Nonperturbative renormalization of the lattice Sommerfield vector model

Vieri Mastropietro

Institute for Advanced Study, Princeton, New Jersey 08540, USA and University of Milan, Department of Mathematics, 20133 Milan, Italy

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The lattice Sommerfield model, describing a massive vector gauge field coupled to a light fermion in two dimensions, is an ideal candidate to verify perturbative conclusions. In contrast with continuum exact solutions, we prove that there is no infinite field renormalization, implying the reduction of the degree of the ultraviolet divergence, and that the anomalies are nonrenormalized. Such features are the counterpart of analog properties at the basis of the Standard Model perturbative renormalizability. The results are nonperturbative in the sense that the averages of invariant observables are expressed in terms of convergent expansions uniformly in the lattice and volume.

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I. INTRODUCTION

Most properties of the Standard Model are known only at a perturbative level with series expansions expected to be generically diverging; in particular its perturbative renormalizability (Refs. [1,2]) relies on two crucial properties; the reduction of the degree of divergence with respect to power counting and the cancellation of the anomalies [3] ensured by the Adler-Bardeen theorem [4]. Such properties are essential to maintain the renormalizability present with massless bosons. The phenomenon of the reduction of the degree of divergence can be already seen in a $U(1)$ gauge theory like QED. Adding a mass to the photon breaks gauge invariance and produces a propagator of the form $\frac{1}{k^2+M^2}(\delta_{\mu\nu}+\frac{k_\mu k_\nu}{M^2})$; due to the lack of decay of the second term the theory becomes dimensionally nonrenormalizable. However the transition in a $U(1)$ gauge theory like QED from a $M=0$ to a $M\neq 0$ case is soft and the theory remains perturbatively renormalizable [5]; the photons are coupled to a conserved current $k_\mu \hat{j}_\mu = 0$ so that the contribution of the nondecaying part of the propagator is vanishing. A similar reduction happens in the electroweak sector, but the fermion mass violates the chiral symmetry and leads to the Higgs introduction; again the renormalizability proof relies on the fact that the $k_\mu k_\nu$ term in the propagator does not contribute [2]. The chiral symmetry is generically violated by anomalies which need to cancel out—such cancellation is based on the Adler-Bardeen property.

All of the above arguments are valid in perturbation theory and nonperturbative effects could be missed. This issue would be solved by a nonperturbative lattice anomaly-free formulation of gauge theory, which is still out of reach, see for instance Refs. [6–8]. In particular one needs to get high cutoff values, exponential in the inverse coupling, a property which is the nonperturbative analog of renormalizabiliy. The implementation of the Adler-Bardeen theorem and of the reduction of the degree of divergence in a nonperturbative context is however a nontrivial issue, as their perturbative derivation uses dimensional regularizations, and functional integral derivations [9] are essentially one-loop results [10].

Therefore, it is convenient to investigate such properties in a simpler context, and the Sommerfield model [11], describing a massive vector gauge field coupled to a light fermion in two dimensions, appears to be the ideal candidate, see also Refs. [12–14]. More exactly, we consider a version of this model with nonzero fermionic mass, but our results are uniform in the mass. The model can be seen as a $d=2$ QED with a massive photon; as in four dimensions, at the level of perturbation theory the transition from $M=0$ to a $M\neq 0$ is soft and the theory remains super-renormalizable. Again, this follows from the conservation of currents, which is ensured at the level of correlations by dimensional regularization; the same regularization provides the anomaly nonrenormalization [12]. In this case however we have access to nonperturbative information and we can check such conclusions. Exact solutions are known in the continuum version of the Sommerfield model (Refs. [11,15–17]). Remarkably, the above perturbative features are not verified; there is an infinite wave function renormalization incompatible with the super-renormalizability, and anomalies have a value depending on the regularization.

In this paper we consider the Sommerfield model on the lattice, and we analyze it using the methods of constructive renormalization. The lattice preserves a number of symmetries, in the form of Ward identities. Our main result is that there is no infinite field renormalization, which is the counterpart of super-renormalizability, and that the Adler-Bardeen theorem holds with finite lattice. Nonperturbative violation of the above perturbative conclusions is therefore excluded. Other two dimension models previously rigorously constructed, see Refs. [18–25], lack of these features. Quantum simulations of two dimension models (Refs. [26–28]) have been also considered in the literature, but they regard mostly the Schwinger model, to which the Sommerfield model reduces when the boson and fermion mass is vanishing. Our results are nonperturbative, in the sense that the averages of gauge-invariant observables are expressed in terms of convergent expansions uniformly in the lattice and volume.

The paper is organized in the following way. In Sec. II we define a lattice version of the Sommerfield model. In Sec. III we derive exact Ward identities for the model. In Sec. IV we integrate the boson field and in Sec. V we perform a nonperturbative-multiscale analysis for the fermionic fields. In Sec. VI we prove the validity of the Adler-Bardeen theorem and in Sec. VII the conclusions are presented.

II. THE LATTICE SOMMERFIELD MODEL

If $\gamma_0 = \sigma_1$, $\gamma_1 = \sigma_2$, we define

$$\langle O \rangle = \frac{1}{Z} \int \prod_x d\bar\psi_x d\psi_x \int_{R^{2|\Lambda|}} \prod_{x,\mu} dA_{\mu,x} e^{-S(A,\psi)} O, \quad (1)$$

where Z the normalization, $x \in \Lambda$, with Λ is a square lattice with step a with antiperiodic boundary conditions and

$$S(A,\psi) = S_A(A) + S_\psi(A,\psi), \quad (2)$$

with

$$S_A(A) = a^2 \sum_x \left[\frac{1}{4} F_{\mu,\nu,x} F_{\mu,\nu,x} + \frac{M^2}{2} A_{\mu,x} A_{\mu,x} \right],$$
$$S_\psi(A,\psi) = a^2 \sum_x [\tilde m \bar\psi_x \psi_x + a^{-1} Z_\psi (\bar\psi_x \gamma^+_\mu e^{iaeA_{\mu,x}} \psi_{x+a_\mu} - \bar\psi_{x+a_\mu} \gamma^-_\mu e^{-iaeA_{\mu,x}} \psi_x)], \quad (3)$$

with $a_\mu = a e_\mu$, $e_0 = (1,0), e_1 = (0,1)$, $\gamma^\pm_\mu = \gamma_\mu \mp r$ $F_{\mu,\nu} = d_\nu A_\mu - d_\mu A_\nu$, and $d_\nu A_\mu = a^{-1}(A_{\mu,x+e_\nu a} - A_{\mu,x})$, $\tilde m = (m + 4r/a)$ and $r = 1$ is the Wilson term. Note that if $1/a$ and L are finite then the integral is finite dimensional.

We generalize the model adding a term $(1-\xi)a^2 \times \sum_x \sum_\mu (d_\mu A_\mu)^2$, $\xi \le 1$ so that the bosonic action is given by $\frac{1}{2} a^2 \sum_x (\sum_{\mu,\nu} (d_\mu A_\nu)^2 + \xi \sum_\mu (d_\mu A_\mu)^2)$. The original model is recovered with $\xi = 1$.

The correlations can be written as derivatives of the generating function,

$$e^{W_\xi(J,B,\phi)} = \int P(d\psi) P(dA) e^{-V(A+J,\psi) + (\psi,\phi) + a^2 \sum_x B_x O}, \quad (4)$$

with $O = O(A+J,\psi)$ an observable, and $P(dA)$ the Gaussian measure with covariance

$$\hat g^A_{\mu,\nu}(k) = \frac{1}{|\sigma|^2 + M^2} \left(\delta_{\mu,\nu} + \frac{\xi \bar\sigma_\mu \sigma_\nu}{(1-\xi)|\sigma|^2 + M^2} \right), \quad (5)$$

with $\sigma_\mu(k) = (e^{ik_\mu a} - 1)a^{-1}$.

$P(d\psi)$ is the fermionic integration with propagator $\hat g^\psi(k) = Z_\psi^{-1} (\tilde k_\mu \gamma_\mu + a^{-1} m(k) I)^{-1}$ with $\tilde k_\mu = \frac{\sin(k_\mu a)}{a}$ and $m(k) = m + r(\cos ak_0 + \cos ak_1 - 2)$; finally,

$$V(A,\psi) = a^2 \sum_x [O^+_{\mu,x} G^+_{\mu,x}(A) + O^-_{\mu,x} G^-_{\mu,x}(A)], \quad (6)$$

with $O^+_\mu = Z_\psi \bar\psi_x (\gamma_\mu - r) \psi_{x+a_\mu}$ and

$$O^-_\mu = -Z_\psi \bar\psi_{x+a_\mu} (\gamma_\mu + r) \psi_x, \quad (7)$$

and $G^\pm_\mu = a^{-1}(e^{\pm ieaA_{\mu,x}} - 1)$.

If $M = 0$ the model (1) is invariant under the gauge transformation $A_{\mu,x} \to A_{\mu,x} + d_\mu \alpha_x$ and $\psi_x \to \psi e^{-ie\alpha_x}$; if $M \neq 0$ the invariance is lost.

III. WARD IDENTITIES AND ξ-INDEPENDENCE

If we restrict to observables such that $O(A,\psi) = O(A + d\alpha, \psi e^{-ie\alpha})$ (which we call invariant observables) there is also gauge invariance in the external fields for $M \neq 0$, that is

$$W_\xi(J + d\alpha, e^{-ie\alpha}\phi, B) = W_\xi(J,\phi,B). \quad (8)$$

This follows by performing in (5) the change of variables $\psi_x \to \psi e^{ie\alpha_x}$, with Jacobian equal to 1 (the integral is finite-dimensional) and noting that $(e^{ie\alpha}\psi, \phi) = (\psi, \phi e^{-ie\alpha})$ and

$$S_\psi(A+J, \psi e^{ie\alpha}) = S_\psi(A+J+d\alpha, \psi) \quad (9)$$

(8) implies that $\partial_\alpha W_\xi(J + d\alpha, e^{-ie\alpha}\phi, B) = 0$. We define $\Gamma_{\mu_1,\ldots,\mu_n,\nu_1,..\nu_m}$ as the derivatives of W_ξ with respect to $J_{\mu_1,x_1}, \ldots, B_{\nu_n,x_n}$. By performing in (8) the derivatives with respect to α and the external fields we get the Ward identities (expressing current conservation)

$$\sum_{\mu_1} \sigma_{\mu_1}(p_1) \hat\Gamma_{\mu_1,\ldots,\nu_n}(p_1,\ldots,p_{n-1}) = 0, \quad (10)$$

and

$$\sigma_\mu(p)\hat{\Gamma}_\mu(p.k) = \hat{S}(k) - \hat{S}(k+p), \tag{11}$$

where $\hat{\Gamma}_\mu(p,k) = \frac{\partial^3 W}{\partial \hat{J}_{\mu,p}\partial\hat{\phi}_k\partial\bar{\phi}_{k-p}}|_0$ is the vertex function and $\hat{S}(k) = \frac{\partial^2 W}{\partial\hat{\phi}_k\partial\bar{\phi}_k}|_0$ is the two-point function.

The conservation of current expressed by the above Ward identity implies that for invariant observables

$$\partial_\xi W_\xi(J,0,B) = 0, \tag{12}$$

i.e., the averages are ξ independent. This follows from $\partial_\xi \int P(dA) \int \prod_x d\psi_x d\bar{\psi}_x O = 0$, with $O(A,\psi)$ invariant; indeed

$$\partial_\xi \int P(dA) \int \prod d\psi_x d\bar{\psi}_x O = \frac{1}{L^2}\sum_p \partial_\xi \hat{g}^{-1}_{\mu,\nu}(p) \int P(dA) A_{\mu,p}A_{\nu,-p} \int \prod d\psi_x d\bar{\psi}_x O, \tag{13}$$

from which we get, using that $A_{\mu,p} = \hat{g}^A_{\mu,\rho}\frac{\partial}{\partial A_{\rho,-p}}$

$$\hat{g}^A_{\mu,\rho'}(p)\partial_\xi(\hat{g}^A(p))^{-1}_{\mu,\nu}\hat{g}^A_{\nu,\rho}(p)\frac{\partial^2}{\partial\hat{J}_{\rho,p}\partial\hat{J}_{\rho',-p}}\int P(dA) \times \int \prod_x d\psi_x d\bar{\psi}_x O(A+J,\psi)|_0. \tag{14}$$

By noting that

$$\partial(\hat{g}^A)^{-1} = -(\hat{g}^A)^{-1}\partial_\xi \hat{g}^A(\hat{g}^A)^{-1}, \tag{15}$$

and $\partial_\xi \hat{g}^A$ is proportional to $\bar{\sigma}_\mu\sigma_\nu$, by using

$$\partial_\alpha \int P(dA) \int \prod_x d\psi_x d\bar{\psi}_x O(A+d\alpha,\psi)|_0 = 0 \tag{16}$$

then (13) vanishes.

Equation (12) ensures that the averages do not depend on ξ, so that one can set $\xi = 0$ in the boson propagator, that is the nondecaying part of the propagator does not contribute. In perturbation theory, the scaling dimension with $\xi = 0$ ($z = 2$) and $\xi = 1$ ($z = 0$) is, if n is the order, n_A the number of A fields and n_ψ the number of ψ fields

$$d + (d-z-2)n/2 - (d-1)n_\psi/2 - (d-z)n_A/2. \tag{17}$$

Hence, in $d = 2$ the theory is dimensionally renormalizable with $\xi = 1$ and super-renormalizable with $\xi = 0$ (in $d = 4$ one pass from nonrenormalizability to renormalizability). The lattice regularization ensures that in the theory remains perturbatively super-renormalizable, as with dimensional regularization. We will investigate the validity of this property at a nonperturbative level.

Finally, we define the axial current as $j^5_\mu = Z^5\bar{\psi}_x\gamma_\mu\gamma_5\psi_x$, where Z^5 is a constant to be chosen so that the electric charge of the chiral and electromagnetic current are the same, defined as the amputated part of the three-point correlation at zero momenta (see [10]); that is

$$\lim_{k,p\to 0}\frac{\partial^3 W}{\partial\hat{B}^5_{\mu,p}\partial\hat{\phi}_k\partial\bar{\phi}_{k-p}}\Bigg|_0 \Bigg/ \frac{\partial^3 W}{\partial\hat{J}_{\mu,p}\partial\hat{\phi}_k\partial\bar{\phi}_{k-p}}\Bigg|_0 = 1, \tag{18}$$

where the source term is (B^5_μ, j^5_μ). The axial current is nonconserved even for $m = 0$, due to the presence of the Wilson term, and one has

$$\sigma_\mu(p)\hat{\Gamma}^5_{\mu,\nu}(p) = H_\nu(p), \tag{19}$$

with $\Gamma^5_{\mu,\nu}$ the derivative of W with respect to B_{μ,x_1}, J_{ν,x_2}. H_ν is called the anomaly and in the noninteracting case $V = 0$ one gets if $m = 0$ $H_\mu = \frac{1}{2\pi}\varepsilon_{\mu,\nu}p_\nu + O(ap^2)$ (lattice or dimensional regularization [12] produce the same result) and $Z^5 = 1$. In the interacting case $H_\nu(p)$ is a series in e and the nonrenormalization property means that all higher-order corrections vanishes.

IV. INTEGRATION OF THE BOSON FIELDS

We can integrate the boson field

$$\int P(dA)e^{-V} = e^{\sum_{n=0}^\infty \frac{(-1)^n}{n!}\mathcal{E}^T_A(V;n)} \equiv e^{V^N(\psi,J)}, \tag{20}$$

where $\mathcal{E}^T_A$ is the truncated expectation, that is the sum of connected diagrams, and $V^N =$

$$a^2\sum_x\sum_\varepsilon a^{-1}e^{-\frac{1}{2}e^2a^2 g^A_{\mu,\mu}(x,x)}e^{iaeJ_{\mu,x}}O^\varepsilon_\mu + \sum_{n,m}a^{n+m}\sum_{\underline{x},\underline{y}}\sum_{\underline{\mu},\underline{\varepsilon}}\left[\prod_{j=1}^n O^{\varepsilon_j}_{\mu_j,x_j}\right]\left[\prod_{k=1}^m G^{\varepsilon_j}_{\mu_j,y_j}(J)\right]W_{n,m}(\underline{x},\underline{y}). \tag{21}$$

Note that $a^2 g^A_{\mu,\mu}(x,x) \le C$. We call $a = \gamma^{-N}$, where $\gamma > 1$ a scaling parameter.

Theorem 1 The kernels in (21) for $n \ge 2$ verify,

$$|W_{n,m}| \le C^{n+m}e^{2(n-1)}\gamma^{N(2-n-m)}(|g^A|_1)^n. \tag{22}$$

Proof.—A convenient representation for $\mathcal{E}^T_A$ is given by the following formula [29]

$$\mathcal{E}^T_A\left(\prod_{k=1}^n e^{ie\varepsilon_k a A_{\mu_k,x_k}}\right) = \sum_{T\in\mathcal{T}}\prod_{i,j\in T}V_{i,j}\int dp_T(s)e^{-V_T(s)}, \tag{23}$$

where $V_{i,j} = e^2a^2\mathcal{E}_A(A_{\mu_i,x_i}A_{\mu_j,x_j})$, $\mathcal{T}$ is the set of tree graphs T on $X = (1..,n)$, $s \in (0,1)$ is an interpolation parameter, $V_T(s)$ is a convex-linear combination of $V(Y) = \sum_{i,j\in Y}\varepsilon_i\varepsilon_j V_{i,j}$, Y are subsets of X, and dp_T is a probability measure. The crucial point is that $V(Y)$ is stable; that is

$$V(Y) = \sum_{i,j\in Y} V_{i,j} = a^2 e^2 \mathcal{E}_A\left(\left[\sum_{i\in Y}\varepsilon_i A_{\mu_i,x_i}\right]^2\right) \geq 0. \tag{24}$$

Therefore, one can bound the exponential $e^{-V_T(s)} \leq 1$ finding

$$\begin{aligned}|W_{n,0}| &\leq C^n a^{-n}\frac{1}{n!}\sum_{T\in\mathcal{T}}\prod_{(i,j)\in T} a^2 e^2 |g^A(x_i,x_j)|_1 \\ &\leq \frac{1}{n!}\sum_{T\in\mathcal{T}} C^n e^{2(n-1)} a^{n-2}(|g^A|_1)^n \\ &\leq C^n e^{2(n-1)}\gamma^{N(2-n)}(|g^A|_1)^n.\end{aligned} \tag{25}$$

With $m \neq 0$ we get an extra a^{-Nm}, so that one recovers the dimensional factor $\gamma^{N(2-l/2-m)}$. ∎

For $\xi = 0$ $|g^A|_1 \leq CM^{-2}$ and $|g^A|_\infty \leq C|\log a|$ while for $\xi = 1$ $|g^A|_1 \leq C|\log a|$ and $|g^A|_\infty \leq Ca^{-2}$. We write $W_{n,m} = \lambda^{n-1}\bar{W}_n$ with $\lambda = e^2$ and $\bar{W}_n$ bounded. The normalization Z_ξ in the analog of (1) is analytic everywhere for finite a, L, and ξ independent; our strategy is to prove that $Z_0 = 1 + O(\lambda)$ and is analytic together with correlations for $|\lambda| \leq \lambda_0$ with λ_0 independent. Hence Z_1 is non vanishing and (1) for $\xi = 1$ is analytic for $|\lambda| \leq \lambda_0$, as the numerator is analytic everywhere for a, L finite: moreover it coincides with the value at $\xi = 0$. It remains then to prove that the correlations with $\xi = 0$ are analytic for $|\lambda| \leq \lambda_0$ and $Z_0 = 1 + O(\lambda)$.

The factor $D = 2 - n - m$ is the scaling dimension, and the terms with $D < 0$ are irrelevant. The marginal term for $\xi = 0$ is $\mathcal{E}_A^T(V;2) =$

$$\sum_{\varepsilon_1,\varepsilon_2} a^4 \sum_{x_1,x_2} e^{i\varepsilon_1 aJ_{\mu,x_1}} O^{\varepsilon_1}_{\mu,x_1} e^{i\varepsilon_2 aJ_{\mu,x_2}} O^{\varepsilon_2}_{\mu,x_2}\lambda v_{\mu,\varepsilon_1,\varepsilon_2}, \tag{26}$$

where $\lambda v_{\mu,\varepsilon_1,\varepsilon_2} = \mathcal{E}_A^T(e^{i\varepsilon_1 eaA_{\mu,x_1}}; e^{i\varepsilon_2 eaA_{\mu,x_2}})$ and, $\lambda = e^2$

$$\begin{aligned}v_{\mu,\varepsilon_1,\varepsilon_2}(x,y) &= e^{-\frac{1}{2}e^2a^2 g^A_{\mu,\mu}(x_1,x_1)} e^{-\frac{1}{2}e^2a^2 g^A_{\mu,\mu}(x_2,x_2)} \\ &\quad\times (e^{-a^2e^2\varepsilon_1\varepsilon_2 g^A_{\mu,\mu}(x_1,x_2)} - 1)e^{-2}a^{-2}.\end{aligned} \tag{27}$$

This can be rewritten as

$$\int_0^1 dt g^A_{\mu,\mu}(x_1,x_2)e^{-\tilde{V}(t)}, \tag{28}$$

with

$$\begin{aligned}2\tilde{V}(t) &= a^2 t\langle(\varepsilon_1 A_\mu(x_1) + \varepsilon_2 A_\mu(x_2))^2\rangle \\ &\quad + a^2(1-t)(g^A_{\mu,\mu}(x_1,x_1) + g^A_{\mu,\mu}(x_2,x_2)),\end{aligned} \tag{29}$$

in agreement with (23). For definiteness we keep only the dimensionally nonirrelevant terms considering

$$e^{W_1(J,B,\phi)} = \int P(d\psi)e^{\mathcal{V}+G(B)+(\psi,\phi)}, \tag{30}$$

with $G(B)$ is a generic source term for gauge invariant observables and $\mathcal{V} =$

$$a^2\sum_x\sum_\varepsilon a^{-1}(e^{-\frac{1}{2}e^2a^2 g^A_{\mu,\mu}(x,x)} e^{iaeJ_{\mu,x}} - 1)O^\varepsilon_\mu + \mathcal{E}^T_A(V;2). \tag{31}$$

Note that $a^2 g^A_{\mu,\mu}$ vanishes as $a \to 0$. In the case of the chiral current

$$G(B^5) = a^2\sum_x Z^5 B^5_{x,\mu}\bar{\psi}_x\gamma_\mu\gamma_5\psi_x. \tag{32}$$

V. INTEGRATION OF THE FERMIONIC FIELDS

Our main result is the following:

Theorem 2 For $|\lambda| \leq \lambda_0 M^2$, with λ_0 independent on a, m, L and $Z_\psi = 1$ the correlations of (30) are analytic in λ; when the fermion mass is vanishing the anomaly is $H_\mu = \frac{\varepsilon_{\mu\nu}p_\nu}{2\pi} + O(ap^2)$.

In order to integrate the fermionic fields we introduce a decomposition of the propagator

$$g^\psi(x) = \sum_{h=-\infty}^{N} g^{(h)}(x), \tag{33}$$

$\hat{g}^h(k) = f^h(k)\hat{g}(k)$ with $f^h(k)$ with support in $\gamma^{h-1} \leq |k| \leq \gamma^{h+1}$. One has to distinguish two regimes, the ultraviolet high energy scales $h \geq h_M$ with $h_M \sim \log M$ the mass scale, and the infrared regime $h \leq h_M$. In the first regime, one uses the nonlocality of the quartic interation (see Refs. [19,30,31]). After the integration of the fields $\psi^N, \psi^{N-1}, \ldots, \psi^h$, $h \geq h_M$ one gets an effective potential with kernels $W^h_{l,m}$ with l fields ($l = 2n$) similar to (21), which can be written as an expansion in λ and in the kernels $W^k_{2,0}$, $W^k_{4,0}$, $W^k_{2,1}$ with $k \geq h+1$. Assuming that, for $k \geq h+1$ one has $|W^k_{2,0}|_1 \leq \gamma^h\lambda/M^2$, $|W^k_{4,0} - v\lambda|_1 \leq \lambda^2/M^2$ and $|W^k_{2,1} - 1|_1 \leq \lambda^2/M^2$ then we get

$$|W^h_{l,m}|_1 \leq C^{l+m}(\lambda/M^2)^{d_{l,m}}\gamma^{h(2-\frac{l}{2}-m)h}, \tag{34}$$

for $d_{l,m} = \max(l/2 - 1, 1)$ if $m = 0$, and $d_{l,m} = \max(l/2 - 1, 0)$ if $m = 1$. The proof of (34) is based on the analog of formula (23) for Grassmann expectations

$$\mathcal{E}^T_\psi\left(\prod_{k=1}^n \tilde{\psi}(P_i)\right) = \sum_{T\in\mathcal{T}}\prod_{i,j\in T} V_{i,j}\int dp_T(s)\det G, \tag{35}$$

and the use of Gram bounds for get an estimate on $\det G$; in addition one uses that $|v|_1 \leq CM^{-2}$, $|g^h|_1 \leq C\gamma^{-h}$,

$|g^h|_\infty \leq C\gamma^h$. We proceed by induction to prove the assumption. One needs to show that there is an improvement in the bounds due to the nonlocality of the boson propagator. The kernel of the two-point function $W^h_{2,0}(x,y)$ can be written as sum over n of truncated expectations; calling $\mathcal{E}^T_{h,N}$ the truncated expectation with respect to $P(d\psi^{[h,N]})$ we have

$$W^h_{2,0}(x,y) = \lambda \sum_n \frac{\partial}{\partial \phi^+_x} \frac{a^4}{(n-1)!} \times \sum_{x_1,x_2} v_{\mu,\varepsilon_1,\varepsilon_2} \mathcal{E}^T_{h,N}\left(\frac{\partial}{\partial \phi^-_y} O^{\varepsilon_1}_{\mu,x_1} O^{\varepsilon_2}_{\mu,x_2}; V; \ldots\right). \quad (36)$$

By using the property, if $\tilde\psi(P) = \prod_{f\in P} \psi_{x_f}$

$$\mathcal{E}^T_{h,N}(\tilde\psi(P_1\cap P_2)\ldots\tilde\psi(P_n)) = \mathcal{E}^T_{h,N}(\tilde\psi(P_1)\tilde\psi(P_2)\ldots\tilde\psi(P_n)) + \sum_{K_1,K_2;K_1\cup K_2=3,\ldots,n} \mathcal{E}^T_{h,N}\left(\tilde\psi(P_1)\prod_{j\in K_1}\tilde\psi(P_i)\right) \times \mathcal{E}^T_{h,N}\left(\tilde\psi(P_2)\prod_{j\in K_2}\tilde\psi(P_i)\right),$$

we get, omitting the ε, μ dependence, $W^h_{2,0}(x,y) =$

$$\lambda a^4 \sum_{z_1,z_2} v(y,z_1) g^{[h,N]}(y+a_\mu, z_2) W^h_{2,0}(z_2,x) W_{0,1}(z_1) + \lambda a^{-1}(e^{-\frac{1}{2}e^2 a^2 g^A_{\mu,\mu}(0))} - 1) a^2 \sum_z g^{[h,N]}(x,z) W^h_{2,0}(z,y) + \lambda a^4 \sum_{z_1,z_2} v(y,z_2) g^{[h,N]}(y+a_\mu, z_1) W^h_{2,1}(z_2;z_1,x). \quad (37)$$

A graphical representation of (37) is in Fig. 1. The second term is bounded by

$$C\lambda/M^2 \gamma^h \gamma^{-h} a \log a \leq \lambda/M^2 \gamma^h/2 \quad (38)$$

for a small enough. The first term contains $\hat W_{1,0}(0) = 0$. Regarding the last term we get a bound

$$\sup_{z_1,z_2} |a^2 \sum_y v(y,z_2) g^{[h,N]}(y+a_\mu, z_1)| a^2 \sum_{z_2,z_1} |W^h_{2,1}(z_2;z_1,0)|. \quad (39)$$

By using the inductive hypothesis $a^2|\sum_{(z_1,z_2)} W(z_2;z_1,0)| \leq C$ we get for (39) the bound

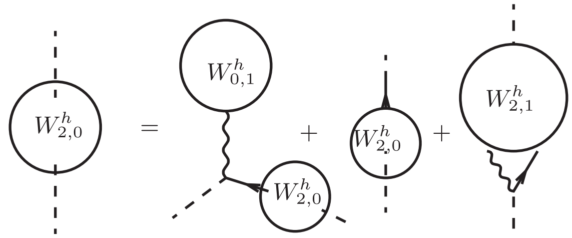

FIG. 1. Graphical representation of (37).

$$\lambda C_1 |a^2 \sum_y |v g^{[h,N]}| \leq \lambda C_1 \left[a^2 \sum_y |v|^3\right]^{\frac{1}{3}} \left[a^2 \sum_y (g^{[h,N]})^{\frac{3}{2}}\right]^{\frac{2}{3}} \leq \lambda M^{-2} C_2 \gamma^h \gamma^{-\frac{4}{3}(h-M)} \leq \lambda M^{-2}\gamma^h/2, \quad (40)$$

for $h \geq h_M$, for $h_M = C \log M$ and C large enough. Note that the above estimates uses crucially that $\xi = 0$; for $\xi = 1$ $[a^2 \sum_y |v|^3]^{\frac{1}{3}}$ would be nonbounded in N.

A similar computation can be repeated for $W_{2,1}$; in particular for the quartic term one uses that the bubble graph is finite, $A = \int dk \mathrm{Tr} g(k)\gamma_\mu g(k)\gamma_\nu$ so that $|W_{2,1}|_1 \leq C\lambda/M^2(\gamma^{h-M} + A|W_{2,1}|_1)$. The above estimates work for $h \geq h_M$ and it says that the theory is super-renormalizable up to that scale.

In the infrared regime $h_m \leq h \leq h_M$, where $h_m = \log_\gamma m$ is the fermion mass scale, the multiscale integration procedure is the same as in the Thirring model with a finite cutoff [23]. The theory is renormalizable in this regime and there is wave function renormalization at each scale $Z_h \sim \gamma^{-\eta h}$, $\eta = O(\lambda^2) > 0$ and an effective coupling with asymptotically vanishing Beta function. The expansions converge therefore uniformly in a, L, m, and the limit $a \to 0, L \to \infty$ can be taken.

VI. ANOMALY NONRENORMALIZATION

The average of the chiral current $\Gamma^5_{\mu,\nu} = \frac{\partial^2 W}{\partial B_\mu \partial J_\nu}|_0$ for $m = 0$ is expressed by a series in λ. It is convenient to introduce a continuum relativistic model $e^{\tilde W(J,B,\phi)} =$

$$\int P_{\tilde Z}(d\psi) e^{-V + \tilde Z^+(J,j) + \tilde Z^-(B,j^5) + (\psi,\phi)}, \quad (41)$$

where $P_{\tilde Z}(d\psi)$ has propagator $\frac{1}{\tilde Z}\frac{\chi(k)}{\gamma_\mu k_\mu}$, with χ a momentum cutoff selecting momenta $\leq \gamma^{\tilde N}$, and

$$V = \tilde Z^2 \tilde\lambda \int dx dy v(x,y) j_{\mu,x} j_{\mu,y}, \quad (42)$$

with v exponentially decaying with rate rate M^{-1} and quartic coupling $\tilde\lambda$. Finally, $j^+_{\mu,x} \equiv j_{\mu,x} = \bar\psi_x \gamma_\mu \psi_x$ and $j^- = \bar\psi_x \gamma_\mu \gamma_5 \psi_x$.

The infrared scales $h \leq h_M$ of the two models differs by irrelevant terms and one can choose $\tilde\lambda$ and $\tilde Z, \tilde Z^-, \tilde Z^+$ as function of λ so that the corresponding running couplings flow to the same fixed point for $h \to -\infty$. As a result, defining

$$\tilde\Gamma^5_{\mu,\nu} = \frac{\partial^2 \tilde W}{\partial B_\mu \partial J_\nu}\Big|_0, \quad (43)$$

we get

$$\hat\Gamma^5_{\mu,\nu}(p) = Z_5 \tilde\Gamma^5_{\mu,\nu}(p) + R_{\mu,\nu}(p), \quad (44)$$

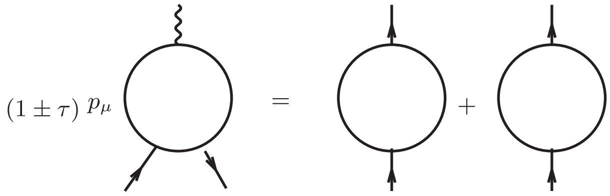

FIG. 2. Graphical representation of (45).

where $R_{\mu,\nu}(p)$ is a continuous function at $p=0$, while $\tilde{\Gamma}^5_{\mu,\nu}(p)$ is not; this provides a relation between the lattice and the continuum model.

The model (41) has two global symmetries, that is $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\alpha\gamma^5}\psi$, but the Ward identity acquires extra terms associated with the momentum regularization [30]. In particular, if $\tau = \tilde{\lambda}\,\hat{v}(0)/4\pi$, in the limit of removed cutoff $\tilde{N} \to \infty$ (a graphical representation of (45) below is in Fig. 2)

$$(1 \mp \tau) p_\mu \tilde{\Gamma}^\pm_\mu(k,p) = \frac{\tilde{Z}^\pm}{\tilde{Z}} \gamma^\pm (\tilde{S}(k) - \tilde{S}(k+p)), \qquad (45)$$

where $\tilde{\Gamma}^\pm_\mu$ are the vertex correlations of (41) of the current $(+)$ and chiral current $(-)$ and $\gamma^+ = I, \gamma^- = \gamma_5$. In the same way the Ward identity for the current is

$$p_\mu \tilde{\Gamma}^5_{\mu,\nu} = \frac{\tilde{Z}^+\tilde{Z}^-}{4\pi\tilde{Z}^2} \frac{\varepsilon_{\mu\nu} p_\mu}{(1+\tau)}, \quad p_\nu \tilde{\Gamma}^5_{\mu,\nu} = \frac{\tilde{Z}^+\tilde{Z}^-}{4\pi\tilde{Z}^2} \frac{\varepsilon_{\nu\mu} p_\nu}{(1-\tau)}. \qquad (46)$$

By comparing (45) with the Ward identity (11), and using that the vertex and the two-point correlations of lattice and continuum model coincide up to subleading term in the momentum, we get a relation between the parameters $\tau, \tilde{Z}^+, \tilde{Z}$

$$\frac{\tilde{Z}^+}{\tilde{Z}(1-\tau)} = 1. \qquad (47)$$

Moreover, the condition on Z^5 (18) and (45) imply

$$\frac{\tilde{Z}^+}{\tilde{Z}(1-\tau)} = Z_5 \frac{\tilde{Z}^-}{\tilde{Z}(1+\tau)} = 1, \qquad (48)$$

from which $Z_5 = (1+\tau)\frac{\tilde{Z}}{\tilde{Z}^-}$. By the Ward identity (10) we get

$$p_\nu \hat{\Gamma}^5_{\mu,\nu}(p) = \frac{\tilde{Z}^+\tilde{Z}^-}{2\pi\tilde{Z}^2} \frac{\varepsilon_{\nu\mu} p_\nu}{(1-\tau)} + p_\nu R_{\mu,\nu}(p) = 0, \qquad (49)$$

so that

$$R_{\mu,\nu}(0) = -\frac{\tilde{Z}^+\tilde{Z}^-}{2\pi Z^2} \frac{\varepsilon_{\nu\mu}}{(1-\tau)} = -(1+\tau)\varepsilon_{\nu\mu}/Z_5. \qquad (50)$$

Finally,

$$\begin{aligned} p_\mu \hat{\Gamma}^5_{\mu,\nu}(p) &= Z_5 p_\mu [\tilde{\Gamma}^5_{\mu,\nu}(p) + R_{\mu,\nu}(p)] \\ &= [(1-\tau)\varepsilon_{\mu,\nu} - (1+\tau)\varepsilon_{\nu,\mu}] p_\mu / 4\pi = 1/2\pi \varepsilon_{\mu,\nu} p_\nu \end{aligned} \qquad (51)$$

that is all the dependence of the coupling disappears. ∎

VII. CONCLUSIONS

We have analyzed a lattice version of the Sommerfield model. Both the reduction of the degree of ultraviolet divergence, manifesting in the finiteness of the field renormalization, and the Adler-Bardeen theorem hold at a nonperturbative level, in contrast with exact solutions in the continuum. Nonperturbative violation of perturbative results are therefore excluded. This provides support to the possibility of a rigorous lattice formulation of the electroweak sector of the Standard Model with exponentially small steps in the inverse coupling, which requires an analogous reduction of degree of divergence. New problems include the fact that a multiscale analysis is necessary also for the boson sector, and the fact that the symmetry is chiral and anomaly cancellation is required; Adler-Bardeen theorem on a lattice is exact for nonchiral theories [32,33] but has subdominant corrections for chiral ones [34].

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