Tadpoles and vacuum bubbles in light-front quantization

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We develop a method by which vacuum transitions may be included in light-front calculations. This allows tadpole contributions which are important for symmetry-breaking effects and yet are missing from standard light-front calculations. These transitions also dictate a nontrivial vacuum and contributions from vacuum bubbles to physical states. In nonperturbative calculations these separate classes of contributions (tadpoles and bubbles) cannot be filtered; instead, we regulate the bubbles and subtract the vacuum energy from the eigenenergy of physical states. The key is replacement of momentum-conserving delta functions with model functions of finite width; the width becomes the regulator and is removed after subtractions. The approach is illustrated in free scalar theory, in quenched scalar Yukawa theory, and in a limited Fock-space truncation of ϕ^4 theory.

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I. INTRODUCTION

Recent calculations of the critical coupling in two-dimensional ϕ^4 theory [1–13] have shown that there is a discrepancy in the nonperturbative equivalence of equal-time and light-front quantization. Although this discrepancy can be explained with a computed shift in the renormalized mass [5,6,8,14], this explanation is a correction to the light-front calculation, rather than a direct calculation. In addition, calculations with coordinates that interpolate between equal-time and light-front quantizations [15,16] indicate that the light-front limit should obtain the same critical coupling as obtained in equal time quantization [17]. The key is the inclusion of tadpole contributions, which on the light front requires zero modes, modes with zero longitudinal momentum, to represent transitions to and from the vacuum, as illustrated in Fig. 1(a).[1] In a nonperturbative calculation, where one cannot pick and choose classes of diagrams, the presence of vacuum transitions necessarily imports (divergent) vacuum bubbles, of a sort shown in Fig. 1(b), as well as tadpoles.[2]

[1]These figures were drawn with JaxoDraw [18].

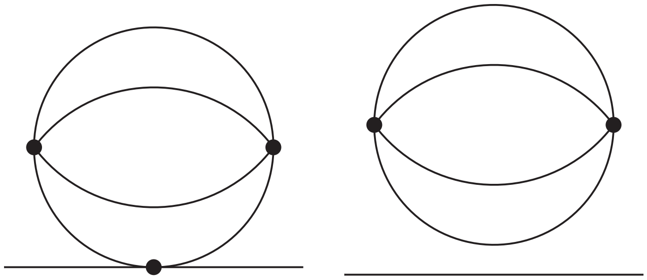
[2]This is a separate question from perturbative equivalence, which has been generally established. For recent discussions, see [19–21].

Zero-mode contributions to physical states and the corresponding nontriviality of the light-front vacuum [19,20,22,23] are of broader interest than just the critical coupling in ϕ^4 theory. They enter into any discussion of symmetry breaking, such as the Higgs mechanism, and of vacuum condensates [24]. More recently they have been identified as possible contributions to higher-twist distribution functions [25]. For these reasons, we explore a possible method for inclusion, within the context of two-dimensional scalar theories; extension to three and four dimensions should be straightforward.[3]

Contributions such as tadpoles and vacuum bubbles that involve transitions to and from the vacuum must rely on terms normally excluded from light-front Hamiltonians. These are terms with only creation operators or only annihilation operators. With light-front longitudinal momenta constrained to be non-negative, momentum conservation requires that the operators create or annihilate zero momentum. On this basis they are always dropped. However, depending on the zero-momentum behavior of the Fock-space wave functions, matrix elements of such terms need not be zero.

For example, consider light-front quantization [28–34] of a two-dimensional scalar theory. We define light-front coordinates [28] and momenta as $x^\pm = t \pm z$ and $p^\pm = E \pm p_z$, with x^+ chosen as the light-front time. The mass-shell condition for the total two-momentum

[3]For alternative methods, see [26,27].



(a) (b)

FIG. 1. (a) Tadpole graph and (b) vacuum bubble in ϕ^4 theory. Note the momentum-conserving transitions to and from the vacuum that imply light-front zero-mode contributions.

(P^-, P^+) is then $M^2 = P^+P^-$, which gives the fundamental bound-state eigenproblem as

$$\mathcal{P}^-|\psi(P^+)\rangle = \frac{M^2}{P^+}|\psi(P^+)\rangle, \tag{1.1}$$

where $\mathcal{P}^-$ is the light-front Hamiltonian. A typical Fock-state wave function for an eigenstate of this Hamiltonian satisfies an equation of the following form:

$$\sum_i^n \frac{\mu^2}{p_i^+}\psi_n(p_1^+,\ldots,p_n^+) + \frac{1}{\sqrt{p_1^+p_2^+p_3^+\cdots}} \times \text{contributions from other Fock sectors} = \frac{M^2}{P^+}\psi_n, \tag{1.2}$$

with n the number of constituents. The symmetry for bosons then requires that the small momentum behavior of this wave function is

$$\psi_n \sim \frac{1}{\sqrt{\prod_i^n p_i^+ \sum_i^n \frac{1}{p_i^+}}}. \tag{1.3}$$

The matrix element of a vacuum transition in ϕ^4 theory reduces to

$$\langle\psi_{n+4}|\int \frac{\prod_i^4 dp_i^+}{\sqrt{\prod_i^4 p_i^+}}\delta\left(\sum_i^4 p_i^+\right)\prod_i^4 a^\dagger(p_i^+)|\psi_n\rangle \tag{1.4}$$

$$\sim \int \frac{\prod_i^{n+4} dp_i^+}{\prod_i^{n+4} p_i^+}\frac{\delta(\sum_i^4 p_{n+i}^+)}{(\sum_i^{n+4}\frac{1}{p_i^+})(\sum_i^n \frac{1}{p_i^+})}\delta\left(P^+ - \sum_i^n p_i^+\right). \tag{1.5}$$

With $Q \equiv \sum_i^4 p_{n+i}^+$, $p_{n+i}^+ = x_i Q$, and $\prod_{i=n+1}^{n+4} dp_i^+ = Q^3 dQ \prod_i^4 dx_i \delta(1-\sum_i^4 x_i)$, this becomes

$$\begin{aligned}\langle\psi_{n+4}|&\int \frac{\prod_i^4 dp_i^+}{\sqrt{\prod_i^4 p_i^+}}\delta\left(\sum_i^4 p_i^+\right)\prod_i^4 a^\dagger(p_i^+)|\psi_n\rangle \\ &\sim \int dQ\delta(Q)\int \frac{\prod_i^4 dx_i\delta(1-\sum_i^4 x_i)}{(\prod_i^4 x_i)(\sum_i^4\frac{1}{x_i})} \\ &\quad\times \int \frac{\prod_i^n dp_i^+\delta(P^+ - \sum_i^n p_i^+)}{(\prod_i^n p_i^+)(\sum_i^n\frac{1}{p_i^+})},\end{aligned} \tag{1.6}$$

which is finite and nonzero. Thus, such vacuum-transition terms cannot be ignored automatically.

Vacuum transitions also generate vacuum bubbles which make contributions proportional to the size L of the spatial dimension as expressed through $4\pi\delta(0) = \int dx^- \equiv L$. A nonperturbative calculation requires a cutoff, to regulate this infinity, and a subtraction of the vacuum energy from any eigenenergy of a physical state. We regulate by replacing delta functions of momentum with model functions δ_ϵ that have a width parameter ϵ and take the limit of $\epsilon \to 0$ at the end of a calculation.[4] For the (nontrivial) vacuum state, we compute a finite energy density, with the model parameter ϵ related to the spatial volume L in a model-dependent way: $4\pi\delta_\epsilon(0) = L$.

For any finite width ϵ, there will be additional modes present in any calculation. We call these ephemeral modes, since they are not zero modes but instead disappear in the limit of zero width. The remaining imprint is essentially a zero-mode contribution, but obtained as a limit. Contributions to massive states, beyond the vacuum-energy shift and tadpoles, are generally negligible for weak coupling; however, for strong coupling, the Fock-state momentum wave functions can become broad enough that they overlap with ephemeral modes. Depending on the zero-momentum behavior of these wave functions, there can be additional contributions from vacuum transitions.

Such contributions cannot be readily captured with the discretized light-cone quantization (DLCQ) formalism [35], and DLCQ calculations with constrained zero modes [36–38] are incomplete. For good resolution of the ephemeral modes, the DLCQ resolution K must satisfy $1/K \ll \epsilon/P^+$. Also, the integrals that must be represented by the rectangular DLCQ grid are highly singular, for which the grid is ill suited. Calculations would be best undertaken with a basis function expansion, for which matrix elements can be computed once and for all with an adaptive Monte Carlo integration, such as is available in the VEGAS package [39].

Even with antiperiodic boundary conditions, the DLCQ approach cannot neglect zero modes. With such boundary

[4] This is equivalent to the regulator used in [19] for the calculation of a time-ordered product. The parameter R in their Eq. (22), introduced as the radius of a circle approaching infinity, is essentially $1/\epsilon$. However, the circle at infinity is something that arises in diagrammatic calculations rather than nonperturbative eigenvalue problems.

conditions one can avoid the constraint equation, but the approximation to the integral operators in $\mathcal{P}^-$ is the midpoint rule with an error no better than the $1/K^2$ for periodic boundary conditions, where the integrals are approximated by the trapezoidal rule.[5] The trouble is that the coefficient of the $1/K^2$ correction is small only if the integrand is slowly varying. If instead there is rapid variation, such as can happen near zero momentum, the approximation becomes quite poor except at very high resolution. In other words, if zero-mode contributions are important, antiperiodic boundary conditions do not provide an approximation any better than periodic boundary conditions.

To explore the inclusion of vacuum transitions, we first consider ϕ^4 theory in more detail; in Sec. II we consider the leading tadpole and vacuum-bubble contributions. Next, in Sec. III, we develop an analytic solution for a free scalar as a generalized coherent state of ephemeral modes. The vacuum bubble contributions replicate the one-loop calculation emphasized by Collins [22]. We also consider the solution for a shifted scalar with nonzero vacuum expectation value. This is done in the continuum, without interpolation from equal-time quantization and without discretization. Finally, we consider quenched scalar Yukawa theory in lowest-order Fock truncation in Sec. IV, to see the subtraction of the vacuum energy of the neutral scalar in the charge-zero sector from the dressed scalar energy in the charge-one sector. Numerical calculations are postponed to future work.

II. LOWEST-ORDER ϕ^4 THEORY

The Lagrangian for two-dimensional ϕ^4 theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{2.1}$$

where μ is the mass of the boson and λ is the coupling constant. The light-front Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{2.2}$$

The mode expansion for the field is[6]

$$\phi(x^+ = 0, x^-) = \int \frac{dp}{\sqrt{4\pi p}}\{a(p)e^{-ipx^-/2} + a^\dagger(p)e^{ipx^-/2}\}. \tag{2.3}$$

The nonzero commutation relation is

$$[a(p), a^\dagger(p')] = \delta(p - p'). \tag{2.4}$$

The light-front Hamiltonian is $\mathcal{P}^- = \mathcal{P}_0^- + \mathcal{P}_{\rm int}^-$, with

$$\mathcal{P}_0^- = \int dp\frac{\mu^2}{p}a^\dagger(p)a(p) + \frac{\mu^2}{2}\int\frac{dp_1dp_2}{\sqrt{p_1p_2}}\delta_\epsilon(p_1+p_2)[a(p_1)a(p_2) + a^\dagger(p_1)a^\dagger(p_2)], \tag{2.5}$$

$$\mathcal{P}_{\rm int}^- = \mathcal{P}_{04}^- + \mathcal{P}_{40}^- + \mathcal{P}_{22}^- + \mathcal{P}_{13}^- + \mathcal{P}_{31}^-, \tag{2.6}$$

where

$$\mathcal{P}_{04}^- = \frac{\lambda}{24}\int\frac{dp_1dp_2dp_3dp_4}{4\pi\sqrt{p_1p_2p_3p_4}}\delta_\epsilon\left(\sum_i^4 p_i\right)a(p_1)a(p_2)a(p_3)a(p_4), \tag{2.7}$$

$$\mathcal{P}_{40}^- = \frac{\lambda}{24}\int\frac{dp_1dp_2dp_3dp_4}{4\pi\sqrt{p_1p_2p_3p_4}}\delta_\epsilon\left(\sum_i^4 p_i\right)a^\dagger(p_1)a^\dagger(p_2)a^\dagger(p_3)a^\dagger(p_4), \tag{2.8}$$

$$\mathcal{P}_{22}^- = \frac{\lambda}{4}\int\frac{dp_1dp_2}{4\pi\sqrt{p_1p_2}}\int\frac{dp_1'dp_2'}{\sqrt{p_1'p_2'}}\delta(p_1+p_2-p_1'-p_2')a^\dagger(p_1)a^\dagger(p_2)a(p_1')a(p_2'), \tag{2.9}$$

$$\mathcal{P}_{13}^- = \frac{\lambda}{6}\int\frac{dp_1dp_2dp_3}{4\pi\sqrt{p_1p_2p_3(p_1+p_2+p_3)}}a^\dagger(p_1+p_2+p_3)a(p_1)a(p_2)a(p_3), \tag{2.10}$$

[5]Without a solution to the constraint equation, the error in DLCQ with periodic boundary conditions is of order $1/K$, unless the end points (the zero modes) make no contribution to the integrals.

[6]For convenience we drop the $+$ superscript and will from here on write light-front momenta such as p^+ as just p.

$$\mathcal{P}_{31}^- = \frac{\lambda}{6}\int \frac{dp_1 dp_2 dp_3}{4\pi\sqrt{p_1p_2p_3(p_1+p_2+p_3)}} a^\dagger(p_1)a^\dagger(p_2)a^\dagger(p_3)a(p_1+p_2+p_3). \tag{2.11}$$

The subscripts indicate the number of creation and annihilation operators in each term.

To isolate the contribution from the tadpole and vacuum bubble in Fig. 1, we consider only two terms in the Fock-state expansion of the eigenstate

$$|\psi(P)\rangle = \psi_1 a^\dagger(P)|0\rangle + \cdots + \int \prod_i^5 dp_i \delta\left(P - \sum_i^5 p_i\right)\psi_5(p_1,\ldots,p_5)\frac{1}{\sqrt{5!}}\prod_i^5 a^\dagger(p_i)|0\rangle + \cdots, \tag{2.12}$$

in order to represent the five constituents in the intermediate states, and we keep only the first term of $\mathcal{P}_0^-$ and the first three terms of $\mathcal{P}_{\rm int}^-$, as the only terms that connect the two Fock sectors. We then consider the eigenvalue problem

$$(\mathcal{P}_0^- + \mathcal{P}_{\rm int}^-)|\psi(P)\rangle = \left(\frac{M^2}{P} + P_{\rm vac}^-\right)|\psi(P)\rangle, \tag{2.13}$$

where we include the shift of vacuum energy $P_{\rm vac}^-$, to be obtained from solving the corresponding vacuum eigenvalue problem

$$(\mathcal{P}_0^- + \mathcal{P}_{\rm int}^-)|{\rm vac}\rangle = P_{\rm vac}^-|{\rm vac}\rangle, \tag{2.14}$$

with $|{\rm vac}\rangle$ the lowest eigenstate. Projection of the eigenvalue problem for the lowest massive state onto Fock sectors yields a system of equations for the Fock-state wave functions

$$\frac{\mu^2}{P}\psi_1 + \frac{\lambda}{\sqrt{24}}\int \frac{\prod_i^4 dp_i}{4\pi\sqrt{\prod_i^4 p_i}}\delta_\epsilon\left(\sum_i^4 p_i\right)\psi_5(p_1,\ldots,p_5) = \left(\frac{M^2}{P} + P_{\rm vac}^-\right)\psi_1, \tag{2.15}$$

$$\begin{aligned}\left(\sum_i^5 \frac{\mu^2}{p_i}\right)\psi_5 &+ \frac{\lambda}{24}\frac{1}{5}\left[\frac{\delta_\epsilon(\sum_i^4 p_i)}{4\pi\sqrt{\prod_i^4 p_i}} + (p_5 \leftrightarrow p_1,p_2,p_3,p_4)\right]\psi_1 \\ &+ 20\frac{\lambda}{4}\int \frac{dp_1' dp_2'}{4\pi\sqrt{p_1p_2p_1'p_2'}}\delta(p_1+p_2-p_1'-p_2')\psi_5(p_1',p_2',p_3,p_4,p_5) = \frac{M^2}{P}\psi_5,\end{aligned} \tag{2.16}$$

where we have invoked a sector-dependent energy shift, with no $P_{\rm vac}^-$ in the top Fock sector.[7]

The second equation can be solved iteratively with respect to the self-coupling of the five-constituent Fock state in the third term of (2.16); this corresponds to a diagrammatic expansion. The leading term in the expansion generates the vacuum bubble in Fig. 1(b) that contributes $P_{\rm vac}^-$ in (2.15). The second term, where the self-interaction acts once, produces the tadpole in Fig. 1(a). Both are written explicitly in (2.17) and (2.18) below. Subtraction of $P_{\rm vac}^-$ from both sides of (2.15) eliminates the divergent bubble.

From (2.15) and (2.16), the contributions take the forms

$$\begin{aligned}\text{bubble} \to &\int \frac{\prod_i^5 dp_i}{\prod_i^4 p_i}\delta\left(P - \sum_i^5 p_i\right)\frac{\delta_\epsilon(\sum_i^4 p_i)^2}{\frac{M^2}{P} - \sum_i^5\frac{\mu^2}{p_i}} \\ &\sim -\int \delta_\epsilon(Q)^2\frac{dQ}{\mu^2}\int\frac{\prod_i^4 dx_i}{\prod_i^4 x_i}\delta\left(1 - \sum_i^4 x_i\right),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\text{tadpole} \to &\int \frac{\prod_i^5 dp_i \delta_\epsilon(\sum_i^4 p_i)}{\sqrt{\prod_i^4 p_i}}\frac{\delta(P - \sum_i^5 p_i)}{\frac{M^2}{P} - \sum_i^5\frac{\mu^2}{p_i}} \\ &\times \int \frac{dp_1' dp_2'}{\sqrt{p_4p_5p_1'p_2'}}\frac{\delta(p_4+p_5-p_1'-p_2')}{\frac{M^2}{P} - \sum_i^3\frac{\mu^2}{p_i} - \frac{\mu^2}{p_1'} - \frac{\mu^2}{p_2'}} \\ &\times \frac{\delta_\epsilon(\sum_i^3 p_i + p_1')}{\sqrt{\prod_i^3 p_i p_1'}}.\end{aligned} \tag{2.18}$$

The expression for the bubble diverges as $\epsilon \to 0$ and is proportional to $\delta(0) = L/4\pi$; however, the same expression is obtained for the nontrivial vacuum energy $P_{\rm vac}^-$ and is subtracted.

The expression for the tadpole contribution can be simplified by noting that $\delta(P - \sum_i^5 p_i)$ reduces to $\delta(P - p_5)$, which can be used to do the p_5 integral, and $\delta(p_4 + p_5 - p_1' - p_2')$ becomes $\delta(p_4 + P - p_1' - p_2')$, which can be used to do the p_2' integral. Finally, $\delta_\epsilon(\sum_i^3 p_i + p_1')$ can be written $\delta_\epsilon(p_4 - p_1')$ and used to do the p_1' integral. These leave

[7]In the top sector, there are, of course, no vacuum corrections from higher Fock sectors.

$$\text{tadpole} \sim \int \frac{\prod_i^4 dp_i}{\prod_i^4 p_i} \frac{1}{p_4 P} \frac{\delta_\epsilon(\sum_i^4 p_i)}{[\frac{M^2}{P} - \sum_i^4 \frac{\mu^2}{p_i} - \frac{\mu^2}{P}]^2}$$
$$\sim \frac{1}{P} \int \delta_\epsilon(Q) dQ \int \frac{\prod_i^4 dx_i}{(\prod_i^4 x_i) x_4 (\sum_i^4 \frac{\mu^2}{x_i})^2}, \quad (2.19)$$

which is finite and inversely proportional to P, the mark of a light-front self-energy correction.

Of course, in a nonperturbative calculation, these contributions cannot be separated. However, with the bubbles regulated, one can solve the eigenproblems for the vacuum and the massive states and then carry out the necessary $P^-_{\rm vac}$ subtraction prior to taking the width parameter ϵ to zero.

III. FREE SCALAR

A. Free vacuum

The free vacuum $|{\rm vac}\rangle$ is an eigenstate of the free scalar Hamiltonian $\mathcal{P}_0^-$ in (2.5):

$$\mathcal{P}_0^- |{\rm vac}\rangle = P^-_{\rm vac} |{\rm vac}\rangle. \quad (3.1)$$

We will show that the vacuum is a generalized coherent state,

$$|{\rm vac}\rangle = \sqrt{Z} e^{A^\dagger} |0\rangle, \quad (3.2)$$

where

$$A^\dagger = \int_0^\infty \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}} a^\dagger(p_1) a^\dagger(p_2). \quad (3.3)$$

For such a state, we have

$$a(p)|{\rm vac}\rangle = 2 \int \frac{dp'}{\sqrt{pp'}} \frac{f(p, p')}{\frac{1}{p} + \frac{1}{p'}} a^\dagger(p') |{\rm vac}\rangle \quad (3.4)$$

and

$$a(p_1) a(p_2) |{\rm vac}\rangle = \frac{2}{\sqrt{p_1 p_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}} |{\rm vac}\rangle + 4 \int \frac{dp_1' dp_2'}{\sqrt{p_1 p_2 p_1' p_2'}} \frac{f(p_1, p_1')}{\frac{1}{p_1} + \frac{1}{p_1'}} \frac{f(p_2, p_2')}{\frac{1}{p_2} + \frac{1}{p_2'}} a^\dagger(p_1') a^\dagger(p_2') |{\rm vac}\rangle. \quad (3.5)$$

With these we can apply $\mathcal{P}_0^-$ to obtain

$$\mathcal{P}_0^- |{\rm vac}\rangle = \frac{\mu^2}{2} \int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \delta_\epsilon(p_1 + p_2) [a^\dagger(p_1) a^\dagger(p_2) + \frac{2}{\sqrt{p_1 p_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}}$$
$$+ 4 \int \frac{dp_1' dp_2'}{\sqrt{p_1 p_2 p_1' p_2'}} \frac{f(p_1, p_1')}{\frac{1}{p_1} + \frac{1}{p_1'}} \frac{f(p_2, p_2')}{\frac{1}{p_2} + \frac{1}{p_2'}} a^\dagger(p_1') a^\dagger(p_2')] |{\rm vac}\rangle$$
$$+ \int dp \frac{\mu^2}{p} a^\dagger(p) \int dp' \frac{2}{\sqrt{pp'}} \frac{f(p, p')}{\frac{1}{p} + \frac{1}{p'}} a^\dagger(p') |{\rm vac}\rangle. \quad (3.6)$$

The solution to $\mathcal{P}_0^- |{\rm vac}\rangle = P^-_{\rm vac} |{\rm vac}\rangle$ is then possible if

$$P^-_{\rm vac} = \frac{\mu^2}{2} \int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \delta_\epsilon(p_1 + p_2) \frac{2}{\sqrt{p_1 p_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}} \quad (3.7)$$

and the symmetrized coefficients of $a^\dagger(p_1) a^\dagger(p_2)$ sum to zero:

$$0 = \frac{\mu^2}{2} \frac{\delta_\epsilon(p_1 + p_2)}{\sqrt{p_1 p_2}} + 2\mu^2 \int \frac{dp_1' dp_2'}{p_1' p_2' \sqrt{p_1 p_2}} \delta_\epsilon(p_1' + p_2')$$
$$\times \frac{f(p_1, p_1')}{\frac{1}{p_1} + \frac{1}{p_1'}} \frac{f(p_2, p_2')}{\frac{1}{p_2} + \frac{1}{p_2'}} + \frac{1}{2} \left[\frac{\mu^2}{p_1} + \frac{\mu^2}{p_2}\right] \frac{2}{\sqrt{p_1 p_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}}. \quad (3.8)$$

In the second term of (3.8), we can compute

$$\int \frac{dp_1' dp_2'}{p_1' p_2'} \delta_\epsilon(p_1' + p_2') \frac{f(p_1, p_1')}{\frac{1}{p_1} + \frac{1}{p_1'}} \frac{f(p_2, p_2')}{\frac{1}{p_2} + \frac{1}{p_2'}}$$
$$= p_1 p_2 \int Q dQ \delta_\epsilon(Q) \int dx \frac{f(p_1, xQ) f(p_2, (1-x)Q)}{(p_1 + xQ)(p_2 + (1-x)Q)}$$
$$= f(p_1) f(p_2) \int Q dQ \delta_\epsilon(Q) = 0. \quad (3.9)$$

The sum of coefficients in (3.8) is then zero if

$$f(p_1,p_2) = -\frac{1}{2}\delta_\epsilon(p_1+p_2). \tag{3.10}$$

This determines the vacuum state.

With this solution for the coherent-state wave function, the energy of the vacuum is

$$\begin{aligned} P^-_{\rm vac} &= -\frac{\mu^2}{2}\int \frac{dp_1dp_2}{p_1p_2}\frac{\delta_\epsilon(p_1+p_2)^2}{\frac{1}{p_1}+\frac{1}{p_2}} \\ &= -\frac{\mu^2}{2}\int \frac{QdQdx}{Q^2x(1-x)}\frac{\delta_\epsilon(Q)^2}{\frac{1}{Q}\frac{1}{x(1-x)}} \\ &= -\frac{\mu^2}{2}\int dQ\delta_\epsilon(Q)^2 = -\frac{\mu^2}{2}\delta_\epsilon(0)\int_0^\infty dQ\delta_\epsilon(Q) \\ &= -\frac{\mu^2}{2}\frac{L}{4\pi}\frac{1}{2} = -\frac{\mu^2 L}{16\pi}. \end{aligned} \tag{3.11}$$

Here L is the (infinite) volume of light-front space; however, δ_ϵ at finite ϵ regulates $P^-_{\rm vac}$ when it is embedded in a nonperturbative calculation.

This result is proportional to the one-loop vacuum bubble computed by Collins [22]. The equivalent perturbative calculation, corresponding to the loop in Fig. 2, is

$$\begin{aligned} &2\frac{\mu^2}{2}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\delta_\epsilon(q_1+q_2)\frac{1}{\frac{M^2}{P}-\frac{\mu^2}{q_1}-\frac{\mu^2}{q_2}}\frac{\mu^2}{2}\frac{\delta_\epsilon(q_1+q_2)}{\sqrt{q_1q_2}} \\ &= 2\frac{\mu^4}{4}\int\frac{dq_1dq_2}{q_1q_2}\frac{\delta_\epsilon(q_1+q_2)^2}{-\frac{\mu^2}{q_1}-\frac{\mu^2}{q_2}}, \end{aligned} \tag{3.12}$$

which matches (3.11). The leading factor of 2 comes from the two possible contractions of the double scalar creation and annihilation operators.

Any massive state in the free theory has a Fock-state wave function that is a product of delta functions of the individual particle momenta p_i. This part of the state will not mix with the ephemeral modes, provided the width parameter ϵ is chosen such that it is much less than all the p_i. For example, the single-particle state with mass μ and momentum P is just $a^\dagger(P)|{\rm vac}\rangle$. It is an eigenstate of $\mathcal{P}_0^- - P^-_{\rm vac}$ with eigenvalue μ^2/P. Weak couplings that broaden the momentum-space wave functions only slightly will produce effectively unmixed contributions, but calculations with strong couplings require more care.

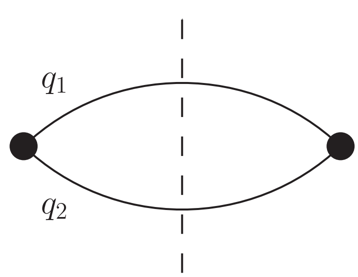


FIG. 2. One-loop self-energy graph as a simple vacuum bubble which contributes to the vacuum state of a free scalar. The dashed line indicates the intermediate state of two ephemeral modes.

B. Shifted scalar

Next we consider the shifted free scalar where $\phi \to \phi + v$. The new Lagrangian is

$$\mathcal{L} = \mathcal{L}_{v=0} - \mu^2 v\phi - \frac{1}{2}\mu^2 v^2, \tag{3.13}$$

and the Hamiltonian is $\mathcal{P}^- = \mathcal{P}_0^- + \mathcal{P}^-_{\rm int}$ with the interaction part

$$\begin{aligned} \mathcal{P}^-_{\rm int} &= \int dx^-\mu^2 v\phi = \sqrt{4\pi}\mu^2 v\int\frac{dp}{\sqrt{p}}\delta_\epsilon(p)[a(p) \\ &\quad + a^\dagger(p)] + \frac{1}{2}\mu^2 v^2 L. \end{aligned} \tag{3.14}$$

The constant term represents the shift in the energy of the vacuum and is therefore proportional to the spatial size L.

The vacuum state $|{\rm vac}\rangle_v$ is now an eigenstate of $\mathcal{P}^-$,

$$(\mathcal{P}_0^- + \mathcal{P}^-_{\rm int})|{\rm vac}\rangle_v = P^-_{\rm vac}|{\rm vac}\rangle_v. \tag{3.15}$$

It can be constructed from the free vacuum as $e^B|{\rm vac}\rangle$ with

$$B \equiv v\int dp\sqrt{4\pi p}\,\delta_\epsilon(p)[a^\dagger(p) - a(p)]. \tag{3.16}$$

This works because

$$e^B\phi(x^-)e^{-B} = \phi(x^-) + v \tag{3.17}$$

and

$$e^B\mathcal{P}_0^- e^{-B} = \mathcal{P}_0^- + \mathcal{P}^-_{\rm int}. \tag{3.18}$$

This then permits

$$\begin{aligned} (\mathcal{P}_0^- + \mathcal{P}^-_{\rm int})|{\rm vac}\rangle_v &= e^B\mathcal{P}_0^- e^{-B}e^B|{\rm vac}\rangle = e^B\mathcal{P}_0^-|{\rm vac}\rangle \\ &= P^-_{\rm vac}e^B|{\rm vac}\rangle = P^-_{\rm vac}|{\rm vac}\rangle_v. \end{aligned} \tag{3.19}$$

Thus, in both the free and the shifted cases, the vacuum is a generalized coherent state of ephemeral modes.

The state is also correctly normalized, because

$$\begin{aligned} {}_v\langle{\rm vac}|{\rm vac}\rangle_v &= \langle{\rm vac}|e^{B^\dagger}e^B|{\rm vac}\rangle = \langle{\rm vac}|e^{-B}e^B|{\rm vac}\rangle \\ &= \langle{\rm vac}|{\rm vac}\rangle = 1. \end{aligned} \tag{3.20}$$

The vacuum expectation value of the field can also be computed:

$$
\begin{aligned}
{}_v\langle \mathrm{vac}|\phi(x^-)|\mathrm{vac}\rangle_v &= \langle \mathrm{vac}|e^{B^\dagger}\phi(x^-)e^B|\mathrm{vac}\rangle \\
&= \langle \mathrm{vac}|e^{-B}\phi(x^-)e^B|\mathrm{vac}\rangle \\
&= \langle \mathrm{vac}|(\phi(x^-)-v)|\mathrm{vac}\rangle = -v. \quad (3.21)
\end{aligned}
$$

This restores the shift.

IV. QUENCHED SCALAR YUKAWA THEORY

In order to look at a case with more structure, we consider scalar Yukawa theory [40], for which the Lagrangian is

$$\mathcal{L} = |\partial_\mu\chi|^2 - m^2|\chi|^2 + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\phi|\chi|^2, \quad (4.1)$$

where χ is a complex scalar field with mass m and ϕ is a real scalar field with mass μ. The two fields are coupled by a Yukawa term with strength g. In two dimensions, the light-front Hamiltonian density is

$$\mathcal{H} = m^2|\chi|^2 + \frac{1}{2}\mu^2\phi^2 + g\phi|\chi|^2. \quad (4.2)$$

The mode expansions for the fields are (2.3) for ϕ and

$$\chi = \int \frac{dp}{\sqrt{4\pi p}}[c_+(p)e^{-ipx^-/2} + c_-^\dagger(p)e^{ipx^-/2}]. \quad (4.3)$$

The nonzero commutation relations of the creation and annihilation operators are (2.4) and

$$[c_\pm(p), c_\pm^\dagger(p')] = \delta(p-p'). \quad (4.4)$$

In terms of these operators, the quenched light-front Hamiltonian $\mathcal{P}^- = \int dx^-\mathcal{H} = \mathcal{P}_0^- + \mathcal{P}_{\mathrm{int}}^-$ is specified by

$$
\begin{aligned}
\mathcal{P}_0^- = &\int dp\frac{m^2}{p}[c_+^\dagger(p)c_+(p) + c_-^\dagger(p)c_-(p)] \\
&+ \int dq\frac{\mu^2}{q}a^\dagger(q)a(q) + \frac{\mu^2}{2}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}} \\
&\times \delta_\epsilon(q_1+q_2)[a(q_1)a(q_2) + a^\dagger(q_1)a^\dagger(q_2)], \quad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{P}_{\mathrm{int}}^- = g\int &\frac{dpdq}{\sqrt{4\pi pq(p+q)}}\{[c_+^\dagger(p+q)c_+(p) \\
&+ c_-^\dagger(p+q)c_-(p)]a(q) + \mathrm{H.c.}\}. \quad (4.6)
\end{aligned}
$$

Pair creation and annihilation terms are suppressed for the complex scalar; without this quenching, the theory is unstable [41]. This also suppresses ephemeral modes for the complex scalar, which would need to appear in pairs to conserve charge, leaving only those of the neutral scalar. The vacuum in the charge-zero sector is that of the free scalar, as given in the previous section; this provides the value of P_{vac}^- for subtraction in the charge-one sector.

We seek eigenstates of $\mathcal{P}^-$, for which the two-dimensional light-front mass eigenvalue problem is

$$\mathcal{P}^-|\psi(P)\rangle = \left(\frac{M^2}{P} + P_{\mathrm{vac}}^-\right)|\psi(P)\rangle. \quad (4.7)$$

We limit this to the charge-one sector. This sector is characterized as a single complex scalar dressed by a cloud of neutrals. For the present purposes we will consider only a severe Fock-space truncation that keeps no more than two neutrals. The Fock-state expansion for the eigenstate is then

$$
\begin{aligned}
|\psi(P)\rangle = &\psi_0c_+^\dagger(P)|0\rangle \\
&+ \int dqdp\delta(P-q-p)\psi_1(q)a^\dagger(q)c_+^\dagger(p)|0\rangle \\
&+ \int dq_1dq_2dp\delta(P-q_1-q_2-p) \\
&\times \psi_2(q_1,q_2)\frac{1}{\sqrt{2}}a^\dagger(q_1)a^\dagger(q_2)c_+^\dagger(p)|0\rangle. \quad (4.8)
\end{aligned}
$$

The normalization condition $\langle\psi(P')|\psi(P)\rangle = \delta(P'-P)$ becomes

$$1 = |\psi_0|^2 + \int dq|\psi_1|^2 + \int dq_1dq_2|\psi_2|^2. \quad (4.9)$$

To construct the eigenvalue problem for the wave functions, we act with $\mathcal{P}_0^-$ and $\mathcal{P}_{\mathrm{int}}^-$ on the eigenstate and then project onto the three Fock sectors included in the truncation. Terms that generate higher Fock sectors are dropped. For $\mathcal{P}_0^-$ we have

$$
\begin{aligned}
\mathcal{P}_0^-|\psi(P)\rangle &= \frac{m^2}{P}\psi_0 c_+^\dagger(P)|0\rangle + \int dq dp \delta(P-q-p)\left(\frac{\mu^2}{q}+\frac{m^2}{p}\right)\psi_1(q)a^\dagger(q)c_+^\dagger(p)|0\rangle \\
&+ \int dq_1 dq_2 dp \delta(P-q_1-q_2-p)\left(\frac{\mu^2}{q_1}+\frac{\mu^2}{q_2}+\frac{m^2}{p}\right)\psi_2(q_1,q_2)\frac{1}{\sqrt{2}}a^\dagger(q_1)a^\dagger(q_2)c_+^\dagger(p)|0\rangle \\
&+ \frac{\mu^2}{2}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\delta_\epsilon(q_1+q_2)\psi_0 a^\dagger(q_1)a^\dagger(q_2)c_+^\dagger(P)|0\rangle + \frac{\mu^2}{2}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\delta_\epsilon(q_1+q_2)\psi_2(q_1,q_2)c_+^\dagger(P)|0\rangle. \quad (4.10)
\end{aligned}
$$

The last two terms violate momentum conservation but only by amounts of order ϵ, the width of δ_ϵ. For $\mathcal{P}_{\rm int}^-$ we find

$$
\begin{aligned}
\mathcal{P}_{\rm int}^-|\psi(P)\rangle &= \frac{g}{\sqrt{4\pi}}\int\frac{dqdp}{\sqrt{qp(p+q)}}\delta(P-p-q)[\psi_1(q)c_+^\dagger(P)+\psi_0 a^\dagger(q)c_+^\dagger(p)]|0\rangle \\
&+ \sqrt{2}\frac{g}{\sqrt{4\pi}}\int\frac{dq_1dq_2dp}{\sqrt{q_2p(p+q_2)}}\delta(P-p-q_1-q_2)[\psi_2(q_1,q_2)a^\dagger(q_1)c_+^\dagger(p+q_2) \\
&+ \psi_1(q_1)a^\dagger(q_1)a^\dagger(q_2)c_+^\dagger(p)]|0\rangle. \quad (4.11)
\end{aligned}
$$

Projection of $\mathcal{P}^-|\psi(P)\rangle = (\frac{M^2}{P}+P_{\rm vac}^-)|\psi(P)\rangle$ onto each of the three Fock sectors yields the following three equations:

$$
\frac{m^2}{P}\psi_0 + \frac{g}{\sqrt{4\pi}}\int_0^P\frac{dq\psi_1(q)}{\sqrt{qP(P-q)}} + \frac{\mu^2}{\sqrt{2}}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\delta_\epsilon(q_1+q_2)\psi_2(q_1,q_2) = \left(\frac{M^2}{P}+P_{\rm vac}^-\right)\psi_0, \quad (4.12)
$$

$$
\left(\frac{\mu^2}{q}+\frac{m^2}{P-q}\right)\psi_1(q) + \frac{g}{\sqrt{4\pi}}\frac{\psi_0}{\sqrt{qP(P-q)}} + \sqrt{2}\frac{g}{\sqrt{4\pi}}\int_0^{P-q}\frac{dq'\psi_2(q,q')}{\sqrt{q'(P-q)(P-q-q')}} = \frac{M^2}{P}\psi_1(q), \quad (4.13)
$$

and

$$
\begin{aligned}
&\left(\frac{\mu^2}{q_1}+\frac{\mu^2}{q_2}+\frac{m^2}{P-q_1-q_2}\right)\psi_2(q_1,q_2) + \frac{\mu^2}{\sqrt{2}}\delta_\epsilon(q_1+q_2)\frac{\psi_0}{\sqrt{q_1q_2}} \\
&+ \frac{1}{\sqrt{2}}\frac{g}{\sqrt{4\pi}}\left[\frac{\psi_1(q_1)}{\sqrt{q_2(P-q_1)(P-q_1-q_2)}}+\frac{\psi_1(q_2)}{\sqrt{q_1(P-q_2)(P-q_1-q_2)}}\right] = \frac{M^2}{P}\psi_2(q_1,q_2). \quad (4.14)
\end{aligned}
$$

The vacuum energy $P_{\rm vac}^-$ appears only in the first equation, because the Fock-space truncation prevents any such correction in all but the lowest Fock sector.

We build a matrix representation for these equations by introducing basis-function expansions

$$
\psi_1(q) = \frac{1}{\sqrt{P}}\sum_n a_n f_n(q), \quad \psi_2(q_1,q_2) = \frac{1}{P}\sum_{nj} b_{nj}g_{nj}(q_1,q_2), \quad (4.15)
$$

with $p = P-q$ and $p = P-q_1-q_2$, respectively, $\tilde{m}\equiv m/\mu$, and

$$
f_{-1}(q) = \frac{C_{-1}}{\sqrt{q(P-q)}}\frac{P\delta_\epsilon(q)}{\frac{1}{q}+\frac{\tilde{m}^2}{P-q}} = C_{-1}\sqrt{qP}\delta_\epsilon(q), \quad (4.16)
$$

$$
f_n(q) = \frac{C_n}{\sqrt{q(P-q)}}\frac{(q/P)^n}{\frac{1}{q}+\frac{\tilde{m}^2}{P-q}}, \quad n\geq 0, \quad (4.17)
$$

$$
\begin{aligned}
g_{-10}(q_1,q_2) &= \frac{D_{-10}\sqrt{P}}{\sqrt{q_1q_2(P-q_1-q_2)}}\frac{P\delta_\epsilon(q_1+q_2)}{\frac{1}{q_1}+\frac{1}{q_2}+\frac{\tilde{m}^2}{P-q_1-q_2}} \\
&= D_{-10}P\sqrt{q_1q_2}\frac{\delta_\epsilon(q_1+q_2)}{q_1+q_2}, \quad (4.18)
\end{aligned}
$$

$$
g_{nj}(q_1,q_2) = \frac{D_{nj}\sqrt{P}}{\sqrt{q_1q_2(P-q_1-q_2)}}\frac{(q_1^jq_2^{(n-j)}+q_1^{(n-j)}q_2^j)/P^n}{\frac{1}{q_1}+\frac{1}{q_2}+\frac{\tilde{m}^2}{P-q_1-q_2}}, \quad n\geq 0, \quad j=0,\ldots,n/2. \quad (4.19)
$$

These have the desired small-momentum behavior shown in (1.3). The negative index $n=-1$ is reserved for the ephemeral-mode contributions. The normalization condition (4.9) reduces to

$$
1 = |\psi_0|^2 + \sum_{nm}B_{nm}^{(1)}a_na_m + \sum_{nj,ml}B_{nj,ml}^{(2)}b_{nj}b_{ml}, \quad (4.20)
$$

where the overlaps between the nonorthogonal basis functions are the symmetric matrices

$$B^{(1)}_{nm} = \frac{1}{P}\int f_n(q) f_m(q) dq,$$
$$B^{(2)}_{nj,ml} = \frac{1}{P^2}\int g_{nj}(q_1,q_2) g_{ml}(q_1,q_2) dq_1 dq_2. \tag{4.21}$$

The normalization coefficients C_n and D_{nj} are fixed by requiring $B^{(1)}_{nn} = 1$ and $B^{(2)}_{nj,nj} = 1$, and one can show that the $n = -1$ basis functions are orthogonal to the others, making $B^{(1)}_{-1,n} = 0$ and $B^{(2)}_{-10,nj} = 0$ for $n \geq 0$.

The system of equations (4.12)–(4.14) becomes, with $\lambda \equiv g/\sqrt{4\pi}\mu^2$ and $\tilde{M} \equiv M/\mu$,

$$\tilde{m}^2\psi_0 - \frac{P}{\mu^2}P^-_{\rm vac}\psi_0 + \lambda\sum_n V^{(0)}_n a_n + \sum_{nj} U^{(0)}_{nj} b_{nj} = \tilde{M}^2\psi_0, \tag{4.22}$$

$$\sum_m T^{(1)}_{nm} a_m + \lambda V^{(0)}_n \psi_0 + \lambda\sum_{ml} V^{(1)}_{n,ml} b_{ml} = \tilde{M}^2 \sum_m B^{(1)}_{nm} a_m, \tag{4.23}$$

$$\sum_{ml} T^{(2)}_{nj,ml} b_{ml} + U^{(0)}_{nj}\psi_0 + \lambda\sum_m V^{(1)}_{m,nj} a_m = \tilde{M}^2 \sum_m B^{(2)}_{nj,ml} b_{ml}. \tag{4.24}$$

The various matrix elements are defined by

$$T^{(1)}_{nm} = \int dq f_n(q)\left(\frac{1}{q} + \frac{\tilde{m}^2}{P-q}\right) f_m(q), \tag{4.25}$$

$$T^{(2)}_{nj,ml} = \frac{1}{P}\int dq_1 dq_2 g_{nj}(q_1,q_2) \times \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{\tilde{m}^2}{P-q_1-q_2}\right) g_{ml}(q_1,q_2), \tag{4.26}$$

$$U^{(0)}_{nj} = \frac{1}{\sqrt{2}}\int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}} \delta_\epsilon(q_1+q_2) g_{nj}(q_1,q_2), \tag{4.27}$$

$$V^{(0)}_n = \sqrt{P}\int \frac{dq f_n(q)}{\sqrt{qP(P-q}},$$
$$V^{(1)}_{n,ml} = \sqrt{\frac{2}{P}}\int \frac{dq dq' f_n(q) g_{ml}(q,q')}{\sqrt{q'(P-q)(P-q-q')}}. \tag{4.28}$$

Details of matrix element computations are left to Appendix A. These include the definition of a factor $\beta \equiv 1/[\int_0^\infty q dq \delta_\epsilon(q)^2]$ which enters the normalization for ephemeral modes.[8] With these matrix elements, the system of equations can be written as

$$\tilde{m}^2\psi_0 - \frac{P}{\mu^2}P^-_{\rm vac}\psi_0 + \lambda\frac{\sqrt{\beta}}{2}a_{-1} + \lambda\sum_{n\geq 0} V^{(0)}_n a_n - \frac{P\sqrt{12\beta}}{\mu^2} b_{-10} + \frac{D_{00}}{2\sqrt{2}} b_{00} = \tilde{M}^2\psi_0, \tag{4.29}$$

$$n=-1:\ -\frac{2P}{\mu^2}\beta P^-_{\rm vac} a_{-1} + \frac{1}{2}\sqrt{\beta} C_0 a_0 + \frac{1}{2}\sqrt{\beta}\lambda\psi_0 + \lambda\sqrt{12} g_{-10} = \tilde{M}^2 a_{-1}, \tag{4.30}$$

$$n=0:\ \frac{1}{2}\sqrt{\beta} C_0 a_{-1} + \sum_{m\geq 0} T^{(1)}_{0m} a_m + \lambda V^{(0)}_0 \psi_0 + \lambda\sum_{ml\geq 0} V^{(1)}_{0,ml} b_{ml} = \tilde{M}^2 \sum_{m\geq 0} B^{(1)}_{0m} a_m, \tag{4.31}$$

$$n>0:\ \sum_{m\geq 0} T^{(1)}_{nm} a_m + \lambda V^{(0)}_n \psi_0 + \lambda\sum_{ml\geq 0} V^{(1)}_{n,ml} b_{ml} = \tilde{M}^2 \sum_{m\geq 0} B^{(1)}_{nm} a_m, \tag{4.32}$$

$$n=-1:\ -\frac{12P}{\mu^2}\beta P^-_{\rm vac} b_{-10} + \frac{1}{2} D_{-10} D_{00} b_{00} - \frac{\sqrt{12}P}{\mu^2}\sqrt{\beta} P^-_{\rm vac}\psi_0 + \lambda\sqrt{12} a_{-1} = \tilde{M}^2 b_{-10}, \tag{4.33}$$

$$n=0:\ \frac{1}{2} D_{-10} D_{00} b_{-10} + \sum_{ml\geq 0} T^{(2)}_{00,ml} b_{ml} + \frac{D_{00}}{2\sqrt{2}}\psi_0 + \lambda\sum_{m\geq 0} V^{(1)}_{m,00} a_m = \tilde{M}^2 \sum_{ml\geq 0} B^{(2)}_{00,ml} b_{ml}, \tag{4.34}$$

$$n>0:\ \sum_{ml\geq 0} T^{(2)}_{nj,ml} b_{ml} + \lambda\sum_{m\geq 0} V^{(1)}_{m,nj} a_m = \tilde{M}^2 \sum_{ml\geq 0} B^{(2)}_{nj,ml} b_{ml}. \tag{4.35}$$

Cancellation of the infinite vacuum energy $P^-_{\rm vac}$ in (4.29), (4.30), and (4.33) is achieved if $a_{-1} = 0$ and $b_{-10} = -\psi_0/\sqrt{12\beta}$. These values correspond to the structure of the vacuum; in other words, as a part of solving the dressed particle state, we have reconstituted the vacuum as the foundation of the physical eigenstate and thereby canceled the (infinite) vacuum energy. The projection onto f_{-1}, which is Eq. (4.30), is no longer needed or used. The factor β disappears in the eigenstate by canceling in the product $b_{-10} f_{-10} \propto b_{-10} D_{-10}$ with $D_{-10} = \sqrt{6\beta}$.

[8]The value of β depends on the model used for δ_ϵ; it is not zero because the integral is over only half the real line. Physical results are independent of β.

This leaves a finite matrix problem with finite corrections due to nonzero matrix elements of vacuum transitions. In particular, there is a finite matrix element $(D_{00}/2\sqrt{2})$ coupling the three-particle sector (b_{00}) to the one-particle sector (ψ_0) between (4.29) and (4.34). The two extra particles are ephemeral modes.

In this severe Fock-space truncation, the matrix elements are simple enough to invoke $\epsilon \to 0$ explicitly. A more general calculation would require a model for δ_ϵ and extrapolation of the limit $\epsilon \to 0$ numerically, in addition to consideration of several δ_ϵ models to confirm model independence.

V. SUMMARY

We have developed a formalism by which vacuum transitions can be included in light-front calculations and have argued that they must be included to have full equivalence with equal-time quantization and to be consistent with the perturbative equivalence of the two quantizations. The latter equivalence follows, at least on a formal level, as a choice of coordinates for evaluation of Feynman diagrams, with proper care as emphasized in [19,20]. In that context, contributions such as nonzero vacuum bubbles and tadpoles are recovered. These have been missing from nonperturbative calculations due to the neglect of vacuum transitions in light-front Hamiltonians. The inclusion of such transitions means that the light-front vacuum is not trivial and instead can be characterized as a generalized coherent state of ephemeral modes, even for a free theory.

The inclusion of vacuum transitions is not important for weakly coupled theories without symmetry breaking. If included, the only impact is to shift $\mathcal{P}^-$ by the vacuum energy $P^-_{\rm vac}$ and to require a simultaneous solution of $\mathcal{P}^-_{\rm full}|{\rm phys}\rangle = (\frac{M^2}{P} + P^-_{\rm vac})|{\rm phys}\rangle$ and $\mathcal{P}^-_{\rm full}|{\rm vac}\rangle = P^-_{\rm vac}|{\rm vac}\rangle$, where $\mathcal{P}^-_{\rm full}$ includes vacuum transitions, instead of solving the traditional light-front problem of $\mathcal{P}^-|{\rm phys}\rangle = \frac{M^2}{P}|{\rm phys}\rangle$. This has been the case for most light-front calculations and is illustrated here for quenched scalar Yukawa theory in Sec. IV, where the extra work of including the vacuum transitions simply disappears in the vacuum subtraction.

For strong coupling, when physical momentum-space wave functions become broad enough to overlap with the near-zero ephemeral modes, and certainly for cases with symmetry breaking, the vacuum transitions need to be kept. Thus, all of the earlier nonperturbative light-front calculations for ϕ_2^4 theory need to be revised, as some had anticipated or approximated, to properly include zero mode contributions that, among other improvements, should resolve the difference in the critical coupling between their results and those from equal-time quantization [1–13]. Our previous work in [5,38], as for many other light-front calculations, suffered from the neglect of vacuum-transition terms in the light-front Hamiltonian. The developments in [38] treated a different aspect of zero modes associated with the DLCQ approximation, but did not address restoration of vacuum transitions. The work in [5] is also based on a Hamiltonian that does not include vacuum transitions but does reintroduce their effects by computing the missing mass renormalization. The focus in this present work is to include vacuum transitions from the beginning of the calculation. Matrix elements of these transitions are regulated by the introduction of the near-zero ephemeral modes, and the light-front Hamiltonian is diagonalized. Vacuum bubbles and tadpoles then contribute to the energies of the eigenstates.

Our approach is based on the realization that vacuum transition matrix elements are nonzero with respect to Fock-state wave functions with the correct small-momentum behavior. These matrix elements lead to tadpole contributions as well as disconnected vacuum bubbles. The vacuum bubbles are regulated by the introduction of a finite width ϵ in momentum-conserving delta functions, so that a bubble's proportionality to $\delta(0)$ is replaced by $\delta_\epsilon(0) = L/4\pi$, where L is the light-front spatial volume. The width ϵ is taken to zero (and L to infinity) after the (infinite) vacuum energy is subtracted. The modes with momentum of order ϵ that are removed in this limit are the ephemeral modes. They represent the accumulation of contributions at zero momentum.

The use of proper basis functions is critical. A standard DLCQ approximation [29,35] cannot capture these effects, partly because the zero-mode contributions form sets of measure zero and partly because the DLCQ grid provides a poor approximation to integral operators with modes of order $\epsilon \ll P^+$, for either periodic or antiperiodic boundary conditions.

There is, of course, an increase in the computational load for any calculation that includes ephemeral modes. The vacuum eigenstate must be computed, which requires a separate matrix diagonalization, though significantly smaller than for physical states. The limit of $\epsilon \to 0$ must be taken by repeating the vacuum and physical-sector calculations several times. In addition, the size of the basis for the physical states must be increased to include ephemeral modes. If N is the maximum number of identical particles in the calculation, we estimate (see Appendix B) that the basis size must increase by a factor of $\sim N/2$ and consequently the matrix size by $N^2/4$.

We have considered several applications of these ideas for inclusion of vacuum transitions. The most basic was to show that the vacuum bubbles and tadpoles expected in ϕ_2^4 theory are in fact reproduced. We next considered the free scalar case in detail, constructing the vacuum state as a generalized coherent state of ephemeral modes and extending this to include the shifted scalar, with recovery of the correct vacuum expectation value. The shifted case can, of course, be handled in DLCQ by inclusion of the constraint equation for the spatial average of the field [36–38]. Here, however, we have an exact analytic solution with no discretization. Also, the analytic solution contains the one-loop vacuum bubble discussed by Collins [22] as a prime example of light-front vacuum structure in perturbation theory.

To illustrate how the approach functions in an interacting theory, we considered the charge-one sector of quenched scalar Yukawa theory. There we have shown how the vacuum subtraction can be implemented and how strong coupling can result in residual effects from ephemeral modes, which in the limit translate to zero-mode effects.

This work was done in two dimensions. The extension to three and four dimensions should be straightforward. The transverse momenta have the full range of $-\infty$ to ∞ and therefore can be balanced without being individually zero. The coherent state for the free scalar vacuum would be built from an operator such as

$$A^\dagger = \int \frac{dp_1^+ dp_2^+ d\vec{p}_\perp}{\sqrt{p_1^+ p_2^+}} \frac{f(p_1^+, p_2^+, \vec{p}_\perp)}{\frac{1}{p_1^+} + \frac{1}{p_2^+}} \times \delta_\epsilon(p_1^+ + p_2^+) a^\dagger(p_1^+, \vec{p}_\perp) a^\dagger(p_2^+, -\vec{p}_\perp) \tag{5.1}$$

that creates two ephemeral modes with opposite transverse momenta.

The ideal demonstration that our approach is useful would be to compute the critical coupling in ϕ^4 theory. The tadpole contributions that were absent previously [5] would now be included. Such a calculation is a natural next step.

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APPENDIX A: MATRIX ELEMENTS FOR SCALAR YUKAWA THEORY

We compute the matrix elements needed to resolve the system of Fock-space equations for scalar Yukawa theory. The matrices are defined in (4.21) through (4.28). With the definition of the model-dependent[9] factor β,

$$\frac{1}{\beta} = \int_0^\infty dq q \delta_\epsilon(q)^2, \tag{A1}$$

the basis function overlaps (4.21) are, for n, $m \geq 0$,

$$B^{(1)}_{-1-1} = \frac{(C_{-1})^2}{P} \int dq q P \delta_\epsilon(q)^2 = \frac{(C_{-1})^2}{\beta}, \tag{A2}$$

$$\begin{aligned} B^{(1)}_{-1n} &= \frac{C_{-1}C_n}{P} \int dq \frac{\sqrt{qP}\delta_\epsilon(q)}{\sqrt{q(P-q)}} \frac{(q/P)^n}{\frac{1}{q} + \frac{\tilde{m}^2}{q}} \\ &= \frac{C_{-1}C_n}{P} \int dq q (q/P)^n \delta_\epsilon(q) \to 0, \end{aligned} \tag{A3}$$

$$\begin{aligned} B^{(1)}_{nm} &= \frac{C_n C_m}{P} \int_0^P \frac{dq}{q(P-q)} \frac{(q/P)^{n+m}}{(\frac{1}{q} + \frac{\tilde{m}^2}{q})^2} \\ &= C_n C_m \int_0^1 \frac{x^{n+m+1}(1-x)dx}{(1-x+\tilde{m}^2 x)^2}, \end{aligned} \tag{A4}$$

$$\begin{aligned} B^{(2)}_{-10,-10} &= (D_{-10})^2 \int dq_1 dq_2 q_1 q_2 \frac{\delta_\epsilon(q_1+q_2)^2}{(q_1+q_2)^2} \\ &= (D_{-10})^2 \int dx x(1-x) \int Q dQ \delta_\epsilon(Q)^2 = \frac{(D_{-10})^2}{6\beta}, \end{aligned} \tag{A5}$$

$$\begin{aligned} B^{(2)}_{-10,nj} &= \frac{D_{-10}D_{nj}}{P} \int dq_1 dq_2 \frac{\delta_\epsilon(q_1+q_2)}{q_1+q_2} \frac{q_1 q_2}{q_1+q_2} \\ &\quad \times \frac{q_1^j q_2^{n-j} + q_1^{n-j} q_2^j}{P^n} \to 0, \end{aligned} \tag{A6}$$

$$\begin{aligned} B^{(2)}_{nj,ml} &= D_{nj} D_{ml} \int_0^1 dx_1 \\ &\quad \times \int_0^{1-x_1} dx_2 \frac{(x_1^j x_2^{n-j} + x_1^{n-j} x_2^j)(x_1^l x_2^{m-l} + x_1^{m-l} x_2^l)}{x_1 x_2 (1-x_1-x_2)(\frac{1}{x_1} + \frac{1}{x_2} + \frac{\tilde{m}^2}{1-x_1-x_2})^2}. \end{aligned} \tag{A7}$$

The normalization conditions, that diagonal elements of $B^{(1)}$ and $B^{(2)}$ be unity, yield $C_{-1} = \sqrt{\beta}$ and $D_{-10} = \sqrt{6\beta}$.

The matrix elements for kinetic energy are, with $P^-_{\rm vac}$ defined in (3.11),

$$\begin{aligned} T^{(1)}_{-1-1} &= (C_{-1})^2 \int dq q P \delta_\epsilon(q)^2 \left[\frac{1}{q} + \frac{\tilde{m}^2}{q}\right] \\ &= (C_{-1})^2 P \int \delta_\epsilon(q)^2 = -(C_{-1})^2 \frac{2P}{\mu^2} P^-_{\rm vac}, \end{aligned} \tag{A8}$$

$$T^{(1)}_{-1n} = C_{-1}C_n \int dq \delta_\epsilon(q)(q/P)^n = \frac{1}{2} C_{-1} C_n \delta_{n0}, \tag{A9}$$

$$T^{(1)}_{nm} = C_n C_m \int \frac{dx x^{n+m}}{1-x+\tilde{m}^2 x}, \tag{A10}$$

$$\begin{aligned} T^{(2)}_{-10,-10} &= \frac{(D_{-10})^2}{P} \int dq_1 dq_2 P^2 q_1 q_2 \frac{\delta_\epsilon(q_1+q_2)^2}{(q_1+q_2)^2} \\ &\quad \times \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{\tilde{m}^2}{P-q_1-q_2}\right) \\ &= -(D_{-10})^2 \frac{2P}{\mu^2} P^-_{\rm vac}, \end{aligned} \tag{A11}$$

[9] For any δ_ϵ model that scales properly with ϵ, β is independent of ϵ.

$$T^{(2)}_{-10,nj} = D_{-10}D_{nj}\int dq_1dq_2\frac{\delta_\epsilon(q_1+q_2)}{q_1+q_2}\frac{q_1^jq_2^{n-j}+q_1^{n-j}q_2^j}{P^n} = \frac{1}{2}D_{-10}D_{00}\delta_{n0}, \tag{A12}$$

$$T^{(2)}_{nj,ml} = 2D_{nj}D_{ml}\times\int dx_1dx_2\frac{x_1^jx_2^{n-j}(x_1^lx_2^{m-l}+x_1^{m-l}x_2^l)}{(x_1+x_2)(1-x_1-x_2)+\tilde{m}^2x_1x_2}. \tag{A13}$$

In $T^{(1)}_{-1n}$ and $T^{(2)}_{-10,nj}$ we have used $\int dq\delta_\epsilon(q)=\frac{1}{2}$, which follows from integrating only over positive q.

The potential terms have the following matrix elements:

$$U^{(0)}_{-10} = \frac{D_{-10}}{\sqrt{2}}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\delta_\epsilon(q_1+q_2)P\sqrt{q_1q_2}\frac{\delta_\epsilon(q_1+q_2)}{q_1+q_2} = -\frac{D_{-10}}{\sqrt{2}}\frac{2P}{\mu^2}P^-_{\mathrm{vac}}, \tag{A14}$$

$$U^{(0)}_{nj} = \frac{1}{\sqrt{2}}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\delta_\epsilon(q_1+q_2)\frac{D_{nj}\sqrt{P}}{\sqrt{q_1q_2(P-q_1-q_2)}}\times\frac{(q_1^jq_2^{n-j}+q^{n-j}q_2^j)/P^n}{\frac{1}{q_1}+\frac{1}{q_2}+\frac{\tilde{m}^2}{P-q_1-q_2}} = \frac{D_{00}}{2\sqrt{2}}\delta_{n0}, \tag{A15}$$

$$V^{(0)}_{-1} = C_{-1}\sqrt{P}\int\frac{dq}{\sqrt{qP(P-q)}}\sqrt{qP}\delta_\epsilon(q) = \frac{1}{2}C_{-1}, \tag{A16}$$

$$V^{(0)}_n = C_n\int\frac{dxx^n}{1-x+\tilde{m}^2x}, \tag{A17}$$

$$V^{(1)}_{-1,-10} = \sqrt{2}C_{-1}D_{-10}\int dqq\delta_\epsilon(q)^2 = \frac{\sqrt{2}}{\beta}C_{-1}D_{-10} = \sqrt{12}, \tag{A18}$$

$$V^{(1)}_{n,-10} = \sqrt{2}C_nD_{-10}\int dqdq'\left(\frac{q}{P}\right)^{n+1}\frac{\delta_\epsilon(q+q')}{q+q'} = \sqrt{2}C_nD_{-10}\int dq\left(\frac{q}{P}\right)^{n+1}\delta_\epsilon(q)\to 0, \tag{A19}$$

$$V^{(1)}_{-1,nj} = \sqrt{2}C_{-1}D_{nj}\int\frac{dqdq'\delta_\epsilon(q)}{q'(P-q')}q\frac{q^jq'^{n-j}+q^{n-j}q'^j}{P^n}\to 0, \tag{A20}$$

$$V^{(1)}_{n,ml} = \sqrt{2}C_nD_{ml}\int\frac{dx_1dx_2x_1^{n+1}}{1-x_1+\tilde{m}^2x_1}\times\frac{x_1^lx_2^{m-l}+x_1^{m-l}x_2^l}{(x_1+x_2)(1-x_1-x_2)+\tilde{m}^2x_1x_2}. \tag{A21}$$

In $V^{(1)}_{n,-10}$ we have used a representation of the Dirac delta function

$$\delta(q) = \int dq'\frac{\delta(q+q')}{q+q'}, \tag{A22}$$

which follows from

$$\int dqf(q)dq'\frac{\delta(q+q')}{q+q'} = \int QdxdQf(xQ)\frac{\delta(Q)}{Q} = \frac{1}{2}f(0). \tag{A23}$$

APPENDIX B: ESTIMATE OF BASIS SIZE

The inclusion of ephemeral modes does increase the basis size required for any numerical calculation. To estimate the increase, we consider the number of basis functions needed in each Fock sector. Let b_n be the number of basis functions used in the nth Fock sector of n particles but with no ephemeral modes, and let e represent the number of particles in an ephemeral mode, for which there would be one basis function proportional to $\delta_\epsilon(\sum_i^e p_i)$. Because each basis function for the entire Fock sector is a product of a basis function for $n-e$ physical particles and this one basis function for the e ephemeral modes, the total number of basis functions for the Fock sector is $\sum_{e=0}^n b_{n-e}$. Summed over all Fock sectors up to a truncation of $n\le N$, we have $\sum_{j=1}^N(N-j+1)b_j$ basis functions. This is to be compared to a calculation without ephemeral modes which would use $\sum_{j=1}^N b_j$ basis functions.

To estimate the ratio, we assume that b_n is approximately constant. In any practical calculation, higher Fock states need to be less important; otherwise, the chosen truncation makes no sense. Being less important they can be assigned fewer basis functions in comparison to the number of particles involved. The lowest Fock sectors need detailed representation in a collection of basis functions which is large in comparison to the number of particles. With b_n constant, the ratio of the two sums reduces to $(N+1)/2$, which for large N we take as simply $N/2$ as discussed in the Summary.

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