A chiral-spin symmetry in QCD in Minkowski spacetime

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In this paper, we look at how to construct in Minkowski spacetime a new type of *chiral-spin* group transformation of the spinor fields, similar to the one discovered by recent works of Glozman *et al.* in the context of high-temperature QCD and truncated studies in lattice calculations. Afterwards, we prove the invariance of free massless fermionic action under such group transformations, as well as the invariance of the Hamiltonian of free massless fermions. At the end, the possible presence of a symmetry driven by such new *chiral-spin* group at high-temperature QCD, also at nonzero chemical potential, is discussed.

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I. INTRODUCTION

In recent works, the emergence of an unexpected symmetry in lattice QCD simulations has been observed, in particular at high-temperature QCD [1–4], right above the chiral phase transition $T > T_c$, but for $T \lesssim 3T_c$, and in truncated studies (see Refs. [5–7] for information on these peculiar works). More specifically, in truncated studies a large degeneracy of hadron masses has been discovered. The symmetry which corresponds to such degeneracy has been explained by the group transformation $SU(2)_{CS}$ (or in words *chiral-spin* group) of the quark fields, first introduced in [5–7], and that contains the axial group $U(1)_A$ as subgroup. However, as we have studied in [8], the mass degeneracy could also be explained, in the Euclidean spacetime, by the group transformation which we have denoted with $SU(2)^{\mathcal{P}}_{CS}$, that is defined in a slightly different manner from $SU(2)_{CS}$, but still has $U(1)_A$ as subgroup. In fact, the two group transformations induce the same transformation in hadron correlators $\langle \mathcal{O}(y)\bar{\mathcal{O}}(x)\rangle$ calculated at fixed reference frame with $x = (\mathbf{0}, x_4)$ and $y = (\mathbf{0}, y_4)$, from which we can still extract the hadron masses, since at large $\mathcal{T} = y_4 - x_4$, we have $\langle \mathcal{O}(y)\bar{\mathcal{O}}(x)\rangle \sim \exp(-m\mathcal{T})$, with m the hadron mass associated with such correlator. Moreover, we have seen that while $SU(2)_{CS}$ is not a symmetry of the free fermionic action, which makes it not compatible with the possibility of deconfinement at extremely high T, $SU(2)^{\mathcal{P}}_{CS}$ is instead a symmetry of the free fermionic action, which makes it more suitable to check its presence at $T \gg T_c$, where QCD is supposed to approach at an almost-free theory.

The work done in Ref. [8] has been considered in Euclidean spacetime. Here, we see that we can define the $SU(2)^{\mathcal{P}}_{CS}$ also in Minkowskian and also prove that it leaves the fermionic action invariant, repeating the same argumentation of [8] (see Sec. III of this paper). For doing so, we need to define a $U(1)$ group starting simply from the parity operator (see Sec. II). Beside this, we also prove the invariance of the Hamiltonian of free massless fermions under $SU(2)^{\mathcal{P}}_{CS}$, giving how the operators of creation and annihilation of quarks and antiquarks (but in general fermions and antifermions) transform under $SU(2)^{\mathcal{P}}_{CS}$ (in Sec. IV). We also briefly discuss what happens when a gauge interaction term is added in the theory, and finally, in Minkowski space the argument made in Ref. [8], regarding the presence of $SU(2)^{\mathcal{P}}_{CS}$ at high T. Moreover, we will see that a possible chemical potential term in the action is $SU(2)^{\mathcal{P}}_{CS}$ invariant. Therefore, we expect that if $SU(2)^{\mathcal{P}}_{CS}$ would be present at high T and zero chemical potential, i.e., $\mu = 0$, it will be present also at $\mu \neq 0$ (Sec. V). At the end, we summarize the main points of this paper in Sec. VI, pointing out that in order to prove the $SU(2)^{\mathcal{P}}_{CS}$ symmetry at high-temperature QCD, it is important to study the presence of our $U(1)$ group derived from the parity operator, defined in Sec. II. This is something that must be checked on lattice simulations and that has not been investigated yet.

We remark that from Secs. II–IV (and including all appendixes), everything is kept general, and the reader can assume that we are considering a theory with whatever gauge group $\mathcal{G}$. Section V is instead specific for QCD, where $\mathcal{G} = SU(3)$, because it is in connection with the lattice results of Refs. [1–7].

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II. FROM PARITY TO A $U(1)$ GROUP

For a spinor field, the parity transformation is defined as $\psi(x) \to P\psi(x)P^\dagger = \gamma^0\psi(\mathcal{P}x)$, with $\mathcal{P} = \mathrm{diag}(1,-1,-1,-1)$ and P the parity operator with properties $P = P^\dagger$ and $P^2 = \mathbb{1}$. The application of two times this transformation gives back again the same spinor field, because $P^2\psi(x)P^{\dagger 2} = \gamma^0 P\psi(\mathcal{P}x)P^\dagger = \gamma^0\gamma^0\psi(\mathcal{P}^2x) = \psi(x)$, since $\mathcal{P}^2 = \mathcal{I}$ and $(\gamma^0)^2 = \mathbb{1}$ [see Eq. (A1) for the representation used for the gamma matrices in this paper]. Therefore, for n applications of parity, we have $P^n\psi(x)P^{\dagger n} = (\gamma^0)^n\psi(\mathcal{P}^n x)$, which is $\psi(x)$ for n even and $\gamma^0\psi(\mathcal{P}x)$ for n odd. Exploiting this fact, we can define the following spinor transformation:

$$U(1)_{\mathrm{P}}\colon \psi(x) \to \psi(x)^{U^\alpha_{\mathrm{P}}} \equiv \sum_{n=0}^{\infty} \frac{(\mathrm{i}\alpha)^n}{n!} P^n\psi(x)P^{\dagger n} = \cos(\alpha)\psi(x) + \mathrm{i}\sin(\alpha)\gamma^0\psi(\mathcal{P}x), \quad (1)$$

where α is some global parameter. We have therefore defined a unitary operator out of P, similar to what we have done in Ref. [8] in Euclidean space, where this has been done also for time reversal.

It is now convenient to introduce a bit of notation. We construct two fields $\psi_\pm(x) = \frac{1}{2}(\psi(x) \pm \psi(\mathcal{P}x))$, that we call "parity partners" and satisfying the properties: $\gamma^0\psi_\pm(x) = \pm P\psi_\pm(x)P^\dagger$ and $\psi_\pm(\mathcal{P}x) = \pm\psi_\pm(x)$. Afterwards we define the two-component field:

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}. \quad (2)$$

Now, we can transform $\psi_+(x)$ and $\psi_-(x)$ via $U(1)_{\mathrm{P}}$ separately, obtaining that $\psi_\pm(x)^{U^\alpha_{\mathrm{P}}} = \frac{1}{2}(\psi(x)^{U^\alpha_{\mathrm{P}}} \pm \psi(\mathcal{P}x)^{U^\alpha_{\mathrm{P}}})$. Therefore, $\Psi(x)$ transforms as

$$U(1)_{\mathrm{P}}\colon \Psi(x) \to \Psi(x)^{U^\alpha_{\mathrm{P}}} = \begin{pmatrix} \psi_+(x)^{U^\alpha_{\mathrm{P}}} \\ \psi_-(x)^{U^\alpha_{\mathrm{P}}} \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\alpha\gamma^0}\psi_+(x) \\ e^{-\mathrm{i}\alpha\gamma^0}\psi_-(x) \end{pmatrix} = e^{\mathrm{i}\alpha(\sigma^3\otimes\gamma^0)}\Psi(x), \quad (3)$$

where $\sigma^3 \otimes \gamma^0$ is Hermitian and traceless. From (3) is evident that $U(1)_{\mathrm{P}}$ transformations, acting on $\Psi(x)$, form a $U(1)$ group.

The transformations (1) also have important consequences on the fermionic actions, as we are going to see now. Let $\Gamma(x)$ be an unspecified matrix function. Under $\psi(x) \to \cos(\alpha)\psi(x) + \mathrm{i}\sin(\alpha)\gamma^0\psi(\mathcal{P}x)$ we observe that

$$\begin{aligned}\bar\psi(x)\Gamma(x)\psi(x) \to{}& \cos(\alpha)^2\bar\psi(x)\Gamma(x)\psi(x) \\ &+ \sin(\alpha)^2\bar\psi(\mathcal{P}x)\gamma^0\Gamma(x)\gamma^0\psi(\mathcal{P}x) \\ &+ \mathrm{i}\sin(\alpha)\cos(\alpha)(\bar\psi(x)\Gamma(x)\gamma^0\psi(\mathcal{P}x) \\ &- \bar\psi(\mathcal{P}x)\gamma^0\Gamma(x)\psi(x)).\end{aligned} \quad (4)$$

Therefore, we can now distinguish two cases.

(1) If $\Gamma(x) = \mathbb{1}$ or $\partial/\partial x^\mu$, then $\gamma^0\Gamma(x) = \Gamma(\mathcal{P}x)\gamma^0$; therefore, $\int d^4x\bar\psi(x)\Gamma(x)\psi(x)$ is invariant after changing $x \to \mathcal{P}x$ in the second and fourth terms in (4). This shows that the action of free massive fermions,

$$S_F(\psi,\bar\psi) = \int d^4x\bar\psi(x)(\mathrm{i}\gamma^\mu\partial^x_\mu - m)\psi(x), \quad (5)$$

with $\partial^x_\mu = \partial/\partial x^\mu$, is invariant under $U(1)_{\mathrm{P}}$ transformations.

(2) If $\Gamma(x) = \gamma^\mu A_\mu(x)$, then $\gamma^0\Gamma(x) = \gamma^\mu\mathcal{P}^\nu_\mu A_\nu(x)\gamma^0$. Therefore after integration d^4x on both sides of (4) and changing $x \to \mathcal{P}x$ in the second and fourth terms, we get the following:

$$\begin{aligned}&\int d^4x\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x) \to \\ &\int d^4x[\bar\psi(x)\gamma^\mu(\cos(\alpha)^2A_\mu(x)+\sin(\alpha)^2A^P_\mu(x))\psi(x) \\ &+\mathrm{i}\sin(\alpha)\cos(\alpha)\bar\psi(x)\gamma^\mu(A_\mu(x)-A^P_\mu(x))\gamma^0\psi(\mathcal{P}x)],\end{aligned} \quad (6)$$

where we defined $A^P_\nu(x) \equiv \mathcal{P}^\mu_\nu A_\mu(\mathcal{P}x) = PA_\nu(x)P^\dagger$. Equation (6) shows how a possible gauge interaction term in the action, namely $S_I(\psi,\bar\psi,A) = g\int d^4x\bar\psi(x)\gamma^\mu \times A_\mu(x)\psi(x)$, transforms. As it is clear from (6), for generic values of α, $S_I(\psi^{U^\alpha_{\mathrm{P}}},\bar\psi^{U^\alpha_{\mathrm{P}}},A) \neq S_I(\psi,\bar\psi,A)$, because in general $A^P_\mu(x) \neq A_\mu(x)$, and a $U(1)_{\mathrm{P}}$ transformation mixes both these fields. Therefore, the interaction term breaks $U(1)_{\mathrm{P}}$. However, if we restrict to particular values of α, we can obtain the invariance of S_I. In particular, we recognize two cases:

(i) $\alpha = \pi k$ with $k = 0,1,2,\ldots \Rightarrow U(1)_{\mathrm{P}}$ reduces to the group $Z_2 \subset U(1)_{\mathrm{P}}$ and of course $S_F + S_I$ is Z_2 invariant.

(ii) $\alpha = \pi k + (\pi/2)$ with $k = 0,1,2,\ldots \Rightarrow$ In this case if we perform also a parity transformation of the gauge field $A_\mu(x) \to A^P_\mu(x)$, we can obtain the invariance of the interaction term, namely $S_I(\psi^{U^\alpha_{\mathrm{P}}},\bar\psi^{U^\alpha_{\mathrm{P}}},A^P) = S_I(\psi,\bar\psi,A)$. In fact, as it is clear from Eq. (1), $U(1)_{\mathrm{P}}$ transformations reduce to parity $\times\mathrm{i}Z_2 \in U(1)_{\mathrm{P}}$ transformations, where for parity $\times\mathrm{i}Z_2$ we mean for example the transformation $\psi(x) \to z\mathrm{i}(P\psi(x)P^\dagger)$, with $z \in Z_2$.

Otherwise, a sufficient condition for the $U(1)_{\rm P}$ invariance of $S_I(\psi,\bar{\psi},A)$ can be obtained restricting ourself to gauge configurations such that

$$A_\mu^P(x) = A_\mu(x), \tag{7}$$

which means $A_0(\mathcal{P}x) = A_0(x)$ and $A_i(\mathcal{P}x) = -A_i(\mathcal{P}x)$.

III. TOWARDS A NEW CHIRAL-SPIN GROUP

Beside $U(1)_{\rm P}$, the other ingredient that we need for constructing our new *chiral-spin* group is to derive the $U(1)_A$ transformations for the field given in (2). $U(1)_A$ transformations are defined on ψ as $\psi(x) \to \psi(x)^{U_A^\alpha} = \exp(-\mathrm{i}\alpha\gamma^5)\psi(x)$. Thus,

$$U(1)_A : \Psi(x) \to \Psi(x)^{U_A^\alpha} = \begin{pmatrix} \psi_+(x)^{U_A^\alpha} \\ \psi_-(x)^{U_A^\alpha} \end{pmatrix} = \begin{pmatrix} e^{-\mathrm{i}\alpha\gamma^5}\psi_+(x) \\ e^{-\mathrm{i}\alpha\gamma^5}\psi_-(x) \end{pmatrix} = e^{\mathrm{i}\alpha(-\mathbb{1}\otimes\gamma^5)}\Psi(x). \tag{8}$$

The generator of $U(1)_A$ for the field Ψ is therefore $-\mathbb{1}\otimes\gamma^5$, which is traceless and Hermitian.

A. New chiral-spin group definition

Taking now the generators of the groups $U(1)_A$ and $U(1)_{\rm P}$, we rename them as $\Sigma_1^{\mathcal{P}} = \sigma^3\otimes\gamma^0$ and $\Sigma_3^{\mathcal{P}} = -\mathbb{1}\otimes\gamma^5$, and we define the third matrix $\Sigma_2^{\mathcal{P}} = \mathrm{i}\Sigma_1^{\mathcal{P}}\Sigma_3^{\mathcal{P}} = \sigma^3\otimes \mathrm{i}\gamma^5\gamma^0$. Now the set of $\Sigma_n^{\mathcal{P}}$s,

$$\vec{\Sigma}^{\mathcal{P}} = (\sigma^3\otimes\gamma^0, \sigma^3\otimes\mathrm{i}\gamma^5\gamma^0, -\mathbb{1}\otimes\gamma^5), \tag{9}$$

which are all traceless and Hermitian, verify the property: $[\Sigma_i^{\mathcal{P}}, \Sigma_j^{\mathcal{P}}] = 2\mathrm{i}\epsilon_{ijk}\Sigma_k^{\mathcal{P}}$. Hence, they are generators of an $su(2)$ algebra. We call the Lie group generated by the $\Sigma_n^{\mathcal{P}}$s as $SU(2)_{CS}^{\mathcal{P}}$. The $SU(2)_{CS}^{\mathcal{P}}$ group transformations on the field Ψ in (1) are given by

$$SU(2)_{CS}^{\mathcal{P}} : \Psi(x) \to \Psi(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} = U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}\Psi(x),$$
$$U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}} = e^{\mathrm{i}\alpha_n\Sigma_n^{\mathcal{P}}} \in SU(2)_{CS}^{\mathcal{P}}, \tag{10}$$

from which for the proper choice of the global vector $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3)$, we can get the group transformations of $U(1)_{\rm P}$ and $U(1)_A$ in (3) and (8), respectively. This means that $U(1)_{\rm P}, U(1)_A \subset SU(2)_{CS}^{\mathcal{P}}$. From the transformations (10) we can get how ψ (and consequently $\bar{\psi} = \psi^\dagger\gamma^0$) transforms, just inverting the definition of $\psi_\pm$ in terms of ψ.

Let us see now how to do it. First of all we recall an important feature which will also be useful later on. As it is well known, every element of a $SU(2)$ group can be written as product of three $U(1)$ matrices, which are subgroups of $SU(2)$. More precisely, if $U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}} \in SU(2)_{CS}^{\mathcal{P}}$, then

$$U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}\Psi(x) = \exp(\mathrm{i}\beta_1\Sigma_1^{\mathcal{P}})\exp(\mathrm{i}\beta_2\Sigma_3^{\mathcal{P}})\exp(\mathrm{i}\beta_3\Sigma_1^{\mathcal{P}})\Psi(x), \tag{11}$$

where β_1, β_2, β_3 are the three Euler angles. Using the generators in (9) and Eq. (2), the previous equation can be used to get the $SU(2)_{CS}^{\mathcal{P}}$ transformations for ψ, namely,

$$\begin{aligned}\psi(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} = {} & \cos(\beta_1)[\exp(-\mathrm{i}\beta_2\gamma^5)(\cos(\beta_3)\psi(x) \\ & + \mathrm{i}\sin(\beta_3)\gamma^0\psi(\mathcal{P}x))] \\ & + \mathrm{i}\sin(\beta_1)\gamma^0[\exp(-\mathrm{i}\beta_2\gamma^5)(\cos(\beta_3)\psi(\mathcal{P}x) \\ & + \mathrm{i}\sin(\beta_3)\gamma^0\psi(x))],\end{aligned} \tag{12}$$

which is what we wanted to get.

A particular case is when in (10), we set $(\alpha_1,\alpha_2,\alpha_3) = (0,\alpha,0)$. In this situation, we obtain another $U(1)$ subgroup of $SU(2)_{CS}^{\mathcal{P}}$, which we call $U(1)_{\rm PA}$ and its generator is therefore $\Sigma_2^{\mathcal{P}} = \sigma^3\otimes\mathrm{i}\gamma^5\gamma^0$. Now a $U(1)_{\rm PA}$ transformation of ψ can be written exploiting the definition of Ψ in (2) and consequently obtaining that

$$\begin{aligned}U(1)_{\rm PA} : \psi(x) \to {} & \psi(x)^{U^\alpha_{\rm PA}} \\ = {} & e^{-\alpha\gamma^5\gamma^0}\psi_+(x) + e^{\alpha\gamma^5\gamma^0}\psi_-(x) \\ = {} & \cos(\alpha)\psi(x) + \mathrm{i}\sin(\alpha)(\mathrm{i}\gamma^5\gamma^0)\psi(\mathcal{P}x),\end{aligned} \tag{13}$$

which also corresponds to set the Euler angles $(\beta_1,\beta_2,\beta_3) = (\pi/4,\alpha,-\pi/4)$ in Eq. (12). As we can see it is similar to the $U(1)_{\rm PA}$ transformations defined in Ref. [8] for the Euclidean case.

We conclude saying that the group $SU(2)_{CS}^{\mathcal{P}}$ as defined by Eq. (7) differently from $SU(2)_{CS}$ in Ref. [7], looks like a rotation in the space of the parity partners $\psi_+(x)$ and $\psi_-(x)$ (which is similar, but not the same, to what we did in Ref. [9] for baryon parity doublets).

B. Consequences on the fermionic action

From how $SU(2)_{CS}^{\mathcal{P}}$ is defined in Eq. (10), we can obtain some consequences on the invariance of the fermionic action, in particular

1. $S_F(\psi,\bar{\psi})$ at $m=0$ is $SU(2)_{CS}^{\mathcal{P}}$ invariant;
2. The mass term of $S_F(\psi,\bar{\psi})$ breaks explicitly $U(1)_{\rm PA}$ and moreover a gauge interaction in the action is not $U(1)_{\rm PA}$ invariant; and
3. A gauge interaction breaks $SU(2)_{CS}^{\mathcal{P}}$. However, if we restrict to gauge fields satisfying the relation given in (7), then $S_I(\psi,\bar{\psi},A)$ is $SU(2)_{CS}^{\mathcal{P}}$ invariant.

The proof of such statements follows by the fact that using the Euler decomposition that we have seen in (11) and (12) any $SU(2)_{CS}^{\mathcal{P}}$ transformation can be written as a product of transformations belonging to $U(1)_\mathrm{P}$ and $U(1)_A$. Therefore, every action which is invariant under $U(1)_\mathrm{P}$ and $U(1)_A$ will be also invariant under $SU(2)_{CS}^{\mathcal{P}}$. If one or both of these two symmetries decays then the action is not invariant anymore under $SU(2)_{CS}^{\mathcal{P}}$. This is valid in particular for $U(1)_\mathrm{PA}$ since the generator $\Sigma_2^{\mathcal{P}}$ is the commutator of $\Sigma_1^{\mathcal{P}}$ and $\Sigma_3^{\mathcal{P}}$. A direct proof of the second statement is also given in Appendix B.

IV. CHIRAL SPIN AND HAMILTONIAN

Another study which we want to add is the invariance of the free fermion Hamiltonian with respect to $U(1)_\mathrm{P}$ and $SU(2)_{CS}^{\mathcal{P}}$ (for the massless case) and derive the $U(1)_\mathrm{P}$ and $SU(2)_{CS}^{\mathcal{P}}$ transformations for creation and annihilation operators for fermions and antifermions. Once we do this, we will briefly discuss the case where a gauge interaction is switched on.

Before we start, we point out here that in this whole Sec. IV, we assume that the spinor field ψ describing free (and eventually massless $m=0$) fermions (or antifermions) is solution of the Dirac equation in the free case, which is [10]

$$\psi(x)=\sum_{r=0}^{1}\int\frac{d^3p}{(2\pi)^{3/2}}[c_r(\boldsymbol{p})u_r(\boldsymbol{p})e^{-\mathrm{i}px}+d_r(\boldsymbol{p})^\dagger v_r(\boldsymbol{p})e^{\mathrm{i}px}], \tag{14}$$

where $u_r(\boldsymbol{p})$ and $v_r(\boldsymbol{p})$ are reported in Eq. (A2), and $c_r(\boldsymbol{p})$, $d_r(\boldsymbol{p})^\dagger$ are the annihilation and creation operators for particles and antiparticles, respectively.

This is the particular situation where the free fermion action $S_F(\psi,\bar{\psi})$, calculated on such spinor field (14), reaches its minimum value, which is zero. From such spinor field (14) we attempt to apply $U(1)_\mathrm{P}$ and $SU(2)_{CS}^{\mathcal{P}}$ transformations, defined in the previous section in case of a totally generic spinor, in order to check the invariance of the free fermion Hamiltonian.

This Hamiltonian, calculated using the spinor field in (14), is given by [10]

$$H_0=\sum_{r=0}^{1}\int d^3pE_{\boldsymbol{p}}[c_r(\boldsymbol{p})^\dagger c_r(\boldsymbol{p})+d_r(\boldsymbol{p})^\dagger d_r(\boldsymbol{p})], \tag{15}$$

which is invariant under parity, i.e., $PH_0P^\dagger=H_0$. This means that calling $c_r^P(\boldsymbol{p})=Pc_r(\boldsymbol{p})P^\dagger$ and $d_r^P(\boldsymbol{p})^\dagger=Pd_r(\boldsymbol{p})^\dagger P^\dagger$, we have that

$$\begin{aligned}H_0&=\frac{1}{2}H_0+\frac{1}{2}PH_0P^\dagger\\&=\frac{1}{2}\sum_{r=0}^{1}\int d^3pE_{\boldsymbol{p}}[C_r(\boldsymbol{p})^\dagger C_r(\boldsymbol{p})+D_r(\boldsymbol{p})^\dagger D_r(\boldsymbol{p})],\end{aligned} \tag{16}$$

where we defined the following parity partners operators:

$$C_r(\boldsymbol{p})=\begin{pmatrix}c_r(\boldsymbol{p})\\c_r^P(\boldsymbol{p})\end{pmatrix},\quad D_r(\boldsymbol{p})^\dagger=(\,d_r(\boldsymbol{p})^\dagger\quad d_r^P(\boldsymbol{p})^\dagger\,). \tag{17}$$

The expressions for $c_r(\boldsymbol{p})$ and $d_r(\boldsymbol{p})^\dagger$ can be obtained by the Fourier transform of $\psi(x)$ [see (A3)], while $c_r^P(\boldsymbol{p})$ and $d_r^P(\boldsymbol{p})^\dagger$ are obtained from the fact that $P\psi(x)P^\dagger=\gamma^0\psi(\mathcal{P}x)$ and given in (A4).

A. $U(1)_\mathrm{P}$ and Hamiltonian

In order to check if H_0 is $U(1)_\mathrm{P}$ invariant, we need to find how $c_r(\boldsymbol{p})$, $d_r(\boldsymbol{p})^\dagger$, $c_r^P(\boldsymbol{p})$, and $d_r^P(\boldsymbol{p})^\dagger$ transform. For this purpose, we just need to use (A3) and (A4) together with the fact that $u_r(\boldsymbol{p})$ and $v_r(\boldsymbol{p})$ given in (A2) transform under parity as

$$\gamma^0u_r(\boldsymbol{p})=u_r(-\boldsymbol{p}),\qquad\gamma^0v_r(\boldsymbol{p})=-v_r(-\boldsymbol{p}). \tag{18}$$

Here, we give the results:

$$\begin{aligned}c_r(\boldsymbol{p})^{U_\mathrm{P}^\alpha}&=\cos(\alpha)c_r(\boldsymbol{p})+\mathrm{i}\sin(\alpha)c_r^P(\boldsymbol{p}),\\c_r^P(\boldsymbol{p})^{U_\mathrm{P}^\alpha}&=\cos(\alpha)c_r^P(\boldsymbol{p})+\mathrm{i}\sin(\alpha)c_r(\boldsymbol{p}),\\(d_r(\boldsymbol{p})^\dagger)^{U_\mathrm{P}^\alpha}&=\cos(\alpha)d_r(\boldsymbol{p})^\dagger+\mathrm{i}\sin(\alpha)d_r^P(\boldsymbol{p})^\dagger,\\(d_r^P(\boldsymbol{p})^\dagger)^{U_\mathrm{P}^\alpha}&=\cos(\alpha)d_r^P(\boldsymbol{p})^\dagger+\mathrm{i}\sin(\alpha)d_r(\boldsymbol{p})^\dagger,\end{aligned} \tag{19}$$

where we have just replaced $\psi\to\psi^{U_\mathrm{P}^\alpha}$ in (A3) and (A4); see Appendix C for the detailed calculation. The result of Eq. (19) can be rewritten using the definition in (17) as

$$\begin{aligned}C_r(\boldsymbol{p})\to C_r(\boldsymbol{p})^{U_\mathrm{P}^\alpha}&\equiv\begin{pmatrix}c_r(\boldsymbol{p})^{U_\mathrm{P}^\alpha}\\c_r^P(\boldsymbol{p})^{U_\mathrm{P}^\alpha}\end{pmatrix}=e^{\mathrm{i}\alpha\sigma^1}C_r(\boldsymbol{p}),\\D_r(\boldsymbol{p})^\dagger\to(D_r(\boldsymbol{p})^\dagger)^{U_\mathrm{P}^\alpha}&\equiv(\,(d_r(\boldsymbol{p})^\dagger)^{U_\mathrm{P}^\alpha}\quad(d_r^P(\boldsymbol{p})^\dagger)^{U_\mathrm{P}^\alpha}\,),\\&=D_r(\boldsymbol{p})^\dagger e^{\mathrm{i}\alpha\sigma^1}\end{aligned} \tag{20}$$

where σ^1 is the first Pauli matrix acting on the two-dimensional space defined in Eq. (17). As it is clear from (16), H_0 is invariant under $U(1)_\mathrm{P}$ transformations of Eq. (20).

This result is actually pretty obvious in the free case, if you consider that H_0 commutes with P. Take, for example, a state of a particle $|\boldsymbol{p}\rangle=c_r(\boldsymbol{p})^\dagger|0\rangle$ that has energy $E_{\boldsymbol{p}}=\langle\boldsymbol{p}|H_0|\boldsymbol{p}\rangle$. The $U(1)_\mathrm{P}$ invariance of H_0 tells us that the state $|\boldsymbol{p}\rangle$ is energetically equivalent to the state $|\tilde{\boldsymbol{p}}\rangle=(c_r(\boldsymbol{p})^{U_\mathrm{P}^\alpha})^\dagger|0\rangle=(\cos(\alpha)c_r(\boldsymbol{p})^\dagger-\mathrm{i}\sin(\alpha)c_r^P(\boldsymbol{p})^\dagger)|0\rangle$, where we used the second of Eq. (19). We rewrite it as $|\tilde{\boldsymbol{p}}\rangle=\cos(\alpha)|\boldsymbol{p}\rangle-\mathrm{i}\sin(\alpha)|-\boldsymbol{p}\rangle$, because $c_r^P(\boldsymbol{p})^\dagger|0\rangle=Pc_r(\boldsymbol{p})^\dagger|0\rangle=P|\boldsymbol{p}\rangle=|-\boldsymbol{p}\rangle$. Hence, we have $\langle\tilde{\boldsymbol{p}}|H_0|\tilde{\boldsymbol{p}}\rangle=\cos(\alpha)^2\langle\boldsymbol{p}|H_0|\boldsymbol{p}\rangle+\mathrm{i}\sin(\alpha)\cos(\alpha)\langle-\boldsymbol{p}|H_0|\boldsymbol{p}\rangle-\mathrm{i}\sin(\alpha)\cos(\alpha)\langle\boldsymbol{p}|H_0|-\boldsymbol{p}\rangle+\sin(\alpha)^2\langle-\boldsymbol{p}|H_0|-\boldsymbol{p}\rangle$. However, $\langle-\boldsymbol{p}|H_0|-\boldsymbol{p}\rangle=\langle\boldsymbol{p}|PH_0P^\dagger|\boldsymbol{p}\rangle=\langle\boldsymbol{p}|H_0|\boldsymbol{p}\rangle=E_{\boldsymbol{p}}$, because $[H_0,P]=0$, and for the same reason $\langle -\boldsymbol{p}|H_0|\boldsymbol{p}\rangle = \langle \boldsymbol{p}|H_0|-\boldsymbol{p}\rangle$. Therefore, $\langle \tilde{\boldsymbol{p}}|H_0|\tilde{\boldsymbol{p}}\rangle = \langle \boldsymbol{p}|H_0|\boldsymbol{p}\rangle = E_{\boldsymbol{p}}$, for whatever value of $\boldsymbol{p}$. Hence, the two states have the same energy.

B. $SU(2)^{\mathcal{P}}_{CS}$ and Hamiltonian

We prove now that the Hamiltonian H_0 is also invariant under $SU(2)^{\mathcal{P}}_{CS}$ transformations for $m=0$. In order to see this point, we give how $c_r(\boldsymbol{p})$, $c^P_r(\boldsymbol{p})$, $d_r(\boldsymbol{p})^\dagger$, $d^P_r(\boldsymbol{p})^\dagger$ transform. At first we express these operators in terms of Ψ instead of ψ, as we write explicitly in Eq. (D4). Secondly, we use that for $m=0$, the vectors u_r and v_r satisfy the properties $\gamma^5 u_r(\boldsymbol{p}) = (\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|)u_r(\boldsymbol{p})$ and $\gamma^5 v_r(\boldsymbol{p}) = (\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|)v_r(\boldsymbol{p})$. This reflects the fact that in the massless case γ^5 coincides with the helicity operator $\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|$. For convenience we choose χ_r and consequently χ'_r, defined by Eq. (A2) in the solution of the Dirac equation, such that they are eigenstates of the helicity operator, i.e., $(\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|)\chi_0 = \chi_0$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|)\chi_1 = -\chi_1$. This means that for $m=0$ we have

$$\gamma^5 u_r(\boldsymbol{p}) = h_r u_r(\boldsymbol{p}), \qquad \gamma^5 v_r(\boldsymbol{p}) = h_{r+1} v_r(\boldsymbol{p}), \tag{21}$$

with $h_r = (-1)^r$, helicity of the particle.

Using these two considerations, and writing a generic $SU(2)^{\mathcal{P}}_{CS}$ element extensively as

$$U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}} = \cos(\alpha) + \mathrm{i}\sin(\alpha)[e_1\Sigma^{\mathcal{P}}_1 + e_2\Sigma^{\mathcal{P}}_2 + e_3\Sigma^{\mathcal{P}}_3], \tag{22}$$

where $(\alpha_1,\alpha_2,\alpha_3) = \alpha(e_1,e_2,e_3)$, with $\sum_{i=1}^3 e_i^2 = 1$, we get that $c_r(\boldsymbol{p})$ and $c^P_r(\boldsymbol{p})$ transform as

$$\begin{aligned} c_r(\boldsymbol{p})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &= \cos(\alpha)c_r(\boldsymbol{p}) \\ &\quad + \mathrm{i}\sin(\alpha)[e_1 c^P_r(\boldsymbol{p}) + e_2 \mathrm{i} h_r c^P_r(\boldsymbol{p}) - e_3 h_r c_r(\boldsymbol{p})], \\ c^P_r(\boldsymbol{p})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &= \cos(\alpha)c^P_r(\boldsymbol{p}) \\ &\quad + \mathrm{i}\sin(\alpha)[e_1 c_r(\boldsymbol{p}) - e_2 \mathrm{i} h_r c_r(\boldsymbol{p}) + e_3 h_r c^P_r(\boldsymbol{p})]. \end{aligned} \tag{23}$$

Full details regarding the derivation of (23) are given in Appendix D.

Equation (23) can be written in a compact way using the notation in (17) as

$$C_r(\boldsymbol{p}) \to C_r(\boldsymbol{p})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} \equiv \begin{pmatrix} c_r(\boldsymbol{p})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} \\ c^P_r(\boldsymbol{p})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} \end{pmatrix} = e^{\mathrm{i}\alpha_n \Sigma^{\mathcal{P}}_{n(c)}} C_r(\boldsymbol{p}), \tag{24}$$

where $\Sigma^{\mathcal{P}}_{n(c)} = \{\sigma^1, -\sigma^2 h_r, -\sigma^3 h_r\}$, are all traceless, Hermitian, and with the property $[\Sigma^{\mathcal{P}}_{i(c)}, \Sigma^{\mathcal{P}}_{j(c)}] = 2\mathrm{i}\epsilon_{ijk}\Sigma^{\mathcal{P}}_{k(c)}$. Therefore, we have found a representation of $SU(2)^{\mathcal{P}}_{CS}$ for the transformations of $C_r(\boldsymbol{p})$ in the massless case. Notice that (24) are basically rotations in the space of the parity partners: $c_r(\boldsymbol{p})$ and $c^P_r(\boldsymbol{p})$, which takes into account the helicity of our particles.

The same can be done for $d_r(\boldsymbol{p})^\dagger$ and $d^P_r(\boldsymbol{p})^\dagger$ and we obtain the following result:

$$\begin{aligned} (d_r(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &= \cos(\alpha)d_r(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha)[e_1 d^P_r(\boldsymbol{p})^\dagger \\ &\quad + e_2 \mathrm{i} h_{r+1} d^P_r(\boldsymbol{p})^\dagger - e_3 h_{r+1} d_r(\boldsymbol{p})^\dagger], \\ (d^P_r(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &= \cos(\alpha)d^P_r(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha)[e_1 d_r(\boldsymbol{p})^\dagger \\ &\quad - e_2 \mathrm{i} h_{r+1} d_r(\boldsymbol{p})^\dagger + e_3 h_{r+1} d^P_r(\boldsymbol{p})^\dagger], \end{aligned} \tag{25}$$

where we used the same procedure as before. Again, details of these calculations are reported in Appendix D. Equation (25) can be given in a compact way as

$$\begin{aligned} D_r(\boldsymbol{p})^\dagger \to (D_r(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &\equiv \left((d_r(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} (d^P_r(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} \right) \\ &= D_r(\boldsymbol{p})^\dagger \left(e^{\mathrm{i}\alpha_n \Sigma^{\mathcal{P}}_{n(d)}}\right)^T, \end{aligned} \tag{26}$$

where $\Sigma^{\mathcal{P}}_{n(d)} = \{\sigma^1, -\sigma^2 h_{r+1}, -\sigma^3 h_{r+1}\}$, are again all traceless, Hermitian, and with the property $[\Sigma^{\mathcal{P}}_{i(d)}, \Sigma^{\mathcal{P}}_{j(d)}] = 2\mathrm{i}\epsilon_{ijk}\Sigma^{\mathcal{P}}_{k(d)}$. Equation (26) expresses the representation of $SU(2)^{\mathcal{P}}_{CS}$ transformations for $D_r(\boldsymbol{p})$ and it is a rotation of the parity partners: $d_r(\boldsymbol{p})^\dagger$ and $d^P_r(\boldsymbol{p})^\dagger$, where we take into account the helicity for antiparticles.

As we can observe, the Hamiltonian in (16) is of course invariant under transformations of $C_r(\boldsymbol{p})$ and $D_r(\boldsymbol{p})^\dagger$ given in (24) and (26). This concludes our proof that H_0 at $m=0$ is $SU(2)^{\mathcal{P}}_{CS}$ invariant.

It remains to see what happens when $m \neq 0$ and in the presence of a gauge interaction. For $m \neq 0$, we already know that H_0 is not invariant under $U(1)_A$ and therefore is not invariant under $SU(2)^{\mathcal{P}}_{CS}$, since $U(1)_A \subset SU(2)^{\mathcal{P}}_{CS}$. The case of a gauge interaction just makes fall the relation (18), valid in the free case. This means that the relations (24) and (26) do not represent anymore $SU(2)^{\mathcal{P}}_{CS}$ transformations. Hence, in this situation we do not expect to have this symmetry, even because, as we have already seen in the previous section, the gauge interaction breaks the invariance also in the action, because $U(1)_{\mathrm{P}}$ is broken explicitly.

V. MASS DEGENERATION AND SYMMETRY

Now we consider the case of QCD, where the fermion fields ψ that we have discussed so far are interpreted as quark fields.

The $SU(2)_{CS}$ group transformations, as have been described in [5–7,11,12], can be defined in Minkowskian spacetime as

$$\begin{aligned} SU(2)_{CS}\colon \psi(x) \to \psi(x)^{U^{\boldsymbol{\alpha}}_{CS}} &= U^{\boldsymbol{\alpha}}_{CS}\psi(x), \\ U^{\boldsymbol{\alpha}}_{CS} &= e^{\mathrm{i}\alpha_n\Sigma_n} \in SU(2)_{CS}, \end{aligned} \tag{27}$$

where $\Sigma_n = \{\gamma^0, \mathrm{i}\gamma^5\gamma^0, -\gamma^5\}$, and we have just substituted $\gamma^4 \to \gamma^0$ from Euclidean to Minkowskian spacetime. We can see from the above definition that we have two important subgroups just tuning the α_ns. One is $U(1)_A$ [which is also a subgroup of $SU(2)^{\mathcal{P}}_{CS}$] and the other is the $U(1)$ group generated by γ^0, from which the group transformation is obtained by choosing $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, 0, 0)$ in (27). Hence, we get

$$U(1)_0 : \psi(x) \to \psi(x)^{U(1)^\alpha_0} = e^{\mathrm{i}\alpha\gamma^0}\psi(x) = \cos(\alpha)\psi(x) + \mathrm{i}\sin(\alpha)\gamma^0\psi(x). \quad (28)$$

Now, using the Euler decomposition, whatever element of $SU(2)_{CS}$ can be obtained by the product of three matrices belonging to $U(1)_A$ and $U(1)_0$. Therefore, the real difference between $SU(2)_{CS}$ and $SU(2)^{\mathcal{P}}_{CS}$ lies in their different subgroups: $U(1)_0$, given in (1), and $U(1)_{\mathrm{P}}$, given in (5), respectively. However, while $U(1)_{\mathrm{P}}$ is a symmetry of the free fermion action S_F in (5), $U(1)_0$ is broken explicitly. This is why $SU(2)_{CS}$ is not a symmetry of free massless quarks. Now we want to show how $SU(2)^{\mathcal{P}}_{CS}$ is related to $SU(2)_{CS}$ and its consequence on hadron correlators and mass degeneration. At first we take the case when $\Psi(x)$ is evaluated at the point $x^{(t)} = (x_0, \mathbf{0})$. In this situation, by definition $\psi_+(x^{(t)}) = \psi(x^{(t)})$ and $\psi_-(x^{(t)}) = 0$. Therefore, the transformation (10) becomes

$$SU(2)^{\mathcal{P}}_{CS} : \Psi(x^{(t)}) \to \Psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} = e^{\mathrm{i}\alpha_n\Sigma^{\mathcal{P}}_n}\begin{pmatrix} \psi(x^{(t)}) \\ 0 \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\alpha_n\Sigma_n}\psi(x^{(t)}) \\ 0 \end{pmatrix}. \quad (29)$$

Now naming $\psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} \equiv \psi_+(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$, Eq. (29) can be rewritten also as

$$SU(2)^{\mathcal{P}}_{CS} : \psi(x^{(t)}) \to \psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} = e^{\mathrm{i}\alpha_n\Sigma_n}\psi(x^{(t)}). \quad (30)$$

It coincides with Eq. (27), which means that $\psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}} = \psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$. Hence, $SU(2)^{\mathcal{P}}_{CS}$ and $SU(2)_{CS}$ are indistinguishable when they act on the spinor $\psi(x^{(t)})$. This has an important consequence, as we start to describe now. Take a hadron observable $O_H(x)$ made by N_q quarks and $\bar{N}_q$ antiquarks, i.e.,

$$O_H(x) = \mathcal{H}_{i_1,\ldots,i_{N_q},j_1,\ldots,j_{\bar{N}_q}} \prod_{l=1}^{N_q} \psi(x)_{i_l} \prod_{k=1}^{\bar{N}_q} \bar{\psi}_{j_k}(x), \quad (31)$$

where $\mathcal{H}$ is a tensor specifying the quantum numbers of the hadron H, and the indices $\{i\}_{l=1,\ldots N_q}$ and $\{j\}_{k=1,\ldots \bar{N}_q}$ enclose Dirac, flavor, and eventually color indices. We now choose to transform it with $SU(2)_{CS}$ and for some choice of the parameters α_ns in (27) we get another hadron observable, i.e.,

$$O_{H'}(x) = \mathcal{H}'_{i_1,\ldots,i_{N_q},j_1,\ldots,j_{\bar{N}_q}} \prod_{l=1}^{N_q} \psi(x)_{i_l} \prod_{k=1}^{\bar{N}_q} \bar{\psi}_{j_k}(x), \quad (32)$$

which is the observable for the hadron H'. $O_H(x)$ and $O_{H'}(x)$ are connected via $SU(2)_{CS}$ if for some $\boldsymbol{\alpha}$ we have $O_{H'}(x) = O_H(x)^{U^{\boldsymbol{\alpha}}_{CS}} \equiv \mathcal{H}_{i_1,\ldots,i_{N_q},j_1,\ldots,j_{\bar{N}_q}} \prod_{l=1}^{N_q} \psi(x)^{U^{\boldsymbol{\alpha}}_{CS}}_{i_l} \times \prod_{k=1}^{\bar{N}_q} \bar{\psi}_{j_k}(x)^{U^{\boldsymbol{\alpha}}_{CS}}$. Now, since for $x = x^{(t)}$ we have $\psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}} = \psi(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$, then $O_H(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$, which is the $SU(2)^{\mathcal{P}}_{CS}$ transformation of $O_H(x)$ with the same set of parameters α_ns, has the property $O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}} = O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$ and consequently $O_{H'}(x^{(t)}) = O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$. Therefore, even if $O_H(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$ could be not associated with an hadron for a generic x, however at $x = x^{(t)}$, it coincides with the hadron operator $O_{H'}(x^{(t)})$. At this point, suppose we consider these two hadron correlators and their expansion in the energy eigenstates using the translational invariance $O_H(x^{(t)}) = \exp(-\mathrm{i}Hx_0)O_H(0)\exp(\mathrm{i}Hx_0)$; we have

$$\langle 0|O_H(y^{(t)})O_H(x^{(t)})^\dagger|0\rangle = \sum_n |\langle 0|O_H(0)|n\rangle|^2 e^{-\mathrm{i}E_n\mathcal{T}},$$
$$\langle 0|O_{H'}(y^{(t)})O_{H'}(x^{(t)})^\dagger|0\rangle = \sum_n |\langle 0|O_{H'}(0)|n\rangle|^2 e^{-\mathrm{i}E'_n\mathcal{T}}, \quad (33)$$

where $y^{(t)} = (y_0, \mathbf{0})$, $\mathcal{T} = y_0 - x_0$, while $m_H = E_0$ and $m_{H'} = E'_0$ are the masses associated with the hadrons H and H', respectively. Now, if $SU(2)_{CS}$ is a symmetry of the theory (which seems to be in truncated studies [5–7]) then $\langle 0|O_H(y^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}}(O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}})^\dagger|0\rangle = \langle 0|O_H(y^{(t)}) \times O_H(x^{(t)})^\dagger|0\rangle$ and therefore, since we have chosen the $SU(2)_{CS}$ transformations such that $O_H(x)^{U^{\boldsymbol{\alpha}}_{CS}} = O_{H'}(x)$, then we have $m_{H'} = m_H$, from (33). This means that a degeneration of masses appears. Let us see the opposite, i.e., we find a degeneration $m_{H'} = m_H$ coming from $\langle 0|O_{H'}(y^{(t)})O_{H'}(x^{(t)})^\dagger|0\rangle = \langle 0|O_H(y^{(t)})O_H(x^{(t)})^\dagger|0\rangle$. In that case at $x = x^{(t)}$ (and also $y = y^{(t)}$), we have $O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} = O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}}$, which means from the correlator side that

$$\begin{aligned}
&\langle 0|O_{H'}(y^{(t)})O_{H'}(x^{(t)})^\dagger|0\rangle \\
&= \langle 0|O_H(y^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}}(O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS}})^\dagger|0\rangle \\
&= \langle 0|O_H(y^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}(O_H(x^{(t)})^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}})^\dagger|0\rangle \\
&= \langle 0|O_H(y^{(t)})O_H(x^{(t)})^\dagger|0\rangle. \quad (34)
\end{aligned}$$

This implies that $SU(2)^{\mathcal{P}}_{CS}$ symmetry can also explain the same mass degeneration. Therefore, looking just at the mass degeneration $m_{H'} = m_H$ does not tell us if the symmetry is $SU(2)_{CS}$ of the truncated studies [1–7,11,12], where at first the mass degeneration has been observed, or $SU(2)^{\mathcal{P}}_{CS}$ of this paper.

However, because $SU(2)^{\mathcal{P}}_{CS}$ is a symmetry of free quarks (in the massless case), its possible presence does not go in contrast with the deconfinement regime at high-temperature QCD, since in Sec. III we have shown that it is a symmetry of the action as well the Hamiltonian of free fermions (let say quarks) in the massless case. On the contrary, $SU(2)_{CS}$ is not a symmetry of S_F at $m = 0$ because $U(1)_0$ is explicitly broken (see Refs. [11,12]). Therefore, we expect $SU(2)^{\mathcal{P}}_{CS}$ to be visible at temperature $T \to \infty$, where quarks will behave as quasifree particles. Hence, while $SU(2)_{CS}$ is just present in the range $T_c < T \lesssim 3T_c$ [1–4], $SU(2)^{\mathcal{P}}_{CS}$ could be visible (at least approximately) also at $T > 3T_c$ and describe the large mass degeneracy which was previously explained by $SU(2)_{CS}$ in the truncated studies [5–7,11,12]. Therefore, a study on the lattice on this point is strongly suggested.

Now, lattice calculations, as we know, are generally performed at zero chemical potential (due to technical difficulties). However, if we suppose to switch on an eventual chemical potential term in the action, this would not spoil $SU(2)^{\mathcal{P}}_{CS}$. In fact, the $SU(2)^{\mathcal{P}}_{CS}$ transformations leave also invariant the chemical potential part of the action (as also $SU(2)_{CS}$ does, see Refs. [11,13]), which is $S_{(\mu)}(\psi, \bar{\psi}) = \mu \int d^4x \psi(x)^\dagger \psi(x)$. The demonstration starts rewriting it as $S_{(\mu)}(\psi, \bar{\psi}) = \mu \int d^4x (\psi_+(x)^\dagger \psi_+(x) + \psi_-(x)^\dagger \psi_-(x))$, since $\psi(x) = \psi_+(x) + \psi_-(x)$. This because the mixing terms give zero under the integration, indeed $\int d^4x \psi_\pm(x)^\dagger \psi_\mp(x) = \int d^4x \psi_\pm(\mathcal{P}x)^\dagger \psi_\mp(\mathcal{P}x) = -\int d^4x \psi_\pm(x)^\dagger \psi_\mp(x) = 0$, where firstly we used the change of variable $x \to \mathcal{P}x$ and $|\det(\mathcal{P})| = 1$ and secondly that $\psi_\pm(\mathcal{P}x) = \pm\psi_\pm(x)$ by definition. Therefore, using (2), then we have $S_{(\mu)}(\psi, \bar{\psi}) = \mu \int d^4x \Psi(x)^\dagger \Psi(x)$. Consequently from Eq. (10), we get $S_{(\mu)}(\psi^{U^{\alpha}_{CS^{\mathcal{P}}}}, \bar{\psi}^{U^{\alpha}_{CS^{\mathcal{P}}}}) = \mu \int d^4x \times (\Psi(x)^{U^{\alpha}_{CS^{\mathcal{P}}}})^\dagger \Psi(x)^{U^{\alpha}_{CS^{\mathcal{P}}}} = \mu \int d^4x \Psi(x)^\dagger \Psi(x) = S_{(\mu)}(\psi, \bar{\psi})$. Hence, the invariance of $S_{(\mu)}(\psi, \bar{\psi})$ under $SU(2)^{\mathcal{P}}_{CS}$ is proven. Consequently, if there is a regime at high temperature where $SU(2)^{\mathcal{P}}_{CS}$ is a symmetry of QCD, then a possible nonzero chemical potential does not break such symmetry.

VI. SUMMARY

In Ref. [8], we have seen how $SU(2)^{\mathcal{P}}_{CS}$ can be a candidate for describing the large degeneracy found on lattice calculations [5–7], and eventually at high-temperature QCD studies [1–4]. Here, we have defined such group in Minkowski space and proved that the fermionic action of free massless fermions is left invariant under such group and a chemical potential term is also $SU(2)^{\mathcal{P}}_{CS}$ invariant.

In the case of a gauge interaction, $SU(2)^{\mathcal{P}}_{CS}$ is explicitly broken except for some special cases [8]. Therefore, a more profound investigation on this is needed. We have also seen that the Hamiltonian of free fermions is $SU(2)^{\mathcal{P}}_{CS}$ invariant and given how the creation and annihilation operators of fermions and antifermions, when organized in parity partners [in the sense of Eq. (17)], would transform in this case; see Eqs. (24) and (26).

Moreover, from the Minkowskian perspective as we have done in Euclidean [8], we have seen that a mass degeneration given by a possible presence of the *chiral-spin* group $SU(2)_{CS}$ symmetry can be either explained by our *chiral-spin* group $SU(2)^{\mathcal{P}}_{CS}$. However, since $SU(2)^{\mathcal{P}}_{CS}$ is a symmetry of free fermions, then it is compatible with the presence of the deconfinement regime in QCD. Therefore, it could be visible (at least *effectively*) at high-temperature QCD, i.e., $T > T_c$ (as, e.g., $U(1)_A$ in Refs. [14–16]), but this is still something to be checked on lattice calculations.

As we have shown in this paper, $SU(2)_{CS}$ and $SU(2)^{\mathcal{P}}_{CS}$ have both $U(1)_A$ as subgroup; nevertheless, they differ from each other by the subgroup generated by γ^0, i.e., $U(1)_{0,} \subset SU(2)_{CS}$ [see Eq. (28)], and $U(1)_{\mathrm{P}} \subset SU(2)^{\mathcal{P}}_{CS}$ [see Eq. (1)]. Now, $U(1)_A$ has been already studied on the lattice, and the suppression of its breaking for extremely high temperature is pretty evident by many lattice studies [14–16]. Therefore, in order to check the presence of $SU(2)^{\mathcal{P}}_{CS}$ in QCD at high T, it is sufficient to verify the presence of $U(1)_{\mathrm{P}}$ [defined by Eq. (1)] in lattice QCD at high temperature, which still has not been done yet.

APPENDIX A: KNOWN FORMULAS

In this paper we have used the Dirac representation for the gamma matrices, namely

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \tau^k \\ -\tau^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \text{(A1)}$$

where τ^k for $k = 1, 2, 3$ are the Pauli matrices. Using such representation the solution of the Dirac equation $(\mathrm{i}\gamma^\mu \partial^x_\mu - m)\psi(x) = 0$, given in (14), contains the 4-component vectors $u_r(\boldsymbol{p})$ and $v_r(\boldsymbol{p})$, which have the following structure [10]:

$$u_r(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}} + m}{2E_{\boldsymbol{p}}}} \begin{pmatrix} \chi_r \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_{\boldsymbol{p}} + m} \chi_r \end{pmatrix},$$

$$v_r(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}} + m}{2E_{\boldsymbol{p}}}} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_{\boldsymbol{p}} + m} \chi'_r \\ \chi'_r \end{pmatrix}, \qquad \text{(A2)}$$

where $E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$, with $\chi'_r = \chi_{r \oplus 1}$ [where $r \oplus 1 = (r+1) \bmod 2$], while χ_0 and χ_1 are two two-dimensional orthogonal vectors, i.e., $\chi^\dagger_r \chi_{r'} = \chi'^\dagger_r \chi'_{r'} = \delta_{rr'}$. Consequently, the normalization is $u_r(\boldsymbol{p})^\dagger u_{r'}(\boldsymbol{p}) = v_r(\boldsymbol{p})^\dagger v_{r'}(\boldsymbol{p}) = \delta_{rr'}$, for whatever $\boldsymbol{p}$.

Other interesting features come from the coefficients $c_r(\boldsymbol{p})$ and $d_r(\boldsymbol{p})^\dagger$ in (14), namely the annihilation and creation operators of particles and antiparticles, respectively. They can be obtained by the Fourier transform

$$c_r(\boldsymbol{p}) = \int \frac{d^3x}{(2\pi)^{3/2}} u_r(\boldsymbol{p})^\dagger \psi(x) e^{\mathrm{i}px},$$
$$d_r(\boldsymbol{p})^\dagger = \int \frac{d^3x}{(2\pi)^{3/2}} v_r(\boldsymbol{p})^\dagger \psi(x) e^{-\mathrm{i}px}. \quad \text{(A3)}$$

Moreover from the parity transformation $P\psi(x)P^\dagger = \gamma^0\psi(\mathcal{P}x)$ and the relations (18), we can express $c_r^P(\boldsymbol{p}) \equiv Pc_r(\boldsymbol{p})P^\dagger$ and $d_r^P(\boldsymbol{p})^\dagger \equiv Pd_r(\boldsymbol{p})^\dagger P^\dagger$ as

$$c_r^P(\boldsymbol{p}) = \int \frac{d^3x}{(2\pi)^{3/2}} u_r(-\boldsymbol{p})^\dagger \psi(x) e^{\mathrm{i}p(\mathcal{P}x)},$$
$$d_r^P(\boldsymbol{p})^\dagger = \int \frac{d^3x}{(2\pi)^{3/2}} (-v_r(-\boldsymbol{p}))^\dagger \psi(x) e^{-\mathrm{i}p(\mathcal{P}x)}, \quad \text{(A4)}$$

where we used that $\gamma^0 = (\gamma^0)^\dagger$, changed the variable $x \to \mathcal{P}x$, used that $\mathcal{P} = \mathrm{diag}(1,-1,-1,-1) = \mathcal{P}^{-1}$, and that the Jacobian $|\det(P)|^2 = 1$.

APPENDIX B: $U(1)_{\mathrm{PA}}$ BREAKING

In this appendix we want to prove that the gauge interaction and also the mass term in the action break explicitly $U(1)_{\mathrm{PA}}$, subgroup of $SU(2)^{\mathcal{P}}_{CS}$. The expression of $U(1)_{\mathrm{PA}}$ transformations for a fermion field ψ is given in Eq. (13), from which the transformation for $\bar\psi$ is $\bar\psi(x)^{U^\alpha_{\mathrm{PA}}} = (\psi(x)^{U^\alpha_{\mathrm{PA}}})^\dagger\gamma^0 = \cos(\alpha)\bar\psi(x) + \mathrm{i}\sin(\alpha)\bar\psi(\mathcal{P}x)\mathrm{i}\gamma^5\gamma^0$.

Let us start with the gauge interaction term of the action $S_I(\psi,\bar\psi,A) = g\int d^4x\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x)$, i.e.,

$$\begin{aligned}
S_I(\psi^{U^\alpha_{\mathrm{PA}}},\bar\psi^{U^\alpha_{\mathrm{PA}}},A)/g &= \int d^4x\bar\psi(x)^{U^\alpha_{\mathrm{PA}}}\gamma^\mu A_\mu(x)\psi(x)^{U^\alpha_{\mathrm{PA}}} \\
&= \cos(\alpha)^2\int d^4x\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x) + \mathrm{i}\sin(\alpha)\cos(\alpha)\int d^4x\bar\psi(x)\gamma^\mu(\mathrm{i}\gamma^5\gamma^0)A_\mu(x)\psi(\mathcal{P}x) \\
&\quad + \mathrm{i}\sin(\alpha)\cos(\alpha)\int d^4x\bar\psi(\mathcal{P}x)(\mathrm{i}\gamma^5\gamma^0)\gamma^\mu A_\mu(x)\psi(x) \\
&\quad - \sin(\alpha)^2\int d^4x\bar\psi(\mathcal{P}x)(\mathrm{i}\gamma^5\gamma^0)\gamma^\mu(\mathrm{i}\gamma^5\gamma^0)A_\mu(x)\psi(\mathcal{P}x) \\
&= \cos(\alpha)^2\int d^4x\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x) + \mathrm{i}\sin(\alpha)\cos(\alpha)\int d^4x\bar\psi(x)\gamma^\mu(\mathrm{i}\gamma^5\gamma^0)A_\mu(x)\psi(\mathcal{P}x) \\
&\quad - \mathrm{i}\sin(\alpha)\cos(\alpha)\int d^4x\bar\psi(x)\mathcal{P}^\mu_\nu\gamma^\nu(\mathrm{i}\gamma^5\gamma^0)A_\mu(\mathcal{P}x)\psi(\mathcal{P}x) + \sin(\alpha)^2\int d^4x\bar\psi(x)\gamma^\nu\mathcal{P}^\mu_\nu A_\mu(\mathcal{P}x)\psi(x) \\
&= \int d^4x[\bar\psi(x)\gamma^\mu(\cos(\alpha)^2A_\mu(x) + \sin(\alpha)^2A^P_\mu(x))\psi(x) \\
&\quad + \mathrm{i}\sin(\alpha)\cos(\alpha)\bar\psi(x)\gamma^\mu(A_\mu(x) - A^P_\mu(x))(\mathrm{i}\gamma^5\gamma^0)\psi(\mathcal{P}x)], \qquad \text{(B1)}
\end{aligned}$$

where we used that $(\mathrm{i}\gamma^5\gamma^0)\gamma^\mu(\mathrm{i}\gamma^5\gamma^0) = -\gamma^\nu\mathcal{P}^\mu_\nu$ and we have changed the variable $x \to \mathcal{P}x$ for the last two terms after the third equality, and used that the Jacobian $|\det(\mathcal{P})| = 1$. Equation (B1) tells us that for a generic value of α, $S_I(\psi^{U^\alpha_{\mathrm{PA}}},\bar\psi^{U^\alpha_{\mathrm{PA}}},A) \neq S_I(\psi,\bar\psi,A)$ because in general $A^P_\mu(x) \equiv \mathcal{P}^\nu_\mu A_\nu(x) \neq A_\mu(x)$ and therefore S_I is not $U(1)_{\mathrm{PA}}$ invariant. Equation (B1) is similar to Eq. (6) for $U(1)_{\mathrm{P}}$; hence, the two consequences below that equation become similar for $U(1)_{\mathrm{PA}}$ in the sense that we can reobtain the invariance of S_I only for particular values of α. In particular, we recognize two cases:

(i) $\alpha = \pi k$ with $k = 0,1,2,\ldots \Rightarrow U(1)_{\mathrm{PA}}$ reduces to the group $Z_2 \subset U(1)_{\mathrm{PA}}$ [see Eq. (13)] and of course $S_F + S_I$ is Z_2 invariant.

(ii) $\alpha = \pi k + (\pi/2)$ with $k = 0,1,2,\ldots \Rightarrow$ In this case if we perform also a parity transformation of the gauge field $A_\mu(x) \to A^P_\mu(x)$, we can obtain the invariance of the interaction term, namely $S_I(\psi^{U^\alpha_{\mathrm{PA}}},\bar\psi^{U^\alpha_{\mathrm{PA}}},A^P) = S_I(\psi,\bar\psi,A)$. In fact, as it is clear from Eq. (13), $U(1)_{\mathrm{PA}}$ group transformations reduce to parity $\times\gamma^5 Z_2 \in U(1)_{\mathrm{PA}}$ transformations, where for parity $\times\gamma^5 Z_2$ we mean the transformation: $\psi(x) \to z\gamma^5(P\psi(x)P^\dagger)$, with $z \in Z_2$.

Otherwise a sufficient condition for the $U(1)_{\mathrm{PA}}$ invariance of $S_I(\psi,\bar\psi,A)$, can be obtained restricting ourself to gauge configurations satisfying the condition in Eq. (7).

Regarding the mass term $m\int d^4x\bar{\psi}(x)\psi(x)$, we have

$$\begin{aligned}\int d^4x\bar{\psi}(x)^{U^\alpha_{\rm PA}}\psi(x)^{U^\alpha_{\rm PA}} &= \cos(\alpha)^2\int d^4x\bar{\psi}(x)\psi(x) + \mathrm{i}\sin(\alpha)\cos(\alpha)\int d^4x\bar{\psi}(x)(\mathrm{i}\gamma^5\gamma^0)\psi(\mathcal{P}x)\\ &\quad + \mathrm{i}\sin(\alpha)\cos(\alpha)\int d^4x\bar{\psi}(\mathcal{P}x)(\mathrm{i}\gamma^5\gamma^0)\psi(x) - \sin(\alpha)^2\int d^4x\bar{\psi}(\mathcal{P}x)(\mathrm{i}\gamma^5\gamma^0)(\mathrm{i}\gamma^5\gamma^0)\psi(\mathcal{P}x)\\ &= \cos(2\alpha)\int d^4x\bar{\psi}(x)\psi(x) + \mathrm{i}\sin(2\alpha)\int d^4x\bar{\psi}(\mathcal{P}x)(\mathrm{i}\gamma^5\gamma^0)\psi(x)\\ &\neq \int d^4x\bar{\psi}(x)\psi(x), \end{aligned} \tag{B2}$$

which is therefore not invariant under $U(1)_{\rm PA}$ transformations. The mass term breaks $U(1)_{\rm PA}$.

APPENDIX C: EXPLICIT CALCULATION OF EQ. (19)

Let us start proving the first of Eq. (19), considering the expression (1) for the $U(1)_{\rm P}$ transformations and the expression of $c_r(\boldsymbol{p})$ in (A3). Therefore, we have

$$\begin{aligned} c_r(\boldsymbol{p})^{U^\alpha_{\rm P}} &= \int\frac{d^3x}{(2\pi)^{3/2}}u_r(\boldsymbol{p})^\dagger\psi(x)^{U^\alpha_{\rm P}}e^{\mathrm{i}px}\\ &= \cos(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}u_r(\boldsymbol{p})^\dagger\psi(x)e^{\mathrm{i}px}\right] + \mathrm{i}\sin(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}u_r(\boldsymbol{p})^\dagger\gamma^0\psi(\mathcal{P}x)e^{\mathrm{i}px}\right]\\ &= \cos(\alpha)c_r(\boldsymbol{p}) + \mathrm{i}\sin(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}(\gamma^0u_r(\boldsymbol{p}))^\dagger\psi(x)e^{\mathrm{i}p(\mathcal{P}x)}\right]\\ &= \cos(\alpha)c_r(\boldsymbol{p}) + \mathrm{i}\sin(\alpha)c_r^P(\boldsymbol{p}), \end{aligned} \tag{C1}$$

where we have changed the variable $x\to\mathcal{P}x$ and used that $|\det(\mathcal{P})| = 1$. We have also used the expression of $c_r^P(\boldsymbol{p})$ in (A4) and Eq. (18).

For $c_r^P(\boldsymbol{p})$ we have

$$\begin{aligned} c_r^P(\boldsymbol{p})^{U^\alpha_{\rm P}} &= \int\frac{d^3x}{(2\pi)^{3/2}}u_r(-\boldsymbol{p})^\dagger\psi(x)^{U^\alpha_{\rm P}}e^{\mathrm{i}p(\mathcal{P}x)}\\ &= \cos(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}u_r(-\boldsymbol{p})^\dagger\psi(x)e^{\mathrm{i}p(\mathcal{P}x)}\right] + \mathrm{i}\sin(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}u_r(-\boldsymbol{p})^\dagger\gamma^0\psi(\mathcal{P}x)e^{\mathrm{i}(\mathcal{P}x)}\right]\\ &= \cos(\alpha)c_r^P(\boldsymbol{p}) + \mathrm{i}\sin(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}(\gamma^0u_r(-\boldsymbol{p}))^\dagger\psi(x)e^{\mathrm{i}px}\right]\\ &= \cos(\alpha)c_r^P(\boldsymbol{p}) + \mathrm{i}\sin(\alpha)c_r(\boldsymbol{p}), \end{aligned} \tag{C2}$$

where we used the same procedure as before. Regarding $d_r(\boldsymbol{p})^\dagger$ we obtain

$$\begin{aligned} (d_r(\boldsymbol{p})^\dagger)^{U^\alpha_{\rm P}} &= \int\frac{d^3x}{(2\pi)^{3/2}}v_r(\boldsymbol{p})^\dagger\psi(x)^{U^\alpha_{\rm P}}e^{-\mathrm{i}px}\\ &= \cos(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}v_r(\boldsymbol{p})^\dagger\psi(x)e^{-\mathrm{i}px}\right] + \mathrm{i}\sin(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}v_r(\boldsymbol{p})^\dagger\gamma^0\psi(\mathcal{P}x)e^{-\mathrm{i}px}\right]\\ &= \cos(\alpha)d_r(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha)\left[\int\frac{d^3x}{(2\pi)^{3/2}}(\gamma^0v_r(\boldsymbol{p}))^\dagger\psi(x)e^{-\mathrm{i}p(\mathcal{P}x)}\right]\\ &= \cos(\alpha)d_r(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha)d_r^P(\boldsymbol{p})^\dagger, \end{aligned} \tag{C3}$$

and similar for $d_r^P(\boldsymbol{p})^\dagger$ we have

$$
\begin{aligned}
(d_r^P(\boldsymbol{p})^\dagger)^{U_{\mathrm{P}}^\alpha} &= \int \frac{d^3x}{(2\pi)^{3/2}} (-v_r(-\boldsymbol{p}))^\dagger \psi(x)^{U_{\mathrm{P}}^\alpha} e^{-\mathrm{i}p(\mathcal{P}x)} \\
&= \cos(\alpha)\left[\int \frac{d^3x}{(2\pi)^{3/2}} (-v_r(-\boldsymbol{p}))^\dagger \psi(x) e^{-\mathrm{i}p(\mathcal{P}x)}\right] - \mathrm{i}\sin(\alpha)\left[\int \frac{d^3x}{(2\pi)^{3/2}} v_r(-\boldsymbol{p})^\dagger \gamma^0 \psi(\mathcal{P}x) e^{-\mathrm{i}p(\mathcal{P}x)}\right] \\
&= \cos(\alpha) d_r^P(\boldsymbol{p})^\dagger - \mathrm{i}\sin(\alpha)\left[\int \frac{d^3x}{(2\pi)^{3/2}} (\gamma^0 v_r(-\boldsymbol{p}))^\dagger \psi(x) e^{-\mathrm{i}px}\right] \\
&= \cos(\alpha) d_r^P(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha) d_r(\boldsymbol{p})^\dagger,
\end{aligned}
\tag{C4}
$$

where we have changed the variable $x \to \mathcal{P}x$ and used that $|\det(\mathcal{P})| = 1$. We have also used the expressions of $d_r(\boldsymbol{p})^\dagger$ and $d_r^P(\boldsymbol{p})^\dagger$ in (A3) and (A4), respectively, and Eq. (18).

APPENDIX D: EXPLICIT CALCULATION OF EQS. (23) AND (25)

In order to prove Eqs. (23) and (25), we introduce two fields,

$$
U_r(\boldsymbol{p}) = \begin{pmatrix} u_r(\boldsymbol{p}) \\ 0 \end{pmatrix}, \qquad V_r(\boldsymbol{p}) = \begin{pmatrix} v_r(\boldsymbol{p}) \\ 0 \end{pmatrix}, \tag{D1}
$$

which are defined in the parity partners space, as $\Psi(x)$ in (2). Hence, we have that $u_r(\boldsymbol{p})^\dagger \psi(x) = U_r(\boldsymbol{p})^\dagger [(\mathbb{1} + \sigma^1) \otimes \mathbb{1}]\Psi(x)$ and $v_r(\boldsymbol{p})^\dagger \psi(x) = V_r(\boldsymbol{p})^\dagger [(\mathbb{1} + \sigma^1) \otimes \mathbb{1}]\Psi(x)$, where $\mathbb{1}$ acts in the Dirac space, $\Psi(x)$ is given in (2), and $(\mathbb{1} + \sigma^1)$ acts on the two-dimensional space of parity partners.

The properties of $u_r(\boldsymbol{p})$ and $v_r(\boldsymbol{p})$ [see Eq. (A2)], given in Eqs. (18) and (21), valid for $m = 0$, can be translated for $U_r(\boldsymbol{p})$ and $V_r(\boldsymbol{p})$ as well, and we obtain

$$
\begin{aligned}
(\mathbb{1} \otimes \gamma^0) U_r(\boldsymbol{p}) &= U_r(-\boldsymbol{p}), \\
(\mathbb{1} \otimes \gamma^0) V_r(\boldsymbol{p}) &= -V_r(-\boldsymbol{p}), \\
(\mathbb{1} \otimes \gamma^5) U_r(\boldsymbol{p}) &= h_r U_r(\boldsymbol{p}), \\
(\mathbb{1} \otimes \gamma^5) V_r(\boldsymbol{p}) &= h_{r\oplus 1} V_r(\boldsymbol{p}),
\end{aligned}
\tag{D2}
$$

where third and fourth equations are valid for $m = 0$, where γ^5 coincides with the helicity operator $\boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$. This implies that for opposite values $-\boldsymbol{p}$, third and fourth equations of (D2) become

$$
\begin{aligned}
(\mathbb{1} \otimes \gamma^5) U_r(-\boldsymbol{p}) &= -h_r U_r(-\boldsymbol{p}), \\
(\mathbb{1} \otimes \gamma^5) V_r(-\boldsymbol{p}) &= -h_{r\oplus 1} V_r(-\boldsymbol{p}).
\end{aligned}
\tag{D3}
$$

Using such notation and defining $\Gamma = [(\mathbb{1} + \sigma^1) \otimes \mathbb{1}]$, we can rewrite $c_r(\boldsymbol{p})$, $c_r^P(\boldsymbol{p})$, $d_r(\boldsymbol{p})^\dagger$, and $d_r^P(\boldsymbol{p})^\dagger$ of Eqs. (A3) and (A4) as

$$
\begin{aligned}
c_r(\boldsymbol{p}) &= \int \frac{d^3x}{(2\pi)^{3/2}} U_r(\boldsymbol{p})^\dagger \Gamma \Psi(x) e^{\mathrm{i}px}, \\
c_r^P(\boldsymbol{p}) &= \int \frac{d^3x}{(2\pi)^{3/2}} U_r(-\boldsymbol{p})^\dagger \Gamma \Psi(x) e^{\mathrm{i}p(\mathcal{P}x)} \\
&= \int \frac{d^3x}{(2\pi)^{3/2}} U_r(-\boldsymbol{p})^\dagger \Gamma \Psi(\mathcal{P}x) e^{\mathrm{i}px}, \\
d_r(\boldsymbol{p})^\dagger &= \int \frac{d^3x}{(2\pi)^{3/2}} V_r(\boldsymbol{p})^\dagger \Gamma \Psi(x) e^{-\mathrm{i}px}, \\
d_r^P(\boldsymbol{p})^\dagger &= \int \frac{d^3x}{(2\pi)^{3/2}} (-V_r(-\boldsymbol{p}))^\dagger \Gamma \Psi(x) e^{-\mathrm{i}p(\mathcal{P}x)} \\
&= \int \frac{d^3x}{(2\pi)^{3/2}} (-V_r(-\boldsymbol{p}))^\dagger \Gamma \Psi(\mathcal{P}x) e^{-\mathrm{i}px},
\end{aligned}
\tag{D4}
$$

where in $c_r^P(\boldsymbol{p})$ and $d_r^P(\boldsymbol{p})^\dagger$, we have changed the variable $x \to \mathcal{P}x$ and used that the Jacobian is $|\det(\mathcal{P})| = 1$. At this point the $SU(2)_{CS}^{\mathcal{P}}$ transformations are given by

$$
\begin{aligned}
c_r(\boldsymbol{p})^{U_{CS^\mathcal{P}}^\alpha} &= \int \frac{d^3x}{(2\pi)^{3/2}} U_r(\boldsymbol{p})^\dagger \Gamma \Psi(x)^{U_{CS^\mathcal{P}}^\alpha} e^{\mathrm{i}px} \\
&= \cos(\alpha) c_r(\boldsymbol{p}) + \mathrm{i}\sin(\alpha) \sum_{i=1}^{3} e_i \int \frac{d^3x}{(2\pi)^{3/2}} U_r(\boldsymbol{p})^\dagger \Gamma \Sigma_i^{\mathcal{P}} \Psi(x) e^{\mathrm{i}px}, \\
c_r^P(\boldsymbol{p})^{U_{CS^\mathcal{P}}^\alpha} &= \int \frac{d^3x}{(2\pi)^{3/2}} U_r(-\boldsymbol{p})^\dagger \Gamma \Psi(x)^{U_{CS^\mathcal{P}}^\alpha} e^{\mathrm{i}p(\mathcal{P}x)} \\
&= \cos(\alpha) c_r^P(\boldsymbol{p}) + \mathrm{i}\sin(\alpha) \sum_{i=1}^{3} e_i \int \frac{d^3x}{(2\pi)^{3/2}} U_r(-\boldsymbol{p})^\dagger \Gamma \Sigma_i^{\mathcal{P}} \Psi(x) e^{\mathrm{i}p(\mathcal{P}x)},
\end{aligned}
$$

$$\begin{aligned}
(d_r(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &= \int \frac{d^3x}{(2\pi)^{3/2}} V_r(\boldsymbol{p})^\dagger \Gamma \Psi(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} e^{-\mathrm{i}px} \\
&= \cos(\alpha) d_r(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha) \sum_{i=1}^{3} e_i \int \frac{d^3x}{(2\pi)^{3/2}} V_r(\boldsymbol{p})^\dagger \Gamma \Sigma_i^{\mathcal{P}} \Psi(x) e^{-\mathrm{i}px}, \\
(d_r^P(\boldsymbol{p})^\dagger)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} &= \int \frac{d^3x}{(2\pi)^{3/2}} (-V_r(-\boldsymbol{p}))^\dagger \Gamma \Psi(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} e^{-\mathrm{i}p(\mathcal{P}x)} \\
&= \cos(\alpha) d_r^P(\boldsymbol{p})^\dagger + \mathrm{i}\sin(\alpha) \sum_{i=1}^{3} e_i \int \frac{d^3x}{(2\pi)^{3/2}} (-V_r(-\boldsymbol{p}))^\dagger \Gamma \Sigma_i^{\mathcal{P}} \Psi(x) e^{-\mathrm{i}p(\mathcal{P}x)}, \qquad \text{(D5)}
\end{aligned}$$

where we used that $\Psi(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}} = U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}} \Psi(x)$ and $U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}$ is given in Eq. (22). From $c_r(\boldsymbol{p})$ and $c_r^P(\boldsymbol{p})$ it is evident that we need to know the terms $U_r(\pm\boldsymbol{p})^\dagger[(\mathbb{1}+\sigma^1)\otimes\mathbb{1}]\Sigma_i^{\mathcal{P}}\Psi(x)$, where $\Sigma_i^{\mathcal{P}} = \{\sigma^3\otimes\gamma^0, \sigma^3\otimes \mathrm{i}\gamma^5\gamma^0, -\mathbb{1}\otimes\gamma^5\}$. At first we notice that by definition $(\sigma^3\otimes\mathbb{1})\Psi(x) = \Psi(\mathcal{P}x)$ and then from Eqs. (D2) and (D3), we get

$$\begin{aligned}
U_r(\pm\boldsymbol{p})^\dagger \Gamma \Sigma_1^{\mathcal{P}} \Psi(x) &= ((\mathbb{1}\otimes\gamma^0) U_r(\pm\boldsymbol{p}))^\dagger \Gamma (\sigma^3\otimes\mathbb{1}) \Psi(x) = U_r(\mp\boldsymbol{p})^\dagger \Gamma \Psi(\mathcal{P}x), \\
U_r(\pm\boldsymbol{p})^\dagger \Gamma \Sigma_2^{\mathcal{P}} \Psi(x) &= ((\mathbb{1}\otimes \mathrm{i}\gamma^5\gamma^0) U_r(\pm\boldsymbol{p}))^\dagger \Gamma (\sigma^3\otimes\mathbb{1}) \Psi(x) = \pm \mathrm{i} h_r U_r(\mp\boldsymbol{p})^\dagger \Gamma \Psi(\mathcal{P}x), \\
U_r(\pm\boldsymbol{p})^\dagger \Gamma \Sigma_3^{\mathcal{P}} \Psi(x) &= ((-\mathbb{1}\otimes\gamma^5) U_r(\pm\boldsymbol{p}))^\dagger \Gamma \Psi(x) = \mp h_r U_r(\pm\boldsymbol{p})^\dagger \Gamma \Psi(x), \qquad \text{(D6)}
\end{aligned}$$

where we have decomposed $\sigma^3\otimes l_i = (\mathbb{1}\otimes l_i)(\sigma^3\otimes\mathbb{1})$, with $l_i = \{\gamma^0, \mathrm{i}\gamma^5\gamma^0\}$ and used that $[\mathbb{1}\otimes l_i, \Gamma] = 0$ for $i = 1, 2$. Now we can plug (D6) in Eq. (D5), having in mind (D4), and finally we obtain the right sides of Eq. (23).

Regarding the terms $V_r(\boldsymbol{p})^\dagger[(\mathbb{1}+\sigma^1)\otimes\mathbb{1}]\Psi(x)^{U^{\boldsymbol{\alpha}}_{CS^{\mathcal{P}}}}$ for $d_r(\boldsymbol{p})^\dagger$ and $d_r^P(\boldsymbol{p})^\dagger$, they can be rewritten using Eqs. (D2) and (D3) as

$$\begin{aligned}
V_r(\pm\boldsymbol{p})^\dagger \Gamma \Sigma_1^{\mathcal{P}} \Psi(x) &= ((\mathbb{1}\otimes\gamma^0) V_r(\pm\boldsymbol{p}))^\dagger \Gamma (\sigma^3\otimes\mathbb{1}) \Psi(x) = -V_r(\mp\boldsymbol{p})^\dagger \Gamma \Psi(\mathcal{P}x), \\
V_r(\pm\boldsymbol{p})^\dagger \Gamma \Sigma_2^{\mathcal{P}} \Psi(x) &= ((\mathbb{1}\otimes \mathrm{i}\gamma^5\gamma^0) V_r(\pm\boldsymbol{p}))^\dagger \Gamma (\sigma^3\otimes\mathbb{1}) \Psi(x) = \mp \mathrm{i} h_{r\oplus 1} V_r(\mp\boldsymbol{p})^\dagger \Gamma \Psi(\mathcal{P}x), \\
V_r(\pm\boldsymbol{p})^\dagger \Gamma \Sigma_3^{\mathcal{P}} \Psi(x) &= ((-\mathbb{1}\otimes\gamma^5) V_r(\pm\boldsymbol{p}))^\dagger \Gamma \Psi(x) = \mp h_{r\oplus 1} V_r(\pm\boldsymbol{p})^\dagger \Gamma \Psi(x), \qquad \text{(D7)}
\end{aligned}$$

where we used the same procedure as in Eq. (D6). Now we plug (D7) in Eq. (D5) and keeping in mind the expression (D4), we obtain the right sides of Eq. (25). This ends our calculation.

[1] C. Rohrhofer, Y. Aoki, G. Cossu, H. Fukaya, L. Ya. Glozman, S. Hashimoto, C. B. Lang, and S. Prelovsek, Approximate degeneracy of $J = 1$ spatial correlators in high temperature QCD, Phys. Rev. D **96**, 094501 (2017); Erratum, Phys. Rev. D **99**, 039901 (2019).

[2] C. Rohrhofer, Y. Aoki, G. Cossu, H. Fukaya, C. Gattringer, L. Ya. Glozman, S. Hashimoto, C. B. Lang, and S. Prelovsek, Symmetries of spatial meson correlators in high temperature QCD, Phys. Rev. D **100**, 014502 (2019).

[3] C. Rohrhofer, Y. Aoki, L. Ya. Glozman, and S. Hashimoto, Chiral-spin symmetry of the meson spectral function above T_c, Phys. Lett. B **802**, 135245 (2020).

[4] L. Ya. Glozman, Y. Aoki, S. Hashimoto, and C. Rohrhofer, Symmetries of temporal correlators and the nature of hot QCD, in *Proceedings of the 38th International Symposium on Lattice Field Theory* LATTICE2021 (2021), 10.22323/1.396.0357.

[5] M. Denissenya, L. Ya. Glozman, and C. B. Lang, Symmetries of mesons after unbreaking of chiral symmetry and their string interpretation, Phys. Rev. D **89**, 077502 (2014).

[6] M. Denissenya, L. Ya. Glozman, and C. B. Lang, Isoscalar mesons upon unbreaking of chiral symmetry, Phys. Rev. D **91**, 034505 (2015).

[7] M. Denissenya, L. Ya. Glozman, and M. Pak, Evidence for a new $SU(4)$ symmetry with $J = 2$ mesons, Phys. Rev. D **91**, 114512 (2015).

[8] Marco Catillo, On $SU(2)_{CS}$-like groups and invariance of the fermionic action in QCD (2021), arXiv:2109.03532v3, [Int. J. Mod. Phys. A (to be published)].

[9] M. Catillo and L. Ya. Glozman, Baryon parity doublets and chiral spin symmetry, Phys. Rev. D **98**, 014030 (2018).

[10] Franz Mandl and Graham Shaw, *Quantum Field Theory, 2nd Edition* (Wiley, 1985), https://www.wiley.com/en-us/Quantum+Field+Theory,+2nd+Edition-p-9780471496830.

[11] L. Ya. Glozman, Chiralspin symmetry and QCD at high temperature, Eur. Phys. J. A **54**, 117 (2018).

[12] L. Ya. Glozman, A hidden classical symmetry of QCD, EPJ Web Conf. **164**, 03002 (2017).

[13] L. Ya. Glozman, Three regimes of QCD, Int. J. Mod. Phys. A **36**, 2044031 (2021).

[14] A. Bazavov *et al.*, Chiral transition and $U(1)_A$ symmetry restoration from lattice QCD using domain wall fermions, Phys. Rev. D **86**, 094503 (2012).

[15] Guido Cossu, Sinya Aoki, Hidenori Fukaya, Shoji Hashimoto, Takashi Kaneko, Hideo Matsufuru, and Jun-Ichi Noaki, Finite temperature study of the axial U(1) symmetry on the lattice with overlap fermion formulation, Phys. Rev. D **87**, 114514 (2013); Erratum, Phys. Rev. D **88**, 019901 (2013).

[16] A. Tomiya, G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, and J. Noaki, Evidence of effective axial U(1) symmetry restoration at high temperature QCD, Phys. Rev. D **96**, 034509 (2017); **96**, 079902(A) (2017).