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Entanglement entropy and flow in two-dimensional QCD: Parton and string duality

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We discuss quantum entanglement between fast and slow degrees of freedom, in a two-dimensional (2D) large N_c gauge theory with Dirac quarks, quantized on the light front. Using the 't Hooft wave functions, we construct the reduced density matrix for an interval in the momentum fraction x space, and calculate its von Neumann entropy in terms of structure functions, that are measured by deep inelastic scattering on mesons (hadrons in general). We found that the entropy is bounded by an area law with logarithmic divergences, proportional to the rapidity of the meson. The evolution of the entanglement entropy with rapidity is fixed by the cumulative singlet parton distribution function (PDF), and bounded from above by a Kolmogorov-Sinai entropy of 1. At low x, the entanglement exhibits an asymptotic expansion, similar to the forward meson-meson scattering amplitude in the Regge limit. The evolution of the entanglement entropy in parton x per unit rapidity measures the meson singlet PDF. The resummed entanglement entropy along the single meson Regge trajectory is stringlike. We suggest that its extension to multimeson states models deep inelastic scattering on a large 2D"nucleus." The result is a large rate of change of the entanglement entropy with rapidity, that matches the current Bekenstein-Bremermann bound for maximum quantum information flow. This mechanism may be at the origin of the large entropy deposition and rapid thermalization, reported in current heavy ion colliders, and may extend to future electron-ion colliders.

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I. INTRODUCTION

Quantum entanglement permeates most of our quantum description of physical laws. It follows from the fact that quantum states are mostly superposition states, and two noncausally related measurements can be correlated, as captured by the famed Einstein–Podolsky–Rosen paradox. A quantitative measure of this correlation is given by the quantum entanglement entropy. The entanglement entropy of quantum many-body system and quantum field theory has been extensively explored in the literature [1–5]. Less known perhaps is the concept of quantum entanglement flow and its relation to quantum information flow and storage. A maximum flow is expected in the most ideal quantum systems, following from the bound in energy change imposed by the uncertainty principle [6,7].

In hadron physics, quantum entanglement is inherent to a hadron undergoing large longitudinal boosts, with its wave function described either by wee partons [8] or string bits [9–11]. Entanglement entropies are currently measured in diffractive pp scattering at large $\sqrt{s}$ in current collider facilities [12–14], and will be measured in ep scattering at low x at future eIC facilities [12–15], with better accuracy. Entanglement entropies in relation to hadronic processes have also been discussed in [16,17].

In ultrarelativistic heavy-ion collisions, these inherently large entanglement entropies are at the origin of the prompt flow of *wee* entropy, likely at the boundary of our quantum laws. They may also explain the almost instantaneous

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thermalization of the current strongly coupled plasma delivered initially at the RHIC facility, and later at the LHC facility [12,13,18]. The duality between the low-x partons and the string bits [13,19,20] explains why their entanglement provides for the most efficient mechanism for scrambling information, matching only that produced by gravitational black holes [9,10].

In this work we discuss entanglement in longitudinal partonic momentum or Bjorken-x space, and also in rapidity space or $\ln\frac{1}{x}$ using two-dimensional QCD. In the large number of colors limit, 2D QCD is solvable with a dual partonic [21] and stringlike description [22]. The purpose of this work is to elucidate the concept of entanglement in single hadron states, or along a fixed Regge trajectory, as probed with deep inelastic scattering (DIS) kinematics. As an example of DIS scattering on a 2D nucleus, we will address the entanglement in a multimeson state (recall that all hadrons are similar on the light front), and show how its growth rate in rapidity saturates the current bound on quantum information flow.

The outline of the paper is as follows: In Sec. II we briefly review the light cone formulation of 2D QCD with Dirac quarks. In the large number of color limit, the two-body sector decouples and solves the 't Hooft equation [21]. In Sec. III, we detail the entangled density matrix in a single meson state, with a single parton-x cut, as probed by DIS scattering. The evolution of the entanglement entropy with rapidity is fixed by the cumulative parton distribution function (PDF), obeys a Kolmogorov-Sinai bound of 1 [23] (and references therein), and reduces to the longitudinal meson structure function at low x. The entanglement entropy is shown to be universal in the 2D scaling limit. In Sec. IV we recast 2D QCD as a string on the light front in the 2-particle sector. We show that the entanglement of the string bits follows by resumming over the one meson Regge trajectory, thanks to duality. We suggest that the resummation over multimeson Regge trajectories may describe DIS scattering on a 2D *nucleus* on the light front. The evolution of the ensuing entanglement entropy with rapidity is extensive in the classical and longitudinal string entropy. The rate of change matches the Bekenstein-Bremermann bound [6,7] for the maximum flow of quantum information. Our conclusions are in Sec. V. More details are given in the appendixes.

II. DISCRETE LIGHT-CONE QUANTIZATION OF 2D QCD

To construct the reduced density matrix we first provide a review of the discrete light-cone quantization of the theory [24–26]. The system is put in a finite box in the light-front space $-\frac{L^-}{2} < x^- < \frac{L^-}{2}$. After choosing antiperiodic boundary condition, the momenta are labeled as

$$k_p^+ = \frac{\pi}{L^-}(2p+1). \tag{1}$$

The good component ψ_{+i} of the fermion field has the mode decomposition as

$$\psi_{+i}(x^-) = \frac{1}{\sqrt{2L^-}}\sum_{p=0}^{N}\left(a_{i,p}e^{-i\frac{\pi(2p+1)}{L^-}x^-} + b_{i,p}^\dagger e^{i\frac{\pi(2p+1)}{L^-}x^-}\right), \tag{2}$$

which satisfies the anticommutation relation

$$[\psi_{+i}(x_1^-), \psi_{+j}^\dagger(x_2^-)]_+ = \delta(x_1^- - x_2^-)\delta_{ij}. \tag{3}$$

Here $i = 1, ..N_c$ is the color indices of the fermion field, which will be omitted below to avoid cluttering. The total number of N for a finite system with lattice cutoff a is given by $N = [\frac{L^-}{2a}] - 1$. Of all the N independent frequencies, half are unfilled (a_p) and half are filled (b_p). In terms of the above light-front (LF) free field, the LF momentum P^+ and LF Hamiltonian are given by [25,26]

$$P^+L^- = 2\pi\sum_{p=0}^{N}\left(p+\frac{1}{2}\right)(a_p^\dagger a_p + b_p^\dagger b_p), \tag{4}$$

and

$$\frac{P^-}{L^-} \equiv H = \frac{M^2}{2\pi}H_0 + \frac{1}{L^-}V, \tag{5}$$

where M^2 is the quark mass square and H_0 reads

$$H_0 = \sum_{p=0}^{N}\frac{a_p^\dagger a_p + b_p^\dagger b_p}{p+\frac{1}{2}}. \tag{6}$$

Here V consists of four-quark contributions which can be computed from the interaction term

$$V = \frac{g_{1+1}^2}{2}\int_{-\frac{L^-}{2}}^{\frac{L^-}{2}} dx^-\psi_+^\dagger\psi_+\frac{1}{(i\partial_-)^2}\psi_+^\dagger\psi_+. \tag{7}$$

Expressed in terms of a_p, b_p, H is independent of L^-. Using the explicit formula of P^+ above, it is clear that to describe a given hadron state with total momentum P^+, not all the modes are required. We only need those p below

$$p \leq \frac{L^-P^+}{2\pi} - \frac{1}{2} \equiv \Lambda^- - \frac{1}{2}. \tag{8}$$

Therefore, Λ^- provides a natural truncation of the Hilbert space. The momentum fractions are labeled by

$$\frac{1}{2\Lambda^-} \leq x_p = \frac{1}{2\Lambda^-}(2p+1) \leq 1. \tag{9}$$

Below, we use the label x for all momenta. For a generic Λ^-, the states described above are purely discrete and break the Lorentz invariance. We expect that for $\Lambda^- \to \infty$, the spectrum of H goes to zero as $\frac{M^2}{\Lambda^-}$, and the Lorentz invariant dispersion relation $P^+P^- = 2M^2$ is restored. In particular, the meson state can be constructed as

$$|n\rangle = \frac{1}{\sqrt{\Lambda^-}} \sum_{0<p<\Lambda^-} \varphi_p a_p^\dagger b_{\Lambda^- - p}^\dagger |0\rangle. \tag{10}$$

At large N_c, the above two-body state closes under the action of P^-. Requiring it to be an eigenstate of P^- leads to the equation

$$(\Lambda^-)^2 m_R^2 \frac{\varphi_p}{(p+\frac{1}{2})(\Lambda^- - p)} + \Lambda^- \frac{g_{1+1}^2 N_c}{\pi} \sum_{l\neq p} \frac{\varphi_p - \varphi_l}{(p-l)^2} = M^2 \varphi_p. \tag{11}$$

In the continuum limit $\Lambda^- \to \infty$, and with the identification $x = \frac{p+\frac{1}{2}}{\Lambda^-}$ and $y = \frac{l+\frac{1}{2}}{\Lambda^-}$, (11) reduces to the 't Hooft integral equation [21] in the continuum

$$\frac{m_R^2}{x\bar{x}} \varphi_n(x) + \frac{g_{1+1}^2 N_c}{\pi} \mathrm{PV} \int_0^1 dy \frac{\varphi_n(x) - \varphi_n(y)}{(x-y)^2} = M_n^2 \varphi(x). \tag{12}$$

The gauge coupling is related to the string tension $g_{1+1}^2 N_c/2 = \sigma_T$ (see below). The renormalized quark mass is $m_R^2 = m_Q^2 - 2\sigma_T/\pi$. The ensuing spectrum is discrete, with eigenvalues and eigenvectors labeled by M_n^2 and $\varphi_n(x)$, respectively. They form a complete set of states in $L^2[0,1]$,

$$\sum_n \varphi_n^\dagger(x) \varphi_n(x') = \delta(x - x'). \tag{13}$$

Their semiclassical and asymptotic behaviors are briefly reviewed in Appendix A.

III. ENTANGLEMENT ENTROPY IN 2D QCD

We now consider how different parts of a meson light-front wave function as a bound quark-antiquark state are entangled in the quark longitudinal momentum $k^+ = xP^+$ [13–15,27]. In particular, we will focus on the entanglement on a single asymmetric cut in longitudinal momentum, by analogy with a DIS experiment where a single parton-x is singled out, say in the segment $x_0 \leq \frac{1}{2}$, including the low-x region. We start by carefully reviewing the structure of the Hilbert space, and then define the pertinent single cut entanglement entropy.

A. Density matrix in longitudinal momentum

Since the color will be always traced out, here we simply omit the color factor. This will not modify our calculation of the entanglement entropy for two-body states. With this in mind, for each x we have the quark and antiquark operators a_x, b_x, and their corresponding 2D Fock space. The total unconstrained Hilbert space is their tensor product

$$\mathcal{H} = \bigotimes_{0<x<1} \mathcal{H}_x \otimes \bar{\mathcal{H}}_x, \tag{14}$$

where

$$\mathcal{H}_x = \mathrm{Span}(|0\rangle_x, a_x^\dagger |0\rangle_x), \quad \bar{\mathcal{H}}_x = \mathrm{Span}(|\bar{0}\rangle_x, b_x^\dagger |\bar{0}\rangle_x). \tag{15}$$

The total dimension of the Hilbert space is then $2^{[\Lambda^- - \frac{1}{2}]+1} \times 2^{[\Lambda^- - \frac{1}{2}]+1}$, spanned by quark and antiquarks. In a confining theory, however, not all of the states in the above Hilbert space are physical. In the 2D QCD, it can be shown that in the large N_c limit, the physical spectrum consists of bound states formed by quarks and antiquarks, more precisely, the meson wave function reads

$$|n\rangle = \frac{1}{\sqrt{\Lambda^-}} \sum_{0<x<1} \varphi_n(x) |x, \bar{x}\rangle, \tag{16}$$

where the basis $|x, \bar{x}\rangle$ can be written in full tensorial form as

$$|x, \bar{x}\rangle = a_x^\dagger |0\rangle_x \otimes b_{\bar{x}}^\dagger |\bar{0}\rangle_{\bar{x}} \otimes_{y\neq x} |0\rangle_y \otimes |\bar{0}|\rangle_{\bar{y}}. \tag{17}$$

For finite Λ^-, $\varphi_n(x, \Lambda^-)$ satisfies a discrete version of the 't Hooft equation, but as $\Lambda^- \to \infty \varphi_n(x, \Lambda^-)$ should converge to its continuum version given above. Below we will always use the continuum version of the wave function. Unlike the free-quark and antiquark states, the total dimension of the Hilbert space spanned by the 't Hooft wave functions is not $\mathcal{H}$, but only the two-quark states spanned by the set of bases $|x, \bar{x}\rangle$ defined above. Indeed, using the completeness equation of the 't Hooft equation one can show that

$$\sum_n |n\rangle\langle n| = \sum_{0<x<1} |x, \bar{x}\rangle\langle x, \bar{x}|, \tag{18}$$

which is nothing but the projection operator into these quark-antiquark two-body states. The total dimension of these states is only Λ^-, but not 4^{Λ^-}.

Given the above meson state, one can construct its density matrix as

$$\rho_n = \frac{1}{\Lambda^-} \sum_{x,x'} \varphi_n^\dagger(x') \varphi_n(x) |x, \bar{x}\rangle\langle x', \bar{x}'|. \tag{19}$$

Below we investigate its entanglement entropy with respect to the tensor product structure in Eq. (14).

B. Reduced density matrix

The entanglement in longitudinal space is captured by the reduced matrix

$$\rho_n(x,x') = \mathrm{tr}_A \rho_n(x,x'), \tag{20}$$

where A denotes the part of the Hilbert space spanned by a_x, b_x, with x lying in one or more subintervals of [0, 1]. How to choose A depends on the probe experiment of interest. For instance, when probing a hadron in a DIS experiment via hard scattering, the virtual photon selects a quark or antiquark with fixed parton-x, say $x_0 < \frac{1}{2}$ in the range $\bar{A} = [0, x_0]$. The DIS event traces the hadron density matrix over the remaining and unobserved longitudinal momentum range $A = [x_0, 1]$. This is particularly clear, when probing a hadron using semi-inclusive DIS production of heavy mesons. In the large N_c or planar approximation, the process is dominated by Reggeon exchange, with the measured parton x, kinematically limited to small $x_0 \ll 1$. This reduction of the density matrix is asymmetric in parton x. A more symmetric but rather *academic* reduction is discussed in Appendix B.

With this in mind, we now perform the partial trace in the tensor product in Eq. (14), over all the $\mathcal{H}_x$ and $\bar{\mathcal{H}}_x$ where $x > x_0$ with $x_0 < \frac{1}{2}$. To carry out the partial trace, it is clear that for $x < x_0$ and $x' < x_0$, we are left with the quark contribution

$$\frac{1}{\Lambda^-}\sum_{x<x_0} |\varphi_n(x)|^2 a_x^\dagger |0\rangle_x \langle 0|_x a_x \bigotimes_{y<x_0, y'<x_0, y\neq x} |0\rangle_y |\bar{0}\rangle_{y'} \langle\bar{0}|_{y'} \langle 0|_y. \tag{21}$$

Similarly, for $x > 1 - x_0$ and $x' > 1 - x_0$, we have the antiquark contribution

$$\frac{1}{\Lambda^-}\sum_{x<x_0} |\varphi_n(\bar{x})|^2 b_x^\dagger |\bar{0}\rangle_x \langle \bar{0}|_x b_x \bigotimes_{y<x_0, y'<x_0, y'\neq x} |0\rangle_y |\bar{0}\rangle_{y'} \langle\bar{0}|_{y'} \langle 0|_y. \tag{22}$$

It is easy to see that in the cases where $x < x_0, x' > 1 - x_0$ or $x > 1 - x_0, x' < x_0$, there are no partial traces that can be formed since in both of these two cases there will be one quark below x_0 and another quark above x_0. The case $x_0 < x < 1 - x_0$ and $x_0 < x' < 1 - x_0$ should be considered, since in this case both the quark and antiquark are above x_0, and should be traced out. This leads to the contribution

$$\frac{1}{\Lambda^-}\sum_{x_0<x<1-x_0} |\varphi_n(\bar{x})|^2 \bigotimes_{y<x_0, y'<x_0} |0\rangle_y |\bar{0}\rangle_{y'} \langle\bar{0}|_{y'} \langle 0|_y. \tag{23}$$

The contribution is proportional to the vacuum contribution $|0\rangle\langle 0|$ for all the momentum modes below x_0 since they should not be traced over. Summing over the above, we found that for the two-body LF wave functions of a meson state, the reduced density matrix is diagonal and can be written schematically as

$$\hat{\rho}_n(x_0) = \frac{1}{\Lambda^-}\sum_{x<x_0} [|\varphi_n(x)|^2 |x\rangle_q \langle x|_q + |\varphi_n(\bar{x})|^2 |x\rangle_{\bar{q}} \langle x|_{\bar{q}}] + \frac{1}{\Lambda^-}\sum_{x_0<x<1-x_0} |\varphi_n(x)|^2 |0\rangle\langle 0|. \tag{24}$$

The first contribution in (24) stems from the valence quark-antiquark pair in an n-meson state, and is expected. The second contribution stems from the vacuum state (zero modes) assumed normalizable, and is unexpected. The trace of the reduced density matrix is 1, using the normalization condition of the wave function

$$\sum_{0<x<1} \frac{\langle x|x\rangle}{\Lambda^-} |\varphi_n(x)|^2 = 1. \tag{25}$$

with the lightlike cutoff Λ^-

$$\langle x|x\rangle = 2\pi\delta(0_x) = 2\pi P^+ \delta(0_{k^+}) = \frac{P^+}{0_{k^+}} \equiv \Lambda^-. \tag{26}$$

From the light cone discretization of 2D QCD, we identify $0_{k^+} = 1/L^-$ as the lowest resolved longitudinal momentum, for a meson with total longitudinal momentum P^+. In the parton model, $N = P^+/0_{k^+}$ counts the number of wee partons, with the larger the momentum, the larger N (see also below). We identify $\chi = \ln \Lambda^-$ with the rapidity, which is fixed by DIS kinematics as $\chi \sim \ln(Q^2/x)$ at low x.

C. Von Neumann entropy

Given the reduced density matrix, the corresponding von Neumann entanglement entropy is given by

$$S_n(x_0) = -\mathrm{tr}\hat{\rho}_n(x_0) \ln \hat{\rho}_n(x_0) = \ln \Lambda^- \int_0^{x_0} dx [|\varphi_n(x)|^2 + |\varphi_n(\bar{x})|^2] - \int_0^{x_0} dx [|\varphi_n(x)|^2 \ln |\varphi_n(x)|^2 + |\varphi_n(\bar{x})|^2 \ln |\varphi_n(\bar{x})|^2] - \int_{x_0}^{1-x_0} dx |\varphi_n(x)|^2 \ln \int_{x_0}^{1-x_0} dx |\varphi_n(x)|^2. \tag{27}$$

Since the n-state quark and antiquark PDF for a meson is given by

$$q_n(x) = \varphi_n^2(x), \qquad \bar{q}_n(x) = \varphi_n^2(\bar{x}), \tag{28}$$

the entanglement entropy is specifically

$$\begin{aligned} S_n(x_0) = \ln\Lambda^- \int_0^{x_0} dx[q_n(x) + \bar{q}_n(x)] \\ - \int_0^{x_0} dx[q_n(x)\ln q_n(x) + \bar{q}_n(x)\ln\bar{q}_n(x)] \\ - \int_{x_0}^{\frac{1}{2}} dx[q_n(x)+\bar{q}_n(x)] \ln \int_{x_0}^{\frac{1}{2}} dx[q_n(x)+\bar{q}_n(x)], \end{aligned} \tag{29}$$

which is symmetric under the exchange of a quark to an antiquark. Note that for $x_0 = \frac{1}{2}$, the result simplifies

$$\begin{aligned} S_n\left(\frac{1}{2}\right) = \ln\Lambda^- - \int_0^1 dx q_n(x)\ln\, q_n(x) \\ \to \ln\Lambda^- - (1-\ln\, 2), \end{aligned} \tag{30}$$

with the rightmost result following from the Wentzel–Kramers–Brillouin (WKB) approximation approximation. We have checked that for other hadrons (nucleons, exotics), the extensive part in (29) with the rapidity is also multiplied by the cumulative probabilities of each parton in that state. Equation (29) is the first major result of this paper.

1. Area law and Kolmogorov-Sinai bound [23]

Since the entanglement entropy depends on the length of the interval $x_0\Lambda^-$ only through logs, it trivially satisfies an area law. Similarly to the spatial entanglement in a 2D gapped system [1], the entanglement contains a log-divergent term $\propto \ln\Lambda^-$ and a finite term. However, unlike the spatial entanglement entropy, the coefficient of the log term depends also on the length of the interval. The logarithmic dependence leads to an evolution in rapidity, and is bounded from above as

$$\frac{dS_n(x_0)}{d\chi} = \int_0^{x_0} dx[q_n(x)+\bar{q}_n(x)] \equiv C(x_0) \leq 1. \tag{31}$$

In a way, the analog of the central charge is played by the *cumulative parton probability* $C(x_0)$, with $C(\frac{1}{2}) = 1$ saturating the bound.

If we identify the logarithmic dependence on P^+ as an evolution in rapidity, then (31) can be viewed as the Kolmogorov-Sinai bound for the entanglement entropy for an n meson in two-dimensional QCD, and we identify the Kolmogorov-Sinai entropy $S_{\rm KS} = 1$ (sum of the positive Lyapunov exponents).

The bound (31) can be understood in the following way. For the 't Hooft wave functions, the reduced density matrix contains only one-body and zero-body (vacuum) terms; therefore, its Schmidt decomposition allows at most $2x_0\Lambda^-$ terms, which implies an upper-bound $S_n \leq \ln\Lambda^- + \ln 2x_0$. However, our result shows that this is an overestimate. For the two-body wave function, it is the finite probability of the zero-mode contribution (vacuum state) that reduces the overestimation. For three- and higher-body wave functions, we show in Appendix D that the naive upper bound $\propto (k-1)\ln\Lambda^-$ for a generic state where k is the maximal number of partons is also an overestimate.

2. Structure function

At low x, (31) is the n-meson F_2^n structure function

$$\frac{dS_n(x_0\sim 0)}{d\chi} \sim x_0(q_n(x_0)+\bar{q}_n(x_0)) = F_2^n(x_0\sim 0), \tag{32}$$

in agreement with the analysis in higher dimensions [12–15]. In 2D (32) measures the low-x partons in the n-meson state

$$\frac{dS_n(x_0\sim 0)}{d\chi} \sim 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta+1}, \tag{33}$$

where we used that at the edges $x=0$ and $x=1$. The 't Hooft wave function has an asymptotic expansion in terms of the dynamically generated coefficient β as

$$\varphi_n(x) = C_n x^\beta, \qquad \pi\beta\cot\pi\beta = -\frac{\pi m_Q^2}{2\sigma_T} + 1. \tag{34}$$

A more refined analysis detailed in Appendix E gives

$$\begin{aligned} S_n(x_0) = 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta+1}\Bigg(\ln(e\Lambda^-) \\ + 2\beta \frac{1+(2\beta+1)\ln\frac{1}{x_0}}{(2\beta+1)} + \mathcal{O}(x_0^2)\Bigg) \end{aligned} \tag{35}$$

for $\beta > 0$. The result is consistent with (32), if we note that the second contribution in (E2) is suppressed in the chiral limit, i.e., $\beta \sim m_Q/\sqrt{\sigma_T}$. In passing, we also note the noncommutativity of the chiral limit with the low-x limit in 2D QCD.

For theories in which there are nontrivial logarithms running in rapidity, for example four-dimensional QCD, (32) measures the growth of low-x partons carried by the quark sea. This is consistent with the forward meson-meson (elastic $n-n \to n-n$) scattering amplitude in the Regge limit in 2D [28]

$$\sigma_n(s) \sim \frac{1}{s}{\rm Im}A_n(s,0) \sim s^{-(2\beta+1)} \sim F_2^n(x_0\sim 0), \tag{36}$$

with a negative Reggeon intercept $\alpha_{\mathbb{R}} = -2\beta$. (In 4D the forward limit is dominated by the pomeron with positive intercept $\alpha_{\mathbb{P}} > 0$, with the stringy relation $\alpha_{\mathbb{R}} + 1 = \alpha_{\mathbb{P}}$.) The forward elastic cross section is a measure of the n-meson F_2^n structure function. It is also consistent with the elastic 2D n-meson form factor $F_n(-q^2) \sim 1/(-q^2)^{\beta+1}$, both of which are dominated by the t-channel *single* Reggeon exchange, which amounts to a full quantum open string exchange after resummation, as we show below.

3. Valence PDF

The longitudinal evolution of the entanglement entropy (29) with parton x is highly nonlinear

$$\frac{dS_n(x_0)}{dx_0} = -(q_n(x_0)\ln q_n(x_0) + \bar{q}_n(x_0)\ln \bar{q}_n(x_0)) + (q_n(x_0)+\bar{q}_n(x_0))\ln\left(\frac{\Lambda^-}{e\int_{x_0}^{\frac{1}{2}} dx(q_n(x)+\bar{q}_n(x))}\right), \tag{37}$$

with most of the nonlinearity arising from the entanglement with the vacuum contribution in (24). For large rapidities χ, the longitudinal growth per unit rapidity is linear, and is a direct measure of the n-meson valence PDF

$$\frac{d^2S_n(x_0)}{d\chi dx_0} = q_n(x_0) + \bar{q}_n(x_0). \tag{38}$$

We expect a similar relation to hold for more general wave functions, e.g., baryons and exotics.

4. Scaling limit

Another interesting limit is the so-called scaling limit, which consists of closing up on the large n-meson states to exhibit the scale invariance of 2D QCD [28,29]. More specifically, consider the limit $\mu_n^2 = M_n^2/m_0^2 \to \infty$ with fixed ratio $\xi = x\mu_n^2$, where $m_0^2 = 2\sigma_T/\pi$. In this limit, the wave function approaches a universal function $\phi(\xi)$:

$$\varphi_n\left(\frac{\xi}{\mu_n^2}\right) \to \phi(\xi). \tag{39}$$

In this case, if we set $x_0 = \frac{\xi_0}{\mu_n^2}$, then the cumulative parton distribution reads

$$\int_0^{x_0} dx(q_n + \bar{q}_n) = \frac{2}{\mu_n^2}\int_0^{\xi_0} d\xi\phi^2(\xi), \tag{40}$$

and

$$\int_0^{x_0} dxq_n \ln\, q_n = \int_0^{x_0} dx\bar{q}_n \ln\, \bar{q}_n = \frac{1}{\mu_n^2}\int_0^{\xi_0} d\xi\phi^2(\xi)\ln\phi^2(\xi). \tag{41}$$

The leading $\mathcal{O}(1/\mu_n^2)$ entanglement entropy in this case is therefore purely expressed in terms of the universal function

$$\mu_n^2 S_n\left(\frac{\xi_0}{\mu_n^2}\right) \to 2\ln\, e\Lambda^- \int_0^{\xi_0} d\xi\phi^2(\xi) - 2\int_0^{\xi_0} d\xi\phi^2(\xi)\ln\phi^2(\xi). \tag{42}$$

For large ξ_0, the first term diverges linearly in ξ_0, while the second term diverges logarithmically.

5. Goldstone-like state

In the limit $m_Q^2/2\sigma_T \ll 1$, 2D QCD admits a massless Goldstone-like mode, with a light-front wave function $\varphi_0(x) = \theta(x\bar{x})$ (modulo the end points). This massless state is not a true Goldstone mode but will kill the chiral condensate in a way similar to the Berezinski-Kosterlitz-Thouless mechanism [30–32], as explained in [33]. In 4D QCD, the pion is a true Goldstone mode, and massless even for a fixed and large constituent mass m_Q. Yet, the pion longitudinal wave function is also totally delocalized in x Bjorken with $\varphi_\pi(x) \approx \theta(x\bar{x})$ in the chiral limit, and for pointlike interactions [34] (and references therein).

With this in mind, the density matrix for the Godstone-like state reads

$$\hat{\rho}_\pi = \frac{1}{\Lambda^-}\sum_{x,x'}\theta(x\bar{x})\theta(x'\bar{x}')|x,\bar{x}\rangle\langle x',\bar{x}'|, \tag{43}$$

modulo the endpoints. When traced over the interval $\bar{x}_0 = 1 - x_0$, the entanglement entropy is

$$S_\pi(x_0) = 2x_0\ln\Lambda^- - (\bar{x}_0 - x_0)\ln(\bar{x}_0 - x_0), \tag{44}$$

which is considerably simpler than (29). The change in rapidity of the pion entanglement entropy reads

$$\frac{dS_\pi(x_0)}{d\chi} = 2x_0.$$

This result is similar to the one we derive below for the entanglement entropy summed over the full Regge trajectory. This is perhaps the signature of the collective nature of the pseudo-Goldstone mode, on the light front. We note that in the massless Schwinger model, the light-front wave function of the "meson" state with mass $m^2 = g^2/\pi$ is

$$|\gamma\rangle = \frac{1}{\sqrt{\Lambda^-}}\sum_{0<x<1}|x,\bar{x}\rangle \tag{45}$$

with the same entanglement entropy (44) as in the pseudo-Goldstone state.

IV. 2D QCD AS A STRING ON THE LIGHT FRONT

Two-dimensional QCD is nonconformal but solvable in the large number of colors limit [21], as we discussed using the discretized light-front quantization earlier. Remarkably, the solution in this limit is identical to that following from a two-dimensional relativistic string with massive endpoints [22]. To show this, we recall that the 2D light-front Hamiltonian (squared mass) for a string with massive ends is [22]

$$H_{\mathrm{LF}} = \frac{m_Q^2}{x\bar{x}} + 2P^+\sigma_T|r^-| \to \frac{m_Q^2}{x\bar{x}} + 2\sigma_T\left|\frac{id}{dx}\right|, \tag{46}$$

with $0 \leq x = k^+/P^+ \leq 1$ the momentum fraction of the quark ($\bar{x} = 1 - x$ is that of the antiquark) in a meson with longitudinal momentum P^+. The relative light-front distance $P^+r^- \to id/dx$ is conjugate to x. The string tension is σ_T. The eigenstates of (46) solve

$$H_{\mathrm{LF}}\varphi_n(x) = \left(\frac{m_Q^2}{x\bar{x}} + 2\sigma_T\left|\frac{id}{dx}\right|\right)\varphi_n(x) = M_n^2\varphi_n(x), \tag{47}$$

with squared radial meson masses as eigenvalues. The confining potential in the Bjorken-x representation is given by the Fourier transform

$$\langle x|P^+|r^-||y\rangle = \int_{-\infty}^{+\infty}\frac{dq}{2\pi}e^{iq(x-y)}|q| \to \mathrm{PV}\frac{-1}{\pi(x-y)^2} + \frac{-1}{\pi x\bar{x}}, \tag{48}$$

with the principal value prescription. Using (48) in (47) yields 't Hooft Eq. (12) with the gauge coupling identified through $\sigma_T = g_{1+1}^2 N_c/2$. A brief semiclassical analysis of the string states is given in Appendix A. In sum, we can regard the even and odd solutions of the 't Hooft equation as the even and odd standing waves of a meson as a string, flying on the light front with either Dirichlet or Neumann boundary conditions modulo the small mass corrections at the edges.

A. Stringy entanglement: Resummed Regge trajectory

In the eikonalized approximation, dipole-dipole (open string) scattering in 2D QCD, sums over all n-meson (Reggeons) exchanges in the t channel. This resummed exchange is stringlike. To describe it, we need to resum over the full meson Regge trajectory in 2D QCD. However, this is not needed as we now show.

Indeed, the full density matrix of the string $\hat{\rho}_{\mathrm{string}}$ can be reconstructed from the n-meson density matrix ρ_n by noting that each of the meson states on the Regge trajectory maps onto a stationary state of the open string with massive endpoints. The orthonormality and completeness of these states implies that the full string density matrix is diagonal in n,

$$\hat{\rho}_{\mathrm{string}} = \frac{1}{\Lambda^-}\sum_{n=1}^{\infty}\sum_{x,x'}\varphi_n^\dagger(x')\varphi_n(x)|x,\bar{x}\rangle\langle x',\bar{x}'|. \tag{49}$$

Using the completeness relation,

$$\sum_n \varphi_n^\dagger(x)\varphi_n(x') = \delta(x - x'), \tag{50}$$

(49) is the projection operator onto the two-body states

$$\hat{\rho}_{\mathrm{string}} = \frac{1}{\Lambda^-}\sum_{0<x<1}|x,\bar{x}\rangle\langle x,\bar{x}|. \tag{51}$$

The reduced density matrix, followed by tracing over the segment $\bar{x}_0 = 1 - x_0$, yields the entanglement entropy

$$S(x_0) = 2x_0\ \ln\Lambda^- - (\bar{x}_0 - x_0)\ln(\bar{x}_0 - x_0), \tag{52}$$

which is independent of the mass at the endpoints of the string. It is surprisingly similar to (44) for the pseudo-Goldstone mode, even though the string density matrix (49) is diagonal in longitudinal space, while the one associated to the pseudo-Goldstone mode (43) is off-diagonal.

B. Stringy entanglement: Multimeson state

The above density matrix takes into account only single meson states. As we argued earlier, this entangled density matrix captures a DIS measurement of the quark distribution in a meson state, in the interval of length x_0 in parton x. Suppose that we want to use a DIS measurement of the quark distribution for the same x_0 interval, in a state composed of many identical hadrons flying on the light front (a 2D nucleus, or a 4D nucleus reduced to its longitudinal components). For that, we extend our analysis to multimeson states, with the corresponding Fock space spanned by all the mesons. Using the completeness relation, it is clear that the corresponding density matrix is now given by

$$\rho = \frac{1}{\mathrm{Dim}}\sum_k \mathrm{Dim}_k\rho_k, \tag{53}$$

where

$$\rho_k = \frac{1}{\mathrm{Dim_k}}\sum_{0<x_1<x_2..<x_k<1}|x_1,x_2,\ldots x_k\rangle\langle x_1,\ldots x_k| \tag{54}$$

are spanned by all k-particle tensor products of the fundamental basis, under the constraint that the same $|x,\bar{x}\rangle$ appears at most N_c times. For large N_c, each can appear infinitely many times.

1. $N_c = 1$ case

To help understand the bookkeeping for general N_c, let us first consider the case with $N_c = 1$, with no 2 mesons allowed to occupy the same longitudinal phase space region. In this case $\mathrm{Dim} = 2^{\Lambda^-}$. After tracing over the segment $(1 - x_0)$, the reduced density matrix for this case is

$$\frac{1}{2^{\Lambda^-}} \sum_{k-(\Lambda^- - N_1)<i<\min(k,N_1)} C^i_{N_1} |x_1, x_2, \ldots x_{k-i}\rangle\langle x_1 \ldots\ldots x_{k-i}|, \tag{55}$$

where $N_1 = (1 - x_0)\Lambda^-$. After summing over all k with the help of the binomial theorem, and replacing $k - i$ by $\tilde{k}$, the result is

$$\rho(x_0) = \frac{1}{2^{\Lambda^-}} \sum_{0\le\tilde{k}\le x_0\Lambda^-} \sum_{k=\tilde{k}}^{N_1+\tilde{k}} C^{k-\tilde{k}}_{N_1} \times \sum_{0\le x_1<x_2\ldots<x_{\tilde{k}}<x_0} |x_1, ..x_{\tilde{k}}\rangle\langle x_1, \ldots.x_{\tilde{k}}|, \tag{56}$$

$$\equiv \frac{1}{2^{x_0\Lambda^-}} \sum_{0\le\tilde{k}\le x_0\Lambda^-} \sum_{0\le x_1<x_2\ldots<x_{\tilde{k}}<x_0} |x_1, ..x_{\tilde{k}}\rangle\langle x_1, \ldots.x_{\tilde{k}}|. \tag{57}$$

This is simply the projection operator onto the subspace with $x_0\Lambda^-$ digits, corresponding to the part of the Hilbert space kept. The dimension of the space is $\mathrm{Dim}(x_0) = 2^{x_0\Lambda^-}$, and the corresponding entanglement entropy is now

$$S_E = \ln\,\mathrm{Dim}(x_0) = \ln 2 \times x_0\Lambda^-. \tag{58}$$

This is the *maximal entropy*, following from the reduction of any density matrix to the small-x interval.

2. General N_c case

For general N_c, and after tracing over the $(1 - x_0)$, we clearly get again the projection operator onto the subspace spanned by all the $|x_1, \ldots x_k\rangle$, with the constraint that $x_k \le x_0$ and that each x_i appears at most N_c times, due to the fermionic character of the underlying quark constituents in any of the colorless meson. The dimension of this Hilbert space is simply $(N_c + 1)^{x_0\Lambda^-}$, hence

$$S_E = \ln(N_c + 1) \times x_0\Lambda^-. \tag{59}$$

The rate of change with rapidity of the string entanglement entropy $S_E(x_0)$, the sum total of all entanglements along each of the exchanged Regge trajectories for fixed $x_0 \le \frac{1}{2}$, is extensive in Λ^-

$$\frac{dS_E(x_0)}{d\chi} = \ln(N_c + 1)x_0\Lambda^-. \tag{60}$$

In the low-x regime, dominated by the vacuum zero modes on the light front, (60) simplifies to

$$\frac{dS_E(x_0 \sim 0)}{d\chi} = \ln(N_c + 1)\frac{1}{2}e^{-\chi}e^{\chi} = \frac{1}{2}\ln(N_c + 1), \tag{61}$$

using the DIS identification $x_0 = \frac{1}{2}e^{-\chi}$.

3. Kolmogorov-Sinai bound [23]

The rate of increase of $S_E(x_0 \sim 0)$ with the rapidity χ saturates the Kolmogorov-Sinai bound at low x, with $S_{\mathrm{KS}} = \frac{1}{2}\ln(N_c + 1)$. The longitudinal quantum entanglement for the resummed mesons (Reggeon) as open strings in 2D is to be compared to the transverse quantum entanglement of $\frac{D_\perp}{6}$ for the resummed glueballs (pomeron) as a closed string exchange in $2 + D_\perp$ dimensions [12–14]. At low x, the entanglement is fixed by the $D_\perp$ transverse quantum vibrations of the string lightlike (analog of Luscher term spacelike).

4. Classical string entropy

Away from low x, the change in $S_E(x_0)$ is extensive in the invariant cutoff Λ^-, e.g.,

$$\frac{dS_E(x_0)}{d\chi} = \ln(N_c + 1)x_0\Lambda^-.$$

This scaling is commensurate with the growth of the string entropy S_S under large boosts. Indeed, a *free* string as a chain undergoing random walks in 1D generates $N_S = 2^{L/l_S}$ states (for a free string backtracking is allowed). The corresponding string entropy $S_S = \ln N_S = \ln 2L/l_S$. Under large longitudinal boosts P^+, the longitudinal length of the string *expands* (recall that the string bits are considered wee [19,35], they carry low momentum, and are oblivious to large boosts). As a result, $L/l_S = P^+/0_{k^+} = x_0\Lambda^-$ counts the number of string bits or wee partons, and the string entropy is $S_S = \ln 2x_0\Lambda^-$, which is seen to scale similarly to (60), in particular

$$\frac{dS_E(x_0)}{d\chi} = \frac{\ln(N_c + 1)}{\ln 2}S_S. \tag{62}$$

This large and quantum wee entropy stored in the longitudinal evolution in rapidity of open strings (Reggeons), when released in a collision, may contribute to the fast scrambling of information in hadronic collisions at ultrarelativistic energies. Perhaps more so, then the quantum wee entropy released from the evolution in rapidity of closed strings (pomerons) [12], provided that x_0 is not asymptotically small as in (61). We note that the string bits interactions may hamper the backtracking, and somehow reduce the entanglement rate in (62).

5. Bekenstein-Bremermann bound [6,7]

Quantum information theory sets a bound on the maximum rate of flow of information I in physical systems, as first noted by Bremermann for single channel systems, based on an argument using Shannon entropy and the quantum uncertainty principle [6]. The bound was revisited by Bekenstein on general grounds, using the maximum entropy storage in a black hole and causality [7]

$$\frac{dS_{\rm max}}{dt} \leq 2\pi E \to 2\pi TS. \tag{63}$$

The rightmost equality follows from the second law. (Here information I is interpreted as entropy in bits units or $I/S = \ln_2 e$.) If we recall that the rapidity χ relates to the Gribov time $t_\chi = \sqrt{\alpha'}\chi$ with $\alpha' = l_S^2$ the open string Regge slope [14,36], then a comparison of (62) with (63) shows that for $N_c = 1$, the Bekenstein-Bremermann bound is saturated, with $T = T_H = 1/(2\pi l_S)$ the Hagedorn temperature (equivalently, the temperature at the Rindler horizon of a black hole). Remarkably, for the multimeson state result with $N_c > 1$ in (62), the bound is still maintained, provided that the temperature exceeds (logarithmically) the Hagedorn temperature.

V. CONCLUSIONS

In the large number of colors, the 2-particle sector of 2D QCD on the light front decouples. The eigenmodes in this sector have a dual description in terms of partons or string modes. We have shown that in the partonic language, the entanglement in longitudinal momentum is captured by an exact reduced density matrix that is a tensor product of both the valence and vacuum states. The entanglement entropy for a single meson with a single cut in parton x, as probed by DIS kinematics, is a nonlinear function of the meson PDF.

For fixed parton x, the evolution in rapidity of the single meson entanglement entropy is the *cumulative* quark single PDF. It is bound by a Kolmogorov-Sinai entropy of 1. At low parton x, it reduces to the longitudinal structure function, as measured in DIS scattering. It is in agreement with the Regge behavior of the pertinent meson-meson scattering in 2D QCD. Alternatively, for fixed rapidity, the evolution in parton x is shown to probe directly the meson singlet PDF.

The sum total of the entanglement entropies for a fixed Regge trajectory is stringlike and extensive with the rapidity, as noted in 4D. We have suggested that DIS scattering on a *nucleus* in 2D can be modeled by DIS scattering on a multihadron state composed of 2D mesons, modulo Fermi statistics (amusingly shared by mesons through longitudinal space exclusion for $N_c = 1$). The evolution in rapidity of the ensuing entanglement entropy is found to be extensive in the longitudinal string entropy in 2D. The rate of change of this entropy matches the maximum rate of quantum information flow, as given by the Bekenstein-Bremermann bound.

A highly boosted multimeson state in 2D (a sort of 2D *nucleus* as all hadrons are similar on the light front) exhibits a growth rate in its wee parton entanglement entropy, that is only matched by the largest information rate flow allowed by the quantum laws of physics, a fit only exhibited by gravitational black holes. Remarkably, this flow exhibits an energy cost which is fixed by the Hagedorn temperature of the underlying longitudinal string.

The highly entangled wee partons in a boosted string as a *mock nucleus* carry an entanglement entropy that is commensurate with the classical string entropy S_S. Their prompt release by smashing, in current colliders at large rapidities $\chi = \ln s$, may explain why a large quantum entanglement entropy of about χS_S is promptly released, over a short time scale $1/l_S$, and at temperatures in (slight) excess of the Hagedorn temperature $T_H = 1/(2\pi l_S)$.

Finally, we note that the t'Hooft model is subcritical in terms of rapidity evolution, with no increase in parton densities for a large rapidity gap. In particular, the linear growth of the entanglement entropy noted here should hold for any general QFT, with subcritical parton evolution in rapidity. In 4D gauge theories with nontrivial parton evolution in rapidity, there are qualitative changes in the evolution of the entanglement in rapidity, and its corresponding entropy [37].

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APPENDIX A: WKB ANALYSIS OF THE STRING STATES

In this appendix, we qualitatively review the semiclassical solutions to 2D QCD, using the dual string form. In particular, the masses are given by the WKB quantization condition

$$\int_{x_-}^{x_+} dx\left(M_n^2 - \frac{m_Q^2}{x\bar{x}}\right) = M_n^2 - m_Q^2 \ln\left(\frac{x_+\bar{x}_-}{x_-\bar{x}_+}\right) = 2\pi\sigma_T n, \tag{A1}$$

with the turning points

$$x_\pm = \frac{1}{2}\left(1 \pm \left(1 - \frac{4m_Q^2}{M_n^2}\right)^{\frac{1}{2}}\right), \tag{A2}$$

and with $M_n \geq 2m_Q$. The mass gap vanishes for $m_Q \to 0$ with a radial Regge trajectory $M_n^2 = n/\alpha'$, and $\alpha' = 1/2\pi\sigma_T$ the slope of the open bosonic string.

A simple understanding of the light-front wave functions can be obtained directly from (47) by noting that for $m_Q^2/2\sigma_T \gg 1$, the mass contribution acts as a confining potential at the endpoints $x = 0$, 1, with $\varphi_n(x)$ standing waves solutions to

$$\left|\frac{id}{dx}\right|\varphi_n(x) \approx \frac{M_n^2}{2\sigma_T}\varphi_n(x), \tag{A3}$$

with Dirichlet boundary conditions. The normalized solutions are $\varphi_n(x) \approx \sqrt{2}\sin((n+1)\pi x)$. A simple estimate of the mass correction for large n follows from first order perturbation theory $M_n^2 \approx n/\alpha' + 2m_Q^2 \ln n$. In the opposite limit of $m_Q^2/2\sigma_T \ll 1$, the confining potential can be ignored to first approximation, in which case the standing waves follow from Neumann boundary conditions, with $\varphi_n(x) \approx \sqrt{2}\cos(n\pi x)$, with an identical Reggeized semiclassical spectrum. The effect of the mass is to cause a rapid distortion of the light-front wave function in a narrow region of x near the endpoints (see below).

APPENDIX B: SYMMETRIC INTERVAL

The reduced density matrix in parton x, was defined by tracing over the length $\bar{x}_0 = 1 - x_0$ for fixed $x_0 \leq \frac{1}{2}$, as motivated by a DIS measurement. This reduction is asymmetric with respect to the quark-antiquark content of the light-front meson wave function. A more *symmetric* but *academic* reduction is to trace over the symmetric length $x_0 < x < \bar{x}_0$. The reduced density matrix is then

$$\hat{\rho}_S(n) = \int_{x_0}^{\bar{x}_0} q_n(x)|0\rangle_S\langle 0|_S + |\tilde{\Phi}\rangle\langle\tilde{\Phi}|, \tag{B1}$$

where one has

$$|\tilde{\Phi}\rangle = \frac{1}{\sqrt{\Lambda^-}}\left(\sum_{0<x<x_0} + \sum_{\bar{x}_0<x<1}\right)\varphi_n(x)|x,\bar{x}\rangle. \tag{B2}$$

The above density matrix represents a binomial distribution, with the independent pair of eigenvalues $(p_n(x_0), 1 - p_n(x_0))$ where

$$p_n(x_0) = \int_0^{x_0} dx(q_n(x) + \bar{q}_n(x)). \tag{B3}$$

The corresponding entanglement entropy is therefore

$$\begin{aligned} S_S(n, x_0) = &-p_n(x_0)\ln p_n(x_0) \\ &- (1 - p_n(x_0))\ln(1 - p_n(x_0)) \end{aligned} \tag{B4}$$

and is independent of Λ^-. As $x_0 \to 0$, one has

$$p_n(x_0) \to \frac{2C_n^2 x_0^{2\beta+1}}{2\beta+1}, \tag{B5}$$

thus

$$\begin{aligned} S_S(n, x_0) = &2C_n^2 x_0^{2\beta+1}\ln\frac{1}{x_0} - \frac{2C_n^2 x_0^{2\beta+1}}{2\beta+1}\ln\frac{2C_n^2}{e(2\beta+1)} \\ &+ \mathcal{O}(x_0^{4\beta+2}). \end{aligned} \tag{B6}$$

The leading contribution is also proportional to $x_0^{2\beta+1}\ln\frac{1}{x_0}$.

APPENDIX C: GENERAL INTERPOLATING INTERVAL

In this appendix, we trace over an *asymmetric* interval centered around $\frac{1}{2}$, that interpolates between the symmetric and asymmetric reduction discussed above. In this case, the reduced density matrix traced over $[x_0, \bar{x}_0 + \delta]$ with $0 < \delta < x_0$ is now

$$\begin{aligned} \hat{\rho} = &\frac{1}{\Lambda^-}\sum_{x_0<x<1-x_0}|\varphi_n(x)|^2|0\rangle_{[x_0,1-x_0]}\langle 0|_{[x_0,1-x_0]} \\ &+ \frac{1}{\Lambda^-}\sum_{x_0-\delta<x<x_0}|\varphi_n(x)|^2|x\rangle\langle x| \\ &+ \frac{1}{\Lambda^-}\sum_{1-x_0<x<1-x_0+\delta}|\varphi_n(x)|^2|\bar{x}\rangle\langle\bar{x}| + |\tilde{\Phi}\rangle\langle\tilde{\Phi}|, \end{aligned} \tag{C1}$$

where the state $|\tilde{\Phi}\rangle$ reads

$$|\tilde{\Phi}\rangle = \frac{1}{\Lambda^-}\left(\sum_{0<x<x_0-\delta} + \sum_{1-x_0+\delta<x<1}\right)\varphi_n(x)|x,\bar{x}\rangle. \tag{C2}$$

The entanglement entropy is therefore given by

$$\begin{aligned} S_n(x_0, \delta) = &\ln\Lambda^-\int_{x_0-\delta}^{x_0} dx(q_n(x) + \bar{q}_n(x)) \\ &- \int_{x_0-\delta}^{x_0} dx(q_n\ln q_n + \bar{q}_n\ln\bar{q}_n) \\ &- \ln\int_{x_0}^{\frac{1}{2}} dx(q_n + \bar{q}_n)\int_{x_0}^{\frac{1}{2}} dx(q_n + \bar{q}_n) \\ &- \ln\int_0^{x_0-\delta} dx(q_n + \bar{q}_n)\int_0^{x_0-\delta} dx(q_n + \bar{q}_n). \end{aligned} \tag{C3}$$

Clearly, it interpolates between the two special cases considered above. When $\delta = x_0$, it reduces to the totally asymmetric case, while for $\delta = 0$, it reduces to the symmetric case. The coefficient of the Λ^- measures this asymmetry

$$\frac{dS_n(x_0,\delta)}{d\ln\Lambda^-} = \int_{x_0-\delta}^{x_0} dx(q_n(x)+\bar{q}_n(x)), \tag{C4}$$

which is always less or equal to 1 but non-negative. It vanishes only for $\delta = 0$.

APPENDIX D: NAIVE BOUND FOR AN *n*-PARTON STATE

Consider a generic wave function with maximally n partons

$$|\Phi\rangle = \sum_{i=1}^{n} \frac{1}{\sqrt{\Lambda^-}^{i}} \sum_{x_1,\ldots x_i} \varphi_i(x_1,\ldots.x_i)|x_1,\ldots.x_i\rangle. \tag{D1}$$

After tracing over $A=[x_0, 1-x_0]$, the reduced density matrix has the form

$$\hat{\rho}_A = \sum_{i,j} \rho_{ij}|i\rangle\langle j|, \tag{D2}$$

with $|i\rangle$ a generic state with i particles. It is important to observe that the diagonal terms follow from tracing $|i\rangle\langle i|$, since partial tracing cannot change the difference in particle numbers. Therefore, the diagonal terms form a reduced density matrix $\hat{\rho}_{\mathrm{A\,dia}}$, which contains more entropy compared to the full reduced density matrix, in general. They could be used to derive a superbound. Specifically, if we retain only the diagonal terms, the reduced density matrix reads

$$\hat{\rho}_{\mathrm{A\,dia}} = \sum_{i=0}^{n-1} \frac{1}{(\Lambda^-)^{i-1}} |x_1,x_2,\ldots.x_i\rangle\langle x_1',x_2',\ldots x_i'| \sum_{j=i+1}^{n} \int_{y\in E_i^j(x,x')} dy \varphi_j^\dagger(x_1,\ldots x_i; y_{i+1},\ldots y_j)\varphi_j(x_1',\ldots x_i'; y_{i+1},..y_j), \tag{D3}$$

with $E_i^j(x,x')$ the region of the j-particle phase space which should be traced over, for a reduction to the i-body phase spaces. Performing another diagonal approximation, the entanglement entropy is bounded by

$$S_E(x_0) \le \ln\Lambda^- \sum_{i=0}^{n-1} i p_i(x_0) + C. \tag{D4}$$

Here, $p_i(x_0)$ is the sum of the cumulative probabilities

$$p_i(x_0) = \sum_{j=i+1}^{n} \int_{x\in A_i^j(x_0)} dx_1,\ldots x_j|\varphi_j(x_1,\ldots x_j)|^2, \tag{D5}$$

where $A_i^j(x_0)$ is the part of the j-body phase space that after tracing, reduces to the i-body state. While the sum over all the probabilities is less then 1, the sum over the i-weighted probabilities is not *a priori* less than 1.

APPENDIX E: LOW-*X* ANALYSIS IN 2D QCD

The low-x analysis of the entanglement entropy in the single and asymmetric cut interval $A=[0, x_0 \le \frac{1}{2}]$ can be carried out exactly for $x_0 \to 0$, in 2D QCD. More specifically, using (34) allows one to unwind each contribution in (29) as

$$\begin{aligned}
\int_0^{x_0} dx[q_n(x)+\bar{q}_n(x)] &= 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta+1} + \mathcal{O}(x_0^2),\\
-\int_0^{x_0} dx[q_n(x)\ln q_n(x)+\bar{q}_n(x)\ln\bar{q}_n(x)] &= 4\beta C_n^2 \frac{1+(2\beta+1)\ln\frac{1}{x_0}}{(2\beta+1)^2} x_0^{2\beta+1} + \mathcal{O}(x_0^2),\\
-\int_{x_0}^{\frac{1}{2}} dx[q_n(x)+\bar{q}_n(x)]\ln\int_{x_0}^{\frac{1}{2}} dx[q_n(x)+\bar{q}_n(x)] &= 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta+1} + \mathcal{O}(x_0^2),
\end{aligned} \tag{E1}$$

and therefore, for $\beta > 0$

$$S_n(x_0) = 2C_n^2 \frac{x_0^{2\beta+1}}{2\beta+1} \left(\ln(e\Lambda^-) + 2\beta \frac{1+(2\beta+1)\ln\frac{1}{x_0}}{(2\beta+1)} + \mathcal{O}(x_0^2)\right), \tag{E2}$$

which is the result quoted in the text.

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