation in the baryon medium at the density $n > n_c > n_0 \simeq 0.5m_\pi^3$ was demonstrated, where n_0 is the nuclear saturation density, m_π is the pion mass.

Pion off-mass-shell effects are of primary importance for description of the pion spectra and a possibility of the pion condensation [2,10,15]. The most important contribution to the pion polarization operator is given by the *p*-wave pion-nucleon and pion- Δ -isobar interactions. For the *s*-wave contribution to π^- polarization operator, Π_S^- , the first works [10,13] employed expression $\Pi_S^{\text{WT},-} \simeq C(n_n - n_p)\omega/m_\pi^2$, with constant C estimated as $C \simeq (1 - 1.4)$, n_n and n_p being the neutron and proton densities, respectively. Such a contribution is usually called the Weinberg-Tomozawa term. After replacements $\omega \rightarrow -\omega$, $\vec{q} \rightarrow -\vec{q}$ the same polarization operator describes π^+ . For the case of the isospin-symmetric matter the mentioned works used $\Pi_S(N \simeq Z) = 0$ taking into account that the experimental value of the pion-nucleon scattering length is tiny. Papers

[11–13,16] studied the problem of the description of pion degrees of freedom in nuclear matter within linear sigma model as the realization of approximate chiral symmetry with partial conservation of axial-vector current (PCAC), as canonical operator equation. Within this consideration $C \simeq m_\pi^2/(2f_\pi^2)$, $f_\pi \simeq 92.4$ MeV is the weak pion decay constant.

First phenomenological optical potential for description of the scattering of on-mass-shell mesons off nuclei was suggested by L. S. Kisslinger in 1955 [19]. Reference [20] introduced a phenomenological optical potential for the description of pion atoms, cf. [17]. Retardation effects were disregarded, whereas they actually play an important role in this problem, cf. [15]. Appropriate fit of ω, \vec{q} dependent optical pion-nucleus potential to the pion-atom data with $N = Z$ and $N \neq Z$ known to that time was performed in [21].

Parametrization of the optical pion-nucleus potential employed in [15,18,21–26] uses the fully off-mass-shell pion-nucleon amplitude, which fulfills the current algebra theorems and the canonical PCAC condition. Oppositely, Refs. [27–31] used the on-mass-shell pion-nucleon amplitude, taking incoming and outgoing pion 4-momenta such that $q^2 = q'^2 = m_\pi^2$, that does not allow to fulfill the so called Adler and Weinberg current algebra conditions.

The reasoning of [27,28] and their followers to put pions on mass shell considering amplitude of the pion-nucleon

scattering and the pion polarization operator in matter goes back to the equivalence theorem that any local change of variables in quantum field theories, which leaves the free field part of the Lagrangian unchanged, does not alter the S matrix [32–34]. Thereby the models dealing with the off-mass-shell amplitude (after variable replacement) and the on-mass-shell one, being employed for description of purely on-mass-shell particle scattering in vacuum, should yield equivalent results. The same statement holds for consideration of the scattering of particles on a number of infinitely heavy centers [35]. The authors [27] spread these statements first to the total amplitude of the pion scattering on a system of massive (but not infinitely massive) centers and then they assumed that the theorem serves “to fully eliminate off-mass-shell effects both in the leading order and in higher order terms,” even when one considers propagation of the classical pion field in the medium. Further, one concluded, cf. [36,37], that off-mass-shell Green functions and all off-mass-shell properties are unobservable.

However in reality the in-medium conserved current j^μ and energy momentum tensor $\Theta^{\mu\nu}$ are expressed in terms of the nonequilibrium fully off shell Green functions G_C and self-energies Σ_C determined on the Schwinger-Keldysh contour. In the thermal equilibrium they are further expressed through the spectral functions $A = -2\text{Im}G^R$, in-medium particle widths $\Gamma = -2\text{Im}\Sigma^R$ and the off shell Fermi/Bose occupations n_ω , where G^R and Σ^R are the retarded Green function and self-energy, cf. [38,39]. Conserved charges, the energy and the momentum are observable quantities. Spectral functions $A = -2\text{Im}G^R$ (and flow B and entropy flow A_S spectral functions) are associated with the density of states and various time delays, and in the virial limit with the measurable phase shifts, cf. [40] and references therein. Observable 3-momentum distributions of the particles radiated from a piece of a nonequilibrium matter are expressed in terms of the nonequilibrium self-energy $i\Pi^{-+}$, the current-current correlator. The 3-momentum distributions of the particles radiated from a piece of the equilibrium matter are expressed in terms of A , Γ , and n_ω , cf. [18,41–43]. Particles radiated to infinity are on mass shell but all internal integrations are performed with the fully off shell Green functions. Authors [37] suggest that the 3-momentum occupations in the medium, \tilde{n}_k^{med} , as they are defined there, are not observable, since these quantities depend on artificially introduced interpolated fields in their example. Here we should stress that not the in-medium occupations \tilde{n}_k^{med} but various frequency integrals of $i\Pi^{-+}$, being calculated in a piece of matter, determine the observable 3-momentum particle distributions at infinity, n_k^{vac} , particle luminosity, etc., cf. [40,44]. Moreover, the Noether and the in-medium four currents coincide only provided some special conditions are fulfilled, e.g., as it occurs in the

Fermi liquid theory and in so-called Φ derivable approximation schemes, cf. [38,39]. At the end let us note that the Landau damping, zero sound and phonon propagation, Landau-Pomeranchuk-Migdal effect, phase transitions in condensed matter and many other effects can be described only dealing with the off-mass-shell particle propagation.

Below it will be explicitly shown that the two mentioned (off shell and on shell) approaches result in essentially different physical consequences, that could be checked experimentally. For example, extrapolation to densities higher than the nuclear saturation density n_0 done within the model used in [15,18,21,22] does not allow for the s -wave pion condensation in the isospin-symmetric nuclear matter, whereas the model employed in [27–30] allows for occurrence of the s -wave pion condensation already at the density $n > (1.4 - 2.5)n_0$ or even for a smaller density, as it will be shown below. Although the latter possibility was mentioned in [45], References [27,28,46] disregarded it arguing that a strong decrease of the effective pion mass is compensated by the ω -dependent range term in the spectrum, whereas, as it will be explicitly shown below, the possibility of the s -wave pion condensation directly follows from their model.

Some works, cf. [47,48], tried to reconcile two mentioned approaches by doing formal replacements of the fields in the Lagrangians, which should not affect physics. The problem is however subtle and some authors changed their position from work to work, whereas in our opinion solution of the puzzle is as follows: from the fact that observables should not depend on the choice of the interpolating fields in the full Lagrangian it does not follow that cancellation of artificially introduced contributions depending on the interpolating fields should occur term by term or in the subgroups of the diagrams. The graphs should be calculated following the ordinary Feynman rules, rather than by a somewhat artificial putting of the ingoing particles on mass shell in each diagram. To keep in mind this circumstance it proves to be especially important in practical schemes, where one deals with the Lagrangians, which differ at least in the high-order contributions in the fields (we further call them reduced Lagrangians). Thereby it is not surprising that such reduced Lagrangians predict different observable effects. Only experimental check of the specific predictions of the models can allow us to choose between them.

The paper is organized as follows. Next section formulates partial conservation of PCAC and current algebra theorems. Then in Sec. III we study conditions for their fulfilment within the linear sigma model and then in Secs. IV and V we consider the Manohar-Georgi and the Gasser-Sainio-Svarc reduced Lagrangians, respectively. All models will be treated at the usage of certain approximations. We construct pion polarization operators in each of models. Section VI discusses two purely phenomenological approaches to construct the s -wave pion-nucleon amplitude

and the s -wave pion polarization operator in the isospin-symmetric nuclear matter. In the first approach one deals with the pion-nucleon amplitude determined off mass shell at arbitrary relations between the variables. The amplitude is constrained by the experimental fact that the s -wave pion-nucleon scattering length, $a_{\pi N}^+$, is very small and by the fulfillment of the current algebra Cheng-Dashen and Weinberg (or Adler) conditions. Correspondingly, the Adler (or Weinberg) condition is then fulfilled identically. In the second approach one puts $q^2 = q'^2 = m_\pi^2$ satisfying the Cheng-Dashen condition and that $a_{\pi N}^+ \simeq 0$ but violating the Adler and Weinberg conditions. Then Sec. VII focuses on a question about presence or absence of the s -wave pion condensation in the isospin-symmetric nuclear matter. It will be shown that the s -wave pion condensation in the isospin-symmetric nuclear matter does not occur in the first approach and it may occur already at $n = (1.4 - 2.5)n_0$, or even at a smaller density, in the second approach. Section VIII contains concluding remarks.

II. PCAC AND CURRENT ALGEBRA

Low-energy pion-nucleon scattering was widely discussed in the 1960s, when ideas of current algebra and PCAC were developed, cf. [49–51]. One introduces the isospin-even pion-nucleon forward scattering amplitude with the pseudovector pole term subtracted,

$$\tilde{D}^+(\nu, t, q^2, q'^2) = D^+(\nu, t, q^2, q'^2) - \frac{\nu_B^2 g^2 \Gamma(q^2) \Gamma(q'^2)}{m_N (\nu_B^2 - \nu^2)}, \quad (1)$$

where $q^2 = \omega^2 - \vec{q}^2$, $q'^2 = \omega'^2 - \vec{q}'^2$, $\nu = (s - u)/(4m_N) = (p + p')(q + q')/(4m_N)$, $\nu_B = (t - q^2 - q'^2)/(4m_N) = -qq'(2m_N)$, $s = (p + q)^2$, $u = (p' - q)^2$, $t = (q - q')^2$ are appropriate kinematical variables; g and Γ are, respectively, the πNN coupling constant and a vertex form factor, and $m_N \simeq 938$ MeV is the nucleon mass in vacuum.

The amplitude \tilde{D}^+ is related to the pion-nucleon sigma term, which is an important measure of chiral symmetry breaking [52],

$$\Sigma(t) = \frac{1}{3} \sum_{i=1}^3 \langle N(p') | [Q_5^i, [Q_5^i, H_{\text{SB}}]] | N(p) \rangle, \quad (2)$$

where N is the nucleon state, H_{SB} is the Hamiltonian density of the symmetry breaking term, and $Q_5^i = \int A_0^i(x) d^3x$ is the $i = 1, 2, 3$ component of the axial-vector charge. On the quark level the axial current is given by $A_\mu^i = \bar{q} \gamma_\mu \gamma_5 \tau^i q$, where q are quark fields, γ_μ , γ_5 are Dirac matrices, τ^i are isospin matrices in SU(2) case. Recent lattice data [53] satisfy PCAC within 5% error bar.

Using the definition [54] of the pion decay constant f_π for the process $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ in vacuum, the hadronic matrix element of the axial current is

$$\langle 0 | A_\mu^i | \pi^j(q) \rangle \equiv i q_\mu f_\pi \delta^{ij}, \quad (3)$$

which yields

$$\langle 0 | \partial^\mu A_\mu^i | \pi^j(q) \rangle = q^2 f_\pi \delta^{ij} = m_\pi^2 f_\pi \delta^{ij} \quad (4)$$

for the pion in vacuum with $q^2 = m_\pi^2$.

Employing convenient choice of the pion field operator normalization, $\langle 0 | \pi^i | \pi^j \rangle = \delta^{ij}$, one gets

$$\partial^\mu A_\mu^i = f_\pi m_\pi^2 \pi^i, \quad (5)$$

where A_μ^i is the hadronic axial vector current.

At the assumption of a smoothly varying amplitude $\tilde{D}^+(\nu, t, q^2, q'^2)$, one arrives at the following relations [49,50] (in variables ν, ν_B, q^2, q'^2 corresponding to $\nu = \nu_B = 0$):

$$\tilde{D}^+(\nu = 0, t = 0, \quad q^2 = 0, \quad q'^2 = 0) = -\Sigma(t = 0)/f_\pi^2 \quad (6)$$

at the Weinberg kinematical point,

$$\tilde{D}^+(0, m_\pi^2, m_\pi^2, 0) = \tilde{D}^+(0, m_\pi^2, 0, m_\pi^2) = 0 \quad (7)$$

at the Adler kinematical point, and

$$\begin{aligned} \tilde{D}^+(0, 2m_\pi^2, m_\pi^2, m_\pi^2) &= +\Sigma(t = 2m_\pi^2)/f_\pi^2 \\ &\simeq (\Sigma(t = 0)/f_\pi^2)(1 + O(m_\pi^2/m_\sigma^2)) \end{aligned} \quad (8)$$

at the Cheng-Dashen kinematical point, cf. [55,56] and Eq. (22) below. Here m_σ is the mass of the sigma meson.

The amplitude in the Weinberg point is repulsive and equal in magnitude to the attractive amplitude in the Cheng-Dashen point. The Cheng-Dashen point is distinguished from the others in the fact that both pions have their momenta on mass shell, i.e., $q^2 = q'^2 = m_\pi^2$. The sigma commutator in (8) can be evaluated either at $t = 2m_\pi^2$ or at $t = 0$, since the difference is a small correction, cf. [52].

One also may assume that putting the final pion back on mass shell (and holding fixed t and ν) should not change the Adler consistency condition much, cf. [24]. If so, one gets an additional condition

$$\tilde{D}^+(0, m_\pi^2, m_\pi^2, 0) \simeq \tilde{D}^+(0, m_\pi^2, m_\pi^2, m_\pi^2) = 0. \quad (9)$$

There exist various estimates of the pion-nucleon Σ term in the literature, which cover a broad range of values. Most

modern estimates yield $\Sigma \simeq 55\text{--}75$ MeV, cf. [57]. Using the Roy-Steiner equations to control the extrapolation of the vanishingly small near threshold πN isoscalar scattering amplitude to zero pion mass, the Bern-Bonn-Jülich group yielded $\Sigma \simeq 53\text{--}63$ MeV, cf. [58]. The values of the Σ term extracted recently from a fit to the pion atoms [59] correspond to $\Sigma \simeq 50\text{--}64$ MeV.

Reference [55] performed calculations in the tree approximation to the linear sigma model. The canonical PCAC relation (5) and the consistency conditions (6)–(8) are fulfilled at specific choices of symmetry breaking terms in the Lagrangian. Some Lagrangians, such as the Manohar-Georgi one [60], including the next-to-leading order terms in the chiral perturbation theory, do not satisfy the canonical PCAC relation and (6) and (7) conditions, cf. discussion in [56].

III. LINEAR SIGMA MODEL

A. Lagrangian and PCAC

The Lagrangian density of the linear sigma model (SM) is given by

$$L_{\text{SM}} = L_{\text{SM}}^{\text{sym}} + L^{\text{sb}}, \quad (10)$$

where the symmetric part of the Lagrangian density (sym) is as follows

$$L_{\text{SM}}^{\text{sym}} = \bar{N}[i\gamma^\mu \partial_\mu - g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]N + [\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi} \partial^\mu \vec{\pi}]/2 - \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2/4. \quad (11)$$

Here π_i , σ , N are the isospin-vector pion, scalar sigma and bispinor neutron and proton fields and v is a positive constant. The symmetry breaking term in the Lagrangian (sb) is taken as

$$L^{\text{sb}} = L_1^{\text{sb}} + L_2^{\text{sb}} + L_3^{\text{sb}} = \epsilon_1 \sigma - \epsilon_2 \vec{\pi} \vec{\pi} - \epsilon_3 \bar{N}N, \quad (12)$$

constants ϵ_i are assumed to be small quantities. Simplifying consideration we did not add terms responsible for the Δ -isobar-pion-nucleon interaction. They also can be included within the SM, cf. [11–13].

With the help of the axial transformations

$$\begin{aligned} \sigma &\rightarrow \sigma + \alpha_i \pi_i, & \pi_i &\rightarrow \pi_i - \alpha_i \sigma, \\ N &\rightarrow N + i\tau_i \alpha_i \gamma_5 N/2 \end{aligned} \quad (13)$$

from (11) one finds expression for the axial-vector current

$$A_\mu^i = \bar{N}\gamma^5 \gamma_\mu \tau^i N/2 + \pi^i \partial_\mu \sigma - \sigma \partial_\mu \pi^i. \quad (14)$$

This expression yields [54]

$$\partial^\mu A_\mu^i = (\epsilon_1 + 2\epsilon_2 \sigma)\pi^i - i\epsilon_3 \bar{N}\gamma_5 \tau^i N. \quad (15)$$

As we see, only for $\epsilon_2 = \epsilon_3 = 0$, $\epsilon_1 \simeq m_\pi^2 f_\pi$, the PCAC condition holds in its canonical form (5). However in the tree approximation it holds also for

$$\epsilon_1 + 2\epsilon_2 \langle \sigma \rangle = m_\pi^2 f_\pi, \quad (16)$$

other terms contribute to loops. The Adler consistency condition is fulfilled provided $\epsilon_3 + 2\epsilon_2 m_N/m_\sigma^2 = 0$, see Eq. (29) below.

B. Mean field, particle masses, and the Σ term

Up to linear terms in ϵ_i minimization of the effective potential yields for the mean field $\langle \sigma \rangle$,

$$\langle \sigma \rangle = v + \epsilon_1/(2\lambda v^2) + \dots, \quad (17)$$

for the nucleon mass,

$$m_N = g\langle \sigma \rangle + \epsilon_3 \simeq gv + g\epsilon_1/(2\lambda v^2) + \epsilon_3 + \dots, \quad (18)$$

for the mass of the $\sigma' = \sigma - \langle \sigma \rangle$ field counted from the expectation value,

$$m_\sigma^2 = \lambda(3\langle \sigma^2 \rangle - v^2) \simeq 2\lambda v^2 + 3\epsilon_1/v + \dots, \quad (19)$$

and for the pion mass

$$m_\pi^2 = \lambda(\langle \sigma \rangle^2 - v^2) + 2\epsilon_2 \simeq \epsilon_1/v + 2\epsilon_2 + \dots \quad (20)$$

Employing (4), (15) we find

$$\begin{aligned} \langle 0|(\epsilon_1 + 2\epsilon_2 \langle 0|\sigma|0\rangle)\pi^i + (2\epsilon_2 \sigma' \pi^i - i\epsilon_3 \bar{N}\gamma_5 \tau^i N)|\pi^j(q)\rangle \\ = f_\pi m_\pi^2 \delta^{ij}. \end{aligned} \quad (21)$$

In the tree approximation only first term contributes and we recover Eq. (16). From Eqs. (16), (20) it follows that $\langle \sigma \rangle = f_\pi$.

For the Σ term in the tree approximation Ref. [55] found expression

$$\begin{aligned} \Sigma(t) &= \frac{(m_N - \epsilon_3)(m_\pi^2 + 2\epsilon_2)}{m_\sigma^2 - t} + \epsilon_3, \\ &\simeq \frac{m_N m_\pi^2}{m_\sigma^2} + \frac{2\epsilon_2 m_N}{m_\sigma^2} + \epsilon_3, \end{aligned} \quad (22)$$

and $\Sigma(t = 2m_\pi^2)$ is close to $\Sigma(t = 0)$ for $2m_\pi^2/m_\sigma^2 \ll 1$.

With $m_N = 939$ MeV, $m_\pi = 139$ MeV, and the ordinary used value $m_\sigma \simeq 600$ MeV for $\epsilon_2 = \epsilon_3 = 0$ we obtain $\Sigma(t = 0) = m_N m_\pi^2/m_\sigma^2 \simeq 50.4$ MeV, $\Sigma(t = 2m_\pi^2) \simeq 56.5$ MeV, cf. (22). These quantities are close to

experimentally motivated values [58,59]. We could consider it as an argument in favor of choosing

$$\frac{2\epsilon_2 m_N}{m_\sigma^2} + \epsilon_3 \simeq 0. \quad (23)$$

C. The πN scattering amplitude

Feynman diagrams contributing to the πN scattering amplitude in the tree approximation to the σ model [55] are shown in Fig. 1. First two diagrams yield the standard pseudoscalar Born pole

$$D_{(1,2)}^+ = \frac{g^2 \nu^2}{m_N (\nu_B^2 - \nu^2)} \quad (24)$$

and the third diagram produces

$$\tilde{D}_{(3)}^+ = \frac{2\lambda \nu g}{m_\sigma^2 - t}. \quad (25)$$

Replacing $\langle \sigma \rangle \simeq f_\pi$ and $2\lambda f_\pi \simeq (m_\sigma^2 - m_\pi^2 + 2\epsilon_2)/f_\pi$, as it follows from (17), (19), (20), one finds

$$\tilde{D}^+ = \frac{g^2}{m_N - \epsilon_3} - \frac{g^2}{m_N} + \frac{g^2}{m_N - \epsilon_3} \frac{t - m_\pi^2 + 2\epsilon_2}{m_\sigma^2 - t}, \quad (26)$$

and in linear approximation in ϵ_i ,

$$f_\pi^2 \tilde{D}^+ \simeq \epsilon_3 + \frac{m_N (t - m_\pi^2 + 2\epsilon_2)}{m_\sigma^2}. \quad (27)$$

Putting $t = 2m_\pi^2$ in this expression yields

$$f_\pi^2 \tilde{D}^+(t = 2m_\pi^2) \simeq \frac{m_N m_\pi^2}{m_\sigma^2} + \frac{2\epsilon_2 m_N}{m_\sigma^2} + \epsilon_3 + \dots \quad (28)$$

Comparing (28) and (22) we obtain $f_\pi^2 \tilde{D}^+(t = 2m_\pi^2) \simeq \Sigma(0) \simeq \Sigma(t = 2m_\pi^2)$. Thus the Cheng-Dashen condition (8) is fulfilled provided $\Sigma(t = 2m_\pi^2) \simeq \Sigma(t = 0) = \Sigma$, i.e., when one neglects the $2m_\pi^2 \Sigma/m_\sigma^2 \ll 1$ correction.

To obtain the amplitude in the Adler point we put $t = m_\pi^2$ and $q' = 0$ in (26) that produces

$$\begin{aligned} \tilde{D}^+(0, m_\pi^2, m_\pi^2, 0) &= \frac{g^2}{m_N - \epsilon_3} - \frac{g^2}{m_N} + \frac{g^2}{m_N - \epsilon_3} \frac{2\epsilon_2}{m_\sigma^2 - m_\pi^2} \\ &\simeq \frac{\epsilon_3 + 2\epsilon_2 m_N/m_\sigma^2}{f_\pi^2} \simeq \frac{\Sigma(0) - m_N m_\pi^2/m_\sigma^2}{f_\pi^2}. \end{aligned} \quad (29)$$

Taking $t = 0$ in (26), in the Weinberg point we have

$$\begin{aligned} \tilde{D}^+(0, 0, 0, 0) &= \frac{g^2}{m_N - \epsilon_3} - \frac{g^2}{m_N} + \frac{g^2}{m_N - \epsilon_3} \frac{2\epsilon_2 - m_\pi^2}{m_\sigma^2} \\ &\simeq -\frac{m_N m_\pi^2}{m_\sigma^2 f_\pi^2} + \frac{\epsilon_3 + 2\epsilon_2 m_N/m_\sigma^2}{f_\pi^2} \\ &\simeq \frac{\Sigma(0) - 2m_N m_\pi^2/m_\sigma^2}{f_\pi^2}. \end{aligned} \quad (30)$$

Note that both the axial current divergence has its canonical form (4) and the conditions (6)–(8), as well as (9), are fulfilled at $\epsilon_2 = \epsilon_3 = 0$, $\epsilon_1 = f_\pi m_\pi^2$. However, as we see, the conditions (6)–(8), as well as (9), cf. (27)–(30), are also satisfied at a weaker assumption that (23) is valid and in the latter case the axial current divergence also gets its canonical form (4), provided one may retain only first term in (21) and put $\langle \sigma \rangle = f_\pi$; i.e., if the tree approximation is valid.

D. S-wave pion polarization operator and mean pion field

Closing the nucleon legs in diagrams for the amplitude of the forward πN scattering shown in Fig. 1 (evaluated in the tree approximation to the πN scattering) one finds the pion polarization operator in the gas approximation, cf. [15,18]. The nonpole part of the pion polarization operator in the isospin-symmetric matter consists of the s -wave and p -wave parts,

$$\Pi_S(\omega, \vec{q}, N = Z) + \delta\Pi_P = -\tilde{D}(\nu, q = q')n + Bn^{1+\alpha}, \quad (31)$$

where $n = n_p + n_n$ is the nucleon density, $\alpha > 0$. Here $\delta\Pi_P(\omega, \vec{q}) \propto \vec{q}\vec{q}'n$ is taken at $\vec{q} = \vec{q}'$. It should be added to the pole NN^{-1} contributions to the p -wave pion polarization operator, Π_P , where N^{-1} is the nucleon hole. $\Delta(1232)$ isobars can be also included, cf. [13,18]. In the gas

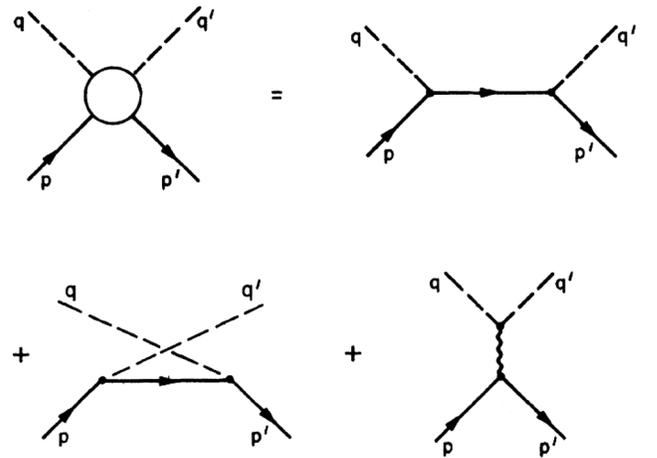


FIG. 1. Feynman diagrams contributing to πN scattering amplitude in tree approximation to σ model.

approximation terms $\propto Bn^{1+\alpha}$, being higher-order in n , should be dropped. We further assume that for all densities of our interest here the correlation effects $\propto B$ remain to be small, i.e., that $|Bn^\alpha| \ll |\tilde{D}|$. A rough estimate [29] gives $\alpha = 1/3$, $B \sim 1/(\pi^2 m_\pi^2)$.

The effective pion Lagrangian density of the classical one-Fourier-component charged pion field $\phi_{\omega, \vec{q}}$ is as follows [15,18],

$$L_{\text{ef}}(\omega, \vec{q}, n_n, n_p, \phi_q) = \langle L(\omega, \vec{q}) \rangle_N,$$

where averaging is done over the nucleon medium and we retained the dependence on the classical pion field up to second-order terms. We have

$$L_{\text{ef}}(\omega, \vec{q} = \vec{q}') = (\omega^2 - \vec{q}^2 - \text{Re}\Pi(\omega, \vec{q}, n)) |\phi_{\omega, \vec{q}}|^2, \quad (32)$$

where $\Pi \simeq \Pi_p(\omega, \vec{q}\vec{q}')|_{\vec{q}=\vec{q}'} + \delta\Pi_p(\vec{q}\vec{q}')|_{\vec{q}=\vec{q}'} + \Pi_s$; i.e., it includes the p -wave and the s -wave terms. Certainly such a calculated polarization operator includes not all possible diagrams. The resummation is done following the Dyson equation $D = D_0 + D_0\Pi D$, where D_0 is the free pion Green function with Π_p which phenomenologically includes the NN correlations in the vertices, cf. [15,18], and with the term $\Pi_s + \delta\Pi_p$, as we evaluated it in the gas approximation. The nucleons are treated, as quasiparticles with the effective Fermi liquid nucleon mass, n is the full baryon density.

The spectrum of quasiparticle excitations with the pion quantum numbers is found from (32) when one puts

$$D^{-1}(\omega, \vec{q}, n) = \omega^2 - \vec{q}^2 - \Pi(\omega, \vec{q}, n) = 0. \quad (33)$$

E. SM1 model

Assuming that (23) is approximately fulfilled, we may following (22) put $\Sigma(t = 2m_\pi^2) \simeq \Sigma(t = 0) = \Sigma_1 = m_N m_\pi^2 / m_\sigma^2$. From (27), (23) we have

$$\begin{aligned} \tilde{D}_{\text{SM1}}^+(\nu, t, q^2, q'^2) &\simeq \frac{m_N(t - m_\pi^2)}{m_\sigma^2 f_\pi^2} \\ &\simeq \frac{\Sigma_1(q^2 + q'^2 - m_\pi^2 - 2\omega^2)}{f_\pi^2 m_\pi^2} + \frac{2\Sigma_1}{f_\pi^2 m_\pi^2} \vec{q}\vec{q}', \end{aligned} \quad (34)$$

where we label the amplitude by subscript 1 provided condition (23) is supposed to be fulfilled. Putting $t = 0$, $q^2 = q'^2 = m_\pi^2$, we find the relation between the πN scattering length and the Σ term in this model: $4\pi a_{\pi N}^+(1 + m_\pi/m_N) = -\Sigma_1/f_\pi^2$. However this relation badly agrees with the experimental value $a_{\pi N}^+ \simeq -0.0083/m_\pi$, which is a tiny quantity. Taking (34) on the mass-shell, i.e., for $q^2 = q'^2 = m_\pi^2$, one would get

$$\begin{aligned} \tilde{D}_{\text{SM1}}^+(\nu, t(q^2 = q'^2 = m_\pi^2)) \\ \simeq \frac{\Sigma_1(m_\pi^2 - 2\omega^2)}{f_\pi^2 m_\pi^2} + \frac{2\Sigma_1 \vec{q}\vec{q}'}{f_\pi^2 m_\pi^2}, \end{aligned} \quad (35)$$

again being in contradiction with the condition that $a_{\pi N}^+$ is a tiny quantity.

Using (34) in the gas approximation (for $B = 0$) we find

$$\Pi_S^{\text{SM1,off}}(\omega, \vec{q}, N = Z) \simeq \Sigma_1(m_\pi^2 + 2\vec{q}^2)n/(m_\pi^2 f_\pi^2) \quad (36)$$

and

$$\delta\Pi_P^{\text{SM1}}(N = Z) \simeq -2\Sigma_1 \vec{q}\vec{q}'n/(m_\pi^2 f_\pi^2) \quad (37)$$

for $\vec{q} = \vec{q}'$. Summing (36), (37) we have

$$\Pi_S^{\text{SM1,off}} + \delta\Pi_P^{\text{SM1}}(N = Z) \simeq \Sigma_1 n / f_\pi^2. \quad (38)$$

The superscript ‘‘off’’ indicates that $\Pi_S^{\text{SM,off}}$ is constructed with the help of the amplitude taken at arbitrary values ω and \vec{q} . Thus in this case (with $\Pi_S^{\text{SM1,off}}$) the s -wave πN interaction in the isospin-symmetric matter derived within the linear sigma model proves to be repulsive and the s -wave pion condensation at $\omega_c = 0$ does not occur at least up to sufficiently high densities. The term $\Sigma_1 n / f_\pi^2$ can be included with the help of the replacement $m_\pi^2 \rightarrow m_\pi^2 + \Sigma_1 n / f_\pi^2$. After this replacement is done, the consideration of the problem of the p -wave pion-nucleon condensation performed in [1–13] and other works, cf. reviews [14–18], remains the same.

The pion spectrum in isospin-symmetric matter calculated using Eqs. (36), (37) (in model ‘‘SM1,off’’) renders

$$\omega_{\text{SM1,off}}^2 = m_\pi^2 + \frac{\Sigma_1 n}{f_\pi^2} + \vec{q}^2 + \Pi_P^{\text{SM}}(\omega, \vec{q}, N = Z). \quad (39)$$

F. SM2 model

Now do not require fulfilment of the condition (23). Then we can satisfy condition $a_{\pi N}^+ \simeq 0$, but at the price of the choosing of a large value for the Σ term in (22), $\Sigma = \Sigma_2 \simeq 2m_N m_\pi^2 / m_\sigma^2$, in this case we label the amplitude by subscript 2. Assuming, as above, that $m_\sigma \simeq 600$ MeV we get $\Sigma_2 \simeq 100$ MeV. Then from (22), (27) we obtain

$$\tilde{D}_{\text{SM2}}^+(\nu, t, q^2, q'^2, a_{\pi N}^+ = 0) = m_N t / (m_\sigma^2 f_\pi^2). \quad (40)$$

Weinberg and Adler conditions (6) and (7), as well as (9), prove to be violated. However note that even in this case the amplitude \tilde{D}_{SM2}^+ as a function of t shows a smooth change from the Cheng-Dashen point through the Adler point, where now $\tilde{D}_{\text{SM2}}^+(0, m_\pi^2, m_\pi^2, 0) = \Sigma_2 / 2f_\pi^2$, to the Weinberg

point where now $\tilde{D}_{\text{SM2}}^+(0, 0, 0, 0) = 0$. The corresponding on-mass-shell limit of the amplitude is given by

$$\begin{aligned} \tilde{D}_{\text{SM2}}^+(\nu, t(q^2 = q'^2 = m_\pi^2), a_{\pi N}^+ = 0) \\ = 2m_N(m_\pi^2 - \omega^2 + \vec{q}\vec{q}')/(m_\sigma^2 f_\pi^2). \end{aligned} \quad (41)$$

Thus within the sigma model choosing $m_\sigma \simeq 600$ MeV we may either satisfy the experimental finding that $\Sigma \simeq (50 - 60)$ MeV violating requirement that $a_{\pi N}^+ \simeq 0$ or we may fulfill condition $a_{\pi N}^+ \simeq 0$ at the price of the usage of an increased value of $\Sigma \simeq 100$ MeV and at violation of the Weinberg and Adler conditions (6), (7). Note that taking a larger value for m_σ we could, at this price, decrease the quantity Σ to the values not contradicting the data.

In case of the model 2 we have

$$\Pi_S^{\text{SM2,off}} + \delta\Pi_P^{\text{SM2,off}} = 0, \quad (42)$$

as it follows from (40) for $q = q'$, i.e., $t = 0$, and as it has been used in the early works [10–13]. As in case of model 1, the consideration of the possibility of the p -wave pion condensation performed in [1–13] and other works, cf. reviews [14–18], remains unchanged.

Employing the amplitude (40) (model “SM2,off”) we arrive at the spectrum

$$\omega_{\text{SM2,off}}^2 = m_\pi^2 + \vec{q}^2 + \Pi_P^{\text{SM}}(\omega, \vec{q}, N = Z). \quad (43)$$

Namely such a spectrum (taken at $\Pi = \Pi_P$) has been studied in [11–13,16] in case of the isospin-symmetric matter.

G. Off shell and on shell treatments of problem of s -wave pion condensation

In case when the Lagrangian density is fixed, here by Eqs. (10)–(12), the off-mass-shell amplitude has certain physical sense, it permits us to construct the pion polarization operator in the nucleon medium in the gas approximation and the effective pion Lagrangian, which allows us to describe specific observable effects, e.g., such as the possibility of the critical phenomena associated with the pion condensate phase transition. The same pion polarization operator follows in the gas approximation, if we use the ordinary Feynman diagrammatic rules. This is the key statement for our further consideration. It was put in doubt in [27,28] and in a number of subsequent works. Constructing Π_S , those authors supposed not only to put $q = q'$ but also take $q^2 = q'^2 = m_\pi^2$ with the argument that the scattering amplitude for particles in vacuum has the meaning only for on-mass-shell particles, i.e., for $q^2 = q'^2 = m_\pi^2$. In reality, pions can be considered as free particles only between collisions with rarely distributed infinitely massive centers, when the nucleon recoil and pion coherence effects can be neglected. The statement should not work for consideration of the classical pion field in a rather

dense matter. Anyhow in the case of the sigma model under consideration with such a on-mass-shell approach, using the amplitude (35) one would arrive at

$$\Pi_S^{\text{SM1,on}}(\omega, \vec{q}, N = Z) \simeq \frac{\Sigma_1(2\omega^2 - m_\pi^2)n}{f_\pi^2 m_\pi^2}, \quad (44)$$

with $\Sigma_1 = m_N m_\pi^2 / f_\pi^2$, and one would get

$$\omega_{\text{SM1,on}}^2 = \frac{m_\pi^2(1 - \frac{\Sigma_1 n}{f_\pi^2} m_\pi^2) + \Pi_P^{\text{SM}}}{1 - 2\Sigma_1 n / (f_\pi^2 m_\pi^2)} + \vec{q}^2, \quad (45)$$

that differs from (39), although only in terms obtained beyond the framework of the validity of the gas approximation.

With (41) one would obtain

$$\Pi_S^{\text{SM2,on}}(\omega, \vec{q}, N = Z) \simeq \frac{\Sigma_2(\omega^2 - m_\pi^2 - \vec{q}\vec{q}')n}{f_\pi^2 m_\pi^2} \quad (46)$$

at $\Sigma_2 = 2m_N m_\pi^2 / f_\pi^2$ and $\vec{q} = \vec{q}'$ and

$$\omega_{\text{SM2,on}}^2 = m_\pi^2 + \frac{\Pi_P^{\text{SM}}}{1 - n\Sigma_2 / (f_\pi^2 m_\pi^2)} + \vec{q}^2 \quad (47)$$

instead of (43). We see that the squared effective pion mass determined, as the quantity entering the spectrum $\omega^2(\vec{q} = 0) = m^{*2}$, in the approximation linear in n , is the same for the models “SM1,off” and “SM1,on” ($m_1^{*2} = m_\pi^2 + \Sigma_1 n / f_\pi^2$), and for the models “SM2,off” and “SM2,on,” $m_2^{*2} = m_\pi^2 \neq m_1^{*2}$. However in nonlinear in n terms the quantities $m_{1,\text{off}}^{*2}$ and $m_{1,\text{on}}^{*2}$ are different, as well as $m_{2,\text{off}}^{*2}$ and $m_{2,\text{on}}^{*2}$. This result does not disagree with the equivalence theorem, cf. [61], since the conditions of applicability of the equivalence theorem are not fulfilled in this case.

Note that Ref. [45] associated occurrence of the s -wave pion condensation with the vanishing of the effective pion mass term in the effective Lagrangian, however in their model ω dependence in Π_S was not included. In case of model “SM1,on” the effective pion mass defined as $m_{\text{ef}}^2 = m_\pi^2 + \Pi(q^2 = 0)$ vanishes for $n > n_{\text{cm}}^{\text{SM1,on}} = f_\pi^2 m_\pi^2 / \Sigma_1 \simeq (2 - 2.5)n_0$. Assuming decrease of the effective pion decay parameter f_π with increasing density Ref. [45] estimated a still smaller value $n_{\text{cm}} \simeq 1.6n_0$. In case of model “SM2,on” such a defined the effective pion mass vanishes for $n > n_{\text{cm}}^{\text{SM2,on}} = f_\pi^2 m_\pi^2 / \Sigma_2$. However then Refs. [27,46] found that a repulsion from the range ω^2 term compensates the attraction in the expression for the pion spectrum. In case “SM1,on” resulting in the spectrum (45), the effective pion mass m^* ($\neq m_{\text{ef}}$) increases with increasing n and in case “SM2,on” resulting in the spectrum (47), m^* stays constant. Also, we see that expressions for spectra contain poles: the spectrum given by Eq. (45) at $n = n_{\text{co}}^{\text{SM1,on}} = f_\pi^2 m_\pi^2 / (2\Sigma_1)$,

and the spectrum given by Eq. (47) for $n = n_{c\omega}^{\text{SM2,on}} = f_\pi^2 m_\pi^2 / \Sigma_2$. With $\Sigma_2 \simeq 2\Sigma_1 \simeq 100$ MeV we estimate $n_{c\omega}^{\text{SM1,on}} = n_{cm}^{\text{SM1,on}} / 2$, $n_{c\omega}^{\text{SM2,on}} = n_{cm}^{\text{SM2,on}} \simeq n_{cm}^{\text{SM1,on}} / 2$, and thus $n_{c\omega}^{\text{SM1,on}} \simeq n_{c\omega}^{\text{SM2,on}} \simeq (1 - 1.3)n_0$ in both models. References [59,62] employed $f_\pi(n)$ instead of f_π in the description of pion atoms. With $f_\pi \rightarrow f_\pi(n)$ the value $n_{c\omega}$ is still decreased. However note that presence of the pion condensation at $n < n_0$ contradicts to the experimental data.

The possibility of the s -wave pion condensation in isospin-symmetric matter was not worked out in [27,46] and in subsequent papers, which employed the s -wave polarization operator found with the help of the on-mass-shell scattering amplitude in their models. Most of the researches focused attention on the s -wave kaon condensation in isospin-asymmetric matter, although consideration of the kaon condensation problem is completely analogous to the consideration of the pion condensation. The point is that Refs. [27,46] associated the possibility of the s -wave pion condensation with vanishing of the squared effective pion mass in the pion spectrum. In their on shell model, for $a_{\pi N}^+$, being small negative quantity, the value $n_{cm} > n_{c\omega}$. The authors [27] wrote that, when the overall coefficient of the terms $\propto \omega^2$ in the expression for the spectrum given by $D^{-1} = 0$ vanishes at $n = n_{c\omega}$, “it forces the squared wave number \vec{q}^2 to be negative for any ω . This situation corresponds to evanescent waves in the medium,” as they stated, rather than to the pion condensation. For $n > n_{cm} > n_{c\omega}$ the quantity m^{*2} is positive in their model, cf. Eq. (26) in [27], not allowing for the s -wave condensation according their argumentation. We do not support this statement of [27]. In the models “SM1,on” and “SM2,on” the term proportional to the squared effective pion mass, m_{ef}^2 , in the effective pion Lagrangian density, $\delta L^{\text{SM,on}} = m_\pi^2 |\phi|^2 (n/n_{cm}^{\text{SM,on}} - 1)/2$, changes the sign for $n > n_{cm}^{\text{SM,on}}$. This means that the s -wave pion condensation associated with the change of the sign of the squared effective pion mass term, m_{ef}^2 , in the energy density is energetically favorable at $n > n_{cm}^{\text{SM,on}}$ in the given models. Also note that in both sigma models, “SM1,on” and “SM2,on,” the energy density acquires the ω^2 -dependent term $\delta E_\omega^{\text{SM,on}} = -\omega^2 |\phi|^2 (n/n_{c\omega}^{\text{SM,on}} - 1)/2$ that may result in appearance of a frequency dependent classical pion field for $n > n_{c\omega}^{\text{SM,on}}$. We continue discussion of the s -wave pion condensation in Sec. VII.

We should stress that Eqs. (36) and (37) for $\Sigma_1 = m_N m_\pi^2 / f_\pi^2$ as well as Eq. (42) for $\Sigma_2 = 2m_N m_\pi^2 / f_\pi^2$ follow in the low n approximation right from the Lagrangian of the model (in the tree approximation for the amplitude) and do not need extra putting $q^2 = q'^2 = m_\pi^2$. To get them we just closed the nucleon legs in the diagrams shown in Fig. 1 that corresponds to the averaging of the Lagrangian over the nucleon degrees of freedom. Although for a small n the probability of the multiple scattering processes is also

small, it is principally nonzero, and its calculation requires integrations over ω and \vec{q} rather than putting $q^2 = m_\pi^2$. Moreover we should say that the gas approximation in calculation of Π_S might be valid up to $n \lesssim$ several n_0 , because the density dependence of the diagrams with multiple integrations in the intermediate states entering Π_S is a weak. Thereby, Refs. [15,18,22] treated expression (31) with $B = 0$, as approximately valid not only for $n \ll n_0$ but in a wider range of densities. Moreover, a numerical smallness of some specific diagrams contributing to Π_S beyond the validity of the gas approximation was demonstrated in [29,30]. Also, a successful fit of the phenomenological pion optical potential to the pion atom data [21,29,59,62] was performed employing smallness of the correlation effects in Π_S . Thus following our argumentation, ω and \vec{q} in (31) should be treated as independent variables, not connected by the on-mass-shell condition.

Some works followed the idea of Ref. [47] that the answer on the question about fulfilment or not fulfilment of the PCAC condition (5) and the current algebra theorems (6)–(8) depends on the choice of the artificial interpolating fields, in which terms one may rewrite the Lagrangian, not changing physics. We should stress that Eqs. (F4), (F5) of [47], which they use to demonstrate their point, do depend on the choice of the interpolating fields, since $\delta L = j_A \phi \neq j_A \tilde{\phi}$. Thereby the Lagrange equations, which follow from such constructed Lagrangians, are different reflecting difference in physical effects associated with presence of even tiny $j_A \neq 0$. Thus the statement on relevance of the off shell effects in the problem of the s -wave pion condensation does not contradict to often mentioned equivalence theorem that any local change of variables in quantum field theories, which leaves the free field part of the Lagrangian unchanged, does not alter the S matrix [33], and it does not contradict to on shell consideration of the scattering of particles on infinitely heavy centers [35]. However, already with the two-particle scattering in vacuum obeying the Bethe-Salpeter equation in the tree-level kernel there appear complications, since solutions in this kernel prove to be dependent on the representation of the theory [63]. In our case nucleons in matter undergo recoil effects and the pion field (especially the classical condensate field) does not obey the free Klein-Gordon equation.

Let us also mention that similar results, as for the “SM2, off,” follow from an extended linear sigma model [64], permitting to reproduce the experimental value of the axial-vector constant g_A . Essentially increasing the parameter m_σ in the ordinary and extended sigma models it is possible to recover values both of the πN scattering length and the sigma-term. However one should notice that the value $m_\sigma \sim 600$ MeV is required to get an appropriate fit of the nuclear equation of state in the relativistic mean-field models, cf. [65]. Also, both the linear sigma model and the extended

linear sigma model hardly reproduce some of the πN scattering amplitude properties predicted by the heavy-baryon chiral perturbation theory, cf. [66].

Concluding this section, we argued that with the once chosen Lagrangian of the model (here the sigma model), at the lowest tree level order, in the low density approximation, the pion polarization operator is unambiguously constructed. The s -wave part of the polarization operator is proportional to the pion-nucleon scattering amplitude at arbitrary relation between kinematical variables, rather than to the on-mass-shell amplitude $\tilde{D}(\nu, t, q^2 = q'^2 = m_\pi^2)$ taken at $\vec{q} = \vec{q}'$. Usage of the on-mass-shell amplitude to construct Π_S^{SM} within the models SM1 and SM2 in the gas approximation over nucleons and for the classical pion mode, may lead to erroneous results, such as occurrence of the s -wave pion condensation in the isospin-symmetric matter at rather low densities. However in application to other models the issue can be more subtle.

IV. MANOHAR-GEORGI MODEL

To continue demonstration of our key point that the knowledge of the scattering amplitude off-mass-shell matters in the problem of the s -wave pion polarization in the medium let us consider two other Lagrangians, introduced in [60,67]. With a reduction to $SU(2)_L \times SU(2)_R$ and the s -wave channel, the $O(Q^2)$ Manohar-Georgi (MG) Lagrangian density renders as

$$L_{\text{MG}} = \bar{N}i v^\mu \partial_\mu N - \Sigma \bar{N}N + \partial_\mu \bar{\pi} \partial^\mu \pi / 2 - m_\pi^2 \bar{\pi}^2 / 2 + L_{\text{WT}} + [\Sigma \bar{\pi}^2 / 2 + c_2 (v^\mu \partial_\mu \bar{\pi})^2 + c_3 \partial_\mu \bar{\pi} \partial^\mu \pi] \bar{N}N / f_\pi^2 \dots, \quad (48)$$

cf. [56], where v^μ is the four velocity of the nucleon and $v^\mu = (1, 0, 0, 0)$ in the center-of-mass reference frame of the nucleon, the values m_π^2 and f_π^2 include $O(Q^2)$ loop corrections to the corresponding quantities at tree level, constants c_2 and c_3 are of the order $O(Q^0)$, and the constant $\Sigma = \Sigma(0)$ is linear in the quark masses and therefore is of order $O(Q^2)$. The Weinberg-Tomozawa (vector) term, $\Pi_S^{\text{WT},-} \simeq (n_n - n_p)\omega / (2f_\pi^2)$, does not enter into the pion self-energy in the case of the isospin-symmetric nuclear matter of our interest here. We should note that the original MG Lagrangian contains an infinite number of terms [labeled by dots in (48)], whereas we shall consider the reduced MG Lagrangian given by Eq. (48), i.e., dropping terms labeled by dots.

The Lagrangian density (48) yields the amplitude

$$\tilde{D}_{\text{MG}}^+ \simeq \frac{2c_2 \omega \omega' + 2c_3 q q' + \Sigma}{f_\pi^2}, \quad (49)$$

i.e., $\tilde{D}^+ \simeq \Sigma / f_\pi^2$ in all three kinematical points (6)–(8) at $\vec{q} = 0$, and thereby conditions (6), (7) are not fulfilled. Only

the lowest order $\bar{\pi}^2 \bar{N}N$ terms were included to derive Eq. (49).

From the reduced Lagrangian density (48) one recovers the isospin-even pion-nucleon scattering length,

$$a_{\pi N}^+ \simeq \frac{(2c_2 + 2c_3)m_\pi^2 + \Sigma}{4\pi f_\pi^2 (1 + m_\pi / m_N)}. \quad (50)$$

We find $(2c_2 + 2c_3)m_\pi^2 \simeq -\Sigma + 4\pi b_+$, $b_+ = a_{\pi N}^+ f_\pi^2 (1 + m_\pi / m_N)$. The tiny value $4\pi b_+$ is further put zero, as we have done above within the sigma model.

Then

$$\tilde{D}_{\text{MG}}^+ \simeq \frac{\Sigma(m_\pi^2 - \omega^2)}{f_\pi^2 m_\pi^2} - \frac{2c_3 \vec{q} \vec{q}'}{f_\pi^2}, \quad (51)$$

and in the gas approximation

$$\Pi_S^{\text{MG}}(\omega, \vec{q}, N = Z) \simeq n \Sigma \frac{\omega^2 - m_\pi^2}{f_\pi^2 m_\pi^2}, \quad (52)$$

and

$$\delta \Pi_P^{\text{MG}} \simeq 2c_3 \vec{q} \vec{q}' n / f_\pi^2 \quad (53)$$

is taken at $\vec{q} = \vec{q}'$. These expressions remind expressions for $\Pi_S^{\text{SM}1,2,\text{on}}$, $\delta \Pi_P^{\text{SM}}$ introduced in previous section. However here expressions (52) and (53) follow right from the Lagrangian and one does not need to do the replacement $\vec{q}^2 = \vec{q}'^2 = m_\pi^2$.

The spectrum of the pion quasiparticle excitations in the isospin-symmetric matter is determined by putting zero the inverse pion quasiparticle propagator (33). Thus from (52), (53) we get

$$\omega^2 = m_\pi^2 + \frac{\vec{q}^2 (1 + \frac{2c_3 n}{f_\pi^2}) + \Pi_P}{1 - n \Sigma / (f_\pi^2 m_\pi^2)}. \quad (54)$$

Note that the same amplitude can be found from the Lagrangian density

$$L_{\text{MG}} \rightarrow L_{\text{MG}} + j_i \pi_i \quad (55)$$

at the pseudoscalar source term considered in the limit $j_i \rightarrow 0$. Then the amplitude is determined by Fig. 2(a) after

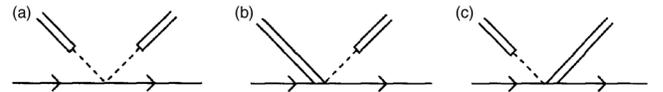


FIG. 2. Feynman diagrams contributing to πN scattering amplitude in tree approximation to MG model (a), and to GSS model, (a), (b), (c). Solid line indicates nucleon, double line relates to the pseudoscalar source j_i , and dashed line to the pion.

amputation of the external legs. The point vertex in Fig. 2(a) is given by $i\tilde{D}_{\text{MG}}^+$.

Finally note that the MG model allows for the s -wave pion condensation in the isospin-symmetric matter for $n > n_c = f_\pi^2 m_\pi^2 / \Sigma$ provided the gas approximation holds up to such densities, cf. discussion of models ‘‘SM1,on’’ and ‘‘SM2,on’’ in previous section and consideration below in Sec. VII.

V. GASSER-SAINIO-SVARC MODEL

Within the functional integral formulation of chiral perturbation theory developed by Gasser and Leutwiller [68], which was extended to include nucleons, Gasser, Sainio, and Svarc [67] introduced the Lagrangian density (GSS),

$$L_{\text{GSS}} = L_{\text{MG}} + j_i \pi_i (1 - \Sigma \bar{N}N / f_\pi^2 m_\pi^2), \quad (56)$$

with the pseudovector source $j_i = 2Bf_\pi p_i$, $B = m_\pi^2 / (m_u + m_d)$ satisfying the canonical PCAC condition (5) and with L_{MG} from (48). Since Green functions are obtained by taking functional derivatives of the generating functional with respect to the source j_i , the nontrivial coupling of the source to the pion field, here in the form $j_i \pi_i (1 - \Sigma \bar{N}N / f_\pi^2 m_\pi^2)$, matters. From the Ward identity one gets [67],

$$\tilde{D}_{\text{GSS}}^+ = \frac{2c_2 \omega \omega' + 2c_3 q q' + \Sigma}{f_\pi^2} + \frac{(q^2 + q'^2 - 2m_\pi^2) \Sigma}{m_\pi^2 f_\pi^2}, \quad (57)$$

cf. [48]. The same expression follows from the diagrams shown in Fig. 2 for the connected $\pi\pi NN$ Green function in the lowest order [56]:

$$\begin{aligned} \tilde{D}_{\text{GSS}}^+ &= i(q^2 - m_\pi^2) A_{\pi N}^{\text{GSS}}(q^2, q'^2)(q'^2 - m_\pi^2), \\ A_{\pi N}^{\text{GSS}} &= \frac{i}{q^2 - m_\pi^2} \frac{i(2c_2 \omega \omega' + 2c_3 q q' + \Sigma)}{f_\pi^2} \frac{i}{q'^2 - m_\pi^2} \\ &\quad - \frac{\Sigma}{f_\pi^2 m_\pi^2} \left(\frac{i}{q'^2 - m_\pi^2} + \frac{i}{q^2 - m_\pi^2} \right). \end{aligned} \quad (58)$$

The first term in the last equality relates to the diagram shown in Fig. 2(a) and the second term is associated with the second term in the Lagrangian (56) and with Figs. 2(b) and 2(c). The off-mass-shell scattering amplitudes satisfy now all three conditions (6)–(8) in difference with the MG case. The condition (9) is fulfilled for $c_3 = 0$. For on-mass-shell variables, $q^2 = q'^2 = m_\pi^2$, the amplitudes \tilde{D}_{GSS}^+ and \tilde{D}_{MG}^+ coincide.

We could obtain the same Eq. (57) in another way. First perform variable replacement $\pi \rightarrow \pi + \alpha\pi(\bar{N}N)$ in the reduced MG Lagrangian dropping terms $O(\alpha^2, (\bar{N}N)^2)$. We get the Lagrangian density

$$\begin{aligned} L_\alpha &= L_{\text{MG}} + \alpha[(\partial_\mu \bar{N} \cdot N + \bar{N} \cdot \partial_\mu N)(\partial^\mu \pi^* \cdot \pi + \pi^* \partial^\mu \pi)] \\ &\quad + 2\alpha[\bar{N}N \partial_\mu \pi^* \partial^\mu \pi - m_\pi^2 \bar{N}N \cdot \pi^* \pi] \\ &\quad + O(\alpha^2, (\bar{N}N)^2, \pi^3), \end{aligned} \quad (59)$$

where the linear in α term follows from the contribution of the free pion Lagrangian. In the Fourier transformation the terms linear in α yield in the amplitude the contribution

$$\alpha(q - q')^2 + 2\alpha(qq' - m_\pi^2) = \alpha(q^2 + q'^2 - 2m_\pi^2).$$

Putting $\alpha = \Sigma / (m_\pi^2 f_\pi^2)$ we recover \tilde{D}_{GSS}^+ and $A_{\pi N}^{\text{GSS}}$ which fulfill the Cheng-Dashen, Adler, and Weinberg conditions. Note that only, if we included all the dropped terms, the two models, labeled α and MG, would lead to identical results. Only in the latter case we could drop α contribution relying on the equivalence theorem. Although the term α is obviously nonzero off mass shell in the Lagrangian cutted at the order $O(\alpha^2, (\bar{N}N)^2, \pi^3)$, it should be compensated by the higher order diagrams corresponding to many-particle scatterings in the calculation of the observables, but only in case, when all the dropped terms are included. If one treats the Lagrangian (59) as it is, i.e., dropping the terms $O(\alpha^2, (\bar{N}N)^2, \pi^3)$, then values of the observables calculated using (59) and (56) for any α and using (48) differ.

Let us consider explicitly example of the static spatially uniform classical charged pion field $\phi = \langle (\pi_1 + i\pi_2) / \sqrt{2} \rangle$. Then $|\pi|^2$ terms in L_α yield $\delta L_\alpha = \delta L_{\text{MG}} - 2\alpha m_\pi^2 \bar{N}N |\phi|^2 + O(\alpha^2, (\bar{N}N)^2, \phi^3)$, where $\delta L_{\text{MG}} = -(m_\pi^2 - \Sigma \bar{N}N / f_\pi^2) |\phi|^2$. The squared effective mass term of the field ϕ is given by $m^{*2} = m_\pi^2 - (\Sigma / f_\pi^2 - 2\alpha m_\pi^2) \bar{N}N$. For $\alpha > \Sigma / (2m_\pi^2 f_\pi^2)$ the term $m^{*2} |\phi|^2$ even changes the sign. Also we can see that with the recovered α^2 contribution the partial term $m_\pi^2 |\phi|^2$ after the variable replacement would yield $m_\pi^2 |\phi|^2 (1 + \alpha \bar{N}N)^2$, whereas in the linear approximation we have $m_\pi^2 |\phi|^2 \rightarrow m_\pi^2 |\phi|^2 (1 + 2\alpha \bar{N}N)$. The former term is always non-negative, whereas the latter one changes the sign for $\alpha < -\Sigma / (2m_\pi^2 f_\pi^2)$. By these examples we showed that it is completely not surprising that the reduced Lagrangians (48) and (59) describe different physics.

Obviously the pion polarization operator, which is recovered from the reduced Lagrangian (59) with the help of the Feynman rule diagrammatics, also differs from that which follows from the reduced MG Lagrangian. Note also that in the uniform gas approximation the term α in the first line (59) is reduced to the full derivative in the effective action and can be dropped. However replacing $t = (q - q')^2 = 2m_\pi^2 - 2qq'$ following the on-mass-shell receipt we would get nonzero contribution to the pion polarization operator in nuclear matter from this full derivative term. This circumstance can be considered as extra argument that the on-mass-shell replacement does not hold for calculation of the diagrams term by term and that the equivalence

theorem does not hold for Lagrangians (48) and (59) provided $O(\alpha^2, (\bar{N}N)^2)$ terms are dropped.

The Lagrangian density L_{GSS} produces the s -wave part of the pion polarization operator in the gas approximation

$$\Pi_S^{\text{GSS}}(\omega, \vec{q}, N = Z) \simeq \frac{\Sigma n(m_\pi^2 - \omega^2 + 2\vec{q}^2)}{f_\pi^2 m_\pi^2}, \quad (60)$$

where, as above, we used $2(c_2 + c_3)m_\pi^2 \simeq -\Sigma$, and we have $\delta\Pi_P^{\text{GSS}} = \delta\Pi_P^{\text{MG}}$. Even for $\omega = m_\pi$ the terms $\propto \vec{q}^2$ in Π_S corresponding to L_{MG} and L_{GSS} are different. Only for $q^2 = m_\pi^2$, i.e., on mass shell, do we get $\Pi_S^{\text{MG,on}} = \Pi_S^{\text{GSS,on}}$.

From (60), (33) we obtain

$$\omega^2 = m_\pi^2 + \frac{\vec{q}^2(1 + \frac{2\Sigma n}{f_\pi^2 m_\pi^2}) + \Pi_P(\omega, \vec{q}\vec{q}') + \delta\Pi_P}{1 + n\Sigma/(f_\pi^2 m_\pi^2)}, \quad (61)$$

at $\vec{q} = \vec{q}'$, which differs from (54) in the correlation terms. It implies existence of physical effects, which are different in the models described by the reduced Lagrangians L_{MG} and L_{GSS} . For example, as follows from (60) at $\omega = 0$, likely the GSS model (with reduced Lagrangian in off-mass-shell treatment) does not allow for the s -wave pion condensation in isospin-symmetric matter, whereas the MG model allows it for $n > n_c \simeq (1.4 - 2.5)n_0$. However, we stress once more that the diagrams in Π_S contributing beyond the gas approximation were omitted in both cases.

VI. PHENOMENOLOGICAL EXPRESSIONS FOR Π_S IN ISOSPIN-SYMMETRIC MATTER

Now consider how one can proceed not employing the microscopic expression for the Lagrangian. One may employ that the nonpole part of the amplitude $\tilde{D}^+(\nu, t, q^2, q'^2)$ is a smooth function of its variables. It can therefore be expanded near the soft point ($q = q' = 0$) as a power series in q^2, q'^2, ν^2 , and t . In the static nucleon limit $\nu^2 \simeq \omega^2$ and $\nu_B = -qq'/(2m_N) \rightarrow 0$. In this limit the pole term vanishes and $D = \tilde{D}^+$. Moreover one has $\nu_B = 0$ in the Weinberg, Adler, and Cheng-Dashen kinematical points, where thereby again $D = \tilde{D}^+$.

Retaining only linear terms in the Taylor expansion of $\tilde{D}^+(\nu, t, q^2, q'^2)$, after the regrouping the terms one arrives at

$$\begin{aligned} \tilde{D}^+(\nu, t, q^2, q'^2) &\simeq \alpha_1 + \alpha_2(q^2 + q'^2)/m_\pi^2 \\ &+ \beta\nu^2 + \gamma[(t - q^2 - q'^2)/2 + \nu^2] \\ &+ \beta_1\nu^4 \dots, \end{aligned} \quad (62)$$

cf. [27]. This expansion, although with differently regrouped terms, coincides with that previously employed in [15,18,21]. For further needs we also explicitly wrote a higher order term $\beta_1\nu^4$.

As we have mentioned, the experimental value of $a_{\pi N}^+$ is a very small quantity. Thereby, simplifying consideration we further continue to put $a_{\pi N}^+ \simeq 0$. Estimated value $\beta_1 \simeq 0.2/m_\pi$ is small [15,69] (of the order of $\sim m_\pi/m_N$). Dropping all terms $\sim m_\pi/m_N$ including the term $\propto \beta_1$, we have

$$\begin{aligned} D^+(m_\pi, 0, m_\pi^2, m_\pi^2) &\simeq \tilde{D}^+(m_\pi, 0, m_\pi^2, m_\pi^2) \\ &\simeq 4\pi(1 + m_\pi/m_N)a_{\pi N}^+ \simeq 0. \end{aligned} \quad (63)$$

Additionally assuming fulfilment of the Cheng-Dashen condition we have

$$\begin{aligned} \beta &\simeq -(\Sigma - 4\pi b_+)/ (f_\pi^2 m_\pi^2) \simeq -\Sigma / (f_\pi^2 m_\pi^2), \\ \alpha_1 + 2\alpha_2 &= \Sigma / f_\pi^2 \end{aligned} \quad (64)$$

and we arrive at

$$\tilde{D}^+ \simeq \alpha_1 + \frac{(\Sigma/f_\pi^2 - \alpha_1)(q^2 + q'^2)}{2m_\pi^2} - \frac{\Sigma\nu^2}{f_\pi^2 m_\pi^2} + \gamma\vec{q}\vec{q}'. \quad (65)$$

Employing the Weinberg condition we get $\alpha_1 = -\Sigma/f_\pi^2$, $\alpha_2 = \Sigma/f_\pi^2$. The Adler condition is then fulfilled automatically. Putting in (62) $\nu^2 \simeq \omega^2$ and $(t - q^2 - q'^2)/2 + \nu^2 \simeq \vec{q}\vec{q}'$, we arrive at

$$\tilde{D}^+(\omega^2, q^2, q'^2, \vec{q}\vec{q}') \simeq -\frac{\Sigma(m_\pi^2 - q^2 - q'^2 + \omega^2)}{f_\pi^2 m_\pi^2} + \gamma\vec{q}\vec{q}'. \quad (66)$$

The condition (9) is satisfied for $\gamma = \Sigma/(f_\pi^2 m_\pi^2)$. Recall that all three Weinberg, Cheng-Dashen, and Adler conditions are satisfied in the GSS model described by the reduced Lagrangian L_α for $\alpha = \Sigma/(m_\pi^2 f_\pi^2)$, and in the ‘‘SM1,off’’ model. The condition (9) is fulfilled in the ‘‘SM1,off’’ model, whereas in the GSS model described by the reduced Lagrangian L_α for $\alpha = \Sigma/(m_\pi^2 f_\pi^2)$ it is satisfied for $c_3 = 0$. Note also that Eq. (66) can be considered as the simplest linear in the $t, q^2 = q'^2$, and ν^2 interpolation expression between the Cheng-Dashen and Weinberg points, satisfying the mass-shell condition (63).

With the πN nonpole amplitude (66), in the gas approximation we arrive at the s -wave pion polarization operator (labeled below as MSTV)

$$\begin{aligned} \Pi_S^{\text{MSTV}}(\omega, \vec{q}, N = Z) &\simeq \frac{\Sigma(m_\pi^2 - \omega^2 + 2\vec{q}^2)}{f_\pi^2 m_\pi^2} n \\ &\simeq \Pi_S^{\text{GSS}}(\omega, \vec{q}, N = Z), \end{aligned} \quad (67)$$

which coincides with that used in [18], cf. also [26], and

$$\delta\Pi_P^{\text{MSTV}} \simeq -\gamma\vec{q}\vec{q}'. \quad (68)$$

The value γ can be constrained from the analysis of the p -wave πN scattering amplitude and from the data on pionic atoms. References [18,21,22] used $\delta\Pi_p^{\text{MSTV}} = 0$ fitting the parameters of the p -wave πN interaction from the analysis of the data on pion atoms. If one takes $\gamma = -2c_3/f_\pi^2$, then one gets $\delta\Pi_p^{\text{MSTV}} = \delta\Pi_p^{\text{MG}} = \delta\Pi_p^{\text{GSS}}$. The condition (9) is satisfied for $\gamma = \Sigma/(f_\pi^2 m_\pi^2)$.

Constructing the pion polarization operator in the gas approximation Refs. [27,28,70] conjectured to put in (62) $q^2 = q'^2 = m_\pi^2$, exploiting that the amplitude of the πN scattering in vacuum has physical sense only for $q^2 = q'^2 = m_\pi^2$. In their approach the amplitude (62) satisfies the condition that $a_{\pi N}^+ \simeq 0$ and the Cheng-Dashen condition (8), whereas Weinberg and Adler conditions, as well as condition (9), are not fulfilled. Then from (65) one finds

$$\tilde{D}^+(\omega^2, q^2 = q'^2 = m_\pi^2, \vec{q}\vec{q}') \simeq \frac{\Sigma(m_\pi^2 - \omega^2)}{f_\pi^2 m_\pi^2} + \gamma \vec{q}\vec{q}', \quad (69)$$

and with this amplitude one gets polarization operator used in [29,30] (labeled below as KKW)

$$\begin{aligned} \Pi_S^{\text{KKW}}(\omega, \vec{q}, N = Z) &\simeq \frac{\Sigma(\omega^2 - m_\pi^2)}{f_\pi^2 m_\pi^2} n \\ &\simeq \Pi_S^{\text{MG}}(\omega, \vec{q}, N = Z), \end{aligned} \quad (70)$$

$\delta\Pi_p$ remains the same as in Eq. (68).

As we have shown, employment of the conjecture of [27,28,70] resulting in expression (70) does not work in the SM and GSS models but it works in the case of the MG model described by the reduced Lagrangian (48), where the off-mass-shell amplitude, as it follows from (48), coincides with the on-mass-shell one. Thus, if one recovers the s -wave pion polarization operator employing the scattering amplitude, rather than the Lagrangian, one cannot say is it better to use the off-mass-shell amplitude or the on-mass-shell one. Only after the Lagrangian of the model is selected and approximation scheme is chosen, arbitrariness in the choice of the amplitude and the polarization operator disappears. They follow directly from the model, although depending on the approximation employed for their calculation. In our examples above, we presented Π_S using the gas approximation. In other cases one can use perturbative expansions up to the given order [71], semiclassical series in the number of loops [72], expansion in number of vertices in Φ derivable models [39], expansion in classical field, as in Ginzburg-Landau model of phase transitions, etc., Confronting various physical effects to the data one may then do a choice in favor of one model relatively others.

VII. S-WAVE PION CONDENSATION: TO BE OR NOT TO BE?

Using Eq. (67) we may recover the corresponding effective pion Lagrangian density, written in the time-space representation. Present it explicitly for the case of the spatially uniform charged pion field for simplicity:

$$\begin{aligned} L_{\text{ef}}^{\text{MSTV}}(\nabla\phi = 0, N = Z) &= |\dot{\phi}|^2(1 + n/n_c) + \beta_1 |\ddot{\phi}|^2 \\ &\quad - m_\pi^2 |\phi|^2(1 + n/n_c) - \Lambda |\phi|^4/2, \end{aligned} \quad (71)$$

where $n_c = f_\pi^2 m_\pi^2 / \Sigma$ and we recovered a small term $\propto \beta_1 > 0$, the last term is responsible for the pion-pion effective interaction and for simplicity we put $\Lambda = \text{const} > 0$. Actually, in the medium $\Lambda = \Lambda(\omega, \vec{q})$. As a typical value, we may take $\Lambda \sim 1$, cf. [15]. Employing Eq. (70) we get

$$\begin{aligned} L_{\text{ef}}^{\text{KKW}}(\nabla\phi = 0, N = Z) &= |\dot{\phi}|^2(1 - n/n_c) + \beta_1 |\ddot{\phi}|^2 \\ &\quad - m_\pi^2 |\phi|^2(1 - n/n_c) - \Lambda |\phi|^4/2. \end{aligned} \quad (72)$$

With $\phi = f e^{-i\omega t}$, where f is the real constant, we recover the energy densities

$$\begin{aligned} E_{\text{ef}}^{\text{MSTV}}(\nabla\phi = 0, N = Z) &= [\omega^2(1 + n/n_c) + 3\beta_1 \omega^4] f^2 \\ &\quad + m_\pi^2(1 + n/n_c) f^2 + \Lambda f^4/2, \end{aligned} \quad (73)$$

and

$$\begin{aligned} E_{\text{ef}}^{\text{KKW}}(\nabla\phi = 0, N = Z) &= [\omega^2(1 - n/n_c) + 3\beta_1 \omega^4] f^2 \\ &\quad + m_\pi^2(1 - n/n_c) f^2 + \Lambda f^4/2. \end{aligned} \quad (74)$$

As it is seen, $E_{\text{ef}}^{\text{MSTV}}(\nabla\phi = 0)$ has a minimum for $f = 0$, i.e., s -wave pion condensation in isospin-symmetric matter does not occur, whereas $E_{\text{ef}}^{\text{KKW}}(\nabla\phi = 0) < 0$ for $n > n_c$. Thus in the latter model the s -wave pion condensation appears for $n > n_c$ (certainly, provided contributions beyond the gas approximation remain small up to $n \sim n_c$).

Further we focus on the KKW model, which allows for the s -wave pion condensation for $n > n_c$. Minimization of the energy density in ω gives

$$\omega_m^2 = (n/n_c - 1)/(6\beta_1), \quad (75)$$

and the equation of motion, $dL/d\phi = 0$, yields

$$f^2 = [(\omega_m^2 + m_\pi^2)(n/n_c - 1) - \beta_1 \omega_m^4] \theta(n - n_c) / \Lambda, \\ \simeq m_\pi^2 (n/n_c - 1) \theta(n - n_c) / \Lambda + O((n - n_c)^2), \quad (76)$$

and $\theta(x)$ is the step function. Thus at least in the vicinity of the critical point the occurring pion field is quasistatic. Notice here that a nonstatic complex field carries electric charge that modifies the initial N/Z ratio. However for n in the vicinity of the critical point the accumulated charge is only tiny and can be neglected. Setting solution (76) back to the energy density, we find

$$E_{\text{ef}}^{\text{KKW}}(N = Z) \simeq -\frac{m_\pi^4 (n/n_c - 1)^2}{2\Lambda} \theta(n - n_c) \\ + O((n - n_c)^4) < 0. \quad (77)$$

Thereby in the KKW model the s -wave pion condensation occurs in the isospin-symmetric nuclear matter by the second-order phase transition at $n > n_c \simeq (2 - 2.5)n_0$ for $\Sigma \simeq 50\text{--}60$ MeV. If one assumes $f_\pi(n)/f_\pi \simeq 1 - 0.1n/n_0$, one gets $n_c \simeq (1.4 - 1.7)n_0$. Note that employing the Gell-Mann-Oakes-Renner relation [73] $f_\pi^2(n)/f_\pi^2 = 1 - \Sigma n / (m_\pi^2 f_\pi^2)$ one would obtain a still smaller value $f_\pi(n)/f_\pi \simeq 1 - 0.18n/n_0$ yielding a smaller value of n_c .

For $n/n_c \simeq 1.3$ we estimate the energy gain per particle due to s -wave pion condensation to be $-\mathcal{E}_{\text{ef}} = -E_{\text{ef}}/n \sim 0.1m_\pi$. In the nonlinear Weinberg model [74] for the case of the static field we estimate a stronger energy gain, $\mathcal{E}_{\text{ef}} = -(n - n_c)\theta(n - n_c)m_\pi^2 f_\pi^2 / (n_c n)$ for $|\phi|^2 = 2f_\pi^2$. Estimated energy gains could be sufficient for formation of metastable (or may be even stable) s -wave pion condensate droplets already in heavy-ion collisions with energies $\lesssim \text{GeV} \cdot A$. In case of the p -wave pion condensation such possibilities have been discussed in 1970s–1980s, cf. [15,18].

Equation of motion for the time-dependent classical field ϕ renders

$$(1 - n/n_c)\ddot{\phi} - \beta_1 \phi + (1 - n/n_c)m_\pi^2 \phi + \Lambda|\phi|^2 \phi = 0.$$

For a slow field, dropping a small term $\propto \beta_1$ we find a partial solution

$$\phi(t) = e^{i\alpha t} \theta(n - n_c) \sqrt{\frac{m^2 (n/n_c - 1)}{\Lambda}} \text{th} \frac{m}{\sqrt{2}} t, \quad (78)$$

where α is arbitrary constant. The states with different α are degenerate. In presence of the interaction term $\delta L = \epsilon_4 \phi \sqrt{\phi^* / \phi} + \text{c.c.}$, for a real value ϵ_4 , one would deal with the first order phase transition permitting metastable and stable states.

Since n is an independent variable, the quantities $\Pi_S(n)$, which we have estimated, may only slightly change with the temperature, T , in a broad range of the temperatures, cf. [16,18]. Indeed, within the gas approximation the T

dependence enters Π_S via $f_\pi(T)$ and $\Sigma(T)$ and it becomes essential only in the vicinity of the critical point of the deconfinement phase transition. Thus, if the KKW model were valid, the s -wave pion condensation in the isospin-symmetric nuclear matter would be expected to occur already for $n > n_c \sim (1.4 - 2.5)n_0$ and one could expect to observe some experimental consequences of the s -wave pion condensation in heavy-ion collisions in this case. Oppositely, with MSTV model for the s -wave pion-nucleon interaction the s -wave pion condensation in the isospin-symmetric nuclear matter does not occur at least up to very high densities.

VIII. CONCLUSION

We studied subtleties of the description of the s -wave pion-nucleon interaction in the isospin-symmetric nuclear matter. First, properties of the s -wave pion-nucleon interaction were studied on explicit examples of the linear sigma model, the Lagrangian (11), and the Manohar-Georgi and Gasser-Sainio-Svarc models with a finite number of terms in the Lagrangians [reduced Lagrangians (48) and (59) at $\alpha = \Sigma / (f_\pi^2 m_\pi^2)$], respectively, provided terms labeled by dots in (48) are dropped). On examples of the linear sigma and Gasser-Sainio-Svarc models we showed that the knowledge of the scattering amplitude, as a function of only the ν variable, for $q^2 = q'^2 = m_\pi^2$, is not sufficient to correctly describe the s -wave part of the pion polarization in matter even at low nucleon density. Even in the lowest-order in n , the so-called gas approximation, the s -wave pion polarization operator proves to be dependent on the values of ν and $q^2 \neq m_\pi^2$ variables, as it straightly follows from the analysis of the Feynman diagrams (shown in Fig. 1 in case of the sigma model) and the vertices of the Lagrangians. The key point here is that the gas approximation is applicable beyond the framework of the approximation of the scattering of free pions on static nucleon centers. Even if complicated many-particle processes occur with only a small probability (for low n), to calculate the probability of such processes one requires the knowledge of the off-mass-shell pion-nucleon amplitude and the pion polarization operator for $q^2 \neq m_\pi^2$. Off-mass-shell information is needed for the description of the zero-sound modes and the Landau damping in Fermi systems, even at low densities. Certainly, description of the phase transition phenomena at a higher density (clustering, Pomeranchuk instability, liquid-gas transition, etc., cf. [75]) also requires the knowledge of the in-medium scattering amplitudes and the dressed Green functions. More generally, particles are permanently produced and absorbed in the medium and do not exist in asymptotically free states, cf. [39,41]. Thereby knowledge of the on-mass-shell amplitudes is not sufficient to calculate relevant physical quantities in all mentioned cases.

It does not contradict to the well known equivalence theorem that any local change of variables in quantum field

theories, which leaves the free field part of the Lagrangian unchanged, does not alter the S matrix [33]. Coming back to the problem considered in this paper, nucleons in matter undergo recoil effects (even if being small at low n) and do not fulfill the free Dirac equation and the pion field (especially the classical condensate field) does not obey the free Klein-Gordon equation. Thus, even in the gas approximation (when effects nonlinear in n , although exist, are small) the pion polarization operator is determined by the off-mass-shell pion-nucleon scattering amplitude rather than by the on-mass-shell one. Thereby, only selection of the Lagrangian of the model allows one to determine unambiguously within the given model and at the given approximation level the physically important quantities. Only in the case of the Manohar-Georgi model determined by the reduced Lagrangian (48) (when terms labeled by dots are dropped) from those models we considered, the s -wave pion polarization operator in the gas approximation proved to be independent on whether one uses off shell or on shell pion-nucleon scattering amplitudes. In other cases for $q^2 = q'^2 = m_\pi^2$ (at arbitrary ν), and for $q^2 = q'^2 \neq m_\pi^2$ even the pion spectra prove to be different in the nonlinear density dependent terms.

We demonstrated that the s -wave pion condensation in the isospin-symmetric matter hardly occurs within the linear sigma model and the model described by the reduced Gasser-Sainio-Svarc Lagrangian, whereas it could occur in these models, if one artificially used the on-mass-shell amplitude to construct the pion polarization operator. However within the model described by the reduced Manohar-Georgi Lagrangian (48), where the pion-nucleon amplitude (49) does not depend explicitly on values of q^2 and q'^2 , the s -wave pion condensation in the isospin-symmetric matter may appear already at $n > (2 - 2.5)n_0$ or even for $n > 1.4n_0$, in the latter case provided the effective pion decay constant f_π^* is decreased with increasing n . Occurrence of the condensation could result in appearance of metastable (or may be even stable) condensate droplets already in heavy-ion collisions with energies $\lesssim \text{GeV} \cdot A$. Further experimental check of presence or absence of these phenomena could help one to choose between employment of those phenomenological Lagrangians in the given problem. Important information on the pion polarization operator can be found from further studies of deeply bound states in the single-pion and double-pion atoms, cf. [76] and references therein.

In evaluation of the s -wave contribution to the pion polarization operator we disregarded correlation effects [we put $B = 0$ in Eq. (31)] presenting only intuitive arguments and mentioning rough estimates in favor of their smallness. However the quantitative consideration should be still done.

In conclusion of this analysis, predictions on presence or absence of the s -wave condensation in the isospin-symmetric matter in the SM, reduced MG and reduced GSS

TABLE I. Predictions on s -wave condensation in the isospin-symmetric matter in the SM1,2, reduced MG and reduced GSS models.

	SM1	SM2	MG	GSS
W (6)	+	-	-	+
A (7)	+	-	-	+
CD (8)	+	+	+	+
π , off	- (38)	- (42)	+ (52)	- (60)
π , on	+ (44)	+ (46)	+ (52)	+ (52)

models are illustrated in Table I, line π , off. The Weinberg, Adler, and Cheng-Dashen conditions are fulfilled within SM1 and GSS models, whereas in SM2 and MG models only the Cheng-Dashen condition is fulfilled. In the full off-mass-shell treatment the s -wave pion condensation in isospin-symmetric matter may occur at $n \sim (1.4 - 2.5)n_0$ only in the reduced MG model. In the latter model the off-mass-shell and on-mass-shell treatments coincide. In the artificial models using the on-mass-shell description, cf. line π , on, the s -wave pion condensation in isospin-symmetric matter may occur at $n \sim (1.4 - 2.5)n_0$ in all considered models.

Further, within a general phenomenological description not focusing on a specific model we constructed the fully off-mass-shell pion-nucleon scattering amplitude fitting parameters to satisfy the current algebra theorems and incorporating smallness of the s -wave pion-nucleon scattering length. The s -wave pion condensation in the isospin-symmetric matter hardly occurs in this model but it may occur already for $n > (1.4 - 2.5)n_0$, provided one constructs the pion polarization operator in the gas approximation employing the on-mass-shell pion-nucleon scattering amplitude and the fact of the smallness of the s -wave pion-nucleon scattering length. As we have mentioned, the latter procedure works in case of the model described by the reduced Manohar-Georgi Lagrangian (48), but not in cases of the linear sigma model and the GSS model described by the reduced Lagrangian L_α with $\alpha = \Sigma/(f_\pi^2 m_\pi^2)$, cf. (59). Manifestation or nonmanifestation of effects of the s -wave pion condensate in experimental investigations of various nuclear systems (atomic nuclei, heavy-ion collisions, neutron stars) could help to distinguish between different models.

Predictions on presence or absence of the s -wave condensation in the isospin-symmetric matter within the phenomenological (Ph) expansion (62) are illustrated in Table II, line “Ph, off.” The Weinberg, Adler, and

TABLE II. Predictions on s -wave pion condensation in the isospin-symmetric matter in phenomenological description.

	W	A	CD	π -cond
Ph, off	+	+	+	- (67)
Ph, on	-	-	+	+ (70)

Cheng-Dashen conditions are fulfilled and the s -wave pion condensation in isospin-symmetric matter does not occur at least up to a high density. Oppositely, in the on-mass-shell treatment, line “Ph, on,” only the Cheng-Dashen condition is fulfilled and the s -wave pion condensation in isospin-symmetric matter may occur already at $n \sim (1.4 - 2.5)n_0$.

We focused on the study of the isospin-symmetric matter, whereas consideration of the s -wave interaction in asymmetric medium is straightforward. For that, as a minimal step, it is sufficient to incorporate the Weinberg-Tomozawa term.

In consideration of the kaon polarization in the matter most of the authors focused on the phenomenon of the s -wave kaon condensation, whereas [70,77,78] considered

possibilities of both s - and p -wave condensations. Many works treated this problem within the relativistic mean field models, cf. [79]. References [77,78] employed the low-energy theorems. Many other works, cf. [46,70], used the on-mass-shell realization of the s -wave kaon-nucleon amplitude in matter putting $q^2 = q'^2 = m_K^2$. All our caveats concerning subtleties of the question about the s -wave pion polarization and condensation hold also for the case of the s -wave kaon polarization and condensation.

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