Modular symmetry anomaly and nonperturbative neutrino mass terms in magnetized orbifold models

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We study the modular symmetry anomaly in magnetized orbifold models. The nonperturbative effects such as D-brane instanton effects can break tree-level symmetry. We study which part of the modular symmetry is broken explicitly by Majorana mass terms with three generations of neutrinos. The modular weight of neutrino mass terms does not match with other coupling terms in the tree-level Lagrangian. In addition, the Z_N symmetry of the modular flavor symmetry is broken and a certain normal subgroup of the modular flavor symmetry remains in neutrino mass terms.

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I. INTRODUCTION

Superstring theory predicts extra six-dimensional (6D) compact space in addition to our four-dimensional (4D) spacetime. Certain compactifications such as the torus and some orbifold compactifications have a kind of geometrical symmetries, called modular symmetry. Then, the modular symmetry appears in 4D low-energy effective field theory [1]. Furthermore, the modular symmetry transforms zero-modes with each other, e.g., in heterotic orbifold models [2–4] and magnetized D-brane models [5–11]. (See also [12–15].) That is, the modular symmetry can include a flavor symmetry among three generations of quarks and leptons in particle physics.[1]

Inspired by the above aspects, the modular flavor symmetric models have recently been studied intensively in the bottom-up approach. (See for early works [20].)[2] Indeed, the modular symmetry includes S_3, A_4, S_4, and A_5 as finite modular groups [22], and these non-Abelian discrete symmetries are often used for the model building for quark and lepton flavor models in the bottom-up approach [23–28].

Symmetries at tree level are broken by quantum effects, that is an anomaly. Then, symmetry breaking terms appear by nonperturbative-instanton effects in field theory. The modular symmetry can also be anomalous. Indeed, the modular symmetry anomaly, which is relevant to the automorphy factor except in flavor symmetry, were studied in 4D low-energy effective field theory derived from heterotic string theory [29,30]. Such anomalies can be canceled by the 4D Green-Schwarz mechanism due to the axionic shift of the dilaton multiplet. That leads to important aspects. The moduli and dilaton mix in one-loop effective field theory. Furthermore, mixed anomalies between the modular symmetry and gauge symmetries should be universal for all of the gauge symmetries in heterotic models. This universality condition on mixed anomalies constrains massless spectra. Several phenomenological applications were carried out, e.g., the gauge-coupling unification, Yukawa couplings, and the hidden sector [30–32]. Moreover, the modular symmetry anomaly relevant to the automorphy factor was studied in intersecting and magnetized D-brane models [33]. Similar to heterotic models, moduli mixing in one-loop effective field theory is required to cancel the anomaly by the 4D Green-Schwarz mechanism.

In addition to the automorphy factor, anomalies of the modular flavor symmetries were studied in 4D effective field theory of magnetized D-brane models [34]. Anomalous subsymmetries in the modular flavor symmetry correspond to discrete symmetries of $U(1)$ gauge groups. Thus, those anomalies can be canceled by the same Green-Schwarz mechanism to cancel the $U(1)$ anomalies.

Anomalies of the flavor symmetries are important. If they are anomalous, the tree-level flavor symmetries are not exact at quantum level, but symmetry-breaking terms appear nonperturbatively and affect the flavor structure. In this paper, we study more about anomalies of the modular symmetries and nonperturbative symmetry-breaking terms in the 4D effective field theory of magnetized D-brane models.

[1]Calabi-Yau compactifications have many moduli, and they have larger symplectic-modular symmetries [16–19].
[2]See (for more recent list) references e.g., in Ref. [21].

Magnetized D-brane models lead to quite interesting low-energy effective field theory [35–47]. Yukawa couplings as well as higher-dimensional couplings can be calculated by the overlap integrations of wave functions on the compact space [39,48,49]. Actually, from such compactifications, realistic quark masses and mixing angles as well as charged lepton masses have been studied [49–52]. In D-brane models, D-brane instanton effects induce new terms such as right-handed Majorana neutrino mass terms [53–55], and explicit forms were also studied in magnetized models [56,57]. We study the modular symmetry anomaly and breaking symmetries due to neutrino mass terms induced by D-brane instanton effects. Such studies have implications on 4D modular flavor symmetric models.

This paper is organized as follows. In Sec. II we briefly review the modular symmetry and its anomaly. In Sec. III we review Majorana neutrino masses generated by D-brane instanton effects in magnetized orbifold models. In Sec. IV we study the modular symmetry anomaly of the Majorana mass terms, generally. In particular, in Sec. V we study modular flavor symmetry anomalies of Majorana mass terms for four types of three generations of right-handed neutrinos with modular symmetry on magnetized T^2/Z_2 orbifold, explicitly. In Sec. VI we discuss more on possible corrections due to D-brane instanton effects. We conclude this study in Sec. VII.

II. MODULAR SYMMETRY AND ANOMALY

In this section we give a brief review on the modular symmetry and its anomalies.

A. Modular symmetry

The modular group $\Gamma \equiv SL(2,\mathbb{Z})$ is the group of (2×2) matrices,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where a, b, c, and d are integers satisfying $ad - bc = 1$. The generators of Γ are given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

and they satisfy the following algebraic relations,

$$S^2 = -\mathbb{I}, \qquad S^4 = (ST)^3 = \mathbb{I}, \tag{3}$$

where $\mathbb{I}$ denotes the unit matrix.

Under the modular symmetry, the modulus τ transforms as

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}. \tag{4}$$

The modular forms are holomorphic functions of τ, which transform as

$$f_i(\tau) \to f_i(\gamma\tau) = J_k(\gamma,\tau)\rho_{ij}(\gamma)f_j(\tau), \tag{5}$$

where $J_k(\gamma,\tau) = (c\tau + d)^k$ denotes the automorphy factor with the modular weight k and $\rho_{ij}(\gamma)$ is unitary matrix. The modular forms transform as

$$f_i(\tau) \to f_i(\gamma'\tau) = J_k(\gamma',\tau)f_i(\tau), \tag{6}$$

for γ' in a certain subgroup such as congruence subgroups.

B. Matter fields and anomalies

Here, we give a brief review on the modular symmetry anomaly in string-derived low-energy effective field theory. In 4D effective field theory, chiral matter fields ϕ_i have the following Kähler metric,

$$\frac{1}{(2\mathrm{Im}\tau)^{k_i}}|\phi_i|^2, \tag{7}$$

and also transform [1]

$$\phi_i \to J_{-k_i}(\gamma,\tau)\rho_{ij}(\gamma)\phi_j, \tag{8}$$

under the modular symmetry. The matrix $\rho_{ij}(\gamma)$ represents the flavor symmetry.

In Refs. [29,30], the modular symmetry anomalies, which are relevant to the automorphy factor $J_{-k_i}(\gamma,\tau)$, were studied, i.e., for γ' satisfying $\rho_{ij}(\gamma') = \delta_{ij}$. The anomaly coefficients of mixed anomalies with the G_a gauge symmetry are written by [29],

$$A_a = -C(G_a) + \sum_i T(R_a^i)(1 + 2k_i), \tag{9}$$

where $C(G_a)$ is the quadratic Casimir of G_a and $T(R_a^i)$ denotes the Dynkin index of the representation R_a^i of the chiral matter field ϕ_i under G_a.

This anomaly can be canceled by the 4D Green-Schwarz mechanism, where other moduli T_α in the gauge kinetic function $f_a(T_\alpha)$ of the gauge group G_a transform under the modular symmetry [29,30],

$$T_\alpha \to T_\alpha + \frac{1}{8\pi^2}\delta^\alpha_{\mathrm{GS}}\ln(c\tau + d). \tag{10}$$

A one-loop correction on the gauge kinetic function, which depends on τ, may also contribute partly to the anomaly cancellation. In heterotic string theory on orbifolds, the dilaton corresponds to the Green-Schwarz field.

The tree-level Kähler potential of moduli,

$$-\ln(2\mathrm{Im}\tau) - \sum_\alpha \ln(T_\alpha + \bar{T}_\alpha), \tag{11}$$

is not invariant under modular symmetry because of the above transformation (10). The modular-invariant Kähler potential is written by

$$-\ln(2\mathrm{Im}\tau) - \sum_\alpha \ln\left(T_\alpha + \bar{T}_\alpha + \frac{1}{8\pi^2}\delta^\alpha_{\mathrm{GS}} \ln \mathrm{Im}\tau\right). \tag{12}$$

Thus, at this level, the modulus τ and other moduli T_α mix each other in the Kähler potential. This study was extended to 4D low-energy effective field theory derived from intersecting and magnetized D-brane models [33].

Furthermore, the anomalies corresponding to $\rho_{ij}(\gamma)$ were studied in Ref. [34]. The matrix $\rho_{ij}(\gamma)$ represents the flavor symmetry corresponding to a non-Abelian discrete group. In field theory with a non-Abelian discrete group, the anomaly-free condition for the mixed anomaly with the non-Abelian gauge group G_a is written by [58–61],

$$(\det \rho_{ij}(\gamma))^{\sum_i 2T_2(R^i_a)} = 1. \tag{13}$$

The subsymmetry corresponding to the element γ with $\det \rho_{ij}(\gamma) = 1$ is always anomaly free. The other part of symmetry corresponding to the element with $\det \rho_{ij}(\gamma) \neq 1$ can be anomalous, although it depends on $\sum_i 2T_2(R^i_a)$. Following this criteria, the anomalies were studied in Ref. [34] in magnetized D-brane models. It was found that the anomalous part can be embedded in discrete part of anomalous $U(1)$ gauge symmetry.

The $U(1)$ anomaly can be canceled by the 4D Green-Schwarz mechanism [62–67], which requires the shift of the moduli T_α

$$T_\alpha \to T_\alpha + A^\alpha \Lambda, \tag{14}$$

under the gauge transformation of the $U(1)$ vector multiplet V,

$$V \to V + \Lambda + \bar{\Lambda}, \tag{15}$$

where Λ denotes the gauge transformation parameter. Since the anomalous symmetries corresponding to $\det \rho_{ij}(\gamma) \neq 1$ is embedded in a discrete part of anomalous $U(1)$ gauge symmetry, anomalies of the modular flavor symmetries are also canceled by the same mechanism.

The flavor symmetry is quite important in low-energy effective field theory. In what follows, we study more about its anomalies. Nonperturbative effects such as D-brane instanton effects break the tree-level symmetry and induce breaking terms in low-energy effective field theory. One of the important terms in low-energy effective field theory is the right-handed Majorana neutrino mass term, which can be induced by D-brane instanton effects. In the following sections we study which part of modular flavor symmetry is broken by neutrino mass terms by D-brane instanton effects in magnetized orbifold models.

III. MAJORANA NEUTRINO MASS TERMS IN MAGNETIZED ORBIFOLD MODELS

In this section we review magnetized orbifold models and Majorana neutrino masses generated by D-brane instanton effects in magnetized orbifold models.

A. Neutrinos in magnetized T^2/Z_2 orbifold compactifications

First, we consider $\mathcal{M}_4 \times (T^2 \times X_4)/Z_2$ as 10D spacetime in IIB superstring theory, where $\mathcal{M}_4$ is our 4D spacetime and X_4 is a 4D compact space. The action of Z_2 for T^2 is given by the Z_2 twist of the T^2 coordinate; thus, it includes toroidal orbifold, T^2/Z_2. We also introduce D-branes wrapping p-cycles on the compact space, $(T^2 \times X_4)/Z_2$. The low-energy effective theory of the open strings stretching between D-branes is given by supersymmetric gauge theory, and magnetic fluxes can be turned on. Suppose that neutrinos N_a correspond to zero modes of open strings between two stacks of D-branes, D_{N1} and D_{N2}, with different quantized magnetic fluxes denoted as $\overline{M}_{N1}$ and $\overline{M}_{N2}$, respectively. For simplicity, we assume that they are D9-branes spreading the whole 10D spacetime. We denote the difference of their magnetic fluxes on T^2 as $M_N \equiv \overline{M}_{N1} - \overline{M}_{N2}$, which appears in the zero-mode equation of neutrinos. The generation number of the neutrinos, N_a, is determined by this M_N as well as boundary conditions on T^2/Z_2 such as the Z_2 parity $m \in \{0, 1\}$ and the Scherk-Schwarz (SS) phases $\alpha_1, \alpha_\tau \in \{0, 1/2\}$ as will be shown. Note that although the total generation number is also affected by degeneracy on X_4 and the Z_2 action on X_4, we assume that the degeneracy on X_4 is just one and the action of Z_2 for the wave function on the X_4 is trivial; hence, the generation number is given by the degeneracy on T^2/Z_2. In this case, the contribution from X_4 to the neutrino sector is just a flavor-independent overall factor in Yukawa couplings and neutrino masses. Hereafter, we concentrate on the 6D $\mathcal{M}_4 \times T^2/Z_2$, where we denote the real coordinate of $\mathcal{M}_4$ and the complex coordinate of T^2/Z_2 as x and z, respectively.

Now, we briefly review magnetized T^2/Z_2 orbifold compactifications [40,41]. the T^2/Z_2 orbifold is constructed by the identification, $z \sim z + 1 \sim z + \tau \sim -z$, where τ is the complex structure modulus of T^2 as well as T^2/Z_2. A two-dimensional (2D) spinor on T^2/Z_2 with $U(1)$ unit charge $q = 1$ under the magnetic flux M, the SS phases (α_1, α_τ), and Z_2 parity m,

$$\psi^{(\alpha_1,\alpha_\tau;m),M}(z) = \begin{pmatrix} \psi_+^{(\alpha_1,\alpha_\tau;m),M}(z) \\ \psi_-^{(\alpha_1,\alpha_\tau;m),M}(z) \end{pmatrix}, \tag{16}$$

should satisfy the following boundary conditions,

$$\begin{aligned} \psi_\pm^{(\alpha_1,\alpha_\tau;m),M}(z+1) &= e^{2\pi i\alpha_1} e^{\pi i M \frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \psi_\pm^{(\alpha_1,\alpha_\tau;m),M}(z), \\ \psi_\pm^{(\alpha_1,\alpha_\tau;m),M}(z+\tau) &= e^{2\pi i\alpha_\tau} e^{\pi i M \frac{\mathrm{Im}(\bar\tau z)}{\mathrm{Im}\tau}} \psi_\pm^{(\alpha_1,\alpha_\tau;m),M}(z), \\ \psi_\pm^{(\alpha_1,\alpha_\tau;m),M}(-z) &= (-1)^m \psi_\pm^{(\alpha_1,\alpha_\tau;m),M}(z). \end{aligned} \tag{17}$$

When M is positive (negative), only ψ_+ (ψ_-) has zero-mode solutions of the Dirac equation $i\not{D}\psi(z) = 0$, where the covariant derivative includes the background $U(1)$ gauge potential which induces the magnetic flux M. Hereafter, we consider only the case with the positive magnetic flux for simplicity and we omit the notation of the chirality, "+". The ath zero-mode solution can be expressed as

$$\begin{aligned} &\psi^{(a+\alpha_1,\alpha_\tau;m),M}(z) \\ &= \mathcal{N}^a e^{\pi i M z \frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \left(e^{2\pi i \frac{(a+\alpha_1)\alpha_\tau}{M}} \vartheta \begin{bmatrix} \frac{a+\alpha_1}{M} \\ -\alpha_\tau \end{bmatrix} (Mz, M\tau) \right. \\ &\left. \quad + (-1)^{m-2\alpha_\tau} e^{2\pi i \frac{M-(a+\alpha_1)}{M}} \vartheta \begin{bmatrix} \frac{M-(a+\alpha_1)}{M} \\ -\alpha_\tau \end{bmatrix} (Mz, M\tau) \right), \end{aligned} \tag{18}$$

$$\mathcal{N}^a = \begin{cases} \frac{1}{2}\left(\frac{M}{\mathcal{A}^2}\right)^{\frac{1}{4}} & (a+\alpha_1 = 0, |M|/2) \\ \frac{1}{\sqrt{2}}\left(\frac{M}{\mathcal{A}^2}\right)^{\frac{1}{4}} & (\text{otherwise}) \end{cases}, \tag{19}$$

where $\mathcal{A}$ denotes the area of T^2 and ϑ denotes the Jacobi theta function given by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{l\in\mathbb{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}. \tag{20}$$

The normalization factor, $\mathcal{N}^a$, is determined by

$$\begin{aligned} &\int_{T^2/Z_2} dz d\bar{z} \psi_+^{(a+\alpha_1,\alpha_\tau;m),M}(z) (\psi_+^{(a'+\alpha_1,\alpha_\tau;m),M}(z))^* \\ &= (2\mathrm{Im}\tau)^{-\frac{1}{2}} \delta_{a,a'}. \end{aligned} \tag{21}$$

Then, the number of zero modes is shown in Table I. Therefore, we can obtain such numbers of chiral fermions from the magnetized T^2/Z_2 orbifold. The ath generation of the neutrinos in the 4D space-time, $N_a(x)$, comes from the ath zero mode on the magnetized T^2/Z_2, $\psi^{(a+\alpha_1,\alpha_\tau;m)_N,M_N}(z)$. In particular, in the following calculations, we study models with three generations of neutrinos.

TABLE I. The number of zero modes.

$(\alpha_1,\alpha_\tau;m)$	$M\in 2\mathbb{Z}$	$M\in 2\mathbb{Z}+1$
(0, 0; 0)	$\frac{M}{2}+1$	$\frac{M+1}{2}$
(0, 0; 1)	$\frac{M}{2}-1$	$\frac{M-1}{2}$
(1/2, 0; 0)	$\frac{M}{2}$	$\frac{M+1}{2}$
(1/2, 0; 1)	$\frac{M}{2}$	$\frac{M-1}{2}$
(0, 1/2; 0)	$\frac{M}{2}$	$\frac{M+1}{2}$
(0, 1/2; 1)	$\frac{M}{2}$	$\frac{M-1}{2}$
(1/2, 1/2; 0)	$\frac{M}{2}$	$\frac{M-1}{2}$
(1/2.1/2; 1)	$\frac{M}{2}$	$\frac{M+1}{2}$

B. Majorana neutrino masses induced by D-brane instanton effects

Next, let us review the flavor structure of the Majorana neutrino masses generated by D-brane instanton effects in the magnetized orbifold models [57]. The D-brane instanton is an instanton-like solution of string theory. It is localized at a point in 4D spacetime, but wrapping a cycle on the compact space. When a D-brane instanton D_{inst} with a magnetic flux $\overline{M}_{\mathrm{inst}}$ exists, zero-modes β_i (γ_j) appear between D_{N1} (D_{N2}) and D_{inst}. We denote the difference of their magnetic fluxes on T^2/Z_2 as $M_\beta \equiv \overline{M}_{N1} - \overline{M}_{\mathrm{inst}}$ ($M_\gamma \equiv \overline{M}_{N2} - \overline{M}_{\mathrm{inst}}$), which appears in the zero-mode equation of β_i (γ_j). The number of the zero modes is determined by the magnetic flux M_β (M_γ) as well as the boundary conditions on T^2/Z_2; the Z_2 parity and the SS phases. Hereafter, we consider that the ith instanton zero mode in the 4D spacetime, $\beta_i(x)$ ($\gamma_i(x)$), which is localized at the point x in 4D space-time, comes from the ith zero mode on the magnetized T^2/Z_2, $\psi^{(i+\alpha_1,\alpha_\tau;m)_\beta,M_\beta}(z)$ ($\psi^{(i+\alpha_1,\alpha_\tau;m)_\gamma,M_\gamma}(z)$).

We give a comment on $U(1)$ gauge symmetries. Each D-brane has a $U(1)$ gauge symmetry. That is, D_{N1} and D_{N2} have $U(1)_1$ and $U(1)_2$ gauge symmetries, respectively. Then, neutrinos have $(1,-1)$ charges under $U(1)_1 \times U(1)_2$, while β_i and γ_i would have $(-1,0)$ and (0,1) charges, respectively. Both of $U(1)_1 \times U(1)_2$ or their linear combination can be anomalous. Such anomalies could be canceled by the 4D Green-Schwarz mechanism as will be shown.

There appears three point couplings of their zero modes and neutrinos,

$$d_a^{ij} \beta_i(x) \gamma_j(x) N_a(x), \tag{22}$$

where d_a^{ij} denotes the coupling coefficients. Due to the three point couplings, Majorana neutrino mass terms, $M_{ab} N_a(x) N_b(x)$, can be induced [53,54] as

$$\begin{aligned} M_{ab}N_a(x)N_b(x) &= e^{-S_{\rm cl}(T_\alpha,\overline{M}_{\rm inst})}\int d^2\beta d^2\gamma e^{-d_a^{ij}\beta_i(x)\gamma_j(x)N_a(x)} \\ &= e^{-S_{\rm cl}(T_\alpha,\overline{M}_{\rm inst})}(\varepsilon_{ij}\varepsilon_{k\ell}d_a^{ik}d_b^{j\ell})N_a(x)N_b(x) \\ &= e^{-S_{\rm cl}(T_\alpha,\overline{M}_{\rm inst})}m_{ab}N_a(x)N_b(x). \end{aligned} \tag{23}$$

Here, we give several comments. First, $S_{\rm cl}(T_\alpha,\overline{M}_{\rm inst})$ denotes the classical action of the D-brane instanton which depends on the moduli T_α through D-brane instanton volume and magnetic flux in the compact space. Note that there appears the axion of T_α, to which the D-brane instanton couples in the imaginary part of $S_{cl}(T_\alpha,\overline{M}_{\rm inst})$. Second, both $\beta_i(x)$ and $\gamma_j(x)$ are two numbers of Grassmann zero modes $(i,j=1,2)$. Here, the Grassmann integration for the Grassmann field $\psi(\psi=\beta_i,\gamma_j)$ satisfies

$$\int d\psi\psi = 1. \tag{24}$$

Thus, the Majorana mass terms can be generated in the only case that the numbers of both zero modes, $\beta_i(x)$ and $\gamma_j(x)$, are two. In order to obtain Majorana masses of three neutrinos from three point couplings, $d_a^{ij}\beta_i(x)\gamma_j(x)N_a(x)$, their magnetic fluxes and SS phases as well as Z_2 parities should satisfy

$$\begin{aligned} &M_N = M_\beta + M_\gamma, \\ &(\alpha_1,\alpha_\tau;m)_N \equiv (\alpha_1,\alpha_\tau;m)_\beta + (\alpha_1,\alpha_\tau;m)_\gamma \pmod 1. \end{aligned} \tag{25}$$

Otherwise, the coupling vanishes, $d_a^{ij}=0$. When the above condition is satisfied, the coupling coefficients d_a^{ij} can be calculated from

$$\begin{aligned} d_a^{ij} = &\int_{T^2/Z_2} dz d\bar{z} \psi^{(i+\alpha_1,\alpha_\tau;m)_\beta,M_\beta}(z)\psi^{(j+\alpha_1,\alpha_\tau;m)_\gamma,M_\gamma}(z) \\ &\times (\psi^{(a+\alpha_1,\alpha_\tau;m)_N,M_N}(z))^*. \end{aligned} \tag{26}$$

This comes from the following decomposition of 6D fields, β^{6D}, γ^{6D}, and N^{6D},

$$\begin{aligned} \beta^{6D} &= \sum_i \beta_i^{(M;\alpha_1,\alpha_\tau;m)_\beta}(x)\otimes\psi^{(i+\alpha_1,\alpha_\tau;m)_\beta,M_\beta}(z), \\ \gamma^{6D} &= \sum_j \gamma_j^{(M;\alpha_1,\alpha_\tau;m)_\gamma}(x)\otimes\psi^{(j+\alpha_1,\alpha_\tau;m)_\gamma,M_\gamma}(z), \\ N^{6D} &= \sum_a N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x)\otimes(\psi^{(a+\alpha_1,\alpha_\tau;m)_N,M_N}(z))^*. \end{aligned} \tag{27}$$

If each number of instanton zero-modes, $\beta_i^{(M;\alpha_1,\alpha_\tau;m)_\beta}$ and $\gamma_j^{(M;\alpha_1,\alpha_\tau;m)_\gamma}$, is two and satisfy the above condition (25), all combinations of such zero-modes can contribute to generating the Majorana neutrino masses. Thus, the total Majorana mass terms can be written as

$$\begin{aligned} &M_{ab}N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x)N_b^{(M;\alpha_1,\alpha_\tau;m)_N}(x) \\ &= \left(\sum_{M_{\rm inst}} e^{-S_{\rm cl}(T_\alpha,\overline{M}_{\rm inst}^{M_{\rm inst}})}\sum_{\boldsymbol{\alpha}_{\rm inst}} m_{ab}^{(M,\boldsymbol{\alpha})_{\rm inst}}\right) \\ &\quad\times N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x)N_b^{(M;\alpha_1,\alpha_\tau;m)_N}(x), \end{aligned} \tag{28}$$

where we denote $\boldsymbol{\alpha}_X\equiv(\alpha_1,\alpha_\tau)_X$, $\boldsymbol{\alpha}_{\rm inst}\equiv(\boldsymbol{\alpha}_\beta,\boldsymbol{\alpha}_\gamma)$, and $M_{\rm inst}=(M_\beta,M_\gamma)$. Hereafter, we denote $(\alpha_1,\alpha_\tau)=(0,0)$, $(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$ as A, B, C, and D, respectively.

In addition to these zero modes, we have neutral zero modes which correspond to the gauge multiplets on the D-brane instanton. The number of the neutral zero modes is also crucial since extra Grassmann integral can eliminate the nonperuturbative superpotential. These neutral zero modes must be absorbed by interaction terms to obtain nonzero nonperturbative effects. On the other hand, the integration of the neutral zero modes does not affect the modular symmetry anomaly since the wave function of gauge multiplets is constant on the compact space and does not transform under the modular group. Thus, we investigate the flavor structure of the nonperturbative Majorana mass term, assuming the integration of the neutral zero modes is properly absorbed by interaction terms in the present paper.

IV. MODULAR SYMMETRY ANOMALY OF MAJORANA NEUTRINO MASS TERMS

In this section let us study modular symmetry anomaly of the Majorana-neutrino mass terms in Eq. (23). First, we briefly review the modular transformation of the wave functions and the coupling coefficients. Under $\gamma\in\Gamma$ transformation, the modulus as well as the coordinate, (z,τ), transforms as

$$\gamma:(z,\tau)\to(z',\tau')=\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right). \tag{29}$$

We call this transformation the modular transformation. In particular, under the S and T transformations defined in Eq. (2), (z,τ) transform as

$$\begin{aligned} S&:(z,\tau)\to(z',\tau')=\left(-\frac{z}{\tau},-\frac{1}{\tau}\right), \\ T&:(z,\tau)\to(z',\tau')=(z,\tau+1), \end{aligned} \tag{30}$$

respectively. Thus, this modular transformation also satisfies Eq. (3).

Now, let us investigate the modular transformation for the wave functions on magnetized T^2/Z_2 in Eq. (18) [8,10]. For this purpose, we introduce the double covering group of Γ, $\tilde{\Gamma}$. (See e.g., Ref. [68] and references therein.) The generators of $\tilde{\Gamma}$, $\tilde{S}$ and $\tilde{T}$, satisfy

$$\tilde{S}^2 = \tilde{Z},\quad \tilde{S}^4 = (\tilde{S}\tilde{T})^3 = \tilde{Z}^2,\quad \tilde{S}^8 = (\tilde{S}\tilde{T})^6 = \tilde{Z}^4 = \mathbb{I},\quad \tilde{Z}\tilde{T} = \tilde{T}\tilde{Z}, \tag{31}$$

where $\tilde{Z}$ expands the center of Γ. $\tilde{S}$ and $\tilde{T}$ transformations of (z,τ) are the same as S and T transformation in Eq. (30). Hence, under $\tilde{\gamma} \in \tilde{\Gamma}$ transformation, the wave functions in Eq. (18) transform as

$$\tilde{\gamma}: \psi^{(a+\alpha_1,\alpha_\tau;m),M}(z,\tau) \to \psi^{(a+\alpha_1,\alpha_\tau;m),M}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \tilde{J}_{1/2}(\tilde{\gamma},\tau)\sum_{a'}\sum_{\boldsymbol{\alpha}'}\rho(\tilde{\gamma})_{aa'}^{\boldsymbol{\alpha}\boldsymbol{\alpha}'}\psi^{(a'+\alpha'_1,\alpha'_\tau;m),M}(z,\tau), \tag{32}$$

where $\tilde{J}_{1/2}$ and ρ denote the automorphy factor with modular weight $1/2$ and the unitary matrix, respectively. For $\tilde{S}$ and $\tilde{T}$ transformations, they can be expressed as

$$\tilde{J}_{1/2}(\tilde{S},\tau) = (-\tau)^{1/2}, \qquad \rho(\tilde{S})_{aa'}^{\boldsymbol{\alpha}\boldsymbol{\alpha}'} = \begin{cases} \mathcal{N}^a\mathcal{N}^{a'}\frac{4e^{\pi i/4}}{\sqrt{M}}\cos\left(\frac{2\pi(a+\alpha_1)(a'+\alpha'_1)}{M}\right)\delta_{(\alpha_\tau,\alpha_1),(\alpha'_1,\alpha'_\tau)} & (m=0) \\ \mathcal{N}^a\mathcal{N}^{a'}\frac{4ie^{\pi i/4}}{\sqrt{M}}\sin\left(\frac{2\pi(a+\alpha_1)(a'+\alpha'_1)}{M}\right)\delta_{(\alpha_\tau,\alpha_1),(\alpha'_1,\alpha'_\tau)} & (m=1) \end{cases}, \tag{33}$$

$$\tilde{J}_{1/2}(\tilde{T},\tau) = 1, \qquad \rho(\tilde{T})_{aa'}^{\boldsymbol{\alpha}\boldsymbol{\alpha}'} = \begin{cases} e^{\frac{\pi i(a+\alpha_1)^2}{M}}\delta_{a,a'}\delta_{(\alpha_1,\alpha_\tau-\alpha_1),(\alpha'_1,\alpha'_\tau)} & (M\in 2\mathbb{Z}) \\ e^{\frac{\pi i(a+\alpha_1)^2}{M}}\delta_{a,a'}\delta_{(\alpha_1,\alpha_\tau-\alpha_1+\frac{1}{2}),(\alpha'_1,\alpha'_\tau)} & (M\in 2\mathbb{Z}+1) \end{cases}, \tag{34}$$

respectively.[3] Thus, under the modular transformation, in general SS phases transform and then the fields such as $\beta^{(M;\alpha_1,\alpha_\tau;m)_\beta}$ ($\gamma^{(M;\alpha_1,\alpha_\tau;m)_\gamma}$) convert into other fields such as $\beta^{(M;\alpha'_1,\alpha'_\tau;m)_\beta}$ ($\gamma^{(M;\alpha'_1,\alpha'_\tau;m)_\gamma}$). However, wave functions with $M\in 2\mathbb{Z}$ and $(\alpha_1,\alpha_\tau)=(0,0)$ and ones with $M\in 2\mathbb{Z}+1$ and $(\alpha_1,\alpha_\tau)=(1/2,1/2)$ are closed under the modular transformation. Then, we consider the models, that the three generations of neutrinos come from such wave functions with the modular symmetry. Namely, from Table I, there are four cases with three generations of neutrinos: $(M;\alpha_1,\alpha_\tau;m)_N = (4;0,0;0)$, $(8;0,0;1)$, $(5;1/2,1/2;1)$, and $(7;1/2,1/2;0)$. In these cases, the unitary matrices become unitary representations of $\tilde{\Delta}(96)\simeq\Delta(48)\rtimes Z_8$, $\tilde{\Delta}(384)\simeq\Delta(192)\rtimes Z_8$, $A_5\times Z_8$, and $PSL(2,Z_7)\times Z_8$, respectively [10].

From Eqs. (27) and (32), the 4D fields, $\beta_i(x)$, $\gamma_j(x)$, and $N_a(x)$, transform under the modular transformation as

$$\begin{aligned} \tilde{\gamma}: \beta_i^{(M;\alpha_1,\alpha_\tau;m)_\beta}(x) &\to \tilde{J}_{-1/2}(\tilde{\gamma},\tau)\sum_{i'}\sum_{\boldsymbol{\alpha}'_\beta}\rho_\beta^{-1}(\tilde{\gamma})_{ii'}^{\boldsymbol{\alpha}_\beta\boldsymbol{\alpha}'_\beta}\beta_{i'}^{(M;\alpha'_1,\alpha'_\tau;m)_\beta}(x), \\ \tilde{\gamma}: \gamma_j^{(M;\alpha_1,\alpha_\tau;m)_\gamma}(x) &\to \tilde{J}_{-1/2}(\tilde{\gamma},\tau)\sum_{j'}\sum_{\boldsymbol{\alpha}'_\gamma}\rho_\gamma^{-1}(\tilde{\gamma})_{jj'}^{\boldsymbol{\alpha}_\gamma\boldsymbol{\alpha}'_\gamma}\gamma_{j'}^{(M;\alpha'_1,\alpha'_\tau;m)_\gamma}(x), \\ \tilde{\gamma}: N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x) &\to (\tilde{J}_{-1/2}(\tilde{\gamma},\tau))^*\sum_{a'}\rho_N^T(\tilde{\gamma})_{aa'}^{\boldsymbol{\alpha}_N\boldsymbol{\alpha}_N}N_{a'}^{(M;\alpha_1,\alpha_\tau;m)_N}(x), \end{aligned} \tag{35}$$

respectively. Note that Eq. (34) satisfies $\rho(\tilde{\gamma})^T = \rho(\tilde{\gamma})$. Therefore, in this case, the 4D three generations of neutrinos, $N_a(x)$, transform nontrivially as triplets under the above discrete modular flavor transformation with the modular weight $-1/2$.[4] In the following section we discuss their modular flavor symmetry anomalies individually.

On the other hand, from the modular transformation for wave functions in Eq. (32), we can find the modular transformation for the three point coupling coefficients, $d_a^{ij,(M,\boldsymbol{\alpha})_{\rm inst}}(\tau)$, in Eq. (26) [47],

$$\begin{aligned} \tilde{\gamma}: d_a^{ij,(M,\boldsymbol{\alpha})_{\rm inst}}(\tau) \to d_a^{ij,(M,\boldsymbol{\alpha})_{\rm inst}}\left(\frac{a\tau+b}{c\tau+d}\right) &= |\tilde{J}_{1/2}(\tilde{\gamma},\tau)|^2\tilde{J}_{1/2}(\tilde{\gamma},\tau)\sum_{(i'j'a')}\sum_{\boldsymbol{\alpha}'_{\rm inst}}\rho_d(\tilde{\gamma})_{(ija)(i'j'a')}^{\boldsymbol{\alpha}_{\rm inst}\boldsymbol{\alpha}'_{\rm inst}}d_{a'}^{i'j',(M,\boldsymbol{\alpha}')_{\rm inst}}(\tau), \\ \rho_d(\tilde{\gamma})_{(ija)(i'j'a')}^{\boldsymbol{\alpha}_{\rm inst}\boldsymbol{\alpha}'_{\rm inst}} &= \rho_\beta(\tilde{\gamma})_{ii'}^{\boldsymbol{\alpha}_\beta\boldsymbol{\alpha}'_\beta}\rho_\gamma(\tilde{\gamma})_{jj'}^{\boldsymbol{\alpha}_\gamma\boldsymbol{\alpha}'_\gamma}(\rho_N(\tilde{\gamma})_{aa'}^{\boldsymbol{\alpha}_N\boldsymbol{\alpha}_N})^*, \end{aligned} \tag{36}$$

[3]Since the definition of wave functions in Eq. (18) is modified from ones in Ref. [10], the matrix forms are also modified from ones in Ref. [10].

[4]This is consistent with the Kähler metric in Eq. (7), obtained from Eq. (21) in Refs. [8,21].

where the term, $|\tilde{J}_{1/2}(\tilde{\gamma},\tau)|^2 = |c\tau+d|$, comes from the factor, $(2\mathrm{Im}\tau)^{-\frac{1}{2}}$, obtained by integration in Eq. (26). Thus, by combining Eqs. (35) and (36), the three point coupling terms transform under the modular transformation as

$$\tilde{\gamma}: d_a^{ij,(M,\boldsymbol{\alpha})_{\mathrm{inst}}}(\tau)\beta_i^{(M;\alpha_1,\alpha_\tau;m)_\beta}(x)\gamma_j^{(M;\alpha_1,\alpha_\tau;m)_\gamma}(x)N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x) \to d_{a'}^{i'j',(M,\boldsymbol{\alpha}')_{\mathrm{inst}}}(\tau)\beta'^{(M;\alpha'_1,\alpha'_\tau;m)_\beta}_{i'}(x)\gamma_{j'}^{(M;\alpha'_1,\alpha'_\tau;m)_\gamma}(x)N_{a'}^{(M;\alpha_1,\alpha_\tau;m)_N}(x). \tag{37}$$

In particular, if instanton zero modes, β and γ, also come from wave functions consistent with the modular symmetry, the above three point couplings are modular invariant. Even if the above term transforms under the modular transformation, the total three point couplings which can generate Majorana masses, $\sum_{(M,\boldsymbol{\alpha})_{\mathrm{inst}}} d_a^{ij,(M,\boldsymbol{\alpha})_{\mathrm{inst}}}(\tau)\beta_i^{(M;\alpha_1,\alpha_\tau;m)_\beta}(x)$ $\gamma_j^{(M;\alpha_1,\alpha_\tau;m)_\gamma}(x)N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x)$, are modular invariant.

Now, let us see the modular transformation for the Majorana mass terms in Eq. (23). Since we obtain the modular transformation for $d_a^{ij,(M,\boldsymbol{\alpha})_{\mathrm{inst}}}$, we can find that the mass matrix elements, $m_{ab}^{(M,\boldsymbol{\alpha})_{\mathrm{inst}}}(\tau)$, transform under the modular transformation as

$$\tilde{\gamma}: m_{ab}^{(M,\boldsymbol{\alpha})_{\mathrm{inst}}}(\tau) \to m_{ab}^{(M,\boldsymbol{\alpha})_{\mathrm{inst}}}\left(\frac{a\tau+b}{c\tau+d}\right) = |\tilde{J}_1(\tilde{\gamma},\tau)|^2\tilde{J}_1(\tilde{\gamma},\tau)\sum_{(a'b')}\sum_{\boldsymbol{\alpha}'_{\mathrm{inst}}}\rho_m(\tilde{\gamma})_{(ab)(a'b')}^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}} m_{a'b'}^{(M,\boldsymbol{\alpha}')_{\mathrm{inst}}}(\tau),$$
$$\rho_m(\tilde{\gamma})_{(ab)(a'b')}^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}} = \det[\rho_{\mathrm{inst}}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}}](\rho_N(\tilde{\gamma})_{aa'}^{\boldsymbol{\alpha}_N\boldsymbol{\alpha}_N})^*(\rho_N(\tilde{\gamma})_{bb'}^{\boldsymbol{\alpha}_N\boldsymbol{\alpha}_N})^*, \tag{38}$$

where we denote $\det[\rho_{\mathrm{inst}}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}}] \equiv \det[\rho_\beta(\tilde{\gamma})^{\boldsymbol{\alpha}_\beta\boldsymbol{\alpha}'_\beta}]\det[\rho_\gamma(\tilde{\gamma})^{\boldsymbol{\alpha}_\gamma,\boldsymbol{\alpha}'_\gamma}]$. Thus, by combining Eqs. (35) and (38), the mass terms transform under the modular transformation as

$$\tilde{\gamma}: m_{ab}^{(M,\boldsymbol{\alpha})_{\mathrm{inst}}}(\tau)N_a^{(M;\alpha_1,\alpha_\tau;m)_N}(x)N_b^{(M;\alpha_1,\alpha_\tau;m)_N}(x) \to \tilde{J}_2(\tilde{\gamma},\tau)\det[\rho_{\mathrm{inst}}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}}]m_{a'b'}^{(M,\boldsymbol{\alpha}')_{\mathrm{inst}}}(\tau)^{\boldsymbol{\alpha}'_{\mathrm{inst}}}N_{a'}^{(M;\alpha_1,\alpha_\tau;m)_N}(x)N_{b'}^{(M;\alpha_1,\alpha_\tau;m)_N}(x). \tag{39}$$

This means that even if we consider instanton zero modes consistent with the modular symmetry, the Majorana mass terms are generally not invariant under the modular transformation. In other words, there appears modular symmetry anomaly, in general, in the Majorana mass terms generated by D-brane instanton effects. Indeed, the anomalous factor, $\tilde{J}_2(\tilde{\gamma},\tau)\det[\rho_{\mathrm{inst}}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}}]$, comes from transformation for measures of instanton zero modes in the path integral in Eq. (23),

$$\tilde{\gamma}: d^2\beta^{(M;\alpha_1,\alpha_\tau;m)_\beta}d^2\gamma^{(M;\alpha_1,\alpha_\tau;m)_\gamma} \to \tilde{J}_2(\tilde{\gamma},\tau)\det[\rho_{\mathrm{inst}}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\mathrm{inst}}\boldsymbol{\alpha}'_{\mathrm{inst}}}]d^2\beta^{(M;\alpha'_1,\alpha'_\tau;m)_\beta}d^2\gamma^{(M;\alpha'_1,\alpha'_\tau;m)_\gamma}. \tag{40}$$

This transformation can be obtained by Eqs. (24) and (35). Namely, the modular symmetry anomalies of Majorana neutrino mass terms are caused by integration of the instanton zero-modes appeared by D-brane instantons. In the following section we discuss the detail structure of the modular symmetry anomaly of Majorana mass terms for individual types of models with three generations of neutrinos.

We comment on the automorphy factor $\tilde{J}_2(\tilde{\gamma},\tau)$ in the anomaly. As reviewed in Sec. II B, such an anomaly of the automorphy factor can be canceled by the Green-Schwarz mechanism due to other moduli T_α. The neutrino mass terms in Eq. (23) include the factor $e^{-S_{\mathrm{cl}}(T_\alpha,\overline{M}_{\mathrm{inst}})}$, and this factor does not depend on the complex structure modulus τ but depends on other moduli T_α, which correspond to Kähler moduli and the dilaton in type IIB string theory. In the Green-Schwarz mechanism, these moduli T_α transform as Eq. (10) to cancel the anomaly of the automorphy factor. Then, it is expected that the modular transformation for $e^{-S_{\mathrm{cl}}(T_\alpha,\bar{M}_{\mathrm{inst}})}$ may cancel the modular weight anomaly, i.e., the factor $\tilde{J}_2(\tilde{\gamma},\tau)$ in Eq. (39). In fact, Eq. (10) implies the shift of the Chern-Simons term in S_{cl}, and a part of automorphy factor can be canceled. At any rate, our purpose is not to show that the 4D Green-Schwarz mechanism works, but to show which part of the modular symmetry is anomalous and is broken by nonperturbative neutrino mass terms. The modular weight of $m_{a'b'}(\tau)N_{a'}N_{b'}$ without $e^{-S_{\mathrm{cl}}}$ does not match with other terms in the tree-level Lagrangian. This point is important for 4D modular flavor models.

Furthermore, the factor $\det \rho_{\rm inst}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\rm inst}\boldsymbol{\alpha}'_{\rm inst}}$ in Eq. (39) can break the flavor symmetry. As discussed in Ref. [60], the anomalous part in any non-Abelian discrete group corresponds to Z_N symmetry. In Ref. [34] it was found that such anomalous Z_N symmetry can be embedded in $U(1)$ gauge symmetry. Note that the neutrino mass terms have $(2,-2)$ charge under $U(1)_1 \times U(1)_2$. That means the D-brane instanton effect breaks $U(1)_1 \times U(1)_2$. As a result, the symmetry $U(1)' = U(1)_1 - U(1)_2$ is broken, while the neutrino mass terms are invariant under $U'' = U(1)_1 + U(1)_2$ and this $U(1)''$ symmetry remains. The Green-Schwarz mechanism requires the moduli T_α in $e^{-S_{\rm cl}(T_\alpha, \overline{M}_{\rm inst})}$ to shift as Eq. (14) in order to cancel the $U(1)'$ anomaly. When the anomalous part of the modular flavor symmetry $\rho_{\rm inst}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\rm inst}\boldsymbol{\alpha}'_{\rm inst}}$ corresponds to a discrete subgroup Z_N of $U(1)'$ as found in Ref. [34], the modular flavor anomaly can be canceled by the same Green-Schwarz mechanism as $U(1)'$. As said above, our purpose is not to show that the 4D Green-Schwarz mechanism works, but to show which part of the modular symmetry is anomalous and is broken by nonperturbative neutrino mass terms. Obviously, after the moduli T_α are stabilized, the factor $e^{-S_{\rm cl}(T_\alpha, \overline{M}_{\rm inst})}$ is just a constant. Then, $U(1)'$ is broken and the subsymmetry of the modular flavor symmetry corresponding to $\det \rho_{\rm inst}(\tilde{\gamma})^{\boldsymbol{\alpha}_{\rm inst}\boldsymbol{\alpha}'_{\rm inst}}$ in Eq. (39) is broken. Which part is broken in the modular flavor symmetry is important. We will study it explicitly by use of concrete models in the following section.

V. MODULAR FLAVOR SYMMETRY ANOMALIES OF MAJORANA MASS TERMS FOR THREE-GENERATION NEUTRINOS

In this section we study modular flavor symmetry anomalies of Majorana mass terms for four types of models with three generations of neutrinos; the neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (4;0,0;0)$, $(8;0,0;1)$, $(5;1/2,1/2;1)$, and $(7;1/2,1/2;0)$. Here, in these models we study nonperturbative neutrino mass terms induced by instanton zero modes consistent with the modular symmetries, i.e., the zero-mode wave functions, which transform ones with the same boundary conditions under the modular transformation. Hereafter, we use the following notations,

$$\begin{aligned} c^{(M_\beta,M_\gamma)} &\equiv (2{\rm Im}\tau)^{-1}\mathcal{A}^{-1}\left(\frac{M_\beta M_\gamma}{M_N}\right)^{1/2}, \\ \eta_N^{(M)} &\equiv \vartheta\begin{bmatrix} \frac{N}{M} \\ 0 \end{bmatrix}(0, M\tau), \\ \zeta^{(M)}_{N,L;\pm} &\equiv \eta_N^{(M)} \pm_1 \eta^{(M)}_{N+L}, \\ \lambda^{(M)}_{(N,L;\pm_1),K;\pm_2} &\equiv \zeta^{(M)}_{N,L;\pm_1} \pm_2 \zeta^{(M)}_{N+K,L;\pm_1} \end{aligned} \tag{41}$$

and we use the following relation,

$$\sum_{k=0}^{g-1} \eta^{(M)}_{gn+(M/g)k} = \eta_n^{(M/g^2)}, \tag{42}$$

where $M/g^2 \in \mathbb{Z}$ for $\exists\, g \in \mathbb{Z}$.

A. Three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (4;0,0;0)$

Here, we study three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (4;0,0;0)$. In this case, the modular transformation matrices for the neutrinos, ρ_N are given as

$$\begin{aligned} \rho_N(\tilde{S}) &= \frac{e^{\pi i/4}}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \\ \rho_N(\tilde{T}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pi i/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \tag{43}$$

which satisfy

$$\begin{aligned} &\rho_N(\tilde{S})^2 = i\mathbb{I}, \\ &\rho_N(\tilde{S})^4 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^3 = -\mathbb{I}, \\ &\rho_N(\tilde{S})^8 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^6 = \rho_N(\tilde{T})^8 = \mathbb{I}, \end{aligned} \tag{44}$$

and also

$$[\rho_N(\tilde{S})^{-1}\rho_N(\tilde{T})^{-1}\rho_N(\tilde{S})\rho_N(\tilde{T})]^3 = \mathbb{I}. \tag{45}$$

They are the unitary representation of $\tilde{\Delta}(96) \simeq \Delta(48) \rtimes Z_8 \simeq (Z_4 \times Z'_4) \rtimes Z_3 \rtimes Z_8$ [10], where the generators of Z_4, Z'_4, Z_3, and Z_8 are given by

$$\begin{aligned} \rho_N(a) &= \rho_N(\tilde{S})\rho_N(\tilde{T})^2\rho_N(\tilde{S})^5\rho_N(\tilde{T})^4, \\ \rho_N(a') &= \rho_N(\tilde{S})\rho_N(\tilde{T})^2\rho_N(\tilde{S})^{-1}\rho_N(\tilde{T})^{-2}, \\ \rho_N(b) &= \rho_N(\tilde{T})^5\rho_N(\tilde{S})^5\rho_N(\tilde{T})^4, \\ \rho_N(c) &= \rho_N(\tilde{S})\rho_N(\tilde{T})^2\rho_N(\tilde{S})\rho_N(\tilde{T})^5, \end{aligned} \tag{46}$$

respectively.

Their Majorana masses can be generated by only one pair of the instanton zero modes [57], $(\beta,\gamma)^T$, with

$$\begin{pmatrix} (M;\alpha_1,\alpha_\tau;m)_\beta \\ (M;\alpha_1,\alpha_\tau;m)_\gamma \end{pmatrix} = \begin{pmatrix} (2;0,0;0) \\ (2;0,0;0) \end{pmatrix}.$$

Here, we denote $(M,\boldsymbol{\alpha})_{\rm inst} = ((2,2),(A,A))$. Then, the mass matrix can be written as

$$\begin{aligned} M_{ab} &= e^{-S_{\rm cl}(T_\alpha,\bar{M}_{\rm inst}^{(2,2)})} m_{ab}^{((2,2),(A,A))}, \\ &= e^{-S_{\rm cl}(T_\alpha,\bar{M}_{\rm inst}^{(2,2)})} c^{(2,2)} \begin{pmatrix} X_3 & 0 & X_1 \\ 0 & -\sqrt{2}X_2 & 0 \\ X_1 & 0 & X_3 \end{pmatrix}, \end{aligned} \tag{47}$$

where $X_I (I = 1, 2, 3)$ are given by

$$X_1 = (\eta_0^{(4)})^2 + (\eta_2^{(4)})^2, \qquad X_2 = 2\sqrt{2}(\eta_1^{(4)})^2, \\ X_3 = 2\eta_0^{(4)}\eta_2^{(4)}. \tag{48}$$

The modular transformation matrices for the mass matrix elements $(X_1, X_2, X_3)^T \equiv \mathbf{X}^T$, ρ_m, are given as

$$\rho_m(\tilde{S}) = \frac{i}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \rho_m(\tilde{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{49}$$

which satisfy

$$\rho_m(\tilde{S})^2 = -\mathbb{I}, \\ \rho_m(\tilde{S})^4 = [\rho_m(\tilde{S})\rho_m(\tilde{T})]^3 = \rho_m(\tilde{T})^4 = \mathbb{I}. \tag{50}$$

They are the unitary representation of $S_4' \simeq \Delta'(24) \simeq \Delta(12) \rtimes Z_4 \simeq (Z_2 \times Z_2') \rtimes Z_3 \rtimes Z_4$ [57], where the generators of Z_2, Z_2', Z_3, and Z_4 are the same as Eq. (46), respectively, by considering Eq. (50) instead of Eq. (44).

On the other hand, since we obtain

$$\det[\rho_{\rm inst}(\tilde{S})^{(A,A)(A,A)}] = \det[\rho_{\rm inst}(\tilde{T})^{(A,A)(A,A)}] = -1, \tag{51}$$

from Eq. (34), we can find that

$$\begin{aligned} \det[\rho_{\rm inst}(a)^{(A,A)(A,A)}] &= \det[\rho_{\rm inst}(a')^{(A,A)(A,A)}] \\ &= \det[\rho_{\rm inst}(b)^{(A,A)(A,A)}] = 1, \\ \det[\rho_{\rm inst}(c)^{(A,A)(A,A)}] &= -1 (\det[\rho_{\rm inst}(c^2)^{(A,A)(A,A)}] = 1). \end{aligned} \tag{52}$$

Thus, the Majorana mass term $M_{ab}(\tau)N_a^{(4;0,0;0)}(x)N_b^{(4;0,0;0)}(x)$ is invariant under a, a', b, and c^2 transformation, while it transforms as

$$M_{ab}(\tau)N_a^{(4;0,0;0)}(x)N_b^{(4;0,0;0)}(x) \\ \to -M_{ab}(\tau)N_a^{(4;0,0;0)}(x)N_b^{(4;0,0;0)}(x),$$

under c transformation. As a result, among the neutrino flavor symmetry $\tilde{\Delta}(96) \simeq \Delta(48) \rtimes Z_8$, there remains $\Delta(48) \times Z_4$ flavor symmetry in the neutrino mass terms, while Z_2 part of Z_8 symmetry is broken.[5] Here, the direct product comes from the reason that $\rho_N(c)^2 = \rho_N(\tilde{S})^6 = -i\mathbb{I}^6$ commutes all of the generators, a, a', and b.

B. Three generations of neutrinos with $(M; \alpha_1, \alpha_\tau; m)_N = (8; 0,0; 1)$

Here, we study three generations of neutrinos with $(M; \alpha_1, \alpha_\tau; m)_N = (8; 0, 0; 1)$. In this case, the modular transformation matrices for the neutrinos, ρ_N are given as

$$\rho_N(\tilde{S}) = \frac{ie^{\pi i/4}}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \\ \rho_N(\tilde{T}) = e^{\pi i/8}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{3\pi i/8} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{53}$$

which satisfy

$$\rho_N(\tilde{S})^2 = -i\mathbb{I}, \\ \rho_N(\tilde{S})^4 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^3 = -\mathbb{I}, \\ \rho_N(\tilde{S})^8 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^6 = \rho_N(\tilde{T})^{16} = \mathbb{I}, \tag{54}$$

and also

$$[\rho_N(\tilde{S})^{-1}\rho_N(\tilde{T})^{-1}\rho_N(\tilde{S})\rho_N(\tilde{T})]^3 = \mathbb{I}. \tag{55}$$

They are the unitary representation of $\tilde{\Delta}(384) \simeq \Delta(192) \rtimes Z_8 \simeq (Z_8 \times Z_8') \rtimes Z_3 \rtimes Z_8$ [10], where the generators of Z_8, Z_8', Z_3, and Z_8 are given by

$$\rho_N(a) = \rho_N(\tilde{S})\rho_N(\tilde{T})^2\rho_N(\tilde{S})^5\rho_N(\tilde{T})^4, \\ \rho_N(a') = \rho_N(\tilde{S})\rho_N(\tilde{T})^2\rho_N(\tilde{S})^{-1}\rho_N(\tilde{T})^{-2}, \\ \rho_N(b) = \rho_N(\tilde{T})^7\rho_N(\tilde{S})^{11}\rho_N(\tilde{T})^8, \\ \rho_N(c) = \rho_N(\tilde{S})\rho_N(\tilde{T})^{-10}\rho_N(\tilde{S})\rho_N(\tilde{T})^{-5}, \tag{56}$$

respectively.

Note that wave functions with $M \in 2\mathbb{Z}$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ and ones with $M \in 2\mathbb{Z}+1$ and $(\alpha_1, \alpha_\tau) = (1/2, 1/2)$ are consistent with the modular symmetry, because they transform to ones with the same boundary conditions. The Majorana masses can be generated by two pairs of the instanton zero modes [57], $(\beta, \gamma)^T$, with

[5] Actually, according to the analysis in Ref. [60], we can find that $\Delta(48)$ transformation is automatically anomaly free.

[6] See Ref. [10].

$$\begin{pmatrix} (M;\alpha_1,\alpha_\tau;m)_\beta \\ (M;\alpha_1,\alpha_\tau;m)_\gamma \end{pmatrix} = \begin{pmatrix} (2;0,0;0) \\ (6;0,0;1) \end{pmatrix}, \begin{pmatrix} (3;1/2,1/2;1) \\ (5;1/2,1/2;0) \end{pmatrix},$$

which are consistent with the modular transformation. However, the latter case includes some complexity. In this section we study only the former, but we will study the latter in the next section. Here, we denote $(M,\boldsymbol{\alpha})_{\text{inst}} = ((2,6),(A,A))$. The mass matrix can be written as

$$\begin{aligned} M_{ab}^{(2,6)} &= e^{-S_{\text{cl}}(T_\alpha,\overline{M}_{\text{inst}}^{(2,6)})} m_{ab}^{((2,6),(A,A))} \\ &= e^{-S_{\text{cl}}(T_\alpha,\overline{M}_{\text{inst}}^{(2,6)})} c^{(2,6)} \begin{pmatrix} X_3 & 0 & X_1 \\ 0 & -\sqrt{2}X_2 & 0 \\ X_1 & 0 & X_3 \end{pmatrix}, \end{aligned} \tag{57}$$

where X_I are given by

$$\begin{aligned} &X_1 = (\zeta_{1,6;-}^{(24)})^2 + (\zeta_{5,6;-}^{(24)})^2, \qquad X_2 = \sqrt{2}(\zeta_{2,12;-}^{(24)})^2, \\ &X_3 = 2\zeta_{1,6;-}^{(24)}\zeta_{5,6;-}^{(24)}. \end{aligned} \tag{58}$$

As discussed in Ref. [57], the modular transformation for the matrix the elements $(X_1,X_2,X_3)^T \equiv \mathbf{X}^T$, $\rho_{m^{(2,6)}}$, is given as

$$\begin{aligned} \rho_{m^{(2,6)}}(\tilde{S}) &= \frac{i}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \\ \rho_{m^{(2,6)}}(\tilde{T}) &= e^{\pi i/12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pi i/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \tag{59}$$

and they satisfy

$$\begin{aligned} &\rho_{m^{(2,6)}}(\tilde{S})^2 = -\mathbb{I}, \qquad \rho_{m^{(2,6)}}(\tilde{T})^8 = e^{2\pi i/3}\mathbb{I}, \\ &\rho_{m^{(2,6)}}(\tilde{S})^4 = [\rho_{m^{(2,6)}}(\tilde{S})\rho_{m^{(2,6)}}(\tilde{T})]^3 = \rho_{m^{(2,6)}}(\tilde{T})^{24} = \mathbb{I}, \end{aligned} \tag{60}$$

and also satisfy

$$[\rho_{m^{(2,6)}}(\tilde{S})^{-1}\rho_{m^{(2,6)}}(\tilde{T})^{-1}\rho_{m^{(2,6)}}(\tilde{S})\rho_{m^{(2,6)}}(\tilde{T})]^3 = \mathbb{I}. \tag{61}$$

They are the unitary representation of $\Delta'(96)\times Z_3 \simeq (\Delta(48)\rtimes Z_4)\times Z_3 \simeq ((Z_4\times Z_4')\rtimes Z_3\rtimes Z_4)\times Z_3$, where the generators of Z_4, Z_4', Z_3, and Z_4 are the same as Eq. (56), respectively, by considering Eq. (60) instead of Eq. (54), while the generator of the last Z_3 is given by

$$\rho_{m^{(2,6)}}(d) = \rho_{m^{(2,6)}}(\tilde{T})^{16}. \tag{62}$$

Note that this transformation for the neutrinos is trivial: $\rho_N(d) = \mathbb{I}$.

On the other hand, since we obtain

$$\begin{aligned} &\det[\rho_{inst^{(2,6)}}(\tilde{S})^{(A,A)(A,A)}] = 1, \\ &\det[\rho_{inst^{(2,6)}}(\tilde{T})^{(A,A)(A,A)}] = e^{4\pi i/3}, \end{aligned} \tag{63}$$

from Eq. (34), we can find that

$$\begin{aligned} &\det[\rho_{\text{inst}^{(2,6)}}(a)^{(A,A)(A,A)}] = \det[\rho_{\text{inst}^{(2,6)}}(a')^{(A,A)(A,A)}] = \det[\rho_{\text{inst}^{(2,6)}}(b)^{(A,A)(A,A)}] = \det[\rho_{\text{inst}^{(2,6)}}(c)^{(A,A)(A,A)}] = 1, \\ &\det[\rho_{\text{inst}^{(2,6)}}(d)^{(A,A)(A,A)}] = e^{4\pi i/3}(\det[\rho_{\text{inst}^{(2,6)}}(d^3)^{(A,A)(A,A)}] = 1). \end{aligned} \tag{64}$$

Thus, the Majorana mass term generated by instanton zero modes with $(M_\beta,M_\gamma) = (2,6)$, $M_{ab}^{(2,6)}(\tau)N_a^{(8;0,0;1)}(x)N_b^{(8;0,0;1)}(x)$ is invariant under a, a', b, c, and d^3 transformation, while it transforms as

$$\begin{aligned} &M_{ab}^{(2,6)}(\tau)N_a^{(8;0,0;1)}(x)N_b^{(8;0,0;1)}(x) \\ &\to e^{4\pi i/3}M_{ab}^{(2,6)}(\tau)N_a^{(8;0,0;1)}(x)N_b^{(8;0,0;1)}(x), \end{aligned}$$

under d transformation. As a result, the full $\tilde{\Delta}(384)$ flavor symmetry of neutrinos remains, although Z_3 transformation for $M_{ab}^{(2,6)}(\tau)$ becomes meaningless in the Lagrangian.

C. Three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (5;1/2,1/2;1)$

Here, we study three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (5;1/2,1/2;1)$. In this case, the modular transformation matrices for the neutrinos, ρ_N are given as

$$\begin{aligned} \rho_N(\tilde{S}) &= \frac{ie^{\pi i/4}}{\sqrt{5}}\begin{pmatrix} 2s(1) & 2s(3) & \sqrt{2} \\ 2s(3) & 2s(1) & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 1 \end{pmatrix}, \\ \rho_N(\tilde{T}) &= \begin{pmatrix} e^{\pi i/20} & 0 & 0 \\ 0 & e^{9\pi i/20} & 0 \\ 0 & 0 & e^{25\pi i/20} \end{pmatrix}, \end{aligned} \tag{65}$$

and they satisfy

$$\begin{aligned}
&\rho_N(\tilde{T})^5 = e^{\pi i/4}\mathbb{I},\\
&\rho_N(\tilde{S})^2 = -i\mathbb{I}, \rho_N(\tilde{T})^{10} = i\mathbb{I},\\
&\rho_N(\tilde{S})^4 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^3 = \rho_N(\tilde{T})^{20} = -\mathbb{I},\\
&\rho_N(\tilde{S})^8 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^6 = \rho_N(\tilde{T})^{40} = \mathbb{I},
\end{aligned} \tag{66}$$

where $s(n) \equiv \sin(n\pi/10)$. They are the unitary representation of $A_5 \times Z_8$ [10], where the generators of A_5 are given by

$$\rho_N(S') = \rho_N(\tilde{S})\rho_N(\tilde{T})^{45}, \qquad \rho_N(T') = \rho_N(\tilde{T})^{-24}, \tag{67}$$

and the generator of Z_8 is given by

$$\rho_N(c) = \rho_N(T)^5. \tag{68}$$

Their Majorana masses can be generated by only one pair of the instanton zero-modes, $(\beta,\gamma)^T$, with

$$\begin{pmatrix} (M;\alpha_1,\alpha_\tau;m)_\beta \\ (M;\alpha_1,\alpha_\tau;m)_\gamma \end{pmatrix} = \begin{pmatrix} (2;0,0;0) \\ (3;1/2,1/2;1) \end{pmatrix}.$$

Here, we denote $(M,\boldsymbol{\alpha})_{\text{inst}} = ((2,3),(A,D))$. Then, the mass matrix can be written as

$$\begin{aligned}
M_{ab} &= e^{-S_{\text{cl}}(T_\alpha,\bar{M}^{(2,3)}_{\text{inst}})} m_{ab}^{((2,3),(A,D))},\\
&= e^{-S_{\text{cl}}(T_\alpha,\bar{M}^{(2,3)}_{\text{inst}})} c^{2,3} \begin{pmatrix} \sqrt{2}X_1 & X_4 & X_5 \\ X_4 & \sqrt{2}X_2 & X_6 \\ X_5 & X_6 & \sqrt{2}X_3 \end{pmatrix},
\end{aligned} \tag{69}$$

where $X_I (I = 1,2,3,4,5,6)$ are given by

$$\begin{aligned}
X_1 &= 2(\zeta^{(30)}_{1,10;+}\eta^{(30)}_6 - \zeta^{(30)}_{4,10;+}\eta^{(30)}_9),\\
X_2 &= 2(\zeta^{(30)}_{7,10;+}\eta^{(30)}_{12} - \zeta^{(30)}_{-2,10;+}\eta^{(30)}_3),\\
X_3 &= 2(\eta^{(30)}_0\eta^{(30)}_5 - \eta^{(30)}_{15}\eta^{(30)}_{10}),\\
X_4 &= \sqrt{2}(\zeta^{(30)}_{-2,10;+}\eta^{(30)}_9 - \zeta^{(30)}_{7,10;+}\eta^{(30)}_6 + \zeta^{(30)}_{4,10;+}\eta^{(30)}_3 - \zeta^{(30)}_{1,10;+}\eta^{(30)}_{12}),\\
X_5 &= (\zeta^{(30)}_{1,10;+}\eta^{(30)}_0 - \zeta^{(30)}_{4,10;+}\eta^{(30)}_{15} + 2\eta^{(30)}_5\eta^{(30)}_6 - 2\eta^{(30)}_{10}\eta^{(30)}_9),\\
X_6 &= (\zeta^{(30)}_{-2,10;+}\eta^{(30)}_{15} - \zeta^{(30)}_{7,10;+}\eta^{(30)}_0 + 2\eta^{(30)}_3\eta^{(30)}_{10} - 2\eta^{(30)}_{12}\eta^{(30)}_5).
\end{aligned} \tag{70}$$

The modular transformation matrices for the mass matrix elements $(X_1,X_2,X_3,X_4,X_5,X_6)^T \equiv \mathbf{X}^T$, ρ_m, are given as

$$\begin{aligned}
\rho_m(\tilde{S}) &= \frac{i}{5}\begin{pmatrix}
4s^2(1) & 4s^2(3) & 2 & \sqrt{2} & 4s(1) & 4s(3)\\
4s^2(3) & 4s^2(1) & 2 & \sqrt{2} & -4s(1) & -4s(3)\\
2 & 2 & 1 & -2\sqrt{2} & 2 & -2\\
\sqrt{2} & \sqrt{2} & -2\sqrt{2} & 3 & \sqrt{2} & -\sqrt{2}\\
4s(1) & -4s(3) & 2 & \sqrt{2} & 2+2s(1) & 2+2s(3)\\
4s(3) & -4s(1) & -2 & \sqrt{2} & -2+2s(3) & 2+2s(1)
\end{pmatrix},\\
\rho_m(\tilde{T}) &= \begin{pmatrix}
e^{-23\pi i/30} & 0 & 0 & 0 & 0 & 0\\
0 & e^{13\pi i/30} & 0 & 0 & 0 & 0\\
0 & 0 & e^{25\pi i/30} & 0 & 0 & 0\\
0 & 0 & 0 & e^{25\pi i/30} & 0 & 0\\
0 & 0 & 0 & 0 & e^{\pi i/30} & 0\\
0 & 0 & 0 & 0 & 0 & e^{-11\pi i/30}
\end{pmatrix},
\end{aligned} \tag{71}$$

which satisfy

$$\rho_m(\tilde{T})^5 = e^{\pi i/6}\mathbb{I},$$
$$\rho_m(\tilde{S})^2 = \rho_m(\tilde{T})^{30} = -\mathbb{I},$$
$$\rho_m(\tilde{S})^4 = [\rho_m(\tilde{S})\rho_m(\tilde{T})]^3 = \rho_m(\tilde{T})^{60} = \mathbb{I}. \quad (72)$$

These are the unitary representation of $A_5 \times Z_{12}$, where the generators of A_5 and Z_{12} are the same as Eqs. (67) and (68), respectively, by considering Eq. (72) instead of Eq. (66).

On the other hand, since we obtain

$$\det[\rho_{\mathrm{inst}}(\tilde{S})^{(A,D)(A,D)}] = 1, \quad \det[\rho_{\mathrm{inst}}(\tilde{T})^{(A,D)(A,D)}] = e^{4\pi i/3}, \quad (73)$$

from Eq. (34), we can find that

$$\det[\rho_{\mathrm{inst}}(S')^{(A,D)(A,D)}] = \det[\rho_{\mathrm{inst}}(T')^{(A,D)(A,D)}] = 1,$$
$$\det[\rho_{\mathrm{inst}}(c)^{(A,D)(A,D)}] = e^{2\pi i/3}(\det[\rho_{\mathrm{inst}}(c^3)^{(A,D)(A,D)}] = 1). \quad (74)$$

Thus, the Majorana mass term $M_{ab}(\tau)N_a^{(5;1/2,1/2;1)}(x) N_b^{(5;1/2,1/2;1)}(x)$ is invariant under S', T', and c^3 transformation, while it transforms as

$$M_{ab}(\tau)N_a^{(5;1/2,1/2;1)}(x)N_b^{(5;1/2,1/2;1)}(x)$$
$$\to e^{2\pi i/3}M_{ab}(\tau)N_a^{(5;1/2,1/2;1)}(x)N_b^{(5;1/2,1/2;1)}(x),$$

under the c transformation. Note that the anomaly free c^3 transformation becomes the generator of Z_8 symmetry for neutrinos. As a result, the full $A_5 \times Z_8$ flavor symmetry of neutrinos remain, although Z_3 transformation of Z_{12} for the mass matrix elements becomes meaningless in the Lagrangian.[7]

D. Three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (7;1/2,1/2;0)$

Here, we study three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (7;1/2,1/2;0)$. In this case, the modular transformation matrix for the neutrinos, ρ_N is given as

[7]Actually, according to the analysis in Ref. [60], we can find that the A_5 transformation is automatically anomaly free.

$$\rho_N(\tilde{S}) = \frac{2e^{\pi i/4}}{\sqrt{7}}\begin{pmatrix} c(1) & c(3) & c(5) \\ c(3) & c(9) & -c(1) \\ c(5) & -c(1) & c(3) \end{pmatrix},$$
$$\rho_N(\tilde{T}) = \begin{pmatrix} e^{\pi i/28} & 0 & 0 \\ 0 & e^{9\pi i/28} & 0 \\ 0 & 0 & e^{25\pi i/28} \end{pmatrix}, \quad (75)$$

where $c(n) \equiv \cos(n\pi/14)$. They satisfy

$$\rho_N(\tilde{T})^7 = e^{\pi i/4}\mathbb{I},$$
$$\rho_N(\tilde{S})^2 = \rho_N(\tilde{T})^{14} = i\mathbb{I},$$
$$\rho_N(\tilde{S})^4 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^3 = \rho_N(\tilde{T})^{28} = -\mathbb{I},$$
$$\rho_N(\tilde{S})^8 = [\rho_N(\tilde{S})\rho_N(\tilde{T})]^6 = \rho_N(\tilde{T})^{56} = \mathbb{I}, \quad (76)$$

and also satisfy

$$[\rho_N(\tilde{S})^{-1}\rho_N(\tilde{T})^{-1}\rho_N(\tilde{S})\rho_N(\tilde{T})]^4 = \mathbb{I}. \quad (77)$$

They are the unitary representation of $PSL(2,Z_7) \times Z_8$ [10], where the generators of $PSL(2,Z_7)$ are given by

$$\rho_N(S') = \rho_N(\tilde{S})\rho_N(\tilde{T})^{21},$$
$$\rho_N(T') = \rho_N(\tilde{T})^{24}, \quad (78)$$

and the generator of Z_8 is given by

$$\rho_N(c) = \rho_N(T)^7. \quad (79)$$

Their Majorana masses can be generated by one pair of the instanton zero-modes, $(\beta,\gamma)^T$, with

$$\begin{pmatrix} (M;\alpha_1,\alpha_\tau;m)_\beta \\ (M;\alpha_1,\alpha_\tau;m)_\gamma \end{pmatrix} = \begin{pmatrix} (2;0,0;0) \\ (5;1/2,1/2;0) \end{pmatrix},$$

which is consistent with the modular transformation. Then, the mass matrix can be written as

$$M_{ab}^{(2,5)} = e^{-S_{\mathrm{cl}}(T_\alpha,\bar{M}_{\mathrm{inst}}^{(2,5)})}m_{ab}^{((2,5),(A,D))} \quad (80)$$

$$= e^{-S_{\mathrm{cl}}(T_\alpha,\bar{M}_{\mathrm{inst}}^{(2,5)})}c^{(2,5)}\begin{pmatrix} \sqrt{2}X_1 & X_4 & X_5 \\ X_4 & \sqrt{2}X_2 & X_6 \\ X_5 & X_6 & \sqrt{2}X_3 \end{pmatrix}, \quad (81)$$

where X_I are given by

$$
\begin{aligned}
X_1 &= \sqrt{2}(\zeta^{(70)}_{-1,30;-}\zeta^{(70)}_{-8,30;-} - \zeta^{(70)}_{13,30;-}\zeta^{(70)}_{6,30;-}),\\
X_2 &= \sqrt{2}(\zeta^{(70)}_{11,20;-}\zeta^{(70)}_{18,20;-} - \zeta^{(70)}_{-3,20;-}\zeta^{(70)}_{4,20;-}),\\
X_3 &= \sqrt{2}(\zeta^{(70)}_{9,10;-}\zeta^{(70)}_{2,10;-} - \zeta^{(70)}_{23,10;-}\zeta^{(70)}_{16,10;-}),\\
X_4 &= (\zeta^{(70)}_{-1,30;-}\zeta^{(70)}_{18,20;-} + \zeta^{(70)}_{-8,30;-}\zeta^{(70)}_{11,20;-} - \zeta^{(70)}_{13,30;-}\zeta^{(70)}_{4,20;-} - \zeta^{(70)}_{6,30;-}\zeta^{(70)}_{-3,20;-}),\\
X_5 &= (\zeta^{(70)}_{-1,30;-}\zeta^{(70)}_{2,10;-} + \zeta^{(70)}_{-8,30;-}\zeta^{(70)}_{9,10;-} - \zeta^{(70)}_{13,30;-}\zeta^{(70)}_{16,10;-} - \zeta^{(70)}_{6,30;-}\zeta^{(70)}_{23,10;-}),\\
X_6 &= (\zeta^{(70)}_{11,20;-}\zeta^{(70)}_{2,10;-} + \zeta^{(70)}_{18,20;-}\zeta^{(70)}_{9,10;-} - \zeta^{(70)}_{-3,20;-}\zeta^{(70)}_{16,10;-} - \zeta^{(70)}_{4,20;-}\zeta^{(70)}_{23,10;-}).
\end{aligned} \tag{82}
$$

The modular transformation matrix for the mass matrix elements $(X_1, X_2, X_3, X_4, X_5, X_6)^T \equiv \mathbf{X}^T$, $\rho_{m^{(2,5)}}$, is given as

$$
\rho_{m^{(2,5)}}(\tilde{S}) = \frac{4i}{7}\begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix},
$$

$$
\rho_{m^{(2,5)}}(\tilde{T}) = \begin{pmatrix}
e^{13\pi i/14} & 0 & 0 & 0 & 0 & 0\\
0 & e^{5\pi i/14} & 0 & 0 & 0 & 0\\
0 & 0 & e^{17\pi i/14} & 0 & 0 & 0\\
0 & 0 & 0 & e^{9\pi i/14} & 0 & 0\\
0 & 0 & 0 & 0 & e^{\pi i/14} & 0\\
0 & 0 & 0 & 0 & 0 & e^{25\pi i/14}
\end{pmatrix}, \tag{83}
$$

where S_i $(i = 1, 2, 3)$ denote

$$
\begin{aligned}
S_1 &= \begin{pmatrix} c^2(1) & c^2(3) & c^2(5)\\ c^2(3) & c^2(5) & c^2(1)\\ c^2(1) & c^2(1) & c^2(3) \end{pmatrix},\\
S_2 &= \begin{pmatrix} \sqrt{2}c(1)c(3) & \sqrt{2}c(1)c(5) & \sqrt{2}c(3)c(5)\\ -\sqrt{2}c(3)c(5) & -\sqrt{2}c(1)c(3) & \sqrt{2}c(1)c(5)\\ -\sqrt{2}c(1)c(5) & \sqrt{2}c(3)c(5) & -\sqrt{2}c(1)s(3) \end{pmatrix},\\
S_3 &= \begin{pmatrix} c^2(3)-c(1)c(5) & -c^2(1)+c(3)c(5) & -c^2(5)-c(1)c(3)\\ -c^2(1)+c(3)c(5) & c^2(5)+c(1)c(3) & c^2(3)-c(1)c(5)\\ -c^2(5)-c(1)c(3) & c^2(3)-c(1)c(5) & c^2(1)-c(3)c(5) \end{pmatrix}.
\end{aligned} \tag{84}
$$

They satisfy

$$
\begin{aligned}
&\rho_{m^{(2,5)}}(\tilde{T})^7 = i\mathbb{I},\\
&\rho_{m^{(2,5)}}(\tilde{S})^2 = \rho_{m^{(2,5)}}(\tilde{T})^{14} = -\mathbb{I},\\
&\rho_{m^{(2,5)}}(\tilde{S})^4 = [\rho_{m^{(2,5)}}(\tilde{S})\rho_{m^{(2,5)}}(\tilde{T})]^3 = \rho_{m^{(2,5)}}(\tilde{T})^{28} = \mathbb{I}.
\end{aligned} \tag{85}
$$

They are the unitary representation of $PSL(2, Z_7) \times Z_4$, where the generators of $PSL(2, Z_7)$ and Z_4 are the same as Eqs. (78) and (79), respectively, by considering Eq. (85) instead of Eq. (76).

On the other hand, since we obtain

$$
\det[\rho_{\mathrm{inst}^{(2,5)}}(\tilde{S})^{(A,D)(A,D)}] = \det[\rho_{\mathrm{inst}^{(2,5)}}(\tilde{T})^{(A,D)(A,D)}] = -1, \tag{86}
$$

from Eq. (34), we can find that

$$
\begin{aligned}
&\det[\rho_{\mathrm{inst}^{(2,5)}}(S')^{(A,D)(A,D)}] = \det[\rho_{\mathrm{inst}^{(2,5)}}(T')^{(A,D)(A,D)}] = 1,\\
&\det[\rho_{\mathrm{inst}^{(2,5)}}(c)^{(A,D)(A,D)}] = -1(\det[\rho_{\mathrm{inst}^{(2,5)}}(c^2)^{(A,D)(A,D)}] = 1).
\end{aligned} \tag{87}
$$

Thus, the Majorana mass term $M_{ab}^{(2,5)}(\tau)N_a^{(7;1/2,1/2;0)}(x)N_b^{(7;1/2,1/2;0)}(x)$ is invariant under the S', T', and c^2 transformations, while it transforms as

$$M_{ab}^{(2,5)}(\tau)N_a^{(7;1/2,1/2;0)}(x)N_b^{(7;1/2,1/2;0)}(x) \to -M_{ab}^{(2,5)}(\tau)N_a^{(7;1/2,1/2;0)}(x)N_b^{(7;1/2,1/2;0)}(x),$$

under c transformation. As a result, among the neutrino flavor symmetry $PSL(2,Z_7)\times Z_8$, there remains $PSL(2,Z_7)\times Z_4$ flavor symmetry in neutrino mass terms, while the Z_2 part of Z_8 symmetry is broken.[8]

E. Results and implications to phenomenology

Here, we examine our results and implications to particle phenomenology. First, the modular weight of neutrino masses terms does not match with other terms in the tree-level Lagrangian. We consider the following superpotential in supersymmetric model,

$$W = AY_{ab}(\tau)L_aH_uN_b + Bm_{ab}(\tau)N_aN_b, \qquad (88)$$

where L_a, N_n, H_u denote superfields of left-handed leptons, right-handed neutrinos, and Higgs field, $Y_{ab}(\tau)$ and $m_{ab}(\tau)$ are modular forms corresponding to Yukawa couplings and neutrino masses, respectively. Here, A and B are just constants, which are written following the convention of recent 4D modular flavor models [20], although A and B may depend on other moduli T_α in string-derived low-energy effective field theory. Supoose that the first Yukawa terms are tree-level terms, while the second neutrino masses terms are induced by nonperturbative effects. In global supersymmetric models, we require the modular invariance of tree-level terms. That is, the Yukawa terms $Y_{ab}(\tau)L_aH_dN_b$ have totally vanishing modular weight. However, our analysis shows that the neutrino mass terms $m_{ab}(\tau)N_aN_b$ have nonvanishing modular weight. In our T^2/Z_2 orbifold models, its modular weight is two. In general, the modular weight of the neutrino mass terms $m_{ab}(\tau)N_aN_b$ would depend on compactification, e.g., the sum of modular weights of zero modes, β_i and γ_i, which is (-2) times the modular weight of N_aN_b.

Next, the tree-level flavor symmetry can break to its normal subgroup in the neutrino mass terms, although there is the example, where the full tree-level flavor symmetry remains. Suppose that there is the flavor symmetry $G\rtimes Z_N$ at tree level. Nonperturbative effects may break Z_N, and only the flavor symmetry G may remain in the neutrino mass terms. For example, suppose that the tree-level flavor symmetry is S_4. It may break to A_4 in neutrino mass terms.[9]

VI. MORE CORRECTIONS

In the previous section, we studied the neutrino mass terms induced by the D-brane instanton whose zero-modes transform the ones with the same boundary conditions under the modular symmetry. Note that wave functions with $M\in 2\mathbb{Z}$ and $(\alpha_1,\alpha_\tau)=(0,0)$ and ones with $M\in 2\mathbb{Z}+1$ and $(\alpha_1,\alpha_\tau)=(1/2,1/2)$, are consistent with the modular symmetry because they transform to ones with the same boundary conditions. Wave functions with other SS phases transform to ones with different SS phases. Such D-brane instanton zero modes are allowed by requiring only the condition (25), although consistency with the modular symmetry may forbid such zero modes. Here, we attempt to investigate contributions due to such zero modes.

For the neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N=(4;0,0;0)$, and $(5;1/2,1/2;1)$, all the possible D-brane instanton zero modes satisfying the condition (25) have been studied in the previous section, and all of them are consistent with the modular transformation. On the other hand, the neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N=(8;0,0;1)$, and $(7;1/2,1/2;0)$ have other possibilities for D-brane instanton zero-modes satisfying the condition (25). Here, we study them.

A. Three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N=(8;0,0;1)$

The possible instanton zero modes satisfying the condition (25) are $(\beta,\gamma)^T$ with [57],

$$\begin{pmatrix}(M;\alpha_1,\alpha_\tau;m)_\beta\\(M;\alpha_1,\alpha_\tau;m)_\gamma\end{pmatrix}=\begin{pmatrix}(2;0,0;0)\\(6;0,0;1)\end{pmatrix},\begin{pmatrix}(3;0,0;0)\\(5;0,0;1)\end{pmatrix},\begin{pmatrix}(3;1/2,0;0)\\(5;1/2,0;1)\end{pmatrix},\begin{pmatrix}(3;0,1/2;0)\\(5;0,1/2;1)\end{pmatrix},$$
$$\begin{pmatrix}(3;1/2,1/2;1)\\(5;1/2,1/2;0)\end{pmatrix},\begin{pmatrix}(4;1/2,0;0)\\(4;1/2,0;1)\end{pmatrix},\begin{pmatrix}(4;0,1/2;0)\\(4;0,1/2;1)\end{pmatrix},\begin{pmatrix}(4;1/2,1/2;0)\\(4;1/2,1/2;1)\end{pmatrix}.$$

Here, we denote

$$(M,\boldsymbol{\alpha})_{\rm inst}=((2,6),(A,A)),((3,5),(A,A)),((3,5),(B,B)),((3,5),(C,C)),$$
$$((3,5),(D,D)),((4,4),(B,B)),((4,4),(C,C)),((4,4),(D,D)),$$

[8] Actually, according to the analysis in Ref. [60], we can find that $PSL(2,Z_7)$ transformation is automatically anomaly free.

[9] A similar scenario was studied in Refs. [69,70], although such breaking effects were included in the Yukawa sector.

respectively. Then, the matrix can be written as

$$M_{ab} = M_{ab}^{(2,6)} + M_{ab}^{(3,5)} + M_{ab}^{(4,4)}, \quad (89)$$

$$M_{ab}^{(3,5)} = e^{-S_{\rm cl}(T_\alpha,\bar{M}_{\rm inst}^{(3,5)})} \sum_{\boldsymbol{\alpha}_{\rm inst}=(A,A),(B,B),(C,C),(D,D)} m_{ab}^{((3,5),\boldsymbol{\alpha}_{\rm inst})}, \\ = e^{-S_{\rm cl}(T_\alpha,\bar{M}_{\rm inst}^{(3,5)})} c^{(3,5)} \begin{pmatrix} Y_3 & 0 & Y_1 \\ 0 & -\sqrt{2}Y_2 & 0 \\ Y_1 & 0 & Y_3 \end{pmatrix}, \quad (90)$$

$$M_{ab}^{(4,4)} = e^{-S_{\rm cl}(T_\alpha,\bar{M}_{\rm inst}^{(4,4)})} \sum_{\boldsymbol{\alpha}_{\rm inst}=(B,B),(C,C),(D,D)} m_{ab}^{((4,4),\boldsymbol{\alpha}_{\rm inst})} \\ = e^{-S_{\rm cl}(T_\alpha,\bar{M}_{\rm inst}^{(4,4)})} c^{(4,4)} \begin{pmatrix} Z_3 & 0 & Z_1 \\ 0 & -\sqrt{2}Z_2 & 0 \\ Z_1 & 0 & Z_3 \end{pmatrix}, \quad (91)$$

where Y_I, $Z_I \equiv Z'_I + Z''_I$ $(I = 1, 2, 3)$ are given by

$$Y_1 = 2\sqrt{2}(\zeta^{(120)}_{9,30;-}\lambda^{(120)}_{(37,30;-),40;+} + \zeta^{(120)}_{33,30;-}\lambda^{(120)}_{(-11,30;-),40;+} + \zeta^{(120)}_{-3,30;-}\lambda^{(120)}_{(1,30;-),40;+} + \zeta^{(120)}_{21,30;-}\lambda^{(120)}_{(-47,30;-),40;+}) \\ Y_2 = 4(\zeta^{(120)}_{-6,60;-}\lambda^{(120)}_{(-2,60;-),40;+} - \zeta^{(120)}_{-18,60;-}\lambda^{(120)}_{(-14,60;-),40;+}), \\ Y_3 = 2\sqrt{2}(\zeta^{(120)}_{9,30;-}\lambda^{(120)}_{(-47,30;-),40;+} + \zeta^{(120)}_{33,30;-}\lambda^{(120)}_{(1,30;-),40;+} + \zeta^{(120)}_{21,30;-}\lambda^{(120)}_{(37,30;-),40;+} + \zeta^{(120)}_{-3,30;-}\lambda^{(120)}_{(-11,30;-),40;+}), \quad (92)$$

$$Z'_1 = (\eta_1^{(8)})^2 + (\eta_3^{(8)})^2, \qquad Z''_1 = (\eta_0^{(8)})^2 + (\eta_2^{(8)})^2, \\ Z'_2 = \sqrt{2}\eta_2^{(8)}(\eta_0^{(8)} + \eta_4^{(8)}), \qquad Z''_2 = \sqrt{2}((\eta_1^{(8)})^2 + (\eta_3^{(8)})^2), \\ Z'_3 = 2\eta_1^{(8)}\eta_3^{(8)}, \qquad Z''_3 = 2(\eta_2^{(8)})^2, \quad (93)$$

respectively. Note that $M_{ab}^{(2,6)}$ is already obtained in the previous section.

1. $(M_\beta, M_\gamma) = (3,5)$ case

Let us study the case of the instanton zero-modes with $(M_\beta, M_\gamma) = (3, 5)$. In this case, the modular transformation for the matrix elements $(Y_1, Y_2, Y_3)^T \equiv \mathbf{Y}^T$, $\rho_{m^{(3,5)}}$, is the same as $\rho_{m^{(2,6)}}$, i.e., $\rho_{m^{(3,5)}} = \rho_{m^{(2,6)}}$. Then, this also becomes the unitary representation of $\Delta'(96) \times Z_3$.

On the other hand, we obtain

$$\det[\rho_{\rm inst^{(3,5)}}(\tilde{S})^{(A,A)(A,A)}] = \det[\rho_{\rm inst^{(3,5)}}(\tilde{S})^{(B,B)(C,C)}] = \det[\rho_{\rm inst^{(3,5)}}(\tilde{S})^{(C,C)(B,B)}] = \det[\rho_{\rm inst^{(3,5)}}(\tilde{S})^{(D,D)(D,D)}] = 1, \\ \det[\rho_{\rm inst^{(3,5)}}(\tilde{T})^{(A,A)(C,C)}] = \det[\rho_{\rm inst^{(3,5)}}(\tilde{T})^{(B,B)(B,B)}] = \det[\rho_{\rm inst^{(3,5)}}(\tilde{T})^{(C,C)(A,A)}] = \det[\rho_{\rm inst^{(3,5)}}(\tilde{T})^{(D,D)(D,D)}] = e^{4\pi i/3}, \quad (94)$$

from Eq. (34), and then we find

$$\det[\rho_{\rm inst^{(3,5)}}(\tilde{S})^{(\rm total)(\rm total)}] = 1, \\ \det[\rho_{\rm inst^{(3,5)}}(\tilde{T})^{(\rm total)(\rm total)}] = e^{4\pi i/3}, \quad (95)$$

which are the same as Eq. (64). Note that this is the reason why the elements of $M_{ab}^{(3,5)}$, $\mathbf{Y}^T$, are closed under the modular transformation. Thus, the results is the same as the case of $(M_\beta, M_\gamma) = (2, 6)$, and the full $\tilde{\Delta}(384)$ flavor symmetry of neutrinos.

2. $(M_\beta, M_\gamma) = (4,4)$ case

Next, let us consider the case of the instanton zero modes with $(M_\beta, M_\gamma) = (4, 4)$. In this case, $\tilde{S}$ transformation for the matrix elements $(Z_1, Z_2, Z_3)^T \equiv \mathbf{Z}^T$, $\rho_{m^{(4,4)}}$, is the same as $\rho_{m^{(3,5)}}$ and $\rho_{m^{(2,6)}}$, i.e., $\rho_{m^{(4,4)}}(\tilde{S}) = \rho_{m^{(3,5)}}(\tilde{S}) = \rho_{m^{(2,6)}}(\tilde{S})$. However, $\tilde{T}$ transformation is not closed in the elements $\mathbf{Z}^T = \mathbf{Z}'^T + \mathbf{Z}''^T$ but closed in each of $\mathbf{Z}'^T = (Z'_1, Z'_2, Z'_3)^T$ and $\mathbf{Z}''^T = (Z''_1, Z''_2, Z''_3)^T$. Their unitary matrices of T transformation, $\rho_{m^{(4,4)\prime}}(\tilde{T})$ and $\rho_{m^{(4,4)\prime\prime}}(\tilde{T})$, are given as

$$\rho_{m^{(4,4)\prime}}(\tilde{T}) = e^{\pi i/4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\pi i/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \rho_{m^{(4,4)\prime}}(\tilde{T})^8 = \mathbb{I}, \\ \rho_{m^{(4,4)\prime\prime}}(\tilde{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \rho_{m^{(4,4)\prime\prime}}(\tilde{T})^4 = \mathbb{I}, \quad (96)$$

respectively.

On the other hand, we obtain

$$\det[\rho_{\mathrm{inst}^{(4,4)}}(\tilde{S})^{(B,B)(C,C)}] = \det[\rho_{\mathrm{inst}^{(4,4)}}(\tilde{S})^{(C,C)(B,B)}] = \det[\rho_{\mathrm{inst}^{(4,4)}}(\tilde{S})^{(D,D)(D,D)}] = 1,$$
$$\det[\rho_{\mathrm{inst}^{(4,4)}}(\tilde{T})^{(B,B)(D,D)}] = \det[\rho_{\mathrm{inst}^{(3,5)}}(\tilde{T})^{(D,D)(B,B)}] = e^{5\pi i/4},$$
$$\det[\rho_{\mathrm{inst}^{(4,4)}}(\tilde{T})^{(C,C)(C,C)}] = e^{-\pi i/2}, \tag{97}$$

from Eq. (34), and then we find

$$\det[\rho_{\mathrm{inst}^{(4,4)}}(\tilde{S})^{(\mathrm{total})(\mathrm{total})}] = 1,$$
$$\det[\rho_{\mathrm{inst}^{(4,4)\prime}}(\tilde{T})^{(\mathbf{Z}')(\mathbf{Z}')}] = e^{5\pi i/4},$$
$$\det[\rho_{\mathrm{inst}^{(4,4)\prime\prime}}(\tilde{T})^{(\mathbf{Z}'')(\mathbf{Z}'')}] = e^{-\pi i/2}. \tag{98}$$

This is the reason why the elements of $M_{ab}^{(4,4)}$, $\mathbf{Z}^T$, are closed under $\tilde{S}$ transformation, while $\tilde{T}$ transformation is not closed in the elements $\mathbf{Z}^T$ but closed in each of $\mathbf{Z}'^T$ and $\mathbf{Z}''^T$. Thus, in this case, the Majorana mass term generated by instanton zero modes with $(M_\beta, M_\gamma) = (4,4)$, $M_{ab}^{(4,4)}(\tau)N_a^{(8;0,0;1)}(x)N_b^{(8;0,0;1)}(x)$ is invariant under $\tilde{S}$ and $\tilde{T}^8$ transformation, while $\tilde{T}$ transformation fully breaks this term unless $\tilde{T}^{8n}$ for $\forall\, n \in \mathbb{Z}$. Here, since $\tilde{S}$ and $\tilde{T}^8$ transformations for neutrinos satisfy

$$\rho_N(\tilde{S})^2 = -i\mathbb{I},$$
$$\rho_N(\tilde{S})^4 = [\rho_N(\tilde{S})\rho_N(\tilde{T}^8)]^4 = -\mathbb{I},$$
$$\rho_N(\tilde{S})^8 = [\rho_N(\tilde{S})\rho_N(\tilde{T}^8)]^8 = \rho_N(\tilde{T}^8)^2 = \mathbb{I}, \tag{99}$$

they are the unitary representation of $\tilde{\Sigma}(8) \equiv (Z_2 \times Z_2) \rtimes Z_8$,[10] where the generators of Z_2, Z_2', and Z_8 are given by

$$\rho_N(p) = \rho_N(\tilde{T}^8),$$
$$\rho_N(q) = \rho_N(\tilde{S})\rho_N(\tilde{T}^8)^2\rho_N(\tilde{S})^{-1},$$
$$\rho_N(r) = \rho_N(\tilde{S}), \tag{100}$$

respectively, and they satisfy

$$\rho_N(p)^2 = \rho_N(q)^2 = \rho_N(r)^8,$$
$$\rho_N(p)\rho_N(q) = \rho_N(q)\rho_N(p),$$
$$\rho_N(r)\rho_N(p)\rho_N(r)^{-1} = \rho_N(q), \rho_N(r)\rho_N(q)\rho_N(r)^{-1} = \rho_N(p). \tag{101}$$

Therefore, there remains $\tilde{\Sigma}(8)$ flavor symmetry among the neutrino flavor symmetry $\tilde{\Delta}(384)$.

This result may be obvious because the instanton zero modes, $\beta_1\beta_2\gamma_1\gamma_2$ in the integral Eqs, (23), and (37) are not consistent with the modular symmetry.

B. Three generations of neutrinos with $(M;\alpha_1,\alpha_\tau;m)_N = (7;1/2,1/2;0)$

The possible instanton zero modes satisfying the condition (25) are $(\beta,\gamma)^T$ with,

$$\begin{pmatrix} (M;\alpha_1,\alpha_\tau;m)_\beta \\ (M;\alpha_1,\alpha_\tau;m)_\gamma \end{pmatrix} = \begin{pmatrix} (2;0,0;0) \\ (5;1/2,1/2;0) \end{pmatrix}, \begin{pmatrix} (3;1/2,0;0) \\ (4;0,1/2;0) \end{pmatrix}, \begin{pmatrix} (3;0,1/2;0) \\ (4;1/2,0;0) \end{pmatrix}, \begin{pmatrix} (3;0,0;0) \\ (4;1/2,1/2;0) \end{pmatrix}.$$

Then, the matrix can be written as

$$M_{ab} = M_{ab}^{(2,5)} + M_{ab}^{(3,4)}. \tag{102}$$

The first term $M_{ab}^{(2,5)}$ is the contribution due to the instanton zero mode consistent with the modular symmetry and it has been obtained in the previous section. The second term $M_{ab}^{(3,4)}$ includes the contributes due to the instanton zero modes, which transform to others with different boundary conditions under the modular symmetry. Similar to the previous case with $(M;\alpha_1,\alpha_\tau;m)_N = (8;0,0;1)$, we can compute $M_{ab}^{(3,4)}$ and investigate its flavor symmetry. As a result, when we include $M_{ab}^{(3,4)}$, only $Z_7 \times Z_4$ symmetry, whose generators are T'^2 in Eq. (78) and c^2 in Eq. (79), respectively, remains among the neutrino flavor symmetry $PSL(2,Z_7) \times Z_8$.

C. Comment

As we have studied in the previous section, the D-brane instanton with zero modes whose boundary conditions are consistent with the modular symmetry breaks a single Z_N subgroup of the modular flavor symmetry, and a certain normal subgroup is unbroken. On the other hand, the D-brane instantons with zero modes, which have the boundary condition inconsistent the modular symmetry, can violate the modular symmetry more severely. That is, for the flavor symmetry $G \rtimes Z_N$, the above instantons break not only Z_N but also G to a smaller group, although G may be anomaly

[10] It becomes the quadruple covering group of $\Sigma(8) \simeq (Z_2 \times Z_2) \rtimes Z_2$.

free through the discussion in Sec. II. That may be obvious because the zero modes, β and γ, in Eq. (23) are not linear representations of the modular group in fixed boundary conditions but they transform into the wave functions with different boundary conditions, as shown in Eq. (37). One exception would be the neutrino mass for $(M;\alpha_1,\alpha_\tau;m)_N = (8;0,0;1)$ due to the instanton with $(M_\beta, M_\gamma) = (3,5)$, i.e., $M_{ab}^{(3,5)}$. It includes the instanton zero modes which are inconsistent with the modular symmetry in terms of the boundary condition. However, the full $\tilde{\Delta}(384)$ remains. Its reason is unclear but may be because it is the summation over all of four SS phases, while the other cases are partial summation over SS phases. Alternatively, this result may be just accidental.

The condition (25) does not prohibit the appearance of instanton zero modes which transform into others with different boundary conditions under the modular symmetry. However, if we require the consistency with the modular symmetry, such instanton zero modes would be forbidden. It is unclear whether such instanton zero modes can appear or not. It may be concerned with the consistency condition with the string theory. Throughout this paper, we have investigated the low-energy field theory aspects of the D-brane models. We have not severely taken into account of the stringy consistency conditions of D-brane instantons such as the number of the neutral zero modes. They may restrict the possible configurations of D-brane instantons which can contribute to the superpotential, and a part of the nonperurbative Majorana mass matrices proposed in Secs. V and VI would be prohibited in full stringy models, and anomaly free part of the modular symmetry may be recovered. Investigating full string model is interesting, but it is beyond our scope in the present work. We would study this issue more elsewhere from the viewpoint of string theory.

VII. CONCLUSION

We have studied the modular symmetry anomaly in magnetized orbifold models. Nonperturbative effects can break the tree level symmetry. The neutrino mass terms are important in particle physics, and can be induced by D-brane instanton effects. They can break the modular symmetry. Thus, we have studied the modular flavor symmetry anomalies of Majorana neutrino mass terms in concrete models with three generations of neutrinos on magnetized T^2/Z_2 orbifold explicitly.

It is found that the modular weight of neutrino mass terms does not match with other terms in the tree-level Lagrangian. The sum over weights of the instanton zero modes $\beta_1\beta_2\gamma_1\gamma_2$ is the origin of this difference. This has significant meaning in model building of 4D modular flavor models although it would be canceled by the shift of the axions through the generalized GS mechanism.

In addition, the neutrino mass terms can break the tree-level modular flavor symmetry to its normal subgroup and a single Z_N symmetry is broken, although there is an example that the full tree-level flavor symmetry remains. This point is also important in model building of 4D modular flavor models. Among the tree-level flavor symmetry in the Yukawa coupling terms, its anomaly-free subgroup may remain in the neutrino mass terms.

When we include the effects due to the instanton zero modes, which transform ones with different boundary conditions, the neutirno mass terms break the flavor symmetry to much smaller groups. It is unclear whether we must include such effects or not. That is beyond our scope, and we would study this issue more elsewhere from the viewpoint of string theory.

It is also important to extend our analysis to active neutrino mass terms through the see-saw mechanism by combining Yukawa couplings as well as charged lepton mass terms. We will study elsewhere those lepton mass terms, related to their masses and lepton flavor mixing, in terms of the modular symmetry.[11]

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[11]As for quark sector, their flavor structure such as their masses and flavor mixing is compatible with texture structure and a kind of texture structure can be obtained by considering the modular symmetry at fixed points for the modulus τ [71].

[1] S. Ferrara, D. Lust, A. D. Shapere, and S. Theisen, Phys. Lett. B **225**, 363 (1989).
[2] S. Ferrara, D. Lust, and S. Theisen, Phys. Lett. B **233**, 147 (1989).
[3] W. Lerche, D. Lust, and N. P. Warner, Phys. Lett. B **231**, 417 (1989).
[4] J. Lauer, J. Mas, and H. P. Nilles, Nucl. Phys. **B351**, 353 (1991).
[5] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, Phys. Rev. D **97**, 116002 (2018).
[6] T. Kobayashi and S. Tamba, Phys. Rev. D **99**, 046001 (2019).

[7] H. Ohki, S. Uemura, and R. Watanabe, Phys. Rev. D **102**, 085008 (2020).
[8] S. Kikuchi, T. Kobayashi, S. Takada, T. H. Tatsuishi, and H. Uchida, Phys. Rev. D **102**, 105010 (2020).
[9] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, and H. Uchida, J. High Energy Phys. 11 (2020) 101.
[10] S. Kikuchi, T. Kobayashi, and H. Uchida, Phys. Rev. D **104**, 065008 (2021).
[11] Y. Almumin, M. C. Chen, V. Knapp-Perez, S. Ramos-Sanchez, M. Ratz, and S. Shukla, J. High Energy Phys. 05 (2021) 078.
[12] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, Nucl. Phys. **B947**, 114737 (2019).
[13] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, Nucl. Phys. **B957**, 115098 (2020).
[14] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, J. High Energy Phys. 02 (2021) 018.
[15] H. P. Nilles, S. Ramos–Sánchez, and P. K. S. Vaudrevange, Nucl. Phys. **B966**, 115367 (2021).
[16] A. Strominger, Commun. Math. Phys. **133**, 163 (1990).
[17] P. Candelas and X. de la Ossa, Nucl. Phys. **B355**, 455 (1991).
[18] K. Ishiguro, T. Kobayashi, and H. Otsuka, Nucl. Phys. **B973**, 115598 (2021).
[19] K. Ishiguro, T. Kobayashi, and H. Otsuka, J. High Energy Phys. 01 (2022) 020.
[20] F. Feruglio, *From My Vast Repertoire* (World Scientific, Singapore, 2018); T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, Phys. Rev. D **98**, 016004 (2018); J. T. Penedo and S. T. Petcov, Nucl. Phys. **B939**, 292 (2019); J. C. Criado and F. Feruglio, SciPost Phys. **5**, 042 (2018); T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, J. High Energy Phys. 11 (2018) 196; P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. 04 (2019) 005; 04 (2019) 174; F. J. de Anda, S. F. King, and E. Perdomo, arXiv:1812.05620; H. Okada and M. Tanimoto, Phys. Lett. B **791**, 54 (2019); T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, Phys. Lett. B **794**, 114 (2019); P. P. Novichkov, S. T. Petcov, and M. Tanimoto, Phys. Lett. B **793**, 247 (2019).
[21] S. Kikuchi, T. Kobayashi, H. Otsuka, M. Tanimoto, H. Uchida, and K. Yamamoto, arXiv:2201.04505.
[22] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Nucl. Phys. **B858**, 437 (2012).
[23] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010).
[24] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Prog. Theor. Phys. Suppl. **183**, 1 (2010).
[25] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, Lect. Notes Phys. **858**, 1 (2012), Springer.
[26] D. Hernandez and A. Y. Smirnov, Phys. Rev. D **86**, 053014 (2012).
[27] S. F. King and C. Luhn, Rep. Prog. Phys. **76**, 056201 (2013).
[28] S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, New J. Phys. **16**, 045018 (2014).
[29] J. P. Derendinger, S. Ferrara, C. Kounnas, and F. Zwirner, Nucl. Phys. **B372**, 145 (1992).
[30] L. E. Ibanez and D. Lust, Nucl. Phys. **B382**, 305 (1992).
[31] T. Kobayashi, Phys. Lett. B **326**, 231 (1994).
[32] H. Kawabe, T. Kobayashi, and N. Ohtsubo, Nucl. Phys. **B434**, 210 (1995).
[33] T. Kobayashi, S. Nagamoto, and S. Uemura, Prog. Theor. Exp. Phys. **2017**, 023B02 (2017).
[34] Y. Kariyazono, T. Kobayashi, S. Takada, S. Tamba, and H. Uchida, Phys. Rev. D **100**, 045014 (2019).
[35] C. Bachas, arXiv:hep-th/9503030.
[36] R. Blumenhagen, L. Goerlich, B. Kors, and D. Lust, J. High Energy Phys. 10 (2000) 006.
[37] C. Angelantonj, I. Antoniadis, E. Dudas, and A. Sagnotti, Phys. Lett. B **489**, 223 (2000).
[38] R. Blumenhagen, B. Kors, and D. Lust, J. High Energy Phys. 02 (2001) 030.
[39] D. Cremades, L. E. Ibanez, and F. Marchesano, J. High Energy Phys. 05 (2004) 079.
[40] H. Abe, T. Kobayashi, and H. Ohki, J. High Energy Phys. 09 (2008) 043.
[41] T. H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, J. High Energy Phys. 01 (2014) 065.
[42] T. h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, Nucl. Phys. **B890**, 442 (2014).
[43] T. Kobayashi and S. Nagamoto, Phys. Rev. D **96**, 096011 (2017).
[44] M. Sakamoto, M. Takeuchi, and Y. Tatsuta, Phys. Rev. D **102**, 025008 (2020).
[45] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, Nucl. Phys. **B814**, 265 (2009).
[46] T. h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Nucl. Phys. **B894**, 374 (2015).
[47] K. Hoshiya, S. Kikuchi, T. Kobayashi, Y. Ogawa, and H. Uchida, Prog. Theor. Exp. Phys. **2021**, 033B05 (2021).
[48] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, J. High Energy Phys. 06 (2009) 080.
[49] Y. Fujimoto, T. Kobayashi, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Phys. Rev. D **94**, 035031 (2016).
[50] H. Abe, T. Kobayashi, H. Ohki, A. Oikawa, and K. Sumita, Nucl. Phys. **B870**, 30 (2013).
[51] H. Abe, T. Kobayashi, K. Sumita, and Y. Tatsuta, Phys. Rev. D **90**, 105006 (2014).
[52] T. Kobayashi, K. Nishiwaki, and Y. Tatsuta, J. High Energy Phys. 04 (2017) 080.
[53] R. Blumenhagen, M. Cvetic, and T. Weigand, Nucl. Phys. **B771**, 113 (2007).
[54] L. E. Ibanez and A. M. Uranga, J. High Energy Phys. 03 (2007) 052.
[55] M. Cvetic, R. Richter, and T. Weigand, Phys. Rev. D **76**, 086002 (2007).
[56] T. Kobayashi, Y. Tatsuta, and S. Uemura, Phys. Rev. D **93**, 065029 (2016).
[57] K. Hoshiya, S. Kikuchi, T. Kobayashi, K. Nasu, H. Uchida, and S. Uemura, Prog. Theor. Exp. Phys. **2022**, 013B04 (2022).
[58] T. Araki, T. Kobayashi, J. Kubo, S. Ramos-Sanchez, M. Ratz, and P. K. S. Vaudrevange, Nucl. Phys. **B805**, 124 (2008).
[59] M. C. Chen, M. Fallbacher, M. Ratz, A. Trautner, and P. K. S. Vaudrevange, Phys. Lett. B **747**, 22 (2015).

[60] T. Kobayashi and H. Uchida, Phys. Rev. D **105**, 036018 (2022).
[61] B. Gripaios, Phys. Rev. D **105**, 105008 (2022).
[62] E. Witten, Phys. Lett. **149B**, 351 (1984).
[63] M. Dine, N. Seiberg, and E. Witten, Nucl. Phys. **B289**, 589 (1987).
[64] W. Lerche, B. E. W. Nilsson, and A. N. Schellekens, Nucl. Phys. **B289**, 609 (1987).
[65] T. Kobayashi and H. Nakano, Nucl. Phys. **B496**, 103 (1997).
[66] L. E. Ibanez, R. Rabadan, and A. M. Uranga, Nucl. Phys. **B542**, 112 (1999).
[67] Z. Lalak, S. Lavignac, and H. P. Nilles, Nucl. Phys. **B559**, 48 (1999).
[68] X. G. Liu, C. Y. Yao, B. Y. Qu, and G. J. Ding, Phys. Rev. D **102**, 115035 (2020).
[69] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, J. High Energy Phys. 02 (2020) 097.
[70] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, Phys. Rev. D **100**, 115045 (2019); **101**, 039904(E) (2020).
[71] S. Kikuchi, T. Kobayashi, Y. Ogawa, and H. Uchida, Prog. Theor. Exp. Phys. **2022**, 033B10 (2022).