Analytic interpolation between the Ji and Jaffe-Manohar definitions of the orbital angular momentum distribution of gluons at small x

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In this work, we first introduced a generalized Wilson line gauge link that reproduces both staple and near straight links along a light cone in different parameter limits. We then studied the gauge-invariant bilocal orbital angular momentum operator with such a general gauge link. At the appropriate combination of limits, the operator reproduces both Jaffe-Manohar's and Ji's operator structures and offers a continuous analytical interpolation between the two.

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I. INTRODUCTION

It has been more than three decades since the discovery of proton *spin crisis* by the European Muon Collaboration [1]. By now, a lot of progress has been achieved on both the theoretical and experimental fronts to understand and validate the proton spin sum rule. Moreover, a resurgence of interest has been triggered in the wake of the planned Electron-Ion Collider (EIC) at Brookhaven National Laboratory [2]. The sum rule for proton spin contains both helicity contribution and orbital angular momentum contribution, each separately for quarks and gluons, as

$$S_{q+\bar{q}} + L_{q+\bar{q}} + S_G + L_G = \frac{1}{2}, \tag{1}$$

This can often be presented in terms of the corresponding distributions as

$$\int_0^1 dx[\Delta\Sigma(x, Q^2) + \Delta G(x, Q^2) + L_{q+\bar{q}}(x, Q^2) + L_G(x, Q^2)] = \frac{1}{2}, \tag{2}$$

where $\Delta\Sigma(x, Q^2)$, $\Delta G(x, Q^2)$ are helicity distributions for quarks and gluons, whereas $L_{q+\bar{q}}(x, Q^2)$ and $L_G(x, Q^2)$ are those for angular momentum, respectively.

Gluon orbital angular momentum (OAM) is essentially a two-point correlation function of the field strength tensors at two space-time points. To make such bilocal operators gauge invariant, one needs to introduce appropriate gauge links between the field strength tensors. Two important choices in this regard are, (a) the *staple gauge links* and (b) the *straight gauge links*. The staple gauge links may show up either as the past-pointing or as the future-pointing links. This leads to two main types of OAMs, namely, the dipole type and the Weizsäcker-Williams (WW) type. The dipole OAM contains both the past and future links (Fig. 1), while the Weizsäcker-Williams–type OAM contains either the past-pointing or the future-pointing links along the light cone (Fig. 2)—nonetheless, both of them lead to the Jaffe-Manohar [3] definition of the OAM operator. The gauge links connecting the two space-time points could also be the straight one as shown in Fig. 3, leading to Ji's [4] definition of OAM.

In this study, we have derived a generalized form of the gauge-invariant OAM operator, presented in Eq. (42), which accommodates all possible geometrics of the gauge links. In the appropriate limit, the operator reproduces both

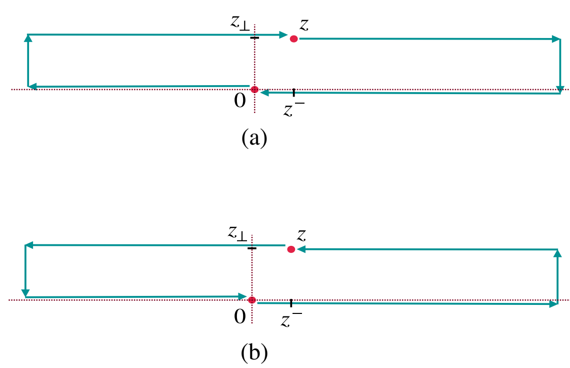

FIG. 1. The gauge link structure $[\eta_1, \eta_2]$ equals $[-\infty, +\infty^\dagger]$ and $[+\infty, -\infty^\dagger]$. Both correspond to the dipole-type Jaffe-Manohar (JM) orbital angular momentum.

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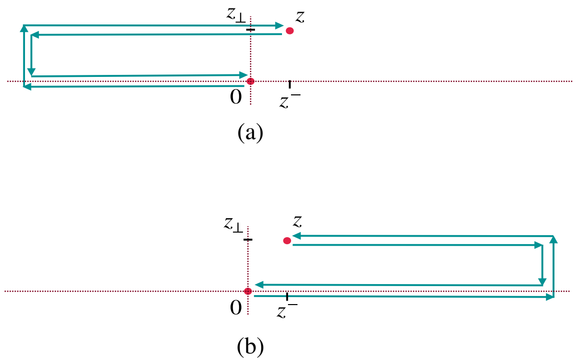

FIG. 2. The gauge link structure $[\eta_1, \eta_2]$ equals $[-\infty, -\infty^\dagger]$ and $[+\infty, +\infty^\dagger]$. Both correspond to the Weizsäcker-Williams–type Jaffe-Manohar (JM) orbital angular momentum.

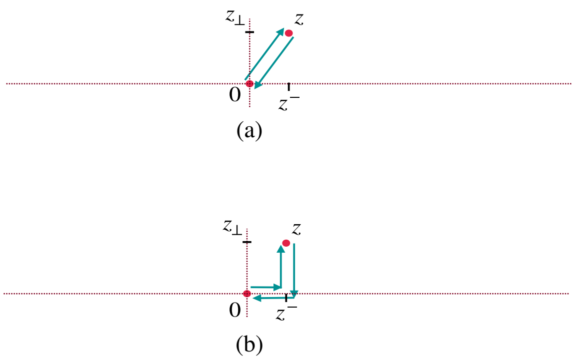

FIG. 3. The straight gauge link structure $[\eta_1, \eta_2]$ equals $[0, 0^\dagger]$. This appears in Ji's orbital angular momentum.

Jaffe-Manohar's and Ji's operator structures and offers a continuous analytical interpolation between the two. Next, we jotted down some of the key points that motivate having a generalized operator and the continuous interpolation like the one we studied in this work.

A. Potential OAM: The integrated torque accumulated by a gluon

A straight Wilson line gauge link yields Ji orbital angular momentum, while an infinite staple-shaped gauge link yields Jaffe-Manohar orbital angular momentum. Unlike the Ji OAM, the Jaffe-Manohar decomposition of OAM is affected by the final- or/and initial-state interactions. Hence, the difference between the two is often interpreted as the integrated torque accumulated by a gluon struck in a deep inelastic scattering (DIS) process as it enters or exits the proton, through initial- or final-state interactions—exerted on the outgoing quark or gluon by the chromomagnetic field produced by the spectators. The difference between the two OAMs is directly connected to the off-forward extension of a Qiu-Sterman term and often cited as potential OAM [5].

B. Quasi-PDF in lattice: Connection to TMDs in DIS and Drell-Yan processes

The operator defining the quasi-parton distribution functions (PDF) involves a straight line link from 0 to z rather than a staple link often appearing in the definitions of transverse momentum dependent parton distribution functions (TMDs) to describe the Drell-Yan and semi-inclusive DIS processes. While the staple link reflects initial- and final-state interactions inherent in these processes, the straight link describes the internal structure of the hadron when it is in the nondisturbed or primordial state. It is unlikely that such a TMD can be measured in a scattering experiment; however, being a well-defined object, they can be measured in the lattice. As the quasi-PDF is defined with space separation connected by a straight gauge link, a proper synergy of two-dimensionsl (2D) Fourier transform and interpolation like the one addressed in this work can connect it to the TMDs that show up in the scattering processes [6].

C. TMD PDFs: Weizsäcker-Williams to dipole

Unintegrated gluon distribution or transverse momentum-dependent gluon distribution functions are one of the key topics to be fully investigated in the upcoming EIC. The TMD PDFs can either be probed in quark-antiquark jet correlation in DIS (the Weizsäcker-Williams distribution) or the direct photon-jet correlation in pA collision. The various TMD PDFs involved in different processes specifically in different dijet channels in pA or eA collisions are related by the two universal ones: the Weizsäcker-Williams TMD and the dipole TMD specifically in small x at the large N_c limit. Both the WW distribution and dipole distribution are essentially dimension-4 two-point correlation functions of classical gluon fields. The operator definitions are different only in the way the gauge links are oriented. An interpolation of the link, the kind we studied here, leads to a universal structure that can reproduce WW and dipole TMD at the appropriate limit.

In Sec. II, we briefly review the essentials of canonical gauge-invariant decomposition, of parton angular momentum by Chen *et al.*, where one has to separate the gauge potential into the so-called *pure* part and *physical* or dynamical parts. In Sec. III, we studied the transverse derivatives of the staple gauge links along the light cone. Here, we introduced two parameters, η_1 and η_2, to make the extent of the gauge link arbitrary with the anticipation that at different combinations of the limit of the parameters it would produce the past-pointing, the future-pointing, or near-straight gauge links. We finally express the transverse derivative of such a general gauge link in terms of the pure and physical gauge components. This also explicitly contains the gauge link extent parameters. In Sec. IV, we recall the connection between the orbital angular momentum of gluons and the gluon Wigner distribution and then derive the OAM operators in terms of pure gauge and the extent parameters. In Sec. V, we move to the small-x eikonal limit and go on to integrate both z^- and $z_\perp$, the position components, which lead to the general gauge-invariant OAM operator structure, valid for arbitrary geometry of the gauge link at small x. In Sec. VI, we briefly note the results for subeikonal cases. We conclude and give our outlook at the end.

II. GAUGE-INVARIANT DECOMPOSITION OF ANGULAR MOMENTUM

Decomposition of angular momentum in a gauge-invariant way can be achieved by separating the gauge field into the pure and physical parts of the field, first proposed by Chen *et al.* [7,8],

$$A_\mu = A_\mu^{\rm pure} + A_\mu^{\rm phys}. \tag{3}$$

Pure gauge field as a differential 1-form defined by identically vanishing the corresponding 2-form as,

$$F_{\mu\nu}^{\rm pure} = \partial_\mu A_\nu^{\rm pure} - \partial_\nu A_\mu^{\rm pure} - ig[A_\mu^{\rm pure}, A_\nu^{\rm pure}] = 0. \tag{4}$$

One can define covariant derivative as well,

$$D_\mu^{\rm pure} = \partial_\mu - igA_\mu^{\rm pure}. \tag{5}$$

The covariant derivative satisfies the commutation relation that vanishes because field tensor vanishes identically,

$$[D_\mu^{\rm pure}, D_\nu^{\rm pure}] = -igF_{\mu\nu}^{\rm pure} = 0. \tag{6}$$

Under gauge transformation, the pure part of the gauge field transforms as

$$A_\mu^{\rm pure} \longmapsto A_\mu^{\rm pure} = U(x)A_\mu^{\rm pure}U^{-1}(x) + \frac{i}{g}U(x)\partial_\mu U^{-1}(x), \tag{7}$$

On the other hand, the physical part of the gauge field transform as

$$A_\mu^{\rm phys} \mapsto A_\mu^{\rm phys} = U(x)A_\mu^{\rm phys}U^{-1}(x) \tag{8}$$

and makes the field strength tensor nonzero,

$$F_{\mu\nu} \equiv \mathcal{D}_\mu^{\rm pure}A_\nu^{\rm phys} - \mathcal{D}_\nu^{\rm pure}A_\mu^{\rm phys} - ig[A_\mu^{\rm phys}, A_\nu^{\rm phys}]. \tag{9}$$

This makes the physical part of the gauge field to be dynamical degrees of freedom of the theory. The covariant derivative in the adjoint representation is defined as

$$\mathcal{D}_\mu^{\rm pure}A_\nu^{\rm phys} = \partial_\mu A_\nu^{\rm phys} - ig[A_\mu^{\rm pure}, A_\nu^{\rm phys}]. \tag{10}$$

One often defines the *natural gauge* as the gauge where $A_\mu^{\rm pure} = 0$ and $A_\mu = A_\mu^{\rm phys}$. Now, this separation of gauge field in terms of pure and physical parts is not unique, as still some gauge freedom remain. This requires further constraint on $A_\mu^{\rm phys}$, which essentially makes it, instead of local, a nonlocal functional of full gauge field, e.g., in the $A_{\rm phys}^+ = 0$ gauge, as

$$A^\mu_{{\rm phys},\pm}(x^-, x_\perp) = \int_{\pm\infty^-}^{x^-} d\omega^- U(x^-, \omega^-; x_\perp) \times F^{+\mu}(\omega^-, x_\perp)U(\omega^-, x^-; x_\perp), \tag{11}$$

where

$$U(a^-, b^-; x_\perp) \equiv \mathcal{P}\left(ig\int_{b^-}^{a^-} dx^- A^+(x^-, x_\perp)\right). \tag{12}$$

For recent review and a comprehensive study within scalar diquark model, see Ref. [9].

III. TRANSVERSE DERIVATIVE OF STAPLE GAUGE LINKS WITH FINITE EXTENT

In this section, we review the derivative of a staple gauge link with respect to transverse coordinate, along the light cone,

$$\mathcal{U}_{[\eta]}(0,z) = U(0,\eta^-;0_\perp)U(\eta^-;0_\perp,z_\perp)U(\eta^-,z^-;z_\perp). \tag{13}$$

In the limit $\eta = +\infty^-$ and $\eta = -\infty^-$, the link above becomes past-pointing and future-pointing staple infinite gauge links along the light cone,

$$\mathcal{U}_{[\eta=-\infty]} = U(0,-\infty^-;0_\perp)U(-\infty^-;0_\perp,z_\perp)U(-\infty^-,z^-;z_\perp), \tag{14}$$

$$\mathcal{U}_{[\eta=+\infty]} = U(0,\infty^-;0_\perp)U(\infty^-;0_\perp,z_\perp)U(\infty^-,z^-;z_\perp). \tag{15}$$

Another interesting limit is when $\eta \to 0$, which is not exactly the straight gauge link but the staple link with the shortest extent. This is partly motivated by the recent work by Engelhardt *et al.*, who consider a general form of the staple-shaped path and make an interpolation between both Ji and Jaffe-Manohar orbital angular momenta when calculating the OAM of quarks on the lattice using the direct derivative method [10].

We now recall the Wilson line from y to z along some arbitrary path $\mathcal{C}$ that can be parametrized via s^μ,

$$U(z,y) = P\exp\left(ig\int_{y,\mathcal{C}}^{z} A_\mu(s)ds^\mu\right). \tag{16}$$

The position derivative of the Wilson line at any point on the path $\mathcal{C}$ is given by

$$\frac{\partial}{\partial x^\lambda}U(z,y) = igU(z,s)A_\mu(s)\frac{\partial s^\mu}{\partial x^\lambda}U(s,y)\Big|_{s=y}^{s=z} + ig\int_y^z ds^\nu U(z,s)F_{\mu\nu}(s)\frac{\partial s^\mu}{\partial x^\lambda}U(s,y). \tag{17}$$

Equation (17) can be used to derive the derivative of $\mathcal{U}_{[\eta]}(0,z)$ as given in Eq. (13) with respect to $z_\perp$ as

$$\begin{aligned}\frac{\partial}{\partial z_\perp^i}\mathcal{U}_{[\eta]}(0,z) = {}& +igU(0^-,\eta^-;0_\perp)A^i(\eta^-,0_\perp)U(\eta^-;0_\perp,z_\perp)U(\eta^-,z^-;z_\perp)\\ & - igU(0^-,\eta^-;0_\perp)U(\eta^-;0_\perp,z_\perp)U(\eta^-,z^-;z_\perp)A^i(z^-,z_\perp)\\ & + igU(0^-,\eta^-;0_\perp)\int_{z_\perp}^{0_\perp} d\omega_\perp U(\eta^-;0_\perp,\omega_\perp)F^{\perp i}(\eta^-,\omega_\perp)U(\eta^-;\omega_\perp,z_\perp)U(\eta^-,z^-;z_\perp)\\ & + igU(0^-,\eta^-;0_\perp)U(\eta^-;0_\perp,z_\perp)\int_{z^-}^{\eta^-} d\omega^- U(\eta^-,\omega^-;z_\perp)F^{+i}(\omega^-,z_\perp)U(\omega^-,z^-;z_\perp). \end{aligned}\tag{18}$$

In the limit $z_\perp^i \to 0$, the expression simplifies further as

$$\begin{aligned}\lim_{z_\perp^i\to 0}\frac{\partial}{\partial z_\perp^i}\mathcal{U}_{[\eta]}(0,z) = {}& igU(0^-,\eta^-;0_\perp)A^i(\eta^-,0_\perp)U(\eta^-,z^-;0_\perp) - igU(0^-,z^-;0_\perp)A^i(z^-,0_\perp)\\ & + ig\int_{z^-}^{\eta^-} d\omega^- U(0^-,\omega^-;0_\perp)F^{+i}(\omega^-,0_\perp)U(\omega^-,z^-;0_\perp). \end{aligned}\tag{19}$$

Special case: $(\eta \to \pm\infty)$

$$\begin{aligned}\lim_{z_\perp^i\to 0}\frac{\partial}{\partial z_\perp^i}\mathcal{U}_{[\eta=\pm\infty]}(0,z)\Big|_{\eta^-\to\pm\infty} = {}& igU(0^-,\pm\infty^-;0_\perp)A^i(\pm\infty^-,0_\perp)U(\pm\infty^-,z^-;0_\perp) - igU(0^-,z^-;0_\perp)A^i(z^-,0_\perp)\\ & + ig\int_{z^-}^{\pm\infty^-} d\omega^- U(0^-,\omega^-;0_\perp)F^{+i}(\omega^-,0_\perp)U(\omega^-,z^-;0_\perp). \end{aligned}\tag{20}$$

We can now express Eq. (19) in terms of the physical gauge field $A^i_{\mathrm{phys},\pm}$ and the pure gauge field A^i_{pure},

$$\begin{aligned}\lim_{z_\perp^i\to 0}\frac{\partial}{\partial z_\perp^i}\mathcal{U}_{[\eta]}(0,z) = {}& igU(0^-,\eta^-;0_\perp)A^i(\eta^-,0_\perp)U(\eta^-,z^-;0_\perp) - igU(0^-,z^-;0_\perp)A^i(z^-,0_\perp)\\ & - U(0^-,\eta^-;0_\perp)A^i_{\mathrm{phys},\pm}(\eta^-,0_\perp)U(\eta^-,z^-;0_\perp) + igU(0^-,z^-;0_\perp)A^i_{\mathrm{phys},\pm}(z^-;0_\perp)\\ = {}& igU(0^-,\eta^-;0_\perp)A^i_{\mathrm{pure}}(\eta^-,0_\perp)U(\eta^-,z^-;0_\perp) - igU(0^-,z^-;0_\perp)A^i_{\mathrm{pure}}(z^-,0_\perp), \end{aligned}\tag{21}$$

where we have used the definition of A^μ_{phys} as in Eq. (11) and also Eq. (3), which are based on the decomposition [7,8] of gauge fields by Chen *et al.* as reviewed in the previous section.

IV. GLUON ORBITAL ANGULAR MOMENTUM

The orbital angular momentum of the gluon can be expressed as the phase-space average of the classical orbital angular momentum weighted with the Wigner distribution of polarized gluons in a longitudinally polarized nucleon. The Wigner distribution is essentially the phase-space distribution of gluons in *transverse momentum* ($k_\perp$)–*impact parameter* ($b_\perp$) space,

$$\begin{aligned}\mathcal{W}_g(x,k_\perp,b_\perp) = {}& \int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{-i\Delta_\perp b_\perp}\int\frac{d^2z_\perp}{(2\pi)^2}e^{iz_\perp k_\perp}\int\frac{dz^-}{2(2\pi)xP^+}e^{-ixP^+z^-}\\ & \times\left\langle P^+,-\frac{\Delta_\perp}{2},S\right|\mathrm{Tr}\,F^{+i}(0)\mathcal{U}_{[\eta_1]}(0;z)F^{+i}(z)\mathcal{U}_{[\eta_2]}(z;0)\left|P^+,\frac{\Delta_\perp}{2},S\right\rangle, \end{aligned}\tag{22}$$

where S is the spin of the target states and the trace is in fundamental representation. The cross-product of transverse momentum and impact parameter returns the orbital angular momentum of gluons, and integrating over them gives orbital angular momentum distribution,

$$L_g(x) = \int d^2b_\perp d^2k_\perp (b_\perp \times k_\perp)\mathcal{W}_g(x, k_\perp, b_\perp), \tag{23}$$

$$= \int d^2b_\perp d^2k_\perp \epsilon^{kj} b_\perp^k k_\perp^j \mathcal{W}_g(x, k_\perp, b_\perp). \tag{24}$$

This can further be written as

$$L_g(x) = \int d^2b_\perp \epsilon^{kj} b_\perp^k \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\Delta_\perp b_\perp} \int \frac{dz^-}{2(2\pi)xP^+} e^{-ixP^+z^-} \times \lim_{z_\perp^j \to 0} \frac{1}{i} \frac{\partial}{\partial z_\perp^j} \left\langle P^+, -\frac{\Delta}{2}, S \middle| \mathrm{Tr} F^{+i}(0)\mathcal{U}_{[\eta_1]}(0;z)F^{+i}(z)\mathcal{U}_{[\eta_2]}(z;0) \middle| P^+, +\frac{\Delta}{2}, S \right\rangle. \tag{25}$$

Now, the transverse derivative in Eq. (25) can be performed using Eq. (21), leading to

$$L_g(x) = \int d^2b_\perp \epsilon^{kj} b_\perp^k \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\Delta_i b_\perp} \int \frac{dz^-}{2(2\pi)xP^+} e^{-ixP^+z^-} \frac{1}{i} \left\langle P^+, -\frac{\Delta}{2}, S \middle| \mathrm{Tr} \sum_{m=1}^{5} \mathcal{O}_m^j \middle| P^+, +\frac{\Delta}{2}, S \right\rangle. \tag{26}$$

Here, the objects $\mathcal{O}_m$ are defined for convenience and are as follows:

$$\begin{aligned}
\mathcal{O}_1^j &= F^{+i}(0) igU(0^-, \eta_1^-; 0_\perp) A_{\mathrm{pure}}^j(\eta_1^-, 0_\perp) U(\eta_1^-, z^-; 0_\perp) F^{+i}(z^-, 0_\perp) \mathcal{U}_{[\eta_2]}(z^-, 0^-; 0_\perp), \\
\mathcal{O}_2^j &= -F^{+i}(0) igU(0^-, z^-; 0_\perp) A_{\mathrm{pure}}^j(z^-, 0_\perp) F^{+i}(z^-, 0_\perp) \mathcal{U}_{[\eta_2]}(z^-, 0^-; 0_\perp), \\
\mathcal{O}_3^j &= F^{+i}(0) \mathcal{U}_{[\eta_1]}(0^-, z^-; 0_\perp) [\partial^j F^{+i}(z^-, z_\perp)]_{z_\perp = 0} \mathcal{U}_{[\eta_2]}(z^-, 0^-; 0_\perp), \\
\mathcal{O}_4^j &= F^{+i}(0) \mathcal{U}_{[\eta_1]}(0, z^-; 0_\perp) F^{+i}(z^-; 0_\perp) igA_{\mathrm{pure}}^j(z^-, 0_\perp) U(z^-, 0^-; 0_\perp), \\
\mathcal{O}_5^j &= -F^{+i}(0) \mathcal{U}_{[\eta_1]}(0, z^-; 0_\perp) F^{+i}(z^-; 0_\perp) igU(z^-, \eta_2^-; 0_\perp) A_{\mathrm{pure}}^j(\eta_2{}^-, 0_\perp) U(\eta_2^-, 0^-; 0_\perp).
\end{aligned}$$

Equation (26) can further be simplified as

$$L_g(x) = \epsilon^{kj} \lim_{\Delta^k \to 0} \frac{\partial}{\partial \Delta^k} \int \frac{dz^-}{2(2\pi)xP^+} e^{-ixP^+z^-} \left\langle P^+, -\frac{\Delta}{2}, S \middle| \mathrm{Tr} \sum_{m=1}^{5} \mathcal{O}_m^j \middle| P^+, +\frac{\Delta}{2}, S \right\rangle. \tag{27}$$

Until now, we have not taken any assumptions specifically for small x. The problem is now reduced to performing the z^- integration in Eq. (27). Now, we expand the exponential in Eq. (27),

$$\begin{aligned}
L_g(x) &= \epsilon^{kj} \lim_{\Delta^k \to 0} \frac{\partial}{\partial \Delta^k} \int \frac{dz^-}{2(2\pi)xP^+} e^{-ixP^+z^-} \left\langle P^+, -\frac{\Delta}{2}, S \middle| \mathrm{Tr} \sum_{m=1}^{5} \mathcal{O}_m^j \middle| P^+, +\frac{\Delta}{2}, S \right\rangle \\
&= \epsilon^{kj} \lim_{\Delta^k \to 0} \frac{\partial}{\partial \Delta^k} \int \frac{dz^-}{2(2\pi)} \sum_{n=0}^{\infty} (-i)^n (xP^+)^{n-1} z^{-n} \left\langle P^+, -\frac{\Delta}{2}, S \middle| \mathrm{Tr} \sum_{m=1}^{5} \mathcal{O}_m^j \middle| P^+, +\frac{\Delta}{2}, S \right\rangle \\
&\equiv \epsilon^{kj} \left[\frac{1}{x} \mathcal{L}_{g,0}^{jk}(x) + \mathcal{L}_{g,1}^{jk}(x) + x\mathcal{L}_{g,2}^{jk}(x) + x^2 \mathcal{L}_{g,3}^{jk}(x) + \cdots \right].
\end{aligned} \tag{28}$$

Here, $\mathcal{L}_{g,0}^{jk}(x)$, $\mathcal{L}_{g,1}^{jk}(x)$, $\mathcal{L}_{g,2}^{jk}(x)$ …, can be interpreted as eikonal, subeikonal, subsubeikonal contributions to the gluon OAM. In this convention, subeikonal refers to the term suppressed by one power of x compared to the eikonal or leading scattering, subsubeikonal refers to suppression by two powers of x, and so on. Any dependence on x, for the terms $\mathcal{L}_{g,0}^{jk}(x)$, $\mathcal{L}_{g,1}^{jk}(x)$, $\mathcal{L}_{g,2}^{jk}(x)$…, can be estimated only by constructing and solving small-x evolution of the object inside the angle brackets $\langle \ldots \rangle$. In the following, we will study the operator structure of eikonal and subeikonal terms in Eq. (28).

V. GLUONS OAM AT EIKONAL LIMIT

In this leading order, we can approximate $\exp(-ixP^+z^-) \simeq 1$ and go ahead to perform the z^- integration in Eq. (27). Now, it is also important to isolate only the polarized contributions, from the object inside the angle bracket $\langle \ldots \rangle$, that would give one more ϵ^{ij}, in addition to the one that is already there originating from the definition of OAM, leading to surviving OAM for gluons. Now, before we proceed, it will be convenient to define the following Parity and Time reversal (PT)-even and PT-odd parts of $A_{\rm phys}$,

$$
\begin{aligned}
A^\mu_{\rm phys,e}(x) &= \frac{1}{2}[A^\mu_{\rm phys,+}(x) + A^\mu_{\rm phys,-}(x)] \\
&= -\frac{1}{2}\int_{-\infty^-}^{+\infty^-} dy^- \epsilon(y^-) U(x^-, y^-; x_\perp) F^{+\mu}(y^-, x_\perp) U(y^-, x^-; x_\perp)
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
A^\mu_{\rm phys,o}(x) &= \frac{1}{2}[A^\mu_{\rm phys,+}(x) - A^\mu_{\rm phys,-}(x)] \\
&= -\frac{1}{2}\int_{-\infty^-}^{+\infty^-} dy^- U(x^-, y^-; x_\perp) \times F^{+\mu}(y^-, x_\perp) U(y^-, x^-; x_\perp),
\end{aligned}
\tag{30}
$$

with $\epsilon(y^-)$ being the sign function. Now, here, we list the integral over z^- of the respective $\mathcal{O}_m$ (details of z^- integrations are in the Appendix),

$$
\begin{aligned}
\int dz^- \mathcal{O}^j_1 &= -2igF^{+i}(0^-, 0_\perp) U(0^-, \eta_1^-; 0_\perp) A^j_{\rm pure}(\eta_1^-, 0_\perp) U(\eta_1^-, \eta_2^-; 0_\perp) A^i_{\rm phys,o}(\eta_2^-, 0_\perp) U(\eta_2^-, 0^-; 0_\perp) \\
\int dz^- \mathcal{O}^j_2 &= 2igF^{+i}(0^-, 0_\perp) A^j_{\rm res}(0^-, 0_\perp) A^i_{\rm phys,o}(0^-, 0_\perp) \\
\int dz^- \mathcal{O}^j_3 &= -2F^{+i}(0^-, 0_\perp) \partial^j A^i_{\rm phys,o}(0^-, 0_\perp) \\
\int dz^- \mathcal{O}^j_4 &= -2igF^{+i}(0^-, 0_\perp) A^i_{\rm phys,o}(0^-, 0_\perp) A^j_{\rm res}(0^-, 0_\perp) \\
\int dz^- \mathcal{O}^j_5 &= 2igF^{+i}(0^-, 0_\perp) U(0^-, \eta_1^-; 0_\perp) A^i_{\rm phys,o}(\eta_1^-, 0_\perp) U(\eta_1^-, \eta_2^-; 0_\perp) A^j_{\rm pure}(\eta_2^-, 0_\perp) U(\eta_2^-, 0^-; 0_\perp).
\end{aligned}
$$

This results can be supplemented back to Eq. (27) to arrive at the final expression of the gluon OAM at small x,

$$
\begin{aligned}
\mathcal{L}^{jk}_{g,0}(x) = &-\lim_{\Delta^k\to 0}\frac{\partial}{\partial \Delta^k}\left\langle P^+, -\frac{\Delta}{2}, S \middle| {\rm Tr} F^{+i}(0^-, 0_\perp)\partial^j A^i_{\rm phys,o}(0^-, 0_\perp) \middle| P^+, +\frac{\Delta}{2}, S\right\rangle \\
&-\lim_{\Delta^k\to 0}\frac{\partial}{\partial \Delta^k}\left\langle P^+, -\frac{\Delta}{2}, S \middle| {\rm Tr} F^{+i}(0^-, 0_\perp) ig[A^i_{\rm phys,o}, A^j_{\rm res}](0^-, 0_\perp) \middle| P^+, +\frac{\Delta}{2}, S\right\rangle \\
&-\lim_{\Delta^k\to 0}\frac{\partial}{\partial \Delta^k}\left\langle P^+, -\frac{\Delta}{2}, S \middle| igF^{+i}(0^-, 0_\perp) U(0^-, \eta_1^-; 0_\perp)[A^j_{\rm pure}(\eta_1^-, 0_\perp) U(\eta_1^-, \eta_2^-; 0_\perp) A^i_{\rm phys}(\eta_2^-, 0_\perp) \right. \\
&\left. - A^i_{\rm phys}(\eta_1^-, 0_\perp) U(\eta_1^-, \eta_2^-; 0_\perp) A^j_{\rm pure}(\eta_2^-, 0_\perp)] U(\eta_2^-, 0^-; 0_\perp) \middle| P^+, +\frac{\Delta}{2}, S\right\rangle.
\end{aligned}
\tag{31}
$$

In the contour gauge, e.g., in the $A^+ = 0$ gauge, $A^{\rm res} = A^{\rm pure}$, which leads to

$$
[A^i_{\rm phys,o}, A^j_{\rm res}] \mapsto [A^i_{\rm phys,o}, A^j_{\rm pure}].
\tag{32}
$$

The eikonal contribution to the gluon OAM operator, $\mathcal{L}^g_0(x)$, using Eq. (30), can then be written as

$$\begin{aligned}\mathcal{L}_{g,0}^{jk}(x,k_\perp,\Delta_\perp,S) = &-\frac{1}{2}\lim_{\Delta^k\to 0}\frac{\partial}{\partial\Delta^k}\left\langle P^+,-\frac{\Delta}{2},S\middle|\mathrm{Tr}F^{+i}(0^-,0_\perp)\mathcal{D}^j_{\mathrm{pure}}(A^i_{\mathrm{phys},+}-A^i_{\mathrm{phys},-})(0^-,0_\perp)\middle|P^+,+\frac{\Delta}{2},S\right\rangle\\ &-\frac{1}{2}\lim_{\Delta^k\to 0}\frac{\partial}{\partial\Delta^k}\left\langle P^+,-\frac{\Delta}{2},S\middle|igF^{+i}(0^-,0_\perp)U(0^-,\eta_1^-;0_\perp)[A^j_{\mathrm{pure}}(\eta_1^-,0_\perp)U(\eta_1^-,\eta_2^-;0_\perp)\right.\\ &\times(A^i_{\mathrm{phys},+}-A^i_{\mathrm{phys},-})(\eta_2^-,0_\perp)-(A^i_{\mathrm{phys},+}-A^j_{\mathrm{phys},-})(\eta_1^-,0_\perp)U(\eta_1^-,\eta_2^-;0_\perp)A^j_{\mathrm{pure}}(\eta_2^-,0_\perp)]\\ &\times\left.U(\eta_2^-,0^-;0_\perp)\middle|P^+,+\frac{\Delta}{2},S\right\rangle,\end{aligned}\tag{33}$$

where we have explicitly shown the presence of $A^i_{\mathrm{phys},\pm}$ in the expression. Now, under PT transformation,

$$\begin{aligned}\Delta_\perp &\to -\Delta_\perp,\\ S &\to -S,\\ F^{\mu\nu} &\to -F^{\mu\nu},\\ A^\mu_{\mathrm{phys},\pm} &\to A^\mu_{\mathrm{phys},\mp}.\end{aligned}\tag{34}$$

This implies that $\mathcal{L}_{g,0}^{jk}(x,k_\perp,\Delta_\perp,S)$ and its PT transformed pair are identical, i.e.,

$$\mathcal{L}_{g,0}^{jk}(x,k_\perp,\Delta_\perp,S) = \mathcal{L}_{g,0}^{jk}(x,-k_\perp,-\Delta_\perp,-S),\tag{35}$$

which ensures that the off-forward matrix elements, in $\mathcal{L}_{g,0}^{jk}(x,k_\perp,\Delta_\perp,S)$, do not have any spin-dependent structure of the following form:

$$i\frac{S^+}{P^+}\epsilon^{lm}\Delta^l_\perp k^m_\perp\ldots\tag{36}$$

In the absence of another totally antisymmetric tensor ϵ inherent to $\mathcal{L}_{g,0}^{jk}$, the eikonal contribution in Eq. (28) vanishes as

$$\epsilon^{kj}\mathcal{L}_{g,0}^{jk} = 0.\tag{37}$$

The nonvanishing spin effect starts showing up only at the subeikonal level, which we will discuss next.

VI. GLUON OAM AT FIRST SUBEIKONAL ORDER

When one moves to next nontrivial order in the expansion of the exponential, as $\exp(-ixP^+z^-)\simeq 1-ixP^+z^-$, the additional i makes all leading T-even terms be T odd, leading to nonzero contribution to gluon OAM for longitudinally polarized targets,

$$\begin{aligned}\mathcal{L}_{g,1}^{jk}(x) = &\,iP^+\lim_{\Delta^k\to 0}\frac{\partial}{\partial\Delta^k}\left\langle P^+,-\frac{\Delta}{2},S\middle|\mathrm{Tr}F^{+i}(0^-,0_\perp)\partial^j\bar{O}^i_{\mathrm{phys},*}(0^-,0_\perp)\middle|P^+,+\frac{\Delta}{2},S\right\rangle\\ &+iP^+\lim_{\Delta^k\to 0}\frac{\partial}{\partial\Delta^k}\left\langle P^+,-\frac{\Delta}{2},S\middle|\mathrm{Tr}F^{+i}(0^-,0_\perp)ig[\bar{O}^i_{\mathrm{phys},*},A^j_{\mathrm{res}}](0^-,0_\perp)\middle|P^+,+\frac{\Delta}{2},S\right\rangle\\ &+iP^+\lim_{\Delta^k\to 0}\frac{\partial}{\partial\Delta^k}\left\langle P^+,-\frac{\Delta}{2},S\middle|igF^{+i}(0^-,0_\perp)U(0^-,\eta_1^-;0_\perp)[A^j_{\mathrm{pure}}(\eta_1^-,0_\perp)U(\eta_1^-,\eta_2^-;0_\perp)\bar{O}^i_{\mathrm{phys},*}(\eta_2^-,0_\perp)\right.\\ &-\left.\bar{O}^i_{\mathrm{phys},*}(\eta_1^-,0_\perp)U(\eta_1^-,\eta_2^-;0_\perp)A^j_{\mathrm{pure}}(\eta_2^-,0_\perp)]U(\eta_2^-,0^-;0_\perp)\middle|P^+,+\frac{\Delta}{2},S\right\rangle\end{aligned}\tag{38}$$

where

$$\bar{O}^i_{\mathrm{phys},*}(x) = -\frac{1}{2}\int_{-\infty^-}^{+\infty^-}dz^-z^-U(x^-,z^-;x_\perp)F^{+i}(z^-,x_\perp)U(z^-,x^-;x_\perp).\tag{39}$$

This is essentially the z^- moment of the physical gauge $A^i_{\mathrm{phys,o}}$ in Eq. (42). However, unlike $A^i_{\mathrm{phys,o}}$, the object $\bar{O}^i_{\mathrm{phys},*}$ in the above expression is PT even, similar to Eq. (29), because of the presence of an extra z^-, in the integrand, which can be written as $z^-=|z^-|\mathrm{sign}(z^-)$. This implies that, unlike $\mathcal{L}_{g,0}^{jk}$, the term $\mathcal{L}_{g,1}^{jk}(x,k_\perp,\Delta_\perp,S)$ is odd under PT transformation as

$$\mathcal{L}_{g,1}^{jk}(x, k_\perp, \Delta_\perp, S) = -\mathcal{L}_{g,1}^{jk}(x, -k_\perp, -\Delta_\perp, -S). \tag{40}$$

One would expect to get a nontrivial contribution to the gluon OAM, in a longitudinally polarized proton, right from this order. As the expression originally stems from a Fourier transform, one can get the intermediate time z^- out of the integral to have an overall energy derivative,

$$\begin{aligned}\bar{O}^i_{\mathrm{phys},*}(x) &= -\frac{1}{2} i\partial_k^+ \int_{-\infty^-}^{+\infty^-} dz^- U(x^-, z^-; x_\perp) F^{+i}(z^-, x_\perp) U(z^-, x^-; x_\perp), \\ &= -\frac{1}{2} i\partial_k^+ (A^i_{\mathrm{phys},+} - A^i_{\mathrm{phys},-}),\end{aligned} \tag{41}$$

which leads to

$$\begin{aligned}\mathcal{L}_{g,1}^{jk}(x, k_\perp, \Delta_\perp, S) = &\frac{1}{2} \lim_{\Delta^k \to 0} \frac{\partial}{\partial \Delta^k} \frac{\partial}{\partial x} \left\langle P^+, -\frac{\Delta}{2}, S_\perp \middle| \mathrm{Tr} F^{+i}(0^-, 0_\perp) \mathcal{D}^j_{\mathrm{pure}} (A^i_{\mathrm{phys},+} - A^i_{\mathrm{phys},-})(0^-, 0_\perp) \middle| P^+, +\frac{\Delta}{2}, S_\perp \right\rangle \\ &+ \frac{1}{2} \lim_{\Delta^k \to 0} \frac{\partial}{\partial \Delta^k} \frac{\partial}{\partial x} \left\langle P^+, -\frac{\Delta}{2}, S_\perp \middle| igF^{+i}(0^-, 0_\perp) U(0^-, \eta_1^-; 0_\perp) [A^j_{\mathrm{pure}}(\eta_1^-, 0_\perp) U(\eta_1^-, \eta_2^-; 0_\perp) \right. \\ &\times (A^i_{\mathrm{phys},+} - A^i_{\mathrm{phys},-})(\eta_2^-, 0_\perp) - (A^i_{\mathrm{phys},+} - A^j_{\mathrm{phys},-})(\eta_1^-, 0_\perp) U(\eta_1^-, \eta_2^-; 0_\perp) A^j_{\mathrm{pure}}(\eta_2^-, 0_\perp)] \\ &\times \left. U(\eta_2^-, 0^-; 0_\perp) \middle| P^+, +\frac{\Delta}{2}, S_\perp \right\rangle.\end{aligned} \tag{42}$$

This also corroborates the fact that in the Taylor expansion of the phase factor $\exp(ixP^+z^-)$ only the odd terms in x can contribute to the gluon OAM for the longitudinally polarized proton [11].

VII. CONCLUSION

Recently, the connection between gluon Wigner distribution and gluon orbital angular momentum has been used to probe the gluon OAM in the hard scattering process at the Electron-Ion Collider. The single longitudinal spin asymmetry in hard diffractive dijet production is found to be sensitive to the gluon orbital angular momentum distribution, at least in the moderate x range [12]. On the theoretical front, specifically at small x, Hatta [11,13] presented a general analysis of the orbital angular momentum of distribution of gluons. Novel operator representation of $L_g(x)$ has been derived at the small-x limit and interestingly found to contain covariant derivatives inserted at some intermediate time between the far past and far future. Moreover, the orbital angular momentum distribution was found to be proportional to the gluon helicity distributions $L_g(x) \approx -\Delta G(x)$ at small x. Subsequently, Kovchegov [14] derived the small-x asymptotics of the gluon orbital angular momentum distribution of proton in the double logarithmic approximation. The procedure adopted was to start with the operator definition of gluon OAM, simplifying it at small x and relating it to the polarized dipole amplitudes for the gluon helicities. The small-x asymptotic of the latter was then utilized to derive the small-x asymptotic of the OAM at the large-N_c limit.

In this work, we studied the transverse derivatives of the staple gauge links, with varying extent along the light cone. This is partly motivated by the recent work by Engelhardt *et al.* [10], who considered a general form of a staple-shaped path and computed both Ji and Jaffe-Manohar orbital angular momentum in lattice using the direct derivative method. The derivative of such a gauge link with respect to the transverse position at the zero transverse position limit has been calculated and expressed in terms of the pure gauge components of the gauge fields within the framework of the decomposition of gauge fields of Chen *et al.*

Wigner distribution of the gluon is essentially the distribution of gluons in impact parameter and transverse momentum space. The connection between the two has been recalled to derive the OAM distribution in terms of the pure or physical gauge components and the gauge link parameters. After performing the integration over both the z^- and $z_\perp$, we derive the general operator form of the OAM, of gluons in a longitudinally polarized proton, which is valid for all possible geometrics of the gauge links. At an appropriate combination of the extent parameters, this correctly reproduces both Jaffe-Manohar and Ji's OAM, and offers a continuous analytical interpolation between the two, for gluon OAM distribution in a longitudinally polarized proton.

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APPENDIX: INTEGRATION OVER z^-

1. For $\mathcal{O}_1^j$ and $\mathcal{O}_5^j$

$$\int dz^- \mathcal{O}_1^j = \int dz^- F^{+i}(0) igU(0^-,\eta_1^-;0_\perp)A^j_{\rm pure}(\eta_1^-,0_\perp)U(\eta_1^-,z^-;0_\perp)F^{+i}(z^-,0_\perp)\mathcal{U}_{[\eta_2]}(z^-,0^-;0_\perp).$$

Now,

$$\mathcal{U}_{[\eta_2]}(z^-,0^-;0_\perp) = U(z^-,0^-;0_\perp), \tag{A1}$$

and

$$\begin{aligned}\int_{-\infty}^{+\infty} dz^- U(\eta_2^-,z^-;0_\perp)F^{+i}(z^-,0_\perp)U(z^-,\eta_2^-;0_\perp) &= -\int_{+\infty}^{\eta_2^-} dz^- U(\eta_2^-,z^-;0_\perp)F^{+i}(z^-,0_\perp)U(z^-,\eta_2^-;0_\perp)\\ &\quad + \int_{-\infty}^{\eta_2^-} dz^- U(\eta_2^-,z^-;0_\perp)F^{+i}(z^-,0_\perp)U(z^-,\eta_2^-;0_\perp)\\ &= -A_{\rm phys,+}(\eta_2^-,0_\perp) + A_{\rm phys,-}(\eta_2^-,0_\perp)\\ &= -2A_{\rm phys,o}(\eta_2^-,0_\perp).\end{aligned}$$

Therefore,

$$\int dz^- \mathcal{O}_1^j = -2igF^{+i}(0^-,0_\perp)U(0^-,\eta_1^-;0_\perp)A^j_{\rm pure}(\eta_1^-,0_\perp)U(\eta_1^-,\eta_2^-;0_\perp)A^i_{\rm phys,o}(\eta_2^-,0_\perp)U(\eta_2^-,0^-;0_\perp).$$

Similarly, one can get

$$\begin{aligned}\int dz^- \mathcal{O}_5^j &= -\int dz^- F^{+i}(0)\mathcal{U}_{[\eta_1]}(0,z^-;0_\perp)F^{+i}(z^-;0_\perp)igU(z^-,\eta_2^-;0_\perp)A^j_{\rm pure}(\eta_2{}^-,0_\perp)U(\eta_2^-,0^-;0_\perp)\\ &= 2igF^{+i}(0^-,0_\perp)U(0^-,\eta_1^-;0_\perp)A^i_{\rm phys,o}(\eta_1^-,0_\perp)U(\eta_1^-,\eta_2^-;0_\perp)A^j_{\rm pure}(\eta_2^-,0_\perp)U(\eta_2^-,0^-;0_\perp).\end{aligned}$$

2. For $\mathcal{O}_2^j$ and $\mathcal{O}_4^j$

$$\int dz^- \mathcal{O}_2^j = -\int dz^- F^{+i}(0^-,0_\perp)igU(0^-,z^-;0_\perp)A^j_{\rm pure}(z^-,0_\perp)F^{+i}(z^-,0_\perp)\mathcal{U}_{[\eta_2]}(z^-,0^-;0_\perp).$$

In the first step, we perform the transformation of $A_{\rm pure}$ at $(z^-,0_\perp)$ to the reference point $(0^-,0_\perp)$,

$$U(0^-,z^-;0_\perp)A^j_{\rm pure}(z^-,0_\perp)U(z^-,0^-;0_\perp) \mapsto A^j_{\rm res}(0^-,0_\perp),$$

and then perform the z^- integration,

$$\int_{-\infty}^{+\infty} dz^- U(0^-, z^-; 0_\perp) F^{+i}(z^-, 0_\perp) U(z^-, 0^-; 0_\perp) = -2A_{\mathrm{phys,o}}(0^-, 0_\perp),$$

which leads to

$$\int dz^- \mathcal{O}_2^j = 2igF^{+i}(0^-, 0_\perp) A_{\mathrm{res}}^j(0^-, 0_\perp) A_{\mathrm{phys,o}}(0^-, 0_\perp),$$

Similarly,

$$\int dz^- \mathcal{O}_4^j = -2igF^{+i}(0^-, 0_\perp) A_{\mathrm{phys,o}}(0^-, 0_\perp) A_{\mathrm{res}}^j(0^-, 0_\perp).$$

3. For $\mathcal{O}_3^j$

$$\int dz^- \mathcal{O}_3^j = \int_{-\infty}^{+\infty} dz^- F^{+i}(0) \mathcal{U}_{[\eta_1]}(0^-, z^-; 0_\perp) \times [\partial^j F^{+i}(z^-, z_\perp)]_{z_\perp = 0} \mathcal{U}_{[\eta_2]}(z^-, 0^-; 0_\perp).$$

To perform integration, we shall first change the order of the integration and differentiation. This is legitimate, as both are linear operators. This will lead to

$$\int dz^- \mathcal{O}_3^j = -2F^{+i}(0^-, 0_\perp) \partial^j A_{\mathrm{phys,o}}^i(0^-, 0_\perp).$$

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