

Four-fermion operators at dimension 6: Dispersion relations and UV completions

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(Received 10 February 2022; accepted 18 May 2022; published 15 June 2022)

A major task in phenomenology today is constraining the parameter space of Standard Model effective field theory and constructing models of fundamental physics from which the Standard Model derives. To this effect, we report an exhaustive list of sum rules for 4-fermion operators of dimension 6, connecting low-energy Wilson coefficients to cross sections in the UV. Unlike their dimension-8 counterparts which are amenable to a positivity bound, the discussion here is more involved due to the weaker convergence and indefinite signs of the dispersion integrals. We illustrate this by providing examples with weakly coupled UV completions leading to opposite signs of the Wilson coefficients for both convergent and non-convergent dispersion integrals. We further decompose dispersion integrals under weak isospin and color groups, which lead to a tighter relation between IR measurements and UV models. These sum rules can become an effective tool for constructing consistent UV completions for Standard Model effective field theory following the prospective measurement of these Wilson coefficients.

DOI: [10.1103/PhysRevD.105.115019](https://doi.org/10.1103/PhysRevD.105.115019)

I. INTRODUCTION

Testing the Standard Model (SM) and searching for new physics are two essential goals of the current and future experimental programs in particle physics. In this respect, all of the measurements can be classified as low-energy (SM scale) and high-energy experiments. For low-energy observables, the Standard Model effective field theory (SMEFT) provides an excellent tool to consistently parametrize new physical perturbations, classified order by order in the form of nonrenormalizable operators with higher dimensions. We expect new physics to kick in above at least the weak scale, and as we approach the regime of high energies greater than this scale, the applicability of effective field theory (EFT) techniques becomes successively questionable. Reliable calculations then require a discussion of the explicit UV completions, and thus it is clear that the connection between UV and IR observables and predictions becomes somewhat model dependent, and

explicit matching is required to infer useful information. In this direction, dispersion relations provide a model-independent way to connect low- and high-energy measurements, in the form of *sum rules* for low-energy Wilson coefficients and high-energy cross sections. This provides a consistent way to match the known and measurable low-energy and speculative high-energy quantities (for a recent reappraisal, see Ref. [1] and for a textbook introduction [2,3]). Their power lies in their generality—they follow from the simple and sacred physical requirements of Poincaré invariance, unitarity, and locality. Recently, there has been significant attention directed toward the application of the dispersion relations and sum rules for SMEFT [4–7]. For the four fermion interactions, most of the effort so far has been focused on the dimension-8 operators [8–11] where the sum rules lead to positivity constraints on the Wilson coefficients in a model-independent way.

On the other hand, from a phenomenological point of view, dimension-8 operators are very hard to measure at experiments, and most likely the new physics will demonstrate itself first via dimension-6 corrections to the SM. Thus, it becomes crucial to understand similar dispersion relations for the dimension-6 operators. The situation here is drastically different from the dimension-8 discussion because the relevant dispersion integral, aside from being possibly nonconvergent, is of indefinite sign and does not admit any simple model-independent positivity bound. However, the situation is far from hopeless, and the

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dispersion relations turn out to be instructive in a different way: instead of being viewed as a constraint on Wilson coefficients, these sum rules are to be used as a *tool to constrain the UV completions* of these operators, given signs to be measured in the IR. Therefore, in a way, we are approaching the IR-UV relationship from the opposite standpoint to what is customary. Our emphasis is on model building for a full theory by taking IR measurements as our input, instead of trying to predict these measurements from general inputs from the UV theory. We will show that different signs of the Wilson coefficients will be related to the dominance of the particle collision cross sections in the various channels and decompose these cross sections as explicitly as possible to indicate the quantum numbers of initial states with dominant cross sections. Moreover, it is crucial to emphasize that sum rules can only be written down for a subspace of the dimension-6 basis, namely, the effective 4 fermion operators that can generate forward amplitudes. Based on these sum rules, we will report examples of the weakly coupled UV completions, which can lead to either sign of the Wilson coefficients. Such information, which we believe was not consistently summarized before, can become a useful guide for the future measurements in case some of the Wilson coefficients are discovered to be nonzero. These measurements, supplemented with the sum rules we derive, will bring us closer to an understanding of the fundamental physics from which the SMEFT derives.

The main limitation of the dispersion relations for dimension-6 operators is that convergence of the integrals generically is not guaranteed. In particular, it is known that elementary vector field exchange [4] in the t channel can lead to a nonvanishing contribution C_∞ from the pole at infinity. Being a characteristic of the amplitudes at high energies, an EFT analysis cannot resolve its sign or strength. However, we still believe it is instructive to study these relations while being agnostic to C_∞ —in fact, this is the philosophy used in context of quantum gravity where the forward graviton pole presents a similar difficulty in terms of convergence. While the situation with gravity requires a more sophisticated treatment invoking Regge towers, in our case, the situation is much simpler, at least for weakly coupled UV completions. In this case, only the neutral vector boson exchange in the t channel can spoil the convergence of the dispersion integrals. Thus, for the rest of the UV completions, the measurements of IR observables directly lead us to infer about the strengths of various UV cross sections and the associated quantum numbers of states contributing to that cross section.

Second, the dispersion relations cannot identify an exact UV completion. At best, this IR information allows us to constrain a hierarchy of couplings and masses in the UV. As an example, suppose a Wilson coefficient receives contributions (positive and negative, respectively) from both scalar and vector exchanges in the UV (see discussion in

Sec. III); then, measuring a certain sign of this coefficient tells us only about the hierarchy of the coupling-to-mass ratio for the scalar and vector pieces. On the UV side of the dispersion relation, this information reflects in the relative sizes of cross sections coming from these two channels, which, of course, is fixed by this very same hierarchy.

The manuscript is organized as follows. In Sec. II, we briefly review dispersion integrals. In Sec. III, we study in detail the operator $(\bar{e}_R \gamma^\mu e_R)^2$ and illustrate the relation between UV completions and signs of the effective operator at tree and one-loop levels. In Sec. IV, we present the whole set of the four fermion operators and identify which of them can be constrained by the dispersion relations. Results are summarized in the Sec. V. Most details of the calculations have been relegated to the Appendixes.

II. REVIEW OF DISPERSION RELATIONS

In this section, we will review dispersion relations and their applications to constraints on EFTs following the discussion in Refs. [1,4,12,13] (readers familiar with the formalism can proceed directly to Sec. III). It is a general principle that the nonanalyticities associated with scattering amplitudes have a physical origin in the form of poles and branch cuts arising from localized particle states and thresholds. The positivity of the spectral function in the Kallen-Lehmann decomposition generalizes to more general cross sections, which can be related to elastic forward scattering amplitudes via a dispersion integral, to be reviewed in a moment. What this means in an EFT context is that, *in perturbation theory*, one can evaluate the two sides of a dispersion integral to a certain order; allowing us to extract information about the effective IR coupling that contributes to that amplitude at low energies on one side of the relation, from general observations about the UV piece of the dispersion relation without any explicit matching.

While unitarity reflects in the positivity of the spectral function and cross sections, we need additional information about the high-energy behavior of the amplitude to control the dispersion integral at the infinite contour. The asymptotics of amplitudes at high energies is a question about the unitarity and locality of the theory. The famous Froissart bound—while technically proved only for theories with a mass gap, but believed to hold true generally—tells us that the behavior of the amplitude $A(s)$ is such that $A(s)/s^2 \rightarrow 0$ as $s \rightarrow \infty$ [14–16]. This, in general, allows us to write down a dispersion relation with two subtractions, i.e., a linear polynomial of the form $a(t) + b(t)s$ supplemented by a contour integral picking up the nonanalytic structure of the amplitude. $a(t), b(t)$ cannot be determined by unitarity alone, but the nonanalytic structure can be related to manifestly positive cross sections via the optical theorem. We can then differentiate this relation with respect to s twice to get rid of the unknown subtractions, and we are left with a manifestly positive integral on the right and the coefficient of s^2 in $A(s)$ on the left—thereby leading to

what are conventionally called ‘‘positivity bounds’’ [1] on EFT parameters.

This prescription, however, cannot be directly applied to dimension-6 operators. Their contribution to $2 \rightarrow 2$ amplitudes scales as p^2 , and so $d^2A(s)/ds^2$ kills information about their couplings, and we cannot constrain them in any way. The best we can do is to look at $dA(0)/ds$ and be left with a dispersion integral of indefinite sign as well as an undetermined subtraction constant (which we will call C_∞ , as it captures the pole of the amplitude at infinity).

Let us briefly derive this dispersion relation from first principles. Consider a process $ab \rightarrow ab$ with the amplitude $A_{ab \rightarrow ab} \equiv A_{ab}(s, t)$, and in the forward limit ($t \rightarrow 0$). This amplitude can be expanded as

$$\begin{aligned} A_{ab}(s, 0) &= \sum_n c_n(\mu^2)(s - \mu^2)^n, \\ c_n(\mu^2) &= \frac{1}{n!} \frac{\partial^n}{\partial s^n} A_{ab}(s, 0)|_{s=\mu^2} \end{aligned} \quad (1)$$

about some arbitrary reference scale μ^2 where the amplitude is analytic. We can now use Cauchy’s theorem to write

$$\begin{aligned} \frac{1}{2\pi i} \oint ds \frac{A_{ab}(s, 0)}{(s - \mu^2)^{n+1}} &= \sum_{s_i, \mu^2} \text{Res} \frac{A_{ab}(s, 0)}{(s - \mu^2)^{n+1}} \\ &= c_n(\mu^2) + \sum_{s_i} \text{Res} \frac{A_{ab}(s, 0)}{(s - \mu^2)^{n+1}}, \end{aligned} \quad (2)$$

where s_i are the physical poles associated with IR stable resonance exchanges in the scattering, and the contour of integration is shown in Fig. 1. The residues at physical poles are IR structures that we will drop henceforth. This can always be done if the scale μ is chosen such that $\mu^2 \gg m_{\text{IR}}^2$, where m_{IR}^2 corresponds to the scale of the s_i poles. Indeed, the last term in Eq. (2) gives corrections of the order $\mathcal{O}(m_{\text{IR}}^2/\mu^2)$, which can be safely ignored.

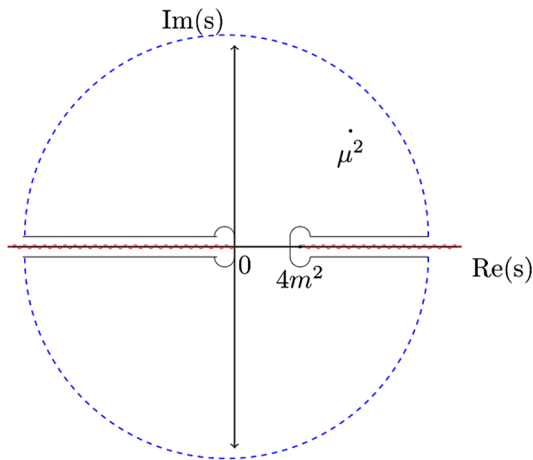


FIG. 1. Analytic structure in complex s plane. The infinite circle is centered at $2m^2$ and will be traversed counterclockwise.

The analytic structure of the amplitude allows us to decompose the integral as a sum of the contributions along the branch cuts and over infinite circle, so that schematically

$$\begin{aligned} \frac{1}{2\pi i} \int ds \frac{A_{ab}(s, 0)}{(s - \mu^2)^{n+1}} &= \text{integrals along cuts} \\ &+ \text{integral on big circle} = C_\infty^n + I_n \\ C_\infty^n &= \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{A_{ab}(|s_\Lambda|e^{i\theta}, 0)}{(|s_\Lambda|e^{i\theta} - \mu^2)^{n+1}} \cdot (|s_\Lambda|e^{i\theta}). \end{aligned} \quad (3)$$

The integration over the branch cuts can be written as a sum of the integrals over discontinuities

$$\begin{aligned} 2\pi i I_n &= \int_{4m^2}^{\infty} \left(\frac{A_{ab}(s + i\epsilon, 0) - A_{ab}(s - i\epsilon, 0)}{(s - \mu^2)^{n+1}} + (-1)^n \right. \\ &\times \left. \frac{A_{ab}(4m^2 - s - i\epsilon, 0) - A_{ab}(4m^2 - s + i\epsilon, 0)}{(s - 4m^2 + \mu^2)^{n+1}} \right). \end{aligned} \quad (4)$$

Since $4m^2 - s = u$ for $t = 0$, the second term is just the u channel crossed amplitude for the process $a\bar{b} \rightarrow a\bar{b}$, i.e., $A_{a\bar{b}}$ (instead of $ab \rightarrow ab$).¹ Using the optical theorem, we can rewrite the discontinuity in terms of cross section, and in the limits $m \rightarrow 0$ and $\mu \rightarrow 0$, we obtain

$$I_n = \int \frac{ds}{\pi s^n} (\sigma_{ab} + (-1)^n \sigma_{a\bar{b}}). \quad (5)$$

For dimension-6 operators, we will be interested in dispersion relations of Eq. (2) for the case $n = 1$,

$$c_1(\mu^2) = \int \frac{ds}{\pi s} (\sigma_{ab} - \sigma_{a\bar{b}}) + C_\infty^{(n=1)}. \quad (6)$$

Note that the quantity $c_n(\mu^2)$ on the left-hand side can be evaluated in IR using the EFT expansion. This introduces an additional source of corrections of the order $\mathcal{O}(\mu^2/\Lambda^2)$, where Λ is the scale suppressing higher-dimensional operators. We can see that the dispersion relations are valid up to corrections of the order $\mathcal{O}(m_{\text{IR}}^2/\mu^2, \mu^2/\Lambda^2)$, and these can be ignored if $\Lambda^2 \gg \mu^2 \gg m_{\text{IR}}^2$.

At last, let us mention that the forward limit $t \rightarrow 0$ must be taken with care and is in principle problematic in the presence of massless particles propagating in the t channel

¹Crossing relations for particles with spin become more nontrivial (see, for example, Refs. [17,18]). However, in the case of the massless spin-1/2 particles, which are the interest of this paper, the usual crossing relations for the forward amplitude remain valid [17], and we will not worry about these issues in the rest of the paper.

of the UV amplitude (see, for example, Refs. [4,13]). In fact, we always have the usual SM Coulomb singularities that lead to the bad behavior in the forward limit. The way out of this problem is by using IR mass regulators to match the known SM contributions to both sides of the dispersion relation and subtract them away.

III. WARM-UP EXERCISE

We consider the simplest case of a fully right-handed operator which is made up of singlet fields e_R , all of the same generation (the dispersion relation for this operator was presented in Ref. [6]),

$$c_{RR}(\bar{e}_R\gamma_\mu e_R)(\bar{e}_R\gamma^\mu e_R). \quad (7)$$

Following the strategy outlined in the previous section, we start by considering the amplitude $A_{e\bar{e}}$ and derive the dispersion relation

$$\left. \frac{dA_{e_R\bar{e}_R}(s,0)}{ds} \right|_{s=0} = \int \frac{ds}{\pi s} (\sigma_{e_R\bar{e}_R} - \sigma_{e_R e_R}) + C_\infty, \quad (8)$$

where we have omitted the ($n=1$) subscript for C_∞ . The amplitude in the IR ($s \rightarrow 0$) limit can be safely calculated using the EFT, and we find (we use helicity amplitudes; for notations and for the explicit conventions, see Appendix A)

$$\begin{aligned} A_{e_R\bar{e}_R}(s,t) &= c_{RR} \cdot 2([2\gamma_\mu 1][3\gamma^\mu 4] - [3\gamma_\mu 1][2\gamma^\mu 4]) \\ &= -8c_{RR}[23]\langle 14 \rangle \end{aligned}$$

$$A_{e_R\bar{e}_R}(s,t)|_{t \rightarrow 0} = -8c_{RR}s \quad (9)$$

so that we arrive at the following sum rule for the c_{RR} Wilson coefficient²:

$$-8c_{RR} = \int \frac{ds}{\pi s} (\sigma_{e_R\bar{e}_R} - \sigma_{e_R e_R}) + C_\infty. \quad (10)$$

Let us see how this equation can be used as guidance for UV completions that lead to the possible signs of the c_{RR} Wilson coefficient.

A. Charge neutral vector exchange

Let us start with the negative sign for c_{RR} . The dispersion relation predicts that this will be generated by the models with resonances in $e\bar{e}$ channel (apart from the C_∞ contribution). The simplest model which can enhance the $\sigma_{e\bar{e}}$ cross section is a simple neutral Z' vector boson with the interaction

$$\mathcal{L}_{Z'} = \lambda Z'_\mu \bar{e}_R \gamma^\mu e_R. \quad (11)$$

Integrating Z' at tree level, we obtain for the Wilson coefficient

$$c_{RR} = -\frac{\lambda^2}{2M_{Z'}^2}, \quad (12)$$

where the sign follows the prediction of the dispersion relations. However, inspecting the amplitudes carefully, we see that the massive vector exchange in the t channel spoils the convergence of the amplitude in the forward region, making the integral over the infinite circle nonvanishing. To this end, let us look at the amplitude $A_{\bar{e}_R e_R}$ in detail:

$$\begin{aligned} iA &= -\lambda^2 \left([2\gamma^\mu 1][3\gamma_\mu 4] \frac{-i}{s - M_{Z'}^2} \right. \\ &\quad \left. - [3\gamma^\mu 1][2\gamma_\mu 4] \frac{-i}{t - M_{Z'}^2} \right) \\ A(s,t) &= -2\lambda^2 [23]\langle 14 \rangle \left(\frac{1}{s - M_{Z'}^2} + \frac{1}{t - M_{Z'}^2} \right). \end{aligned} \quad (13)$$

In the forward limit, this amplitude goes as

$$A(s,t)|_{t \rightarrow 0} = -2\lambda^2 s \left(\frac{1}{s - M_{Z'}^2} + \frac{1}{-M_{Z'}^2} \right). \quad (14)$$

We can see that the integral over infinite contour becomes nonzero and is equal to

$$C_\infty^{(Z')} = \frac{2\lambda^2}{M_{Z'}^2}. \quad (15)$$

We see that, even though the contribution from the infinite contour is nonzero, it turns out to be of the same sign and size as the cross section part of the dispersion relation

$$\left[\int \frac{ds}{\pi s} (\sigma_{e_R\bar{e}_R} - \sigma_{e_R e_R}) \right]^{(Z')} = \frac{2\lambda^2}{M_{Z'}^2} \quad (16)$$

(see Appendix B for details of the calculation). The fact that exchange of the elementary vector boson spoils the convergence of the amplitude in the forward limit at large s is not new and was observed, for example, in Ref. [4] in the discussion of the other dimension-6 operators.

Let us extend the discussion for the operators with two fermion flavors. For example, $c_{e\mu}(\bar{e}_R\gamma^\mu e_R)(\bar{\mu}_R\gamma_\mu\mu_R)$ contributes $e\bar{\mu} \rightarrow e\bar{\mu}$ in the IR. This operator can be generated by two kinds of UV completions with a charge neutral vector boson:

$$\begin{aligned} \mathcal{L}_{UV}^{(1)} &= \lambda Z_{(1)}^\mu (\bar{e}_R\gamma^\mu\mu_R + \text{H.c.}) \\ \mathcal{L}_{UV}^{(2)} &= (\lambda_1 Z_{(2)}^\mu \bar{e}_R\gamma^\mu e_R + \lambda_2 Z_{(2)}^\mu \bar{\mu}_R\gamma_\mu\mu_R). \end{aligned} \quad (17)$$

²In this expression, we should take the value of the Wilson coefficient at the scale $\mu \rightarrow 0$. The renormalization group equation (RGE) evolution of the Wilson coefficients from the EFT cutoff scale to μ can lead to the modification of the Eq. (10) (see Ref. [19] for a recent discussion). In this paper, we will assume that these running effects are subleading and can be safely ignored.

The analysis in both cases is very similar to the single flavor discussion; however, in the first case ($\mathcal{L}_{UV}^{(1)}$), the integral over the infinite contour vanishes, since there is no amplitude with $Z_{(1)}$ in the t channel. Writing down the dispersion relations for the $e\bar{\mu} \rightarrow e\bar{\mu}$ scattering, we will obtain [note that there is a different numerical prefactor compared to Eq. (10) due to combinatorics]:

$$c_{e\mu} = -\frac{1}{2} \left[\int \frac{ds}{\pi s} (\sigma_{e_R\bar{\mu}_R} - \sigma_{e_R\mu_R}) \right] = -\frac{|\lambda|^2}{M_{(1)}^2}. \quad (18)$$

In the second case ($\mathcal{L}_{UV}^{(2)}$), we are in the opposite situation since both cross sections $\sigma_{e\bar{\mu}(\mu)} = 0$ vanish at leading order in perturbation theory. However, there is a forward amplitude for this process, which comes from t -channel diagram, and it contributes only to C_∞ . In other words, the pole at infinity saturates the dispersion relation, and even though no corresponding UV cross section can be measured to constrain this coefficient, it can be nonzero because of this pole. In fact, a simple calculation yields

$$c_{e\mu} = -\frac{C_\infty}{2} = -\frac{\lambda_1\lambda_2}{M_{(2)}^2}, \quad (19)$$

which can be either positive or negative depending on the values of the λ_1, λ_2 couplings. Let us continue with our examination of the UV completions for the various signs of the c_{RR} .

B. Charge-2 scalar

What about the positive sign of c_{RR} ? The dispersion relation in Eq. (10) predicts that this happens for UV completions that generate only σ_{ee} cross section. The simplest possibility is a charge-2 scalar with the interaction

$$\mathcal{L} = \kappa\phi\bar{e}_R^c e_R + \text{H.c.} \quad (20)$$

Then, at the order $\mathcal{O}(\kappa^2)$, only σ_{ee} will be nonvanishing, so the Wilson coefficient must be positive. Indeed, integrating out the scalar field at tree level gives

$$c_{RR} = \frac{|\kappa|^2}{2M_\phi^2}, \quad (21)$$

which is manifestly positive. In this case, the forward amplitude converges quickly enough, so that $C_\infty = 0$ —this is just the statement that a scalar cannot be exchanged in the t channel for the forward amplitudes. We see that the both signs of the Wilson coefficient are possible with a weakly coupled UV completion. One can still wonder whether the negative sign of the c_{RR} interactions in the Eq. (12) is related to the t channel pole and nonconvergence of the

amplitude in the UV. To quell any doubts, in the next subsection, we will build a weakly coupled UV completion without new vector bosons and with convergent forward amplitudes.

C. UV completion at one loop

Let us extend the SM with vectorlike fermion Ψ of charge 1 and a charge-2 complex scalar ϕ with a Yukawa interaction,

$$\mathcal{L} = |D\phi|^2 + i\bar{\Psi}\not{D}\Psi + M_\Psi\bar{\Psi}\Psi - M_\phi^2|\phi|^2 + y\bar{e}_R\phi\Psi. \quad (22)$$

This generates an effective operator at the order $\mathcal{O}(y^4)$, and at this order, the only cross section available is $\sigma_{e\bar{e}}$. The dispersion relation predicts that the Wilson coefficient must be negative. Moreover, $C_\infty = 0$ here as the amplitude scales slowly enough with s . Indeed, integrating out heavy fields at one loop, we obtain

$$c_{RR} = -\frac{|y|^4}{128\pi^2 M_\Psi M_\phi} f(x), \quad x \equiv \frac{M_\Psi}{M_\phi}$$

$$f(x) = \frac{(x + 4x^3 \log x - x^5)}{(1 - x^2)^3}, \quad \lim_{x \rightarrow 1} f(x) = 1/3, \quad (23)$$

where one can see that the function $f(x)$ is always positive. See Appendix B for explicit verification of the dispersion integral in the case $M_\Psi = M_\phi$.

In summary, this warm-up exercise shows us that both signs of the Wilson coefficients are possible within weakly coupled theories. Contribution of the infinite contours is important for the t -channel exchange of the vector resonances. Interestingly, both signs of the Wilson coefficient are possible even for the weakly coupled models with vanishing C_∞ .

We can see that dispersion relations depend on the value of C_∞ , so a completely model-independent relation between the Wilson coefficients is possible only for the UV completions satisfying super-Froissart conditions $\lim_{s \rightarrow \infty} |A(s)| < s$ [20]. In weakly coupled UV completions, only the amplitudes mediated by the neutral vector boson exchange in the t channel (see the discussion in Sec. III A) can violate the convergence $\lim_{s \rightarrow \infty} |A(s)| < s$ requirement. However, even in this case for the operators with four identical fermions, as for the operator in Eq. (7), the sign of the pole at infinity is fixed, and it coincides with the sign of the cross section term in the dispersion relations [see Eq. (10)]. Recently, Ref. [21] presented an analysis of the dispersion relations for four fermion operators using Jacob-Wick decomposition. In agreement with our findings, for the amplitudes satisfying the $\lim_{s \rightarrow \infty} |A(s)| < s$ condition, the authors have shown that for scalar-dominated (vector-dominated) UV completions we have $c_{RR} > 0 (< 0)$.

In the following, we will derive the set of the dispersion relations for the whole set of four fermion operators and identify the UV completions leading to the various signs of the Wilson coefficients.

IV. FOUR FERMION OPERATORS

First of all, let us define a complete basis of the four fermion operators, and we will do this following the notations of Refs. [22,23]:

purely left handed:

$$\begin{aligned} O_{ll}^{ijkm} &= (\bar{l}_L^i \gamma_\mu l_L^j) (\bar{l}_L^k \gamma^\mu l_L^m), & O_{qq}^{(1)ijkm} &= (\bar{q}_L^i \gamma_\mu q_L^j) (\bar{q}_L^k \gamma^\mu q_L^m), \\ O_{qq}^{(3)ijkm} &= (\bar{q}_L^i \gamma_\mu \sigma_a q_L^j) (\bar{q}_L^k \gamma^\mu \sigma_a q_L^m), & O_{ql}^{(1)ijkm} &= (\bar{l}_L^i \gamma_\mu l_L^j) (\bar{q}_L^k \gamma^\mu q_L^m), \\ O_{ql}^{(3)ijkm} &= (\bar{l}_L^i \gamma_\mu \sigma_a l_L^j) (\bar{q}_L^k \gamma^\mu \sigma_a q_L^m), \end{aligned}$$

purely right handed:

$$\begin{aligned} O_{ee}^{ijkm} &= (\bar{e}_R^i \gamma_\mu e_R^j) (\bar{e}_R^k \gamma^\mu e_R^m), & O_{uu}^{ijkm} &= (\bar{u}_R^i \gamma_\mu u_R^j) (\bar{u}_R^k \gamma^\mu u_R^m), \\ O_{dd} &= (\bar{d}_R^i \gamma_\mu d_R^j) (\bar{d}_R^k \gamma^\mu d_R^m), & O_{ud} &= (\bar{u}_R^i \gamma_\mu u_R^j) (\bar{d}_R^k \gamma^\mu d_R^m), \\ O_{ud}^{(8)} &= (\bar{u}_R^i \gamma_\mu T_A u_R^j) (\bar{d}_R^k \gamma^\mu T_A d_R^m), & O_{eu} &= (\bar{e}_R^i \gamma_\mu e_R^j) (\bar{u}_R^k \gamma^\mu u_R^m), \\ O_{ed} &= (\bar{e}_R^i \gamma_\mu e_R^j) (\bar{d}_R^k \gamma^\mu d_R^m), \end{aligned} \quad (24)$$

left – right:

$$\begin{aligned} O_{le} &= (\bar{l}_L^i \gamma_\mu l_L^j) (\bar{e}_R^k \gamma^\mu e_R^m), & O_{qqe} &= (\bar{q}_L^i \gamma_\mu q_L^j) (\bar{e}_R^k \gamma^\mu e_R^m), \\ O_{lu} &= (\bar{l}_L^i \gamma_\mu l_L^j) (\bar{u}_R^k \gamma^\mu u_R^m), & O_{ld} &= (\bar{l}_L^i \gamma_\mu l_L^j) (\bar{d}_R^k \gamma^\mu d_R^m), \\ O_{qu}^{(1)} &= (\bar{q}_L^i \gamma_\mu q_L^j) (\bar{u}_R^k \gamma^\mu u_R^m), & O_{qu}^{(8)} &= (\bar{q}_L^i \gamma_\mu T_A q_L^j) (\bar{u}_R^k \gamma^\mu T_A u_R^m), \\ O_{qd}^{(1)} &= (\bar{q}_L^i \gamma_\mu q_L^j) (\bar{d}_R^k \gamma^\mu d_R^m), & O_{qd}^{(8)} &= (\bar{q}_L^i \gamma_\mu T_A q_L^j) (\bar{d}_R^k \gamma^\mu T_A d_R^m), \\ O_{ledq} &= (\bar{l}_L^i e_R^j) (\bar{d}_R^k q_L^m), & O_{quqd}^{(1)} &= (\bar{q}_L^i u_R^j) i\sigma_2 (\bar{q}_L^k d_R^m)^T, \\ O_{lequ}^{(1)} &= (\bar{l}_L^i e_R^j) i\sigma_2 (\bar{q}_L^k u_R^m)^T, & O_{lequ}^{(3)} &= (\bar{l}_L^i \sigma_{\mu\nu} e_R^j) i\sigma_2 (\bar{q}_L^k \sigma^{\mu\nu} u_R^m)^T, \\ O_{quqd}^{(8)} &= (\bar{q}_L^i T_A u_R^j) i\sigma_2 (\bar{q}_L^k T_A d_R^m)^T, \end{aligned} \quad (25)$$

baryon number violating:

$$\begin{aligned} O_{duq} &= \epsilon_{ABC} (\bar{d}_R^A u_R^B) (\bar{q}_L^C i\sigma_2 l_L), & O_{qqu} &= \epsilon_{ABC} (\bar{q}_L^A i\sigma_2 q_L^B) (\bar{u}_R^C e_R), \\ O_{duu} &= \epsilon_{ABC} (\bar{d}_R^A u_R^B) (\bar{u}_R^C e_R), & O_{qqq} &= \epsilon_{ABC} (i\sigma_2)_{\alpha\delta} (i\sigma_2)_{\beta\gamma} (\bar{q}_L^{C\alpha} q_L^{B\beta}) (\bar{q}_L^{\delta\gamma} l_L^\beta). \end{aligned} \quad (26)$$

The rest of the operators can be reduced to the basis of Eqs. (24)–(26), using the Fierz identities and completeness relations for the $SU(2)$ and $SU(3)$ generators

$$\sum_{a=1}^3 (\sigma^a)_{ij} (\sigma^a)_{kl} = 2 \left(\delta_{il} \delta_{kj} - \frac{1}{2} \delta_{ij} \delta_{kl} \right) \quad (27)$$

$$\sum_{A=1}^8 (T^A)_{ij} (T^A)_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{1}{3} \delta_{ij} \delta_{kl} \right). \quad (28)$$

As we have seen in the previous section, the dispersion relations are effective in the case of forward scattering i.e., when

the initial and final states are the same.³ Therefore, only the following subspace of operators can be subject to sum rules,

$$\begin{aligned} O_{ll}^{iijj}, O_{ll}^{ijji}, O_{qq}^{(1,3)iijj}, O_{qq}^{(1,3)ijji}, O_{ql}^{(1,3)iikk}, O_{ee,uu,dd}^{iijj}, O_{ee,uu,dd}^{ijji}, \\ O_{ud}^{(1),(8)iijj}, O_{ed}^{iijj}, O_{eu}^{iijj}, O_{le,qe,lu,ld}^{iijj}, O_{qu}^{(1),(8)iijj}, O_{qd}^{(1),(8)iijj}, \end{aligned} \quad (29)$$

³Recently, it was shown [8,9] that for dimension-8 operators, the scattering of mixed (entangled) flavor states can lead to additional constraints on the Wilson coefficients. In the case of dimension-6 operators, performing a similar analysis requires additional assumptions on the UV completion [specifically; in Eq. (8), one has to suppose that either the charge 0 or charge 2 cross section dominates the dispersion relation for all the fermion flavors) since the dispersion relations are of indefinite sign. We do not investigate this direction further.

which will be the focus of this paper. For the operators in Eq. (29), the discussion follows closely the results reported for O_{ee} above. Therefore, we will henceforth report only the results and the examples of UV completions leading to various signs.

A. Experimental constraints

Having defined the operators which we will consider in our discussion, let us briefly mention the status of the experimental bounds based on the discussion in Refs. [24,25]. Current experimental bounds on four lepton and two-lepton, two-quark operator are obtained from measurements involving combinations of the Z , W pole observables, fermion production at LEP (large electron-positron collider), low-energy neutrino scatterings, parity violating electron scatterings, and parity violation in atoms. One of the challenges in deriving these bounds comes from the modifications of W , Z vertices which, too, can contribute to the same low-energy observables, so that the global fit including the W , Z pole observables becomes necessary. For example, for two-lepton, two-quark operators Ref. [25] has found nine flat directions unbounded experimentally. Current combinations of the low-energy experimental constraints as well as LHC measurements bound the various Wilson coefficients in the range 10^{-2} – 10^{-3} (where the operators are assumed to be suppressed by the v_{ew}^2 scale), which means sensitivity to the scales \mathcal{O} (few TeV). Just to be specific, for example, the four-electron operator discussed in Eq. (7) is bounded by the Bhabha scattering measurements at LEP-2 [26] and SLAC E158 experiment for the Møller scattering ($e^-e^- \rightarrow e^-e^-$) [27], where both experiments are testing the complementary combinations of the Wilson coefficients leading to the net sensitivity of $\sim 4 \times 10^{-3} v_{ew}^{-2}$ on the value of the Wilson coefficient. LHC measurements of the dilepton production in pp scattering leads to additional strong constraints on the two quark-two lepton operators [28,29], where for some operators we will become sensitive to new physics up to the scale of ~ 50 TeV. So far, all of the measurements are consistent with SM predictions.

B. Fully right handed

1. O_{ee}

This operator has already been discussed in Sec. III, and we would just like to emphasize that there are no sum rules for more than two flavors of fermions. Following the notations of Eqs. (24) and (25), the dispersion relations can be summarized as

$$-8c_{ee}^{iiii} = \int \frac{ds}{\pi s} (\sigma_{\bar{e}_i e_i} - \sigma_{e_i \bar{e}_i}) + C_\infty \quad (30)$$

$$-2(c_{ee}^{ijjj})_{i \neq j} = \int \frac{ds}{\pi s} (\sigma_{\bar{e}_i e_j} - \sigma_{\bar{e}_j e_i}) + C_\infty. \quad (31)$$

Note that in this simple case where the fields are singlets the operators O_{ee}^{ijjj} and O_{ee}^{jjji} are identical after Fierzing, and O_{ee}^{ijjj} and O_{ee}^{jjji} are just trivially identical by symmetrization, so we report the dispersion relation only in terms of c_{ee}^{ijjj} in order to not double-count the operators. Summarizing the discussion about UV completions in Sec. III, we have the following:

- cee<0: (a): neutral Z' (at tree level); (b) vectorlike singlet fermion Ψ and a heavy singlet complex scalar Φ with $Q[\Phi\Psi] = -1$ at one loop.
- cee>0: (a) charge-2 scalar (at tree level); (b) For operators with different flavours of fermions ($O_{ee}^{ijjj}|_{i \neq j}$), Z' can lead to a possibly positive sign if it's couplings to the different flavours of the fermions are of opposite signs.

2. O_{uu} , O_{dd}

Let us proceed with our investigation of the four fermion quark operators. The discussion proceeds exactly in the same way as for the leptons, except for new color structure. Fierzing them into the basis of Eq. (24) and (25), there are only four structures of the operators $O_{uu,dd}^{ijjj}$, $O_{uu,dd}^{jjji}$, which are in this case not related by a Fierz identity because of an implicit contraction of color indices. Let us start with the operators where all of the quarks have the same hypercharge and focus on the operator O_{uu}^{iiii} . Denoting by α, β the color indices and considering same and different color scatterings, we will obtain the following relations:

$$\begin{aligned} -8c_{uu}^{iiii} &= \int \frac{ds}{\pi s} (\sigma_{u_\alpha \bar{u}_\alpha} - \sigma_{u_\alpha u_\alpha}) + C_\infty^{\alpha\alpha} \\ &= \int \frac{ds}{\pi s} \left(\frac{2\sigma_{u\bar{u}}^{(8)} + \sigma_{u\bar{u}}^{(1)}}{3} - \sigma_{uu}^{(6)} \right) - C_\infty^{uu,(6)} \\ -4c_{uu}^{iiii} &= \left[\int \frac{ds}{\pi s} (\sigma_{u_\alpha \bar{u}_\beta} - \sigma_{u_\alpha u_\beta}) + C_\infty^{\alpha\beta} \right]_{\alpha \neq \beta} \\ &= \int \frac{ds}{\pi s} \left(\sigma_{u\bar{u}}^{(8)} - \frac{\sigma_{uu}^{(3)} + \sigma_{uu}^{(6)}}{2} \right) + C_\infty^{u\bar{u},(8)}. \end{aligned} \quad (32)$$

In the last step, we have decomposed the various possibilities of the initial-state fermions in terms of the $SU(3)$ QCD representations. This is convenient, since the Wigner-Eckart theorem requires the amplitudes to remain the same for all of the components of the irreducible representation. In particular, for the quark-antiquark scattering, the initial state will always be decomposed as a singlet and octet of $SU(3)$. Note that we can calculate the integral over the infinite contour using amplitude $A_{u\bar{u}}$ or its crossed version A_{uu} and the values of these integrals will satisfy (see Appendix C for details):

$$-C_\infty^{uu,(6)} = \frac{2C_\infty^{u\bar{u},(8)}}{3} + \frac{C_\infty^{u\bar{u},(1)}}{3} - \frac{C_\infty^{uu,(6)} + C_\infty^{uu,(3)}}{2} = C_\infty^{u\bar{u},(8)}. \quad (33)$$

Finally, let us note that the measurements of the quantities on the right-hand side of Eq. (32) [$\sigma^{(8)}$ and $\sigma^{(1)}$] at collider

experiments are practically impossible. However, we can combine the equations in Eq. (32) and express the dispersion integral in terms of the color averaged cross sections

$$-\frac{16}{3}c_{uu}^{iiii} = \int \frac{ds}{\pi s} (\sigma_{u\bar{u}} - \sigma_{uu}) - \frac{1}{3}C_{\infty}^{uu(6)} + \frac{2}{3}C_{\infty}^{u\bar{u}(8)}. \quad (34)$$

These types of relations can become useful for experimental verification of dispersion relations. In the rest of the paper, for the sake of completeness, we will always report the dispersion relations with and without color averaging. Again, C_{∞} can be nonvanishing, for example, in UV models with charge neutral vector resonances exchange in the t channel, but unlike the four-electron case, here this resonance can be either singlet or octet of $SU(3)$ QCD. Extending this analysis to the case of different flavor of the up quarks, we will obtain

$$\begin{aligned} -2c_u^{iijj} &= \int \frac{ds}{\pi s} \left(\frac{2\sigma_{u\bar{u}}^{(8)} + \sigma_{u\bar{u}}^{(1)}}{3} - \sigma_{uu}^{(6)} \right) - C_{\infty}^{uu(6)} \\ -2(c_u^{iijj} + c_u^{ijji}) &= \int \frac{ds}{\pi s} \left(\sigma_{u\bar{u}}^{(8)} - \frac{\sigma_{uu}^{(3)} + \sigma_{uu}^{(6)}}{2} \right) \\ &+ C_{\infty}^{u\bar{u}(8)}. \end{aligned} \quad (35)$$

We again mention that the operators O_{uu}^{iijj} and O_{uu}^{jjii} (similarly for O_{uu}^{ijji} and O_{uu}^{jiji}) are trivially identical, so it is important that we do not double-count them. As before, expressing everything in terms of uncolored cross sections, we find

$$-2c_{uu}^{iijj} - \frac{2}{3}c_{uu}^{ijji} = \int \frac{ds}{\pi s} (\sigma_{u\bar{u}} - \sigma_{uu}) + \frac{8}{9}C_{\infty}^{u\bar{u}(8)} + \frac{1}{9}C_{\infty}^{u\bar{u}(1)}, \quad (36)$$

and exactly the same relations hold for the down quarks.

Let us look at the possible UV completions. In the case of c_{uu}^{iiii} , we will have a negative sign of the Wilson coefficient with Z' and a positive sign for the charge $-4/3$ scalar which is in $\mathbf{6}$ of QCD $SU(3)$. Similar to the lepton case, we can generate a negative Wilson coefficient by adding vectorlike fermions and a complex scalar with $Q[\Phi\Psi] = 2/3$ and $(\Phi\Psi)$ fundamental of QCD. The discussion of two fermion flavors is almost identical to the lepton case. At last, for the four quark operators, there is a possibility of UV completion with QCD vector octet field with interactions of the type $gV_{\mu}^A(\bar{u}_i\gamma^{\mu}T^A u_i)$. In this case, similarly to Z' , integrals over infinite contours are not vanishing; see for details the discussion in Appendix B 2.

3. $O_{ud}^{(1),(8)}$

Just as in the previous section, we obtain (we will omit here flavor indices as these do not play any role, since the two up quarks and two down quarks should be the same to form sum rules)

$$\begin{aligned} -2\left(c_{ud}^{(1)} + \frac{1}{3}c_{ud}^{(8)}\right) &= \int \frac{ds}{\pi s} \left(\frac{2\sigma_{u\bar{d}}^{(8)} + \sigma_{u\bar{d}}^{(1)}}{3} - \sigma_{ud}^{(6)} \right) \\ &+ \frac{1}{3}(2C_{\infty}^{u\bar{d}(8)} + C_{\infty}^{u\bar{d}(1)}) \\ &- 2\left(c_{ud}^{(1)} - \frac{1}{6}c_{ud}^{(8)}\right) \\ &= \int \frac{ds}{\pi s} \left(\sigma_{u\bar{d}}^{(8)} - \frac{\sigma_{ud}^{(3)} + \sigma_{ud}^{(6)}}{2} \right) + C_{\infty}^{u\bar{d}(8)}. \end{aligned} \quad (37)$$

Rewriting the result in terms of an uncolored cross section, we will obtain

$$-2c_{ud}^{(1)} = \int \frac{ds}{\pi s} (\sigma_{u\bar{d}} - \sigma_{ud}) + \frac{8}{9}C_{\infty}^{u\bar{d}(8)} + \frac{1}{9}C_{\infty}^{u\bar{d}(1)}. \quad (38)$$

Interestingly, we see that only $c_{ud}^{(1)}$ can be reexpressed in terms of the color averaged cross sections.

4. O_{eu}, O_{ed}

The only operators with sum rule are of the form

$$(\bar{e}_{Ri}\gamma^{\mu}e_{Ri})(\bar{u}_{Rja}\gamma_{\mu}u_{Rja}), \quad (39)$$

where no summation over i, j is assumed. The sum rule is identical for both u and d quarks and is given by

$$-2c_{eu}^{iijj} = \int \frac{ds}{\pi s} (\sigma_{\bar{e}_i u_j} - \sigma_{\bar{e}_i \bar{u}_j}) + C_{\infty}^{\bar{e}u} \quad \text{and} \quad u \leftrightarrow d. \quad (40)$$

UV completions are as before, with a positive sign for $u(d)$ coming from a charge $1/3(4/3)$ scalar which is antifundamental of QCD and a negative sign from a charge $5/3(2/3)$ vector field V which is QCD triplet; note that the amplitude is convergent in the forward limit and the infinite integrals do vanish. Neutral charge Z' can lead to the arbitrary sign of the Wilson coefficient; again, in this case, the dispersion relations are saturated by the integrals at infinity.

C. Sum rules for ew doublets

In the next two subsections, we study operators that contribute to doublet-singlet scattering.

1. $O_{le}, O_{lu}, O_{ld}, O_{qe}$

Let us start with the fully leptonic operator and study the forward scattering of $l^p e$ where $p = 1, 2$ is the isospin index, in which case the sum rules are of the form

$$\begin{aligned}
-2c_{le}^{ijj} &= \int \frac{ds}{\pi s} (\sigma_{l_L^i \bar{e}_R^j} - \sigma_{l_L^i e_R^j}) + C_\infty^{l_i e_j} \\
&= \int \frac{ds}{\pi s} (\sigma_{e_L^i \bar{e}_R^j} - \sigma_{e_L^i e_R^j}) + C_\infty^{l_i e_j} \\
&= \int \frac{ds}{\pi s} (\sigma_{\nu_L^i \bar{e}_R^j} - \sigma_{\nu_L^i e_R^j}) + C_\infty^{l_i e_j}. \quad (41)
\end{aligned}$$

Similarly, we can write down the sum rules for the quark lepton operators,

$$\begin{aligned}
-2c_{lu}^{ijj} &= \int \frac{ds}{\pi s} (\sigma_{l_i^p u_j} - \sigma_{l_i^p u_j}) + C_\infty^{l_i u_j} \quad \text{and} \quad u \leftrightarrow d \\
-2c_{qe}^{ijj} &= \int \frac{ds}{\pi s} (\sigma_{\bar{q}_i^p e_j} - \sigma_{q_i^p e_j}) + C_\infty^{q_i e_j}, \quad (42)
\end{aligned}$$

where again p stand for the $SU(2)_L$ index. Note that these sum rules hold true for any isospin for the lepton and any color of the quark.

2. $O_{qu}^{(1),(8)}, O_{qd}^{(1),(8)}$

In this case, the discussion follows closely the one for the quark singlets, and we arrive at two sum rules (we again suppress the flavor index for brevity)

$$\begin{aligned}
-2 \left(c_{qd(u)}^{(1)} + \frac{1}{3} c_{qd(u)}^{(8)} \right) &= \int \frac{ds}{\pi s} \left(\frac{2\sigma_{q\bar{d}(\bar{u})}^{(8)} + \sigma_{q\bar{d}(\bar{u})}^{(1)}}{3} - \sigma_{qd(u)}^{(6)} \right) \\
&\quad + \frac{1}{3} (2C_\infty^{q\bar{d}(\bar{u})(8)} + C_\infty^{q\bar{d}(\bar{u})(1)}) \\
&\quad - 2 \left(c_{qd(u)}^{(1)} - \frac{1}{6} c_{qd(u)}^{(8)} \right) \\
&= \int \frac{ds}{\pi s} \left(\sigma_{q\bar{d}(\bar{u})}^{(8)} - \frac{\sigma_{qd(u)}^{(3)} + \sigma_{qd(u)}^{(6)}}{2} \right) \\
&\quad + C_\infty^{q\bar{d}(\bar{u})(8)}. \quad (43)
\end{aligned}$$

Note that σ_q stands for σ_{q^p} , where p is a $SU(2)$ index and cross sections on the right-hand side of the Eq. (43) can be taken for any component of the quark doublet. Rewriting the result in terms of an uncolored cross section, we will obtain

$$-2c_{qd(u)}^{(1)} = \int \frac{ds}{\pi s} (\sigma_{q\bar{d}(\bar{u})} - \sigma_{qd(u)}) + \frac{8}{9} C_\infty^{q\bar{d}(\bar{u})(8)} + \frac{1}{9} C_\infty^{q\bar{d}(\bar{u})(1)}. \quad (44)$$

Finally, we now study the left-handed operators that contribute to doublet-doublet scattering, where the doublet is that of weak isospin.

3. O_{ll}

Let us start with the four lepton operator $O_{ll}^{(ijj,ijji)}$. Expanding in components, the following sum rules can be derived [we assume $i \neq j$, and we do not write the operators obtained by interchange of $i \leftrightarrow j$ which are identical, just as in the discussion for up quarks; see Eq. (B11)]:

$$\begin{aligned}
-2c_{ll}^{ijj} - 2c_{ll}^{ijji} &= \int \frac{ds}{\pi s} (\sigma_{\bar{e}_i e_j} - \sigma_{e_i e_j}) + C_\infty^{ee, ev} \\
&= \int \frac{ds}{\pi s} (\sigma_{\bar{\nu}_i \nu_j} - \sigma_{\nu_i \bar{\nu}_j}) + C_\infty^{ee, ev} \\
-2c_{ll}^{ijj} &= \int \frac{ds}{\pi s} (\sigma_{\bar{e}_i \nu_j} - \sigma_{e_i \nu_j}) + C_\infty^{ev}. \quad (45)
\end{aligned}$$

We can decompose the amplitude into the weak isospin amplitudes (see Appendix C for details) to obtain the dispersion relations

$$\begin{aligned}
-2c_{ll}^{ijj} - 2c_{ll}^{ijji} &= \int \frac{ds}{\pi s} \left[\frac{1}{2} (\sigma_{i\bar{j}}^{(1)} + \sigma_{i\bar{j}}^{(3)}) - \sigma_{ij}^{(3)} \right] - C_\infty^{ij(3)} \\
-2c_{ll}^{ijj} &= \int \frac{ds}{\pi s} \left[\sigma_{i\bar{j}}^{(3)} - \frac{1}{2} (\sigma_{ij}^{(3)} + \sigma_{ij}^{(1)}) \right] + C_\infty^{(i\bar{j})(3)}, \quad (46)
\end{aligned}$$

where (i, j) and (i, \bar{j}) refer to the leptons from $l_i, l_j(\bar{l}_j)$ doublets and $\sigma_{ij, (i\bar{j})}^{(3,1)}$ refers to cross section from the triplet and singlet initial state formed by ij or $i\bar{j}$. In the case of an operator formed by just one lepton family, we will obtain

$$\begin{aligned}
-8c_{ll} &= \int \frac{ds}{\pi s} [\sigma_{e\bar{e}(\nu\bar{\nu})} - \sigma_{ee,(\nu\nu)}] + C_\infty^{ee} \\
&= \int \frac{ds}{\pi s} \left[\frac{1}{2} (\sigma_{ll}^{(1)} + \sigma_{ll}^{(3)}) - \sigma_{ll}^{(3)} \right] - C_\infty^{ll(3)} \\
-4c_{ll} &= \int \frac{ds}{\pi s} [\sigma_{e\bar{\nu}} - \sigma_{e\nu}] + C_\infty^{e\nu} \\
&= \int \frac{ds}{\pi s} \left[\sigma_{ll}^{(3)} - \frac{1}{2} (\sigma_{ll}^{(3)} + \sigma_{ll}^{(1)}) \right] + C_\infty^{(l\bar{l})(3)}. \quad (47)
\end{aligned}$$

4. $O_{lq}^{(3),(1)}$

In this case, only the operators with ijj flavor structure can contribute, and we arrive at the following dispersion relations:

$$\begin{aligned}
-2c_{lq}^{(1)} - 2c_{lq}^{(3)} &= \int \frac{ds}{\pi s} [\sigma_{e\bar{d}(\nu\bar{u})} - \sigma_{ed(\nu u)}] + C_\infty^{e\bar{d}(\nu\bar{u})} \\
-2c_{lq}^{(1)} + 2c_{lq}^{(3)} &= \int \frac{ds}{\pi s} [\sigma_{e\bar{u}(\nu\bar{d})} - \sigma_{eu(\nu d)}] + C_\infty^{e\bar{u}(\nu\bar{d})}. \quad (48)
\end{aligned}$$

As before, decomposing the cross section under isospin, we will obtain

$$\begin{aligned} -2c_{lq}^{(1)} - 2c_{lq}^{(3)} &= \int \frac{ds}{\pi s} \left[\frac{1}{2} (\sigma_{l\bar{q}}^{(1)} + \sigma_{l\bar{q}}^{(3)}) - \sigma_{lq}^{(3)} \right] - C_{\infty}^{lq(3)} \\ -2c_{lq}^{(1)} + 2c_{lq}^{(3)} &= \int \frac{ds}{\pi s} \left[\sigma_{l\bar{q}}^{(3)} - \frac{1}{2} (\sigma_{ql}^{(1)} + \sigma_{ql}^{(3)}) \right] + C_{\infty}^{l\bar{q}(3)}. \end{aligned} \quad (49)$$

5. O_{qq} and $O_{qq}^{(3)}$

Now, let us proceed to the four quark operators; we will start with the case where all quark doublets belong to the same generation. Then, the dispersion relations in terms of the octet and singlet cross sections will be given by

$$\begin{aligned} -8(c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\frac{2\sigma_{u\bar{u}}^{(8)} + \sigma_{u\bar{u}}^{(1)}}{3} - \sigma_{uu}^{(6)} \right] - C_{\infty}^{(6)uu} \\ -4(c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\sigma_{u\bar{u}}^{(8)} - \frac{\sigma_{uu}^{\bar{3}} + \sigma_{uu}^{(6)}}{2} \right] + C_{\infty}^{(8)u\bar{u}} \\ -4(c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\frac{2\sigma_{u\bar{d}}^{(8)} + \sigma_{u\bar{d}}^{(1)}}{3} - \sigma_{ud}^{(6)} \right] - C_{\infty}^{ud(6)} \\ -4(c_{qq}^{(1)} - c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\sigma_{u\bar{d}}^{(8)} - \frac{\sigma_{ud}^{(6)} + \sigma_{ud}^{\bar{3}}}{2} \right] + C_{\infty}^{ud(8)}. \end{aligned} \quad (50)$$

We can proceed further by performing the decomposition in terms of the $SU(2)_L$ multiplets using the relations

$$\begin{aligned} \sigma_{u\bar{u}} &= \frac{1}{2} (\sigma_{q\bar{q}}^{(1)} + \sigma_{q\bar{q}}^{(3)}), & \sigma_{u\bar{d}} &= \sigma_{q\bar{q}}^{(3)} \\ \sigma_{uu} &= \sigma_{qq}^{(3)}, & \sigma_{ud} &= \frac{1}{2} (\sigma_{qq}^{(1)} + \sigma_{qq}^{(3)}). \end{aligned} \quad (51)$$

Then, we will obtain (the first index will refer now to the QCD multiplet, and the second one will refer to electroweak)

$$\begin{aligned} -8(c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\frac{1}{6} ((2\sigma_{q\bar{q}}^{(8,1)} + \sigma_{q\bar{q}}^{(1,1)} + 2\sigma_{q\bar{q}}^{(8,3)} + \sigma_{q\bar{q}}^{(1,3)})) - \sigma_{qq}^{(6,3)} \right] - C_{qq\infty}^{(6,3)} \\ -4(c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\frac{1}{2} (\sigma_{q\bar{q}}^{(8,1)} + \sigma_{q\bar{q}}^{(8,3)}) - \frac{1}{2} (\sigma_{qq}^{\bar{3},3} + \sigma_{qq}^{(6,3)}) \right] + \frac{C_{qq\infty}^{(8,1)} + C_{qq\infty}^{(8,3)}}{2} \\ -4(c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\frac{1}{3} (2\sigma_{q\bar{q}}^{(8,3)} + \sigma_{q\bar{q}}^{(1,3)}) - \frac{1}{2} (\sigma_{qq}^{(6,1)} + \sigma_{qq}^{(6,3)}) \right] - \frac{C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)}}{2} \\ -4(c_{qq}^{(1)} - c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left[\sigma_{q\bar{q}}^{(8,3)} - \frac{1}{4} (\sigma_{qq}^{\bar{3},1} + \sigma_{qq}^{(6,1)} + \sigma_{qq}^{\bar{3},3} + \sigma_{qq}^{(6,3)}) \right] + C_{qq\infty}^{(8,3)}. \end{aligned} \quad (52)$$

In terms of the color averaged cross sections,

$$\begin{aligned} \frac{16}{3} (c_{qq}^{(1)} + c_{qq}^{(3)}) &= \int \frac{ds}{\pi s} \left(\frac{\sigma_{q\bar{q}}^{(3)} + \sigma_{q\bar{q}}^{(1)}}{2} - \sigma_{qq}^{(3)} \right) - \frac{C_{qq\infty}^{(6,3)}}{3} + \frac{C_{qq\infty}^{(8,1)} + C_{qq\infty}^{(8,3)}}{3} \\ -4 \left(c_{qq}^{(1)} - \frac{c_{qq}^{(3)}}{3} \right) &= \int \frac{ds}{\pi s} \left(\sigma_{q\bar{q}}^{(3)} - \frac{\sigma_{qq}^{(1)} + \sigma_{qq}^{(3)}}{2} \right) - \frac{C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)}}{6} + \frac{2C_{qq\infty}^{(8,3)}}{3}. \end{aligned} \quad (53)$$

In the case of two flavors, the dispersion relations become

$$\begin{aligned}
-2(c_{qq}^{iijj} + c_{qq}^{ijji} + c_{qq}^{(3)iijj} + c_{qq}^{(3)ijji}) &= \int \frac{ds}{\pi s} \left[\frac{1}{6} (2\sigma_{q\bar{q}}^{(8,1)} + \sigma_{q\bar{q}}^{(1,1)} + 2\sigma_{q\bar{q}}^{(8,3)} + \sigma_{q\bar{q}}^{(1,3)}) - \sigma_{qq}^{(6,3)} \right] - C_{qq\infty}^{(6,3)} \\
-2(c_{qq}^{iijj} + c_{qq}^{(3)iijj}) &= \int \frac{ds}{\pi s} \left[\frac{1}{2} (\sigma_{q\bar{q}}^{(8,1)} + \sigma_{q\bar{q}}^{(8,3)}) - \frac{1}{2} (\sigma_{qq}^{(\bar{3},3)} + \sigma_{qq}^{(6,3)}) \right] + \frac{C_{q\bar{q}\infty}^{(8,1)} + C_{q\bar{q}\infty}^{(8,3)}}{2} \\
-2(c_{qq}^{iijj} - c_{qq}^{(3)iijj} + 2c_{qq}^{(3)ijji}) &= \int \frac{ds}{\pi s} \left[\frac{1}{3} (2\sigma_{q\bar{q}}^{(8,3)} + \sigma_{q\bar{q}}^{(1,3)}) - \frac{1}{2} (\sigma_{qq}^{(6,1)} + \sigma_{qq}^{(6,3)}) \right] \\
&\quad - \frac{C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)}}{2} \\
-2(c_{qq}^{iijj} - c_{qq}^{(3)iijj}) &= \int \frac{ds}{\pi s} \left[\sigma_{q\bar{q}}^{(8,3)} - \frac{1}{4} (\sigma_{qq}^{(\bar{3},1)} + \sigma_{qq}^{(6,1)} + \sigma_{qq}^{(\bar{3},3)} + \sigma_{qq}^{(6,3)}) \right] - C_{q\bar{q}\infty}^{(8,3)}.
\end{aligned} \tag{54}$$

The power of these relations allows us to understand immediately the signs of the Wilson coefficients in the various UV completions. For example, for a scalar diquark which is in $(\bar{6}, 1, -1/3)$ representation under $SU(3) \times SU(2) \times U(1)_Y$, we will get

$$c_{qq,6}^{iijj} = c_{qq,6}^{(3)ijji} = -c_{qq,6}^{(3)iijj} = -c_{qq,6}^{ijji} > 0. \tag{55}$$

Similarly, for a scalar diquark which is in $(3, 1, -1/3)$, we will get

$$c_{qq,3}^{iijj} = c_{qq,3}^{ijji} = -c_{qq,3}^{(3)ijji} = -c_{qq,3}^{(3)iijj} > 0. \tag{56}$$

Finally, we can sum and report these sum rules in terms of color averaged cross sections, which yield two equations depending on whether the initial and final states form $SU(2)_L$ triplets or singlets,

$$\begin{aligned}
&-2 \left(c_{qq}^{iijj} + c_{qq}^{(3)iijj} + \frac{1}{3} c_{qq}^{ijji} + \frac{1}{3} c_{qq}^{(3)ijji} \right) \\
&= \int \frac{ds}{\pi s} \left[\frac{\sigma_{q\bar{q}}^{(3)} + \sigma_{q\bar{q}}^{(1)}}{2} - \sigma_{qq}^{(3)} \right] - \frac{C_{qq\infty}^{(6,3)}}{3} + \frac{C_{q\bar{q}\infty}^{(8,1)} + C_{q\bar{q}\infty}^{(8,3)}}{3}, \\
&-2 \left(c_{qq}^{iijj} - c_{qq}^{(3)iijj} + \frac{2}{3} c_{qq}^{(3)ijji} \right) \\
&= \int \frac{ds}{\pi s} \left[\sigma_{q\bar{q}}^{(3)} - \frac{\sigma_{qq}^{(1)} + \sigma_{qq}^{(3)}}{2} \right] - \frac{C_{qq\infty}^{(6,1)} + C_{qq\infty}^{(6,3)}}{6} + \frac{2C_{q\bar{q}\infty}^{(8,3)}}{3}.
\end{aligned} \tag{57}$$

V. SUMMARY

In this work, we explored the sum rules for four-fermion operators at dimension-6 level. As expected, the convergence of the dispersion integrals leading to the dimension-6 Wilson coefficients is not guaranteed and in particular is spoiled by the t-channel exchange of the vector bosons.

This additional feature can modify the predictions of the dispersion relations for sign and strength of IR interactions, and for some UV completions, the value of the Wilson coefficients can be even saturated by the pole at infinity. However, we find that this ambiguity of IR couplings is not related to the (non)convergence of the dispersion integrals, and as an example, we have constructed, in addition to tree level, one-loop weakly coupled models (see Sec. III C) where both signs become available even when the integral over the infinite circle vanishes.

We presented forward dispersion relations for all possible four-fermion dimension-6 operators. To facilitate the connection between the values of the Wilson coefficients and new physics scenarios, we have performed the decomposition in terms of the $SU(2)$ and $SU(3)$ multiplets. In the case of the forward amplitudes satisfying super-Froissart convergence criteria $|A(s)|_{s \rightarrow \infty} < s$, these relations predict in a model-independent way processes with enhanced total cross section in the case of discoveries in low-energy experiments. We carefully indicate all the relevant quantum numbers of the quantities involved in our dispersion relations in order to provide a convenient dictionary for future measurements, where the precise structure of initial states is often unavailable. This can have curious consequences; for example, Eq. (38) tells us that the Wilson coefficients $c_{ud,(qu),(qd)}^{(8)}$ do not enter the dispersion relations with color averaged cross sections.

We emphasize that these sum rules are to be interpreted as a model-independent link between UV and IR measurements (up to C_∞), as opposed to the usual positivity bounds. Even though they are less constraining on the EFT parameter space, these relations can instead be used as a powerful tool for model building to unearth the underlying, fundamental physics that is to be explored in the coming years.

ACKNOWLEDGMENTS

A. A. was supported in part by the MIUR Contract No. 2017L5W2PT. D. G. acknowledges support through

the Ramanujan Fellowship and MATRICS Grant of the Department of Science and Technology, Government of India. We would like to thank Joan Elias Miro for discussion and comments.

APPENDIX A: MASSLESS SPINOR HELICITY CONVENTIONS

We will briefly summarize the key results relevant to us (for a pedagogical introduction, see Ref. [30]) in the $(+, -, -, -)$ signature metric (we will follow the conventions discussed in Refs. [31–33]). We have the two-component spinors $v_{L/R}, u_{L/R}$ and their barred versions. They are related by crossing symmetry, $u_{L/R} = v_{R/L}, \bar{u}_{L/R} = \bar{v}_{R/L}$. It is important to realize that for antiparticles the spinor has opposite handedness to the field that describes it. For instance, a right chiral field e_R has an antiparticle which has the spinor v_L , while the particle carries the spinor u_R . In other words, both u_R and v_L correspond to a right chiral field, whereas v_R and u_L correspond to a left chiral field. To be absolutely clear, we will just refer to the handedness of the relevant spinor as opposed to the helicity of a particle/antiparticle wherever necessary. Operationally, we will assign the brackets

$$\bar{v}_L = \bar{u}_R \equiv [, \quad \bar{v}_R = \bar{u}_L \equiv \langle , \quad v_L = u_R \equiv \rangle, \quad v_R = u_L \equiv \equiv]. \quad (\text{A1})$$

The inner product is antisymmetric as is expected for Grassman-valued quantities

$$\langle pq \rangle = -\langle qp \rangle \quad [pq] = -[qp]. \quad (\text{A2})$$

Note that this also means that $\langle pp \rangle = 0 = [pp]$. Mixed brackets vanish. The formalism encodes a lot of power—for example, it tells us that a \langle and \rangle type spinor cannot occur at a vertex unless there is a γ^μ involved—a vector connects opposite helicity particles. Similarly, the same helicity spinors making up a vertex indicate a scalar is involved. Analytic continuation to the negative momenta can be done using the following prescription:

$$| - p \rangle = i | p \rangle \quad | - p] = i [p]. \quad (\text{A3})$$

These brackets satisfy the following properties:

$$\begin{aligned} \langle 1 | \gamma^\mu | 2 \rangle &= [2 | \gamma^\mu | 1] \\ [i | \gamma_\mu | i \rangle &= 2 p_i \quad \langle i j \rangle [i j] = -2 p_i \cdot p_j = (p_i - p_j)^2. \end{aligned} \quad (\text{A4})$$

The Mandelstam variables will be given by

$$\begin{aligned} s &= 2 p_1 \cdot p_2 = -[12] \langle 12 \rangle & t &= -2 p_3 \cdot p_1 = [13] \langle 13 \rangle \\ u &= -2 p_4 \cdot p_1 = [14] \langle 14 \rangle. \end{aligned} \quad (\text{A5})$$

Finally, the Fierz rearrangement can be written as the following identity:

$$[1 | \gamma^\mu | 2 \rangle [3 | \gamma_\mu | 4 \rangle = -2 [13] \langle 24 \rangle. \quad (\text{A6})$$

APPENDIX B: DETAILS ABOUT CROSS SECTIONS AND LOOP AMPLITUDES

In this Appendix, we will give details about explicit verification of the dispersion relations presented in the text for various models.

1. Z' at tree level

Let us start with neutral vector Z' coupled to right-handed current via $\lambda Z'_\mu \bar{e}_R \gamma^\mu e_R$. It generates $e_R \bar{e}_R$ scattering through the diagrams given in Fig. 2. The full amplitude will be given by

$$A_{e\bar{e}} = -2\lambda^2 [14] \langle 23 \rangle \left(\frac{1}{s-m^2} + \frac{1}{t-m^2} \right). \quad (\text{B1})$$

Matching the IR and UV amplitudes at low energies, we will obtain

$$\begin{aligned} -8c_{ee}^{1111} [14] \langle 23 \rangle &= -2\lambda^2 [14] \langle 23 \rangle \left(\frac{1}{-m^2} + \frac{1}{-m^2} \right) \\ \Rightarrow c_{ee}^{1111} &= -\frac{\lambda^2}{2m^2}. \end{aligned} \quad (\text{B2})$$

Let us verify that this is consistent with our dispersion relation. With a vector Z' at order $O(\lambda^2)$ in perturbation theory, we have $\sigma_{e\bar{e}} \neq 0$ and $\sigma_{ee} = 0$. To calculate the cross sections, note that by the optical theorem we have

$$\text{Im}(e\bar{e} \rightarrow e\bar{e}) = s\sigma_{e\bar{e}}^{\text{tot}}. \quad (\text{B3})$$

We use the fact that $\text{Im}\left(\frac{1}{p^2-m^2+ie}\right) = -\pi\delta(p^2-m^2)$, which, when substituted in the amplitude (14), gives us

$$\text{Im}(e_L^+ e_R^- \rightarrow e_L^+ e_R^-) = 2\lambda^2 \pi s \delta(s-m^2). \quad (\text{B4})$$

Starting from the dispersion relation in Eq. (10), we will get

$$\begin{aligned} -8c_{ee}^{1111} &= \int \frac{ds}{\pi s} (\sigma_{e\bar{e}} - 0) + C_\infty = \int \frac{ds}{\pi s^2} \text{Im}(e\bar{e} \rightarrow e\bar{e}) \\ &+ C_\infty = \frac{2\lambda^2}{m^2} + C_\infty. \end{aligned} \quad (\text{B5})$$

Calculating explicitly C_∞ , we will obtain

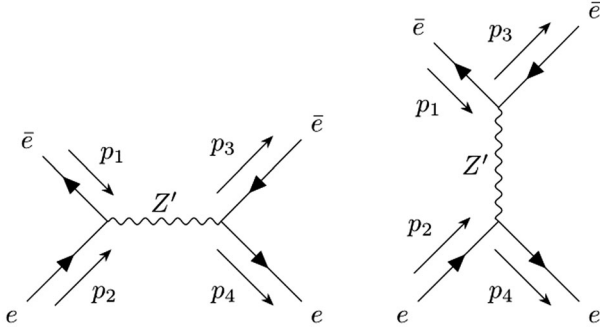


FIG. 2. In accordance with fermion statistics, these diagrams subtract.

$$C_\infty = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{A(|s_\Lambda|e^{i\theta}, 0)}{(|s_\Lambda|e^{i\theta})^2} \cdot (|s_\Lambda|e^{i\theta}) = \frac{2\lambda^2}{m^2}$$

$$A|_{t \rightarrow 0} = -2\lambda^2 s \left(\frac{1}{s-m^2} + \frac{1}{-m^2} \right), \quad (\text{B6})$$

which is of the same sign as the dispersion integral, and therefore we find

$$-8c_{ee}^{1111} = 4\lambda^2/m^2 \Rightarrow c_{ee}^{1111} = -\lambda^2/2m^2 \quad (\text{B7})$$

as claimed in (12), and our dispersion relation is explicitly verified.

2. Integrating out color octet

Very similarly to the charge neutral Z' , we can consider effects coming from integrating out color octet V which has zero electric charge. Let us look, for example, on an octet interacting with a right-handed up-quark current:

$$g_{ij} V_\mu^A (\bar{u}_i \gamma^\mu T^A u_j) \Rightarrow c_{uu}^{ijkl} = \frac{-g_{kj} g_{il}}{M_V^2} + \frac{g_{ij} g_{kl}}{3M_V^2}. \quad (\text{B8})$$

Let us assume that the octet couplings are universal and flavor diagonal; then, $g_{ij} = g\delta_{ij}$, and the Wilson coefficients are equal to

$$c_{uu}^{iijj} = \frac{2g^2}{3M_V^2}, \quad c_{uu}^{ijji} = \frac{-2g^2}{M_V^2}. \quad (\text{B9})$$

Now, let us look at dispersion relations for $i \neq j$; then similar to the discussion in Eq. (19), the cross sections will vanish at $O(g^2)$, and the right-hand side of Eq. (B11) will be controlled by the contribution of the integrals over infinite contours,

$$C_\infty^{(8)} = C_\infty^{\alpha\neq\beta} = \frac{8g^2}{3M_V^2}, \quad C_\infty^{\alpha\alpha} = -C_\infty^{(6)} = -\frac{4g^2}{3M_V^2}, \quad (\text{B10})$$

which confirm the dispersion relations

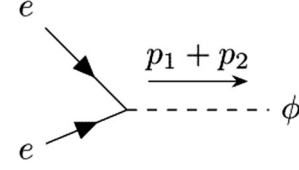


FIG. 3. Scalar production in ee collision; note that the vertex is $= 2!(-i\kappa)$.

$$-2c_u^{iijj} = -C_\infty^{uu,(6)} - 2(c_u^{iijj} + c_u^{ijji}) = C_\infty^{u\bar{u},(8)}. \quad (\text{B11})$$

3. Charge-2 scalar at tree level

Let us build a model where only $\sigma_{ee(\bar{e}\bar{e})}$ is present at the lowest order in perturbation theory. This can be done with a charge-2 scalar, which interacts as follows ($\kappa\phi\bar{e}_R e_R^c + \text{H.c.}$), where the c subscript stands for charge conjugation. Matching the amplitudes in EFT and UV theory, we will obtain

$$-8c_{ee}^{1111}[14]\langle 23 \rangle = -2!2!\kappa^2 \frac{[14]\langle 32 \rangle}{-m^2} \Rightarrow c_{ee}^{1111} = +\frac{\kappa^2}{2m^2}. \quad (\text{B12})$$

At $\mathcal{O}(\kappa^2)$, the only process contributing to the right-hand side of dispersion relations is $ee \rightarrow \phi$, see Fig. 3 for the Feynman diagram. If we treat ϕ as a very narrow resonance, then the cross section is equal to:

$$\sigma_{ee}^{\text{tot}} = 4\kappa^2 \pi \delta(s-m^2). \quad (\text{B13})$$

So, the dispersion relation becomes

$$-8c_{ee}^{1111} = \int \left(0 - \frac{ds}{\pi s} \sigma_{--} \right) = -\frac{4\kappa^2}{m^2}, \quad (\text{B14})$$

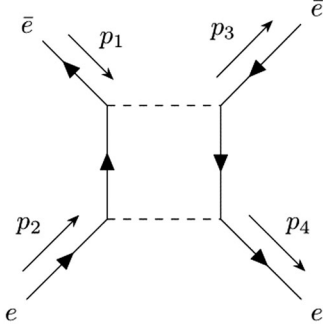
and as expected, we find $c_{ee}^{1111} = +\frac{\kappa^2}{2m^2}$.

4. Dispersion relation at one loop

At last, let us consider the following UV completion for the $(\bar{e}\gamma_\mu e)(\bar{e}\gamma_\mu e)$ operator. It will demonstrate that it is possible to have a negative Wilson coefficient with vanishing integrals over infinite circles. Let us extend the SM with a new heavy scalar Φ and vectorlike fermion Ψ with interactions

$$\lambda(\Phi\bar{e}_R\Psi) + \text{H.c.}, \quad (\text{B15})$$

where electric charges of new fields satisfy $Q[\Phi] + Q[\Psi] = -1$. Let us start by deriving the c_{ee} Wilson coefficient. We can consider $e\bar{e} \rightarrow e\bar{e}$ scattering; then, the amplitude will be given by a box diagram (see Fig. 4) and its crossed version. To match with EFT predictions, we can focus on the processes where external particles have vanishing momentum, in which case the amplitude will be given by

FIG. 4. Forward amplitude at $O(\lambda^4)$ order.

$$iM = \lambda^4 [1|\gamma_\mu|2][4|\gamma_\nu|3] \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - m^2)^4} - (2 \leftrightarrow 3), \quad (\text{B16})$$

where we have assumed that the masses of the new fields are equal $m[\Phi] = m[\Psi] = m$; the loop function for arbitrary masses is reported in the main text. Performing the integral, which is finite, and doing the Fierz rearrangements, we will obtain

$$M = \frac{1}{3} \frac{\lambda^4}{16\pi^2 m^2} [14]\langle 23 \rangle, \Rightarrow c_{ee}^{1111} = -\frac{1}{3} \frac{\lambda^4}{128\pi^2 m^2}. \quad (\text{B17})$$

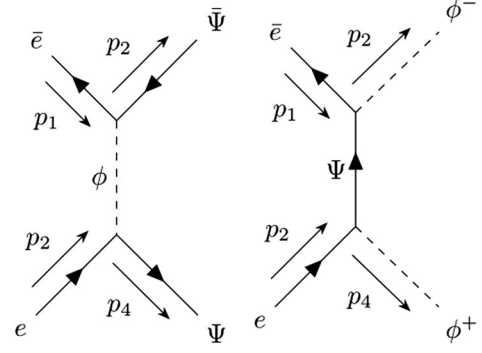
So, we see that sign of the Wilson coefficient is indeed negative. We have checked explicitly that the amplitude satisfies $A(s)/s \rightarrow 0$ at an infinite circle, so all we need to know is the cross section for $e\bar{e}$ scattering to verify the dispersion relations. The total cross section at order $O(\lambda^4)$ receives contribution from two processes: $e\bar{e} \rightarrow \Psi\bar{\Psi}$ and $e\bar{e} \rightarrow \Phi\Phi^*$ (see Fig. 5). Performing the calculation, we obtain

$$\begin{aligned} \sigma(e\bar{e} \rightarrow \Psi\bar{\Psi}) &= \frac{\lambda^4}{16\pi s^2} \sqrt{s(s-4m^2)} \\ \sigma(e\bar{e} \rightarrow \Phi\Phi^*) &= \frac{\lambda^4}{64\pi s^2} \left(-8\sqrt{s(s-4m^2)} \right. \\ &\quad \left. - 4s \log \left(\frac{s - \sqrt{s(s-4m^2)}}{s + \sqrt{s(s-4m^2)}} \right) \right). \end{aligned} \quad (\text{B18})$$

Performing the calculation for the dispersion integral, we will obtain

$$\begin{aligned} &\int \frac{ds}{\pi s} (\sigma(e\bar{e} \rightarrow \Psi\bar{\Psi}) + \sigma(e\bar{e} \rightarrow \Phi\Phi^*)) \\ &= \frac{\lambda^4}{\pi^2 m^2} (1/96 + 1/96) = \frac{\lambda^4}{48m^2 \pi^2} = -8c_{ee}^{1111}, \end{aligned} \quad (\text{B19})$$

satisfying the identity of Eq. (10).

FIG. 5. Processes contributing to the $\sigma_{e\bar{e}}$ at $O(\lambda^4)$ order.

APPENDIX C: DECOMPOSITION OF CROSS SECTIONS IN TERMS OF $SU(2)$ AND $SU(3)$ IRREDUCIBLE REPRESENTATIONS

In this Appendix, we will give details of the decomposition of amplitudes in terms of the irreducible representations of the electroweak $SU(2)$ and QCD $SU(3)$ groups. The Wigner-Eckart theorem tells us that the resulting amplitudes and cross sections will depend only on representations of the initial state (For a similar decomposition of isospins, see [34,35]; for custodial isospin see [4,36], and see [5,13] for other groups). Let us start with two-lepton doublet scattering $L_1 L_2 \rightarrow L_1 L_2$, where L_1, L_2 are $SU(2)_L$ doublet leptons, for, e.g., $(\nu_e, e)^T$. Then, the initial state can be decomposed as a singlet and a triplet under $SU(2)$: $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$. The singlet and triplet states are defined as

$$\begin{aligned} S = \text{singlet} &= \frac{1}{\sqrt{2}} (|\nu e\rangle - |e\nu\rangle) \\ T = \text{triplet} &= \begin{cases} |\nu\nu\rangle \\ \frac{1}{\sqrt{2}} (|\nu e\rangle + |e\nu\rangle), \\ |ee\rangle \end{cases} \end{aligned} \quad (\text{C1})$$

where (ν, e) are the components of EW doublet. Similarly, we can decompose the states for the lepton and antilepton scattering, where we find

$$L_1 = (\nu_1, e_1)^T, \quad \bar{L}_2 = (-\bar{e}_2, \bar{\nu}_2)^T \quad (\text{C2})$$

$$\tilde{S} = \text{singlet} = \frac{1}{\sqrt{2}} (|e\bar{e}\rangle + |\nu\bar{\nu}\rangle) \quad (\text{C3})$$

$$\tilde{T} = \text{triplet} = \begin{cases} -|\nu\bar{e}\rangle \\ \frac{1}{\sqrt{2}} (|\nu\bar{\nu}\rangle - |e\bar{e}\rangle). \\ |e\bar{e}\rangle \end{cases} \quad (\text{C4})$$

Using this decomposition, we can immediately see that the amplitude for the forward scatterings of the various components of the doublets will be decomposed as

$$\begin{aligned}
A_{ee} &= A_{LL}^{(3)}, & A_{e\bar{e}} &= \frac{A_{LL}^{(3)} + A_{LL}^{(1)}}{2}, \\
A_{\nu e} &= \frac{A_{LL}^{(1)} + A_{LL}^{(3)}}{2}, & A_{\bar{\nu}e} &= A_{LL}^{(3)}, \\
A_{\nu\nu} &= A_{LL}^{(3)}, & A_{\nu\bar{\nu}} &= \frac{A_{LL}^{(3)} + A_{LL}^{(1)}}{2}.
\end{aligned} \tag{C5}$$

Similarly, we can decompose the cross sections for quark lepton doublet scatterings. Note that forward amplitudes will satisfy the following crossing relations:

$$\begin{aligned}
A_{LL}^{(1)}(s, u) &= \frac{3A_{LL}^{(3)}(u, s) - A_{LL}^{(1)}(u, s)}{2}, \\
A_{LL}^{(3)}(s, u) &= \frac{A_{LL}^{(1)}(u, s) + A_{LL}^{(3)}(u, s)}{2}, \\
A_{L\bar{L}}^{(1)}(u, s) &= \frac{3A_{LL}^{(3)}(s, u) - A_{LL}^{(1)}(s, u)}{2}, \\
A_{L\bar{L}}^{(3)}(u, s) &= \frac{A_{LL}^{(1)}(s, u) + A_{LL}^{(3)}(s, u)}{2}.
\end{aligned} \tag{C6}$$

These relations can be easily obtained starting from crossing relations presented in Ref. [5]. Generically, the amplitude for $2 \rightarrow 2$ scattering can be decomposed as

$$A^{abcd} = \sum_{\alpha} P_{\alpha}^{abcd} A_{\alpha}, \tag{C7}$$

where a, b, c, d label the components of the individual particles, α stands for the irreducible representations of the product, and P_{α} are the corresponding projectors. Crossing symmetry relates

$$A_{cd}^{ab}(s, u) = A_{dc}^{a'b'}(u, s); \tag{C8}$$

then, using the crossing relations for projectors reported in Ref. [5], one can easily derive the expressions in Eq. (C5). Since we are looking at the dispersion relations for dimension-6 operators and the amplitudes in IR scale linearly with s , the integrals over infinite circle contours must satisfy

$$\begin{aligned}
C_{\infty}^{LL(L\bar{L})} &\equiv \int_{\text{infinite circle}} \frac{ds}{s^2} A^{LL(L\bar{L})}(s), \\
&- C_{\infty}^{LL(3)} = \frac{C_{\infty}^{L\bar{L}(3)} + C_{\infty}^{L\bar{L}(1)}}{2}, - \frac{C_{\infty}^{LL(3)} + C_{\infty}^{LL(1)}}{2} \\
&= C_{\infty}^{L\bar{L}(3)}.
\end{aligned} \tag{C9}$$

The situation is very similar for the quark-quark doublet scattering, but there we can decompose the initial state in the representations of the color $SU(3)$ as well (see Ref. [5] for an example).

1. $SU(3)$ decomposition

Let us consider for simplicity quark-(anti)quark scattering, where the particles are singlets under electroweak $SU(2)$; in this case, initial state can be decomposed as follows:

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}, \quad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \tag{C10}$$

In the case of two-particle scattering, there are only two possibilities: initial particles can have the same or different colors. For the quark-antiquark scattering various initial color states can be decomposed as

$$\begin{aligned}
|1\bar{1}\rangle &= \frac{S}{\sqrt{3}} + \frac{\lambda_8}{\sqrt{6}} + \frac{\lambda_2}{\sqrt{2}}, & |2\bar{2}\rangle &= \frac{S}{\sqrt{3}} + \frac{\lambda_8}{\sqrt{6}} - \frac{\lambda_2}{\sqrt{2}} \\
|3\bar{3}\rangle &= \frac{S - \sqrt{2}\lambda_8}{\sqrt{3}}, & |1\bar{2}\rangle &= \frac{\lambda_1 + i\lambda_2}{\sqrt{2}}, & |2\bar{1}\rangle &= \frac{\lambda_1 - i\lambda_2}{\sqrt{2}} \\
|1\bar{3}\rangle &= \frac{\lambda_4 + i\lambda_5}{\sqrt{2}}, & |3\bar{1}\rangle &= \frac{\lambda_4 - i\lambda_5}{\sqrt{2}}, & |2\bar{3}\rangle &= \frac{\lambda_6 + i\lambda_7}{\sqrt{2}}, \\
|3\bar{2}\rangle &= \frac{\lambda_6 - i\lambda_7}{\sqrt{2}},
\end{aligned} \tag{C11}$$

where $S = \frac{|1\bar{1}\rangle + |2\bar{2}\rangle + |3\bar{3}\rangle}{\sqrt{3}}$ is a $SU(3)$ singlet state and $(\lambda_1 \dots \lambda_8)$ are components of an octet, which can be formed using Gell-Mann matrices (our normalization is $\langle \lambda_i | \lambda_j \rangle = \delta_{ij}$). Similarly, we can decompose the quark-quark initial state in terms of the $\mathbf{6}$ and $\bar{\mathbf{3}}$. Note that in this case the same and different color initial states can be schematically decomposed as

$$|\alpha\alpha\rangle = \mathbf{6}_{\alpha\alpha}, \quad |\alpha\beta\rangle_{\alpha\neq\beta} = \frac{\mathbf{6}_{\alpha\beta} \pm \bar{\mathbf{3}}_{\alpha\beta}}{\sqrt{2}}. \tag{C12}$$

Then, the Wigner-Eckart theorem tells us that the total cross sections and forward scattering amplitudes will satisfy the relations

$$\sigma_{\alpha\alpha} = \sigma^{(6)}, \quad \sigma_{\alpha\beta}|_{\alpha\neq\beta} = \frac{1}{2}(\sigma^{(\bar{3})} + \sigma^{(6)}), \tag{C13}$$

$$\sigma_{\alpha\bar{\alpha}} = \frac{\sigma^{(1)} + 2\sigma^{(8)}}{3}, \quad \sigma_{\alpha\bar{\beta}}|_{\alpha\neq\beta} = \sigma^{(8)}, \tag{C14}$$

where $\alpha(\bar{\beta})$ indices indicate whether we are looking at the same or different color scatterings in qq or $q\bar{q}$ channels (q here stands for a quark, which can be either up or down type). In case we are interested in the color averaged cross sections, these will be related to the above as follows:

$$\begin{aligned}
\sigma_{qq} &\equiv (\sigma_{qq})_{\text{col.aver.}} = \frac{2}{3}\sigma^{(6)} + \frac{1}{3}\sigma^{(\bar{3})} \\
\sigma_{q\bar{q}} &\equiv (\sigma_{q\bar{q}})_{\text{col.aver.}} = \frac{1}{9}\sigma^{(1)} + \frac{8}{9}\sigma^{(8)}.
\end{aligned} \tag{C15}$$

At last, forward amplitudes decomposed under QCD representations will satisfy the following crossing relations:

$$\begin{aligned}
 A_{q\bar{q}}^{(\bar{3})}(s, u) &= \frac{-A_{q\bar{q}}^{(1)}(u, s) + 4A_{q\bar{q}}^{(8)}(u, s)}{3}, & A_{q\bar{q}}^{(6)}(s, u) &= \frac{A_{q\bar{q}}^{(1)}(u, s) + 2A_{q\bar{q}}^{(8)}(u, s)}{3} \\
 A_{q\bar{q}}^{(1)}(u, s) &= 2A_{q\bar{q}}^{(6)}(s, u) - A_{q\bar{q}}^{(\bar{3})}(s, u), & A_{q\bar{q}}^{(8)}(u, s) &= \frac{A_{q\bar{q}}^{(\bar{3})}(s, u) + A_{q\bar{q}}^{(6)}(s, u)}{2}.
 \end{aligned}
 \tag{C16}$$

Similarly, the contours over the infinite circles will be related as follows:

$$-C_{\infty}^{qq(6)} = \frac{C_{\infty}^{q\bar{q}(1)} + 2C_{\infty}^{q\bar{q}(8)}}{3}, \quad -\frac{C_{\infty}^{qq(\bar{3})} + C_{\infty}^{qq(6)}}{2} = C_{\infty}^{q\bar{q}(8)}.
 \tag{C17}$$

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