

Magnetic field induced corrections to the NJL model coupling constant from vacuum polarization

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Magnetic field dependent corrections for the coupling constant of the Nambu-Jona-Lasinio model are calculated by considering the one-loop background field method. These coupling constants turn out to break chiral and flavor symmetries and they lead to a slight improvement of the numerical values of the up and down quark condensates when compared to results from lattice QCD. The corresponding magnetic field dependencies of the neutral pion and kaon masses are also presented and compared with available lattice QCD calculations. The resulting magnetic field correction to the $\eta - \eta'$ mixing angle is also estimated.

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I. INTRODUCTION

Strong magnetic fields are expected to show up in noncentral heavy ion collisions (h.i.c.) and also in astrophysical systems such as dense stars or magnetars. In h.i.c. they may reach $eB_0 \sim 10^{18} \text{ G} \sim m_\pi^2$ or $eB_0 \sim 0.04\text{--}0.3 \text{ GeV}^2$ from RHIC to LHC [1–4], even if this is within a short time interval in a limited spatial region [5–7] with recent indications that magnetic fields may be weaker than previously estimated (although still strong) [8]. In the early Universe [9,10] and in magnetars/neutron stars [11–13], magnetic fields were estimated to have been of the order of $eB_0 \sim 10^{21}$ and 10^{15} G , respectively. Although the geometry of the magnetic fields might be time dependent and extremely complicated, a first theoretical analysis, by considering constant magnetic fields, can be very useful to understand their role in strong interactions systems, which are themselves very complicated to be treated. Strong magnetic fields might lead to many different effects in different aspects of hadron dynamics, both at the quark and gluon level and at the (lower energies) hadron and nuclear levels, and experimental evidences are currently searched in different types of experiments. The validity of the semiclassical description of magnetic field in aspects of h.i.c. has been tested for example in [14]. Among these properties that might receive large contributions from the presence of the magnetic fields, a mechanism of mass generation was predicted earlier: the so-called magnetic catalysis for which, even in the chiral limit, particles

develop mass and which has been identified with the high degeneracy of the lowest Landau level [15–18].

Usually, global properties of low energies hadrons can be suitably investigated by means of hadron effective models and effective field theories whose use have been extended, more recently, for hadrons in strong magnetic fields [1,16,19–23]. Among the successful QCD effective models, the Nambu-Jona-Lasinio (NJL) model is known to reproduce, and eventually to predict, many observables for the hadron structure and dynamics under different conditions [20,24–27]. Several approaches have been already employed to describe how the NJL model coupling constant might be obtained in terms of QCD degrees of freedom in the vacuum [28–33] or to understand further how those degrees of freedom contribute for the NJL-model parameters [34–36]. Lately, lattice QCD provided results for hadron observables in a finite strong magnetic field were also used to test the predictions of NJL model in such conditions and, eventually, this type of comparison may favor an improvement of its the predictive power [37,38]. With such comparisons, one might also obtain knowledge on how the parameters of the effective models might be related to more fundamental degrees of freedom from QCD. It has been envisaged that the NJL coupling constant might receive magnetic field contributions, $G(B)$ [39], because of quark and gluon interactions [40]. This effect contributes to the improvement of the description of the quark-antiquark chiral condensate as a function of the magnetic field [38]. Among important hadron observables, the light pseudoscalar mesons have a special role in the strong interactions since they are the quasi-Goldstone bosons of the dynamical chiral symmetry breaking (DChSB). Their masses in the vacuum are associated basically to the explicit breaking of chiral symmetry and its amplification due to the DChSB. Their behavior under

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strong magnetic fields was investigated in the last few years. Several results were obtained from calculations with the NJL model in strong magnetic fields by assuming $G(B)$ or not [39,41–50]. Lattice QCD has provided few estimations for the behavior of hadron properties under strong magnetic fields, being that a small difference was found when comparing earlier different lattice fermions [37,38,51]. Although this dependence of $G(B)$ has been attributed to the quark-gluon running coupling constant dependence on the magnetic field [40], we believe that other mechanisms can contribute. Besides that, one might be interested in understanding more precisely the role of needed degrees of freedom of the more fundamental theory for defining the NJL-model parameters by articulating further the model itself and their parameters.

In the present work we employ the background field method [23,52] to compute the contribution of the quark polarization under a strong magnetic field for the NJL-coupling constant. This work extends the more restricted calculation for weak magnetic field and $SU(N_f = 2)$ presented in [53] and, besides that, estimations for its effects on the neutral pion and kaon masses are presented. The dependencies of the u, d, and s quark condensates and of the $\eta - \eta'$ mixing angle [35,36,54,55] on the magnetic field are also calculated. The auxiliary field method will be considered, as usual, as that the scalar field allows for the DChSB although a chiral rotation is performed to eliminate the corresponding meson degree of freedom, which seems absent in the light hadron spectrum. For zero magnetic field it has been shown that the choice of the regularization method is little important for the light hadron observables [56] and an investigation for the role of different regularization schemes under finite B has also been carried out [57,58]. A magnetic field independent regularization is chosen for the covariant four-dimensional cutoff regularization. We make use of the (more convenient) proper-time representation for the magnetic field contribution for the quark propagator that is ultraviolet (UV) finite. The B dependence of the results are guided strictly by the behavior of the effective masses from the gap equations, and eventually B -dependent coupling constants. However, since we are concerned with the relative role of the magnetic field dependent coupling constants with respect to the original NJL-coupling constant, G_0 , i.e., to analyze the relative influence due to the magnetic field, the role of the choice of the regularization scheme (in particular for the vacuum part of the equations) may be expected to be relatively small. By resolving the coefficients of a large quark mass expansion of the quark determinant in the background quark currents, for a zero order derivative expansion, the magnetic-field dependent corrections for the NJL-coupling constant, $G_{ij}(B)$, are obtained mostly analytically. These coupling constants turn out to be strongly flavor dependent. The fitting of the

parameters of the parameters of the resulting model, with G_{ij} , is done by means of the usual observables in the vacuum, neutral mesons masses M_{π^0}, M_{K^0} , and the decay constants F_π and F_K . The work is organized as follows. In the next section the sea-quark determinant is presented in the presence of background scalar and pseudoscalar quark currents and local pseudoscalar and scalar auxiliary fields. In Sec. III the corrected NJL model, with $G_{ij}(B)$, is considered for the calculation of the neutral pion and kaon masses as functions of the magnetic field. In Sec. IV numerical results are presented for the quark effective masses, scalar and pseudoscalar magnetic field dependent (corrected) coupling constants, quark-antiquark chiral condensates, and neutral pion and kaon masses. Besides that, a magnetic field correction to the $\eta - \eta'$ mixing angle will be also calculated for different behaviors of the magnetic field dependencies of the $\eta - \eta'$ mass difference that is, currently, also unknown. Finally in Sec. V a summary with a discussion is presented.

II. BACKGROUND FIELD METHOD, SEA-QUARK DETERMINANT, AND GAP EQUATION

The following generating functional will be considered:

$$Z[J, \bar{J}] = N \int \mathcal{D}[\bar{\psi}, \psi] e^{i \int_x (\mathcal{L} + \bar{\psi} J + \bar{J} \psi)},$$

where the NJL-model Lagrangian density for the minimal coupling for a background electromagnetic field can be written as

$$\mathcal{L} = \bar{\psi} (i\gamma \cdot D - m_f) \psi + \frac{G_0}{2} [(\bar{\psi} \lambda^i \psi)^2 + (\bar{\psi} i\gamma_5 \lambda^i \psi)^2], \quad (1)$$

where $i, j, k = 0, \dots, (N_f^2 - 1)$ stand for flavor indices in the adjoint representation, m_f stand for the current quark mass matrix element wherein $f = u, d,$ and s for the fundamental representation, and the sums in color, flavor, and Dirac indices are implicit. The covariant quark derivative is $D = D_\mu = \partial_\mu \delta_{ij} - ie Q_{ij} A_\mu$ for the diagonal matrix $\hat{Q} = \text{diag}(2/3, -1/3, -1/3)$.

Next we apply the one loop background field method [23,52] according to which bilinears of the quark field, $\bar{\psi} \Gamma \psi$ where Γ stands for Dirac, color, or flavor operators, are split into the (constituent quark) background field (ψ_1) that will become quasiparticles of the model and the quantum quark field (ψ_2) that will form mesons and the chiral condensates and which will be integrated out. It can be written as

$$\bar{\psi} \Gamma^q \psi \rightarrow (\bar{\psi} \Gamma^q \psi)_2 + (\bar{\psi} \Gamma^q \psi)_1. \quad (2)$$

This separation preserves chiral symmetry and it may not correspond to a simple mode separation of low and high

energies, which might be a very restrictive assumption and would involve an energy separation scale. Whereas the overall method employed is inspired in the usual constant background field method, one step further can be given with the derivative expansion that allows us to compute a whole effective action [59–61].

The interaction term of the NJL model, Ω , is split into separated terms for ψ_1 and ψ_2 (Ω_1 , Ω_2) and those with both bilinear of ψ_1 and ψ_2 . The interaction Ω_2 will be treated by making use of the auxiliary field method by introducing a set of auxiliary scalar and pseudoscalar fields, $S_i \equiv S_i(x)$ and $P_i \equiv P_i(x)$ for $i = 0, 1, \dots, N_f^2 - 1$, [28]. These auxiliary fields might be introduced by multiplying the generating functional by the following normalized Gaussian integrals:

$$1 = N \int D[S]D[P_i] e^{-\frac{i}{2G_0} \int_x [(S_i - G_0 j_{i(2)}^S)^2 + (P_i - G_0 j_{i(2)}^P)^2]}, \quad (3)$$

where $\int_x = \int d^4x$ and the scalar and pseudoscalar currents were defined as $j_S^{i(2)} = \bar{\psi} \lambda_i \psi$ and $j_P^{i(2)} = \bar{\psi} \lambda_i i \gamma_5 \psi$.

With these auxiliary fields, the quark field ψ_2 , $\bar{\psi}_2$ can be quantized, and an effective action for background quarks and auxiliary fields canonically normalized is obtained. From here on, we can omit the index for a quark background field. By considering the identity $\det A = \exp \text{Tr} \ln(A)$, the resulting model can be written as

$$\begin{aligned} S_{\text{eff}} = & -i \text{Tr} \ln \{ -i [S_0^{B-1} + \Xi \\ & + G_0 \lambda^i [(\bar{\psi} \lambda_i \psi) + i \gamma_5 (\bar{\psi} i \gamma_5 \lambda_i \psi)]] \} \\ & + \int_x \left\{ \bar{\psi} (i \gamma_\mu D^\mu - m) \psi + \frac{G_0}{2} [(\bar{\psi} \lambda^i \psi)^2 \right. \\ & \left. + (\bar{\psi} i \gamma_5 \lambda^i \psi)^2] - \frac{1}{2G_0} [S_i^2 + P_i^2] \right\}, \quad (4) \end{aligned}$$

where Tr stands for traces of discrete internal indices and integration of space-time coordinates and the following quantities have been defined:

$$S_0^{B-1} = (i \not{D} - m_f), \quad (5)$$

$$\Xi = (S \cdot \lambda + i P \cdot \lambda), \quad (6)$$

where S_0^B is the free quark propagator with its coupling to the electromagnetic field, with $\not{D} = \gamma^\mu \cdot D^\mu$. Therefore Ξ provides the auxiliary fields coupling to quarks.

Since the auxiliary fields are unknown, an extremization of Eq. (4) yields the usual gap equations and provides a determination of the auxiliary fields at a mean field level, \bar{S}_i . The solutions of the scalar fields for the gap equations have been investigated in many works both in the vacuum and under constant weak and strong magnetic

fields, to quote few works: [16,62–64]. The magnetic field is known to increase the effective mass, even if the current Lagrangian quark mass is zero, which is known as the magnetic catalysis effect [15–18]. The resulting gap equations for the set of scalar auxiliary fields corresponding to the diagonal flavor generators, S_0 , S_3 , and S_8 , can be written as

$$S_i \equiv \bar{S}_i = -i G_0 \text{Tr} \lambda_i S^{(B)}, \quad (7)$$

where $S^{(B)}$ (defined below) takes into account possible a nonzero expected value in the vacuum for the auxiliary fields. The corresponding equations for the pseudoscalar fields, at zero magnetic field, must be a trivial one to enforce the scalar nature of the vacuum. The scalar auxiliary field mean field makes possible the generation of (effective) mass for the constituent quarks, such that in the fundamental representation one has $M_f^* = m_f + \bar{S}_f$. The quark propagator in a background magnetic field was calculated by considering the Schwinger proper time method and it is shown explicitly in Appendix A. In the absence of (background) quark currents and auxiliary fields for mesons the celebrated Euler Heisenberg effective action can be recovered from Eq. (4) [64–67].

A. GAP equation in a magnetic field

The nontrivial solution for the scalar variables lead to diagonal contributions for the fundamental representation, i.e., \bar{S}_u , \bar{S}_d , and \bar{S}_s . From here on the quark masses become effective masses such that one can write

$$S_f^{B-1} = (i \not{D}_f - M_f^*), \quad (8)$$

where $M_f^* = m_f + \bar{S}_f$ and the different minimal photon couplings to u, d, and s quarks were written above in \not{D}_f . The gap equation for the effective quark masses can be written as

$$M_f^* = m_f - 2G_0 \langle \bar{\psi}_f \psi_f \rangle, \quad (9)$$

where $\langle \bar{\psi}_f \psi_f \rangle = -i \text{tr}_{DC} S_f^B(0)$ is the chiral condensate in the mean field approximation. Here tr_{DC} denotes the trace over Dirac and color indices, and $S_f^B(x-y)$ stands for the quark propagator of flavor f in the presence of a uniform magnetic field. A magnetic field independent regularization will be adopted [57,58]. The UV divergent part can be separated to correspond to the vacuum contribution, whereas the explicitly magnetic field dependent part is UV finite. The regularization scheme considered for the UV divergent part will be the four-momentum cutoff (Λ) in Euclidean space. By using the proper time representation for the magnetic field dependent part we find

$$\begin{aligned}
 M_f^* &= m_f + \frac{G_0 N_c M_f^*}{2\pi^2} \left[\Lambda^2 - M_f^{*2} \ln \left(\frac{\Lambda^2 + M_f^{*2}}{M_f^{*2}} \right) \right] \\
 &+ \frac{G_0 N_c M_f^*}{2\pi^2} \left[M_f^{*2} \left(1 - \ln \frac{M_f^{*2}}{2|q_f B|} \right) \right. \\
 &\left. + |q_f B| \ln \frac{M_f^{*2}}{4\pi|q_f B|} + 2|q_f B| \ln \Gamma \left(\frac{M_f^{*2}}{2|q_f B|} \right) \right] \\
 &= M_{f,0}^* + \frac{G_0 N_c M_f^*}{2\pi^2} \left[M_f^{*2} \left(1 - \ln \frac{M_f^{*2}}{2|q_f B|} \right) \right. \\
 &\left. + |q_f B| \ln \frac{M_f^{*2}}{4\pi|q_f B|} + 2|q_f B| \ln \Gamma \left(\frac{M_f^{*2}}{2|q_f B|} \right) \right]. \quad (10)
 \end{aligned}$$

We remark that the divergences were isolated into the vacuum term before introducing the regularization parameter. Although our departure point was the proper time representation for the propagator, isolating the divergences into the vacuum contribution allowed us to use other regularization schemes than the regularization in proper time since the pure magnetic contribution introduces no new divergences.

B. Magnetic field-dependent corrections to the coupling constant

Since we are interested in the dynamics of quarks by means of their currents, from here on the auxiliary fields will be neglected. By expanding the quark determinant in a large quark effective mass expansion in terms of the quark field bilinears, in a zero order derivative expansion, we find the first order term to be given by

$$S_{\text{det}}^{(1)} = -2G_0 \sum_{f=u,d,s} \int_x \text{tr}_{DC} [iS_f^B(p)] \bar{\psi} \psi, \quad (11)$$

with $S_f^B(p)$ representing the quark propagator in momentum space in the presence of the uniform magnetic field B , which is exhibited in Appendix A. These terms produce a correction to the quark masses that is the same as the gap equation, Eq. (10).

The second order terms of the large quark mass expansion provides fourth order quark interactions. After resolving coupling constants in the very long-wavelength limit for the zero order derivative expansion, results are the following:

$$\begin{aligned}
 \mathcal{L}_{\text{1loop}} &= \frac{\bar{G}_s^{ij}(B)}{2} (\bar{\psi} \lambda_i \psi) (\bar{\psi} \lambda_j \psi) \\
 &+ \frac{\bar{G}_{ps}^{ij}(B)}{2} (\bar{\psi} i\gamma_5 \lambda_i \psi) (\bar{\psi} i\gamma_5 \lambda_j \psi), \quad (12)
 \end{aligned}$$

where

$$\bar{G}_s^{ij}(B) \equiv G_0^2 \Pi_{ij}^s(B) = iG_0^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} [S_f^B(p) \lambda^i S_g^B(p) \lambda^j], \quad (13)$$

$$\begin{aligned}
 \bar{G}_{ps}^{ij}(B) &\equiv G_0^2 \Pi_{ij}^{ps}(B) \\
 &= iG_0^2 \int \frac{d^4 p}{(2\pi)^4} \text{tr} [S_f^B(p) \lambda^i i\gamma_5 S_g^B(p) \lambda^j i\gamma_5]. \quad (14)
 \end{aligned}$$

All these coupling constants are written as combinations of integrals of each quark propagator for which a change of representation for the coupling constants is presented in Appendix B. These coupling constants obviously break chiral and flavor symmetries and they have the same dimension of the NJL-coupling constant, GeV^{-2} . By using the proper time representation for the quark propagator in momentum space and considering only the polarization functions that involve quark flavors with the same electric charge, being that in those cases the Schwinger phases cancel out, it is possible to separate the contributions from the vacuum (zero magnetic field) and the B -dependent contributions similarly to the quark propagator. By separating each of the couplings for given i, j in terms of the related contributions from internal quark propagators f, g it is obtained for each component with equal electric charges $q_f = q_g$:

$$\begin{aligned}
 \Pi_{fg}^s(B) &= \Pi_{fg}^s(B=0) + \tilde{\Pi}_{fg}^s(B) = \Pi_{fg}^s(B=0) \\
 &+ \frac{N_c |q_f B|}{2\pi^2} \int_0^\infty \int_0^\infty ds dr \frac{e^{-sM_f^{*2} - rM_g^{*2}}}{s+r} \\
 &\times \left[\frac{1 \mp M_f^* M_g^* (s+r)}{(s+r) \tanh(|q_f B|(s+r))} \right. \\
 &\left. + \frac{|q_f B|}{\sinh^2(|q_f B|(s+r))} - \frac{2 \mp M_f^* M_g^* (s+r)}{|q_f B|(s+r)^2} \right]. \quad (15)
 \end{aligned}$$

The vacuum contributions for the flavor symmetric model were analyzed in [33,68] and for nondegenerate quark masses in [35,36].

By making the change of variables

$$s = \frac{u}{2}(1+v), \quad r = \frac{u}{2}(1-v), \quad (16)$$

with $0 \leq u < \infty$ and $-1 \leq v \leq 1$, so that $ds dr = (u/2) du dv$, we obtain the following:

$$\begin{aligned}
 \tilde{\Pi}_{fg}^s(B) &= \frac{N_c |q_f B|}{2\pi^2} \int_0^\infty du \int_{-1}^1 dv \frac{e^{-\frac{u}{2}(1+v)M_f^{*2} - \frac{u}{2}(1-v)M_g^{*2}}}{2} \\
 &\times \left[\frac{1 \mp uM_f^* M_g^*}{u \tanh(|q_f B|u)} + \frac{|q_f B|}{\sinh^2(|q_f B|u)} \right. \\
 &\left. - \frac{2 \mp uM_f^* M_g^*}{|q_f B|u^2} \right] \quad (17)
 \end{aligned}$$

for the pure magnetic contribution to the polarization functions. The proper time integrals can be computed in closed form for the diagonal couplings $f = g$, yielding

$$\begin{aligned}\tilde{\Pi}_{ff}^s(B) &= \frac{N_c M_f^{*2}}{2\pi^2} \left[1 + \frac{|q_f B|}{M_f^{*2}} \ln \left(\frac{M_f^{*2}}{4\pi |q_f B|} \right) + \frac{2|q_f B|}{M_f^{*2}} \ln \Gamma \left(\frac{M_f^{*2}}{2|q_f B|} \right) \right. \\ &\quad \left. + (1 \pm 1) \psi \left(\frac{M_f^{*2}}{2|q_f B|} \right) - (2 \pm 1) \ln \left(\frac{M_f^{*2}}{2|q_f B|} \right) + (1 \pm 1) \frac{|q_f B|}{M_f^{*2}} \right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\bar{G}_{ff}^s(B) &= \frac{G_0^2 N_c M_f^{*2}}{2\pi^2} \left[1 + \frac{|q_f B|}{M_f^{*2}} \ln \left(\frac{M_f^{*2}}{4\pi |q_f B|} \right) + \frac{2|q_f B|}{M_f^{*2}} \ln \Gamma \left(\frac{M_f^{*2}}{2|q_f B|} \right) \right. \\ &\quad \left. + 2\psi \left(\frac{M_f^{*2}}{2|q_f B|} \right) - 3 \ln \left(\frac{M_f^{*2}}{2|q_f B|} \right) + 2 \frac{|q_f B|}{M_f^{*2}} \right],\end{aligned}\quad (18)$$

$$\bar{G}_{ff}^{\text{ps}}(B) = \frac{G_0^2 N_c M_f^{*2}}{2\pi^2} \left[1 + \frac{|q_f B|}{M_f^{*2}} \ln \left(\frac{M_f^{*2}}{4\pi |q_f B|} \right) + \frac{2|q_f B|}{M_f^{*2}} \ln \Gamma \left(\frac{M_f^{*2}}{2|q_f B|} \right) - \ln \left(\frac{M_f^{*2}}{2|q_f B|} \right) \right],\quad (19)$$

where $\Gamma(x)$ is the gamma function and $\psi(x)$ is the Euler psi function.

Another case of interest is the one of the couplings $G_{ds}^s(B)$ and $G_{ds}^{\text{ps}}(B)$, which involve quarks of different flavors but with the same electric charge. We have

$$\begin{aligned}\bar{G}_{ds}^s(B) &= \frac{G_0^2 N_c |q_d B|}{2\pi^2} \int_0^\infty du \int_{-1}^1 dv \frac{e^{-\frac{u}{2}(1+v)M_d^{*2} - \frac{u}{2}(1-v)M_s^{*2}}}{2} \\ &\quad \times \left[\frac{1 \mp u M_d^* M_s^*}{u \tanh(|q_d B|u)} + \frac{|q_d B|}{\sinh^2(|q_d B|u)} - \frac{2 \mp u M_d^* M_s^*}{q_d B u^2} \right],\end{aligned}\quad (20)$$

where now the proper time integrals need to be solved by using numerical methods. Note that the divergence of the integrals above, parametrized in the UV cutoff Λ , appears only in the vacuum contributions $G(B=0)$ or $\Pi(B=0)$. The difference between the scalar and pseudoscalar couplings is directly a consequence of chiral symmetry breaking effect in the coupling constants at the one loop level. For the cases addressed above it follows:

$$\begin{aligned}\bar{G}_{csb}^{fg}(B) &\equiv \bar{G}_{ps}^{fg}(B) - \bar{G}_s^{fg}(B), \\ &= 2G_0^2 \frac{N_c |q_f B|}{2\pi^2} \int_0^\infty \int_0^\infty ds dr \frac{e^{-sM_f^{*2} - rM_g^{*2}}}{s+r} \\ &\quad \times \left[\frac{M_f^* M_g^*(s+r)}{(s+r) \tanh(|q_f B|(s+r))} - \frac{M_f^* M_g^*(s+r)}{|q_f B|(s+r)^2} \right].\end{aligned}\quad (21)$$

Although the magnetic field dependence of this quantity is not necessarily small, it will be neglected in most of

calculations as explained below. The overall behavior of the pseudoscalar coupling constants is very different from the needed behavior that describes results from lattice QCD.

C. Fitting of the model parameters at $B=0$ and contributions from $B \neq 0$

The parameters of the model are G_0 , m_u , m_d , and m_s with the additional need of fixing the UV cutoff Λ . By adopting a coupling constant $G_0 = 9.76 \text{ GeV}^{-2}$ the following masses and weak decay constants were considered in the vacuum to fix these parameters M_{π^0} , M_{K^0} , F_π , and F_K , which are written below.

The pseudoscalar mesons masses (M_{ps}) in the framework of the standard NJL model (1) are obtained from the Bethe Salpeter equation at the Born approximation by means of the following equation:

$$1 - G_0 \Pi_{\text{ps}}^{ij}(P^2)|_{P^2=M_{\text{ps}}^2} = 0,\quad (22)$$

where neutral pion and kaon are obtained, respectively, with $ij = 33$ and $ij = 66, 77$. The polarization tensor was rotated back to the Minkowski space and it needs to be computed for the on shell meson, in the limit of zero three-momentum. The corresponding integrals are given by

$$\Pi_{\text{ps}}^{33}(P^2) = \frac{1}{2} [\Pi_{\text{ps}}^{uu}(P^2) + \Pi_{\text{ps}}^{dd}(P^2)],\quad (23)$$

$$\Pi_{\text{ps}}^{66}(P^2) = \Pi_{\text{ps}}^{ds}(P^2).\quad (24)$$

These pseudoscalar polarization tensors as functions of the energy of the meson can be written as

$$\begin{aligned}
 \Pi_{\text{ps}}^{fg}(P^2) = & \frac{1}{2} \left[\frac{M_{f,0}^* - m_f}{M_f^*} + \frac{M_{g,0}^* - m_g}{M_g^*} \right] \\
 & + \frac{N_c}{4\pi^2} [P^2 - (M_f^* - M_g^*)^2] \int_0^1 dx \left\{ \ln \left[\frac{\Lambda^2 + D_{fg}^2(P^2)}{D_{fg}^2(P^2)} \right] + \frac{D_{fg}^2(P^2)}{\Lambda^2 + D_{fg}^2(P^2)} - 1 \right\} \\
 & + \frac{N_c |q_f B|}{2\pi^2} \int_0^\infty \int_0^\infty ds dr \frac{e^{-sM_f^* - rM_g^* + \frac{sr}{s+r}P^2}}{s+r} \\
 & \times \left[\frac{1 + M_f^* M_g^* (s+r) + \frac{sr}{s+r} P^2}{(s+r) \tanh(|q_f B|(s+r))} + \frac{|q_f B|}{\sinh^2(|q_f B|(s+r))} - \frac{2 + M_f^* M_g^* (s+r) + \frac{sr}{s+r} P^2}{(s+r)^3} \right], \quad (25)
 \end{aligned}$$

where $M_{f,0}^*$ was defined in Eq. (10) and $D_{fg}^2(P^2) = -x(1-x)P^2 + xM_f^{*2} + (1-x)M_g^{*2}$. Note that, again, the UV divergent part is written separately, as the vacuum term, from the magnetic field contributions.

The charged pion and kaon weak decay constants, for a meson structure of quark antiquark f, g , are given by [25,26]

$$\begin{aligned}
 F_{\text{ps}} = & \frac{N_c G_{qqPS}}{4} \int \frac{d^4 q}{(2\pi)^4} \\
 & \times \text{Tr}_{F,D} [\gamma_\mu \gamma_5 \lambda_i S_f(q + P/2) \lambda_j S_g(q - P/2)], \quad (26)
 \end{aligned}$$

where i, j are the associated flavor indices as discussed for Eq. (22). The meson-quark coupling can be obtained as the residue of the pole of the bound state equation (BSE) will be calculated in the limit of zero four momentum as

$$G_{qqPS} = \left(\frac{\partial \Pi_{ij}(P^2)}{\partial P_0^2} \right)_{(P_0, \vec{P})=0}^{-2}, \quad (27)$$

where the flavor indices are tied with the quantum numbers of the meson PS , π^+ with $i, j = (1, 1), (2, 2)$ and K^+ with $i, j = (4, 4), (5, 5)$.

III. CORRECTED NJL MODEL

Now consider the NJL model corrected with magnetic field dependent coupling constants found above as given by

$$\begin{aligned}
 \mathcal{L} = & \bar{\psi} (i\gamma \cdot D - m) \psi + \frac{G_0 \delta_{ij} + \bar{G}_s^{ij}(B)}{2} \\
 & \times [(\bar{\psi} \lambda^i \psi)(\bar{\psi} \lambda^j \psi) + (\bar{\psi} i\gamma_5 \lambda^i \psi)(\bar{\psi} i\gamma_5 \lambda^j \psi)] + \mathcal{O}_{chsb}, \quad (28)
 \end{aligned}$$

where $\bar{G}_{ij}^s(B)$ and $\bar{G}_{csb}^{ij}(B)$ were written in Eqs. (18) and (19) and \mathcal{O}_{chsb} are the chiral symmetry breaking corrections for the pseudoscalar coupling constants discussed above and neglected from here on. However, the zero magnetic field limit of $G_{ij}^{s,\text{ps}}(B=0)$ contributes for G_0 making the overall normalization of the coupling constant

ambiguous and this would have consequences for the calculation of observables for which we are interested, however, in investigating only the magnetic field dependence. Therefore the B -dependent coupling constant considered in the Lagrangian above will be exclusively the B -dependent part of Eq. (18). In principle the scalar interactions contribute for the gap equations and the pseudo-scalar couplings G_{ij}^{ps} can be expected to be those by which the bound state pseudoscalar mesons are formed according to the BSE. Because of the completely different behavior of the scalar and pseudoscalar couplings and the fact that the scalar coupling constant helps to improve the behavior of the quark-antiquark condensates with the B field shown below, we decided to analyze rather the effects of the scalar coupling by neglecting the difference between them. So we assume that, for some unknown reason maybe related to the level of approximation in which the one loop quark determinant and its expansion rely, the different between the scalar and pseudoscalar couplings G_{ij}^s and G_{ij}^{ps} should be considerably smaller, and the pseudoscalar coupling constants would have its behavior changed considerably to be similar to the scalar ones, i.e., $\bar{G}_{ij}^s(B) \sim \bar{G}_{ij}^{\text{ps}}(B)$ given by Eq. (18). Next, the usual logics applied to the NJL model must be used again and so the gap equations must be recalculated. This procedure can be done repeatedly until the resulting effective masses, M_f^* , and coupling constants, $G_{ij}^{s,\text{ps}}$, converge.

The auxiliary field method for the corrected model will be presented by neglecting all the mixing-type interactions $G_{i \neq j}$ that are considerably smaller than the diagonal ones. The corresponding unit integral of the auxiliary fields can be written as

$$\begin{aligned}
 1 = & N \int D[S_i] D[P_i] \\
 & \times e^{-\frac{i}{2G_{ij}} \int_x [(S_i - G_{ik} J_k^s)(S_j - G_{jm} J_m^s) + (P_i - G_{ik} J_k^p)(P_j - G_{jm} J_m^p)]}, \quad (29)
 \end{aligned}$$

where $G_{ij} = G_{ij}^s = G_0 \delta_{ij} + \bar{G}_{ij}^s(B)$. The resulting gap equations for the scalar fields in the fundamental representation can be written as

$$M_f^* = m_f - 2G_{ff}^s \langle \bar{\psi}_f \psi_f \rangle, \quad (30)$$

where the relation of G_{ij} with G_{ff} and of S_i with S_f are presented in Appendix B. The resulting BSE for the neutral and charged pion and kaon can be written as

$$\pi^0: \quad 1 - \frac{G_{33}}{2} [\Pi_{\text{ps}}^{uu}(P^2 = M_{\pi^0}^2) + \Pi_{\text{ps}}^{dd}(P^2 = M_{\pi^0}^2)] = 0, \quad (31)$$

$$K^0: \quad 1 - G_{66} \Pi_{\text{ps}}^{ds}(P^2 = M_{K^0}^2) = 0. \quad (32)$$

To provide a more strict comparison with lattice calculations, below we also present the pion mass calculated separately with $\bar{u}u$ structure or $\bar{d}d$ structure. In these cases, the coupling constant G_{33} was also redefined accordingly, as obtained in Appendix B in Eq. (B1c). It yields

$$\begin{aligned} \pi^{\bar{u}u}: \quad & 1 - \frac{G_{uu}}{2} \Pi_{\text{ps}}^{uu}(P^2 = M_{\pi^0}^2) = 0, \\ \pi^{\bar{d}d}: \quad & 1 - \frac{G_{dd}}{2} \Pi_{\text{ps}}^{dd}(P^2 = M_{\pi^0}^2) = 0. \end{aligned} \quad (33)$$

A. Mixing angles

The mixing type interactions $G_{08}(B)$ give rise to the η, η' mesons mixings. This mixing emerges already by considering the contribution of nondegenerate quark masses $M_u \neq M_s$ [35,36], and we will present exclusively the effect of the magnetic field on the mixing angle.

For this the auxiliary fields must be introduced in such a way to account for the mixing interactions. This will be done by means of functional delta functions in the generating functional [69,70] such that in the limit of zero mixing previous results are reproduced. This functional delta function can be written as

$$1 = \int D[P_i] \delta(P_i - G_{ik} j_{\text{ps}}^k), \quad (34)$$

where $j_{\text{ps}}^k = \bar{\psi} \lambda^k i \gamma_5 \psi$, $i, k = 0, 3, 8$ provides components that produce the mesons η, η' , and π_0 . Eventual nonfactorizations [71] may be expected to be small. Consider the following pseudoscalar auxiliary fields quadratic terms:

$$\begin{aligned} \mathcal{L}_{\text{mix}} = & -\frac{M_{88}^2(B)}{2} P_8^2 - \frac{M_{00}^2(B)}{2} P_0^2 \\ & + 2G_{08}(B) \bar{G}_{08} P_0 P_8 + \mathcal{O}(P_3, P_3^2), \dots \end{aligned} \quad (35)$$

where M_{ii}^2 include the contributions from $G_{i=j}$ derived above, and

$$\bar{G}_{08}(B) = \frac{2}{G_{00}(B)(G_{88}(B) - \frac{G_{08}(B)^2}{G_{00}(B)}}). \quad (36)$$

The flavor dependent coupling constants $G_{ij} \propto N_c$, as $N_c \rightarrow \infty$, η and η' become degenerate [72].

A change of basis state can be done to the mass eigenstates η, η' by starting from the singlet flavor states basis with $|\bar{q}q\rangle$ ($q = u, d, s$), P_3, P_8, P_0 . By neglecting the neutral pion mixings, according to the convention from [55], it can be written

$$\begin{aligned} |\eta\rangle &= \cos \theta_{\text{ps}} |P_8\rangle - \sin \theta_{\text{ps}} |P_0\rangle, \\ |\eta'\rangle &= \sin \theta_{\text{ps}} |P_8\rangle + \cos \theta_{\text{ps}} |P_0\rangle. \end{aligned} \quad (37)$$

To describe completely both masses, η, η' one needs two parameters/angles [73]; however, in this work only the mass difference will be considered. By rewriting \mathcal{L}_{mix} in this mass eigenstates basis, the following magnetic field induced deviation of the $\eta - \eta'$ mixing angle is obtained:

$$\Delta \theta_{\text{ps}} = \theta_{\text{ps}}(B) - \theta_{\text{ps}}(B=0) = \frac{1}{2} \arcsin \left(\frac{4G_{08}(B) \bar{G}_{08}(B)}{(M_{\eta}^2 - M_{\eta'}^2)} \right). \quad (38)$$

Let us consider $M_{\eta}(B=0) = 548$ MeV and $M_{\eta'}(B=0) = 958$ MeV [55]. Some results with the NJL method suggest some difference in the magnetic field dependence of $M_{\eta}(B)$ and $M_{\eta'}(B)$ [49]. So we will present an estimation for the mixing angle as a function of the magnetic field by assuming the following different behaviors for the $\eta - \eta'$ mass difference (Δ_B):

$$\begin{aligned} D_1 \equiv \Delta_B^{(1)} &= \sqrt{(M_{\eta}^2(B) - M_{\eta'}^2(B))} \\ &\simeq 786 \text{ MeV} \quad \text{is a constant,} \\ D_2 \equiv \Delta_B^{(2)} &= \sqrt{(M_{\eta}^2(B) - M_{\eta'}^2(B))} \\ &\times \left(1 + \frac{eB}{b_0} \right), \quad \text{for } b_0 = 2 \text{ GeV}^2, \\ D_3 \equiv \Delta_B^{(3)} &= \sqrt{(M_{\eta}^2(B) - M_{\eta'}^2(B))} \\ &\times \left(1 + \frac{eB}{b_0} \right), \quad \text{for } b_0 = 2 \text{ GeV}^2. \end{aligned} \quad (39)$$

IV. NUMERICAL RESULTS

The result of the fitting procedure to fix the parameters of the model with the coupling constant G_0 is presented in Table I, where these were the parameters found to reproduce $m_{\pi} = 135.0$ MeV, $m_K = 498.0$ MeV, $f_{\pi} = 93.0$ MeV, and $f_K = 111.0$ MeV at $B = 0$. It is interesting to note that the current quark masses fixed by the fitting procedure are, as usual, somewhat different than the measured values in Particle Data Group (PDG) tables [55]. In the NJL method the Lagrangian quark masses are free parameters, and therefore they are free to be varied. However, it turns

TABLE I. Set of parameters was fixed to describe correctly neutral pion and kaon masses and decay constants in the vacuum ($B = 0$).

Parameters	Set
Λ	914.6 MeV
G_0	9.76 GeV ⁻²
m_{ud}	6.0 MeV
m_s	165.7 MeV

out that the needed values for these parameters are close to the measured values in PDG and this can be seen as a feature of the model. The difference with respect to the values of PDG can be attributed to, at least, two issues that might be connected. First, the current quark masses in [55] are fixed with respect to an energy scale of the Standard Model and a different energy scale may be more suitable for the dynamics of hadrons within the NJL. Second, the mass generation mechanism in the NJL model involves the solution of (transcendental) gap equations for which the current quark masses contribute nonlinearly, and, in this process, it might need a larger current quark mass that somehow is modified by the presence of the quark condensate.

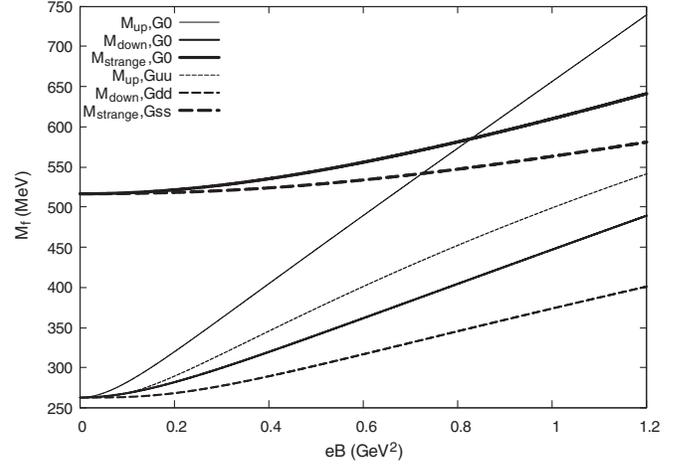
The effect of the derived magnetic field dependence of the NJL coupling constant, Eqs. (18) and (19), will be compared to the effect of parametrizations considered in the literature. For instance, the following two shapes will be considered below [43,74]

$$G_1(eB) = \alpha + \beta e^{-\gamma(eB)^2}, \quad (40)$$

$$G_2(eB) = G_0 \left(\frac{1 + a\xi^2 + b\xi^3}{1 + c\xi^2 + d\xi^4} \right), \quad (41)$$

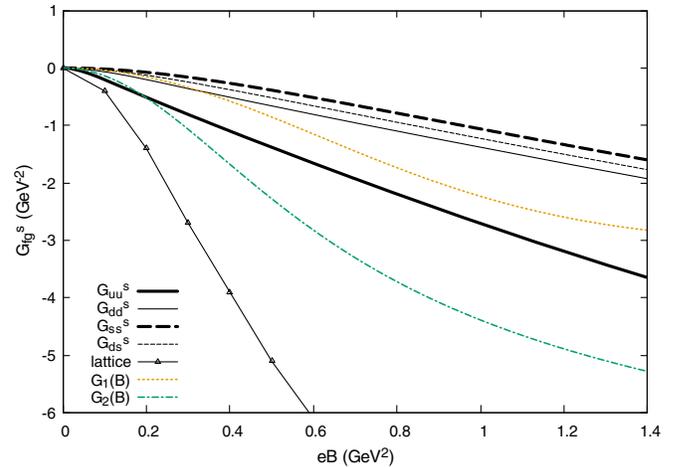
where $\alpha = 6.70 \text{ GeV}^{-2}$ and $\beta = 3.06 \text{ GeV}^{-2}$ such that $G_2(0) = G_0$, and $\gamma = 1.31 \text{ GeV}^{-2}$ —being that β and γ have the same values as used in Ref. [43]. G_0 is normalized by the value presented in the Table I, $\Lambda_{\text{QCD}} = 300 \text{ MeV}$, $\xi = (eB)/\Lambda_{\text{QCD}}^2$, and $a = 0.01088$, $b = -1.0133 \times 10^{-4}$, $c = 0.02228$, and $d = 1.84558 \times 10^{-4}$ [74].

The quark effective masses, as solutions of the first gap equations for G_0 , Eq. (10), and for $G_{ff}^s(B)$, the second gap equation Eq. (30), are presented in Fig. 1. The corrected gap equations, with G_{ff}^s , are solved self-consistently with the coupling constants $G_{ij}^s(B)$. The magnetic field corrections for the scalar coupling constants, which contribute in the corrected gap equations, are shown in the following figure. It is seen that the effect of the magnetic field corrections to the coupling constants in the gap equations is to reduce the effective masses. The deviation with respect to the solution of the gap equations with G_0 is progressively larger for larger magnetic fields. The largest deviation in the effective mass is obtained for the up quark effective mass


 FIG. 1. Effective quark masses as solutions of the gap equations (10) with G_0 (solid lines) and (30) with G_{ff}^s (dashed lines). Thicker (thinner) lines for M_s^* (M_u^*) and intermediary thickness for M_d^* .

and independent of corresponding sign of the quark electric charge.

The resulting magnetic field dependencies of some of the scalar and pseudoscalar coupling constants, $G_{uu}^{s,ps}(B)$, $G_{dd}^{s,ps}(B)$, $G_{ss}^{s,ps}(B)$, and $G_{ds}^{s,ps}(B)$, are shown in Figs. 2 and 3 for the set of parameters above. The pseudoscalar magnetic field corrections are positive and they increase with the magnetic field, whereas the corrections to the scalar coupling constant are negative and decrease with the magnetic field. Together with the scalar coupling constants in Fig. 2, the following are also presented: the parametrizations of Eqs. (40) and (41) from Refs. [43,74],


 FIG. 2. Magnetic field correction to the scalar NJL coupling constant, $G_{ff}^s(eB)$, as functions of the magnetic field. The parametrizations of Eqs. (40) and (41) from Refs. [43,74], respectively, in dotted (yellow) and dot-dashed (green) lines—and extrapolation $G(B)$ from lattice QCD of Ref. [38] in a continuous line with triangles are also shown.

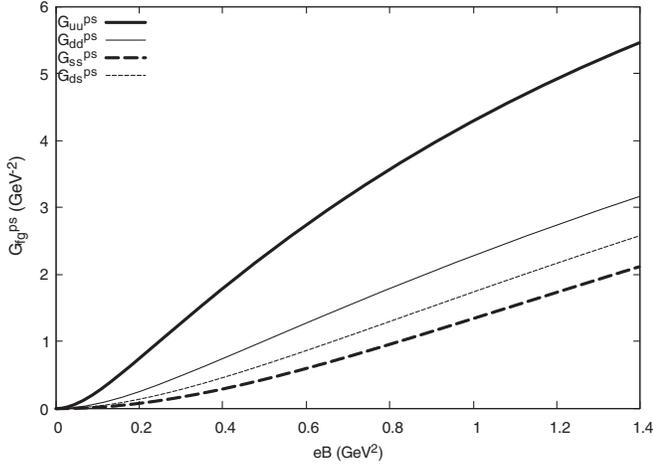


FIG. 3. Magnetic field correction to the pseudoscalar NJL coupling constant, $G_{ff}^{ps}(eB)$, as functions of the magnetic field.

respectively, in dotted (yellow) and dot-dashed (green) lines—and a set of points for a particular definition from lattice QCD to make contact with the NJL model from Ref. [38].

The resulting up and down quark-antiquark chiral scalar condensates will be compared to results from lattice QCD from Ref. [63] by means of their average and their difference. To do this comparison, we define the following quantities:

$$\Sigma_f(B) = \frac{2m_{ud}}{m_\pi^2 f_\pi^2} |\langle \bar{\psi}_f \psi_f \rangle_B - \langle \bar{\psi}_f \psi_f \rangle_{B=0}| + 1, \quad f = u, d, s, \quad (42)$$

where m_π and f_π are the zero magnetic field pion mass and decay constant, respectively, here taken as $m_\pi = 135$ MeV and $f_\pi = 86$ MeV. In Fig. 4 the magnetic field dependent

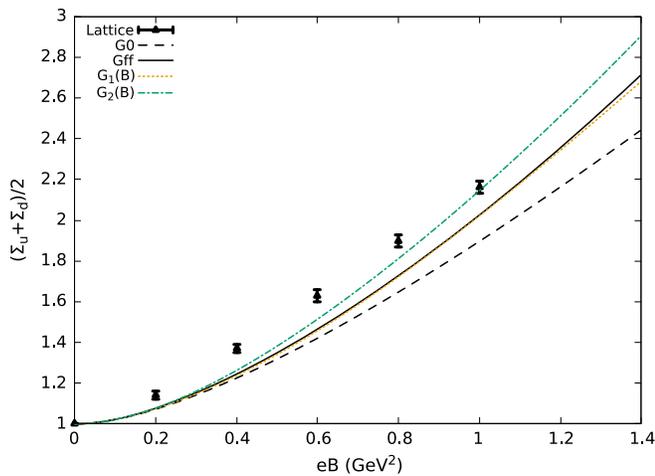


FIG. 4. Magnetic field dependent part of the averaged up and down quark condensates, $(\Sigma_u + \Sigma_d)/2$ for the set of parameters shown above by using G_0 and G_{ff} , Eqs. (10) and (30), and also points from lattice QCD from Ref. [63].

part of the average of the up and down quark condensates $(\Sigma_u + \Sigma_d)(B)/2$, without the vacuum contribution, is shown as a function of the magnetic field. The curves present a comparison for the results obtained with solutions of the two gap equations, namely for the coupling constant G_0 and for coupling constants G_{ff}^s , respectively, Eqs. (10) and (30). Two different lattice calculations [51,75] for these chiral condensates present the same behavior with a nearly linear behavior with the magnetic field for stronger magnetic fields [75]. Points obtained from lattice calculation from Ref. [63]. Besides that, estimations with two different parametrizations for the magnetic field dependence of the NJL coupling constant, Eqs. (40) and (41), are presented. The curve for the magnetic field dependent coupling constant from polarization $G_{ff}(B)$ is basically the same as the one from $G_1(B)$.

In Fig. 5 the magnetic field dependent part of the difference between the up and down quark condensates $(\Sigma_u - \Sigma_d)(B)$, without the vacuum contribution, is shown as a function of the magnetic field for the same cases presented for the previous figure including the lattice results from Ref. [63].

It is interesting that there is an improvement of the difference between the up and down quark condensates due to the use of $G_{ff}^s(B)$ with respect to the use of G_0 for the regime of weak magnetic field, although overall there is no systematic behavior. Points obtained from lattice calculation from Ref. [63] are also shown, being in agreement other estimations [75]. Also, estimations with two different parametrizations for the magnetic field dependence of the NJL coupling constant, Eqs. (40) and (41), are presented. The resulting curve for the coupling constant $G_{ff}(B)$ is between the curves for $G_1(B)$ and $G_2(B)$.

In Fig. 6 the magnetic field dependence of the strange quark-antiquark condensate by means of the quantities (42)

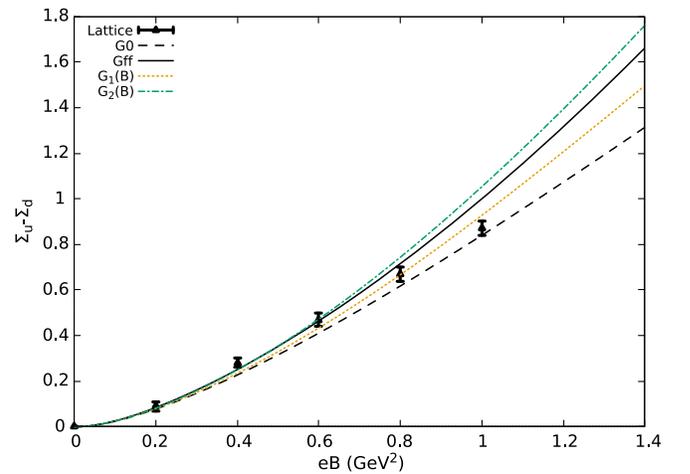


FIG. 5. Magnetic field dependent part of the difference between the up and down quark condensates, $(\Sigma_u - \Sigma_d)(B)$ for the set of parameters shown above, by using G_0 and G_{ff} , Eqs. (10) and (30), and also points from lattice QCD from Ref. [63].

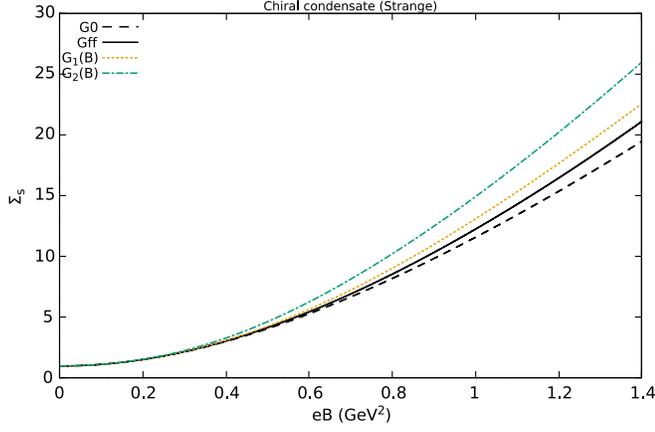


FIG. 6. Magnetic field dependent part of the strange quark condensate, $\Sigma_s(B)$ for the solutions of the gap equation with G_0 (dashed), Eq. (10), and G_{ss} (solid), Eq. (30). Results by considering parametrizations (40) and (41), respectively, dotted (yellow) and dot dashed (green), are also shown.

is exhibited as functions of the magnetic field. The case in which gap equation is solved with G_0 (dashed lines), and also the case in which the magnetic field dependent coupling constant is used, $G_{ff}^s(B)$ (solid lines), are shown. For the sake of comparison, results for the two parametrizations (40) and (41), respectively, dotted (yellow) and dot dashed (green), are also exhibited. The magnetic field dependent coupling constant increases the quark condensates, mostly because the effective masses are slightly reduced as shown in the previous figures. The parametrization (41) yields stronger enhancement due to the magnetic field and parametrization (40) yields results nearly compatible with the calculation with the magnetic field dependent coupling constants G_s^{fg} .

The magnetic field dependence of the three definitions of neutral pion mass are presented in Figs. 7–9—respectively, for π^0 (complete pion state), π^{uu} , and π^{dd} —see Eqs. (31)–(33). The magnetic field behavior of the pseudoscalar coupling constants does not lead to magnetic field behavior of the neutral pion and kaon masses compatible with lattice estimations for strong magnetic fields. This can be understood by analyzing the very different behavior of $G^s(eB)$ and $G^{ps}(eB)$, the former is a decreasing function of the magnetic field and the second an increasing function. Therefore whereas $G^s(eB)$ yields a neutral pion mass that decreases with eB , $G^{ps}(eB)$ yields an increasing neutral pion mass, as shown below. To make possible a more detailed comparison of the effect of the magnetic field dependent coupling constants the pion mass was calculated in different ways: First, for the effective mass from the gap equations [$G(G_0)$] and BSE [$B(G_0)$] by considering G_0 . Second, for the gap equations (G) with $G_{ff}^s(B)$ and BSE (B) with $G_{33}^s(B)$. The same result obtained by considering $G_{ff}^{ps}(B)$ was also plotted. The two parametrizations of Eqs. (40) and (41) were also employed. Besides that, in

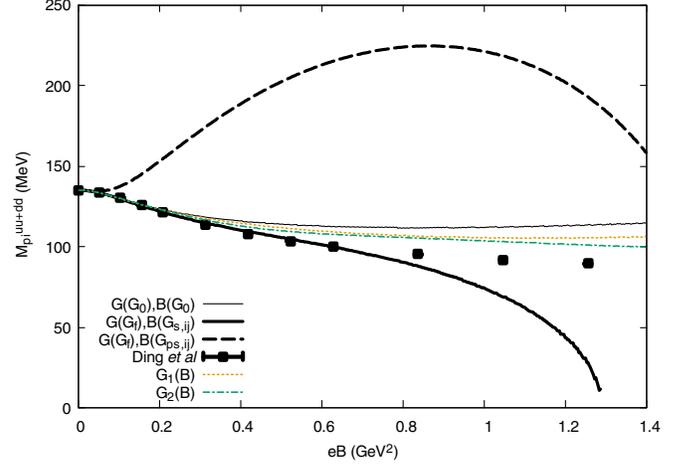


FIG. 7. Neutral pion masses (complete state) for the different cases discussed in the text compared with lattice results from Ref. [51].

most lattice QCD the pion masses have been calculated for the separated states π^{uu} and π^{dd} as discussed for Eq. (33). Lattice QCD results from Refs. [37,51] are also shown in the figures below. The magnetic field coupling constants reduce the values of the pion mass in all cases. The pion mass decreases also because of the behavior of the quark effective masses. The considerable role of magnetic field coupling constant for the complete neutral pion state mass (thick solid line), which reduces the pion mass with respect to the unique coupling constant G_0 (thin solid line) in Fig. 7, is noted. However, due to the nonlinearity of the BSE and to the behavior of the quark effective masses with eB , the pion mass (complete state) drops too fast for magnetic fields stronger than $eB \sim 0.65$ GeV². Eventually the pion mass go to zero by $eB \sim 1.3$ GeV², and there is no more solution for the corresponding BSE. For the cases of π^{uu} and π^{dd} states, exhibited in Figs. 8 and 9, respectively,

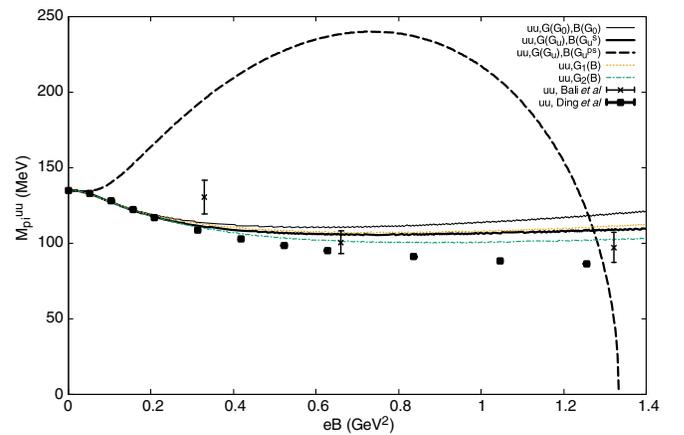


FIG. 8. Neutral pion masses π^{uu} for the different cases discussed in the text compared with lattice results from Refs. [37,51].

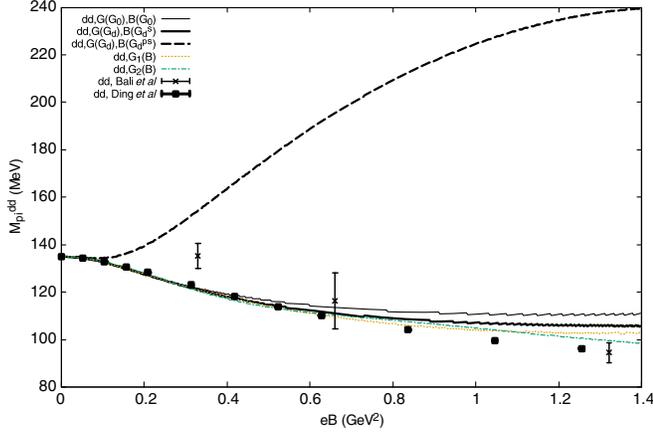


FIG. 9. Neutral pion masses π^{dd} for the different cases discussed in the text compared with lattice results from Refs. [37,51].

for lower values of magnetic fields the magnetic field dependent scalar $G_{ij}^s(eB)$ improves the agreement with lattice QCD. However the magnetic field dependent coupling constants $G_{ff}^s(B)$ are not enough to reproduce lattice QCD data for quite strong magnetic fields, nearly at the same point for the complete pion state and for the $\bar{u}u$ or $\bar{d}d$ states. Note that the BSE for the complete neutral pion state is not consistent with an assumption such that the complete neutral pion mass would be the average of the $\bar{u}u$ and $\bar{d}d$ states. When comparing the BSE Eqs. (31) and (33) for the complete and $\bar{u}u/\bar{d}d$ pion states, it can be seen that G_{33} is an averaged of G_{uu} and G_{dd} , and also $\Pi^{\text{complete}}(P^2 = M_\pi^2)$ is an average of the separated polarization tensors for u and d quarks. Since all the polarization tensors are nonlinear functions of the pion mass (in the limit of zero pion three-momentum), it turns out that the two averages taken to compute the complete pion mass in $G_{33}\Pi(M_\pi^2)$ varies considerably faster than the separated quantities $G_{uu}\Pi_{uu}(M_\pi^2)$ and $G_{dd}\Pi_{dd}(M_\pi^2)$. These behaviors lead to the unexpected faster variation of the complete pion mass with the magnetic field. Of course the separated dependencies of all the three polarization tensors on eB and on $P^2 = M_\pi^2$ produce this unexpected behavior. However further investigation is seemingly needed to certify, first of all, that different lattice calculations provide results in agreement with each other.

In Fig. 10 the magnetic field dependence of the neutral kaon mass is presented for the different cases discussed above: by using gap equations and BSE with G_0 (thin solid line) and gap equations with G_{ff}^s and BSE with G_{66}^s, G_{77}^s (thick solid line) and also $G_{66}^{\text{ps}}, G_{77}^{\text{ps}}$ (dashed line). The parametrizations (40) and (41) were also used, respectively, dotted (yellow) and dot-dashed (green) lines. The pseudo-scalar coupling constant $G_{ij}^{\text{ps}}(eB)$ does not make neutral kaon mass to increase, as it happens in the neutral pion case, although it makes results worsen when compared to

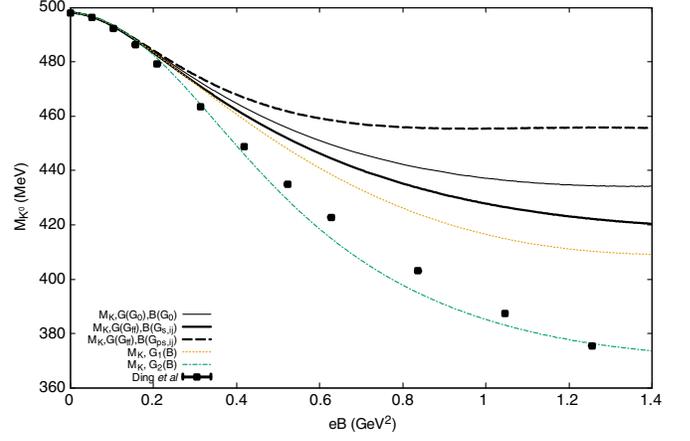


FIG. 10. Neutral kaon masses for the following cases: gap equations and BSE with G_0 (thin line), and gap equations and BSE, respectively, with G_{ff}^s and G_{66}^s (thick line). The use of G_{66}^{ps} is also considered (dashed line). By using $G_1(B)$ and $G_2(B)$ results are exhibited, respectively, in dotted yellow and dot-dashed green lines.

results with G_0 . It is seen that the magnetic field deviation due to the magnetic field dependent coupling constant is not enough to reproduce lattice QCD results although it improves agreement when compared with results obtained with G_0 .

In Fig. 11 the deviation of the $\eta - \eta'$ mixing angle due to the magnetic field, Eq. (38), is presented for three different *ad hoc* prescriptions for the behavior of the $\eta - \eta'$ mass difference with the magnetic field shown in Eq. (39). Again the coupling constants G_{ij}^s were used. The decrease of the $\eta - \eta'$ mass difference, D_3 , contributes for a further increase of the modulus of the mixing angle that is favored by an increase of the coupling constant $G_{08}(B)$ with the magnetic field. The magnetic field dependencies of $G_{00}(B)$ and $G_{88}(B)$ are less relevant than $G_{08}(B)$ for the resulting

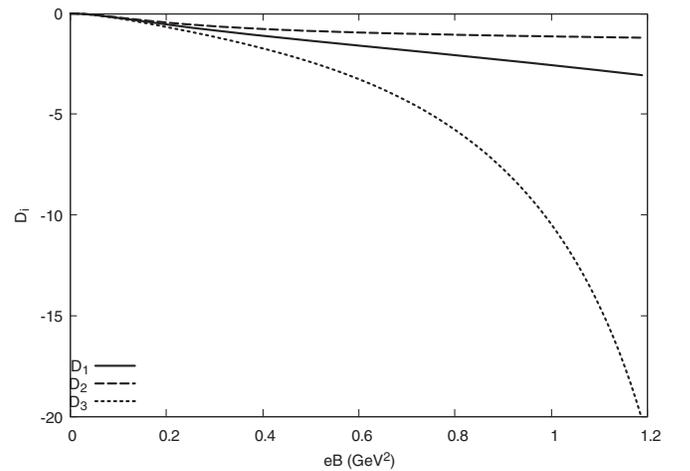


FIG. 11. Magnetic field induced deviation for the $\eta - \eta'$ mixing angle given by Eq. (38) and the three prescriptions for the $\eta - \eta'$ mass differences of Eq. (39).

mixing angle. Results with the use of prescription D_3 are more sensitive to the magnetic field because D_3 considers a reduction of the mass different with the magnetic field in the argument of the arcsin in Eq. (38).

V. SUMMARY AND DISCUSSION

Effects of quark polarization in a constant background magnetic field on the NJL-coupling constant were analyzed firstly in the resulting gap equations, and therefore in the quark-antiquark chiral condensates, and mass generation for constituent quarks. Secondly their effects were analyzed in the BSE for the neutral pion and kaon masses and the $\eta - \eta'$ mixing angle. The one loop level calculation under magnetic field breaks chiral and flavor symmetries inducing different contributions for the scalar and pseudoscalar channels and flavor dependency of the coupling constants. Besides the diagonal coupling constants G_{ii} , mixing type interactions $G_{i\neq j}$ (for $i, j = 0, 3, 8$) also emerge and they contribute to neutral mesons mixings. These mixing interactions have two sources: the magnetic field coupling to quarks and the nondegenerate quark masses; this second effect was also analyzed separately in Ref. [35]. The resulting mixing-type interactions are proportional to the different quark mass differences, $\propto (M_f - M_g)$ and $(M_f^2 - M_g^2)$ for $f \neq g = u, d, s$, and they were mostly considered for an estimate of the magnetic field correction to the $\eta - \eta'$ mixing angle. The magnetic field dependence of the up and down quark-antiquark condensates from the gap equations depend on the scalar coupling constants, $G_0 + G_{ii}^s$, and these results can be said to be slightly improved with respect to results available from lattice QCD calculations although the averaged value may be well reproduced. It indicates, however, that further flavor or magnetic field dependencies of parameters may be needed mainly to reproduce correctly the lattice results for the difference of the up and down quark condensates. The strange quark-antiquark condensate also receives corrections.

Although the corrected scalar coupling constants have a magnetic field dependence with nearly the same behavior of the coupling constant behavior needed to reproduce lattice QCD results, the corrected pseudoscalar coupling constants in this one loop fermion calculation, $G_0 + G_{ii}^{\text{ps}}$, has the opposite magnetic field dependence and they do not lead to results with the behavior found in lattice QCD results. Therefore the pseudoscalar coupling constants were not employed extensively for calculating observables. This suggests that there may have a further different mechanism in the pseudoscalar channel that could generate a strongly decreasing behavior for $G_{ij}^{\text{ps}}(B)$ that should compensate the behavior obtained from polarization process. Therefore, by simply adopting the scalar coupling constant to compute the neutral pion bound states, results receive corrections that somewhat improve the agreement with data from lattice QCD. This comparison presents some subtleties because

lattice QCD calculations have few points for finite magnetic field and they provided neutral pion mass mostly for separated $\bar{u}u$ or $\bar{d}d$ structures. Therefore, to make possible a more detailed comparison among different calculations, we presented calculations for the complete neutral pion state mass and for the $\bar{u}u$ or $\bar{d}d$ states. Neutral pion mass as calculated for G_0 and for the separated states $\bar{u}u$ or $\bar{d}d$ present a similar behavior: for lower magnetic fields there is a decrease of the masses and NJL-predictions yield, for $eB \geq 0.5\text{--}0.9 \text{ GeV}^2$, an increase of masses. A different behavior is obtained for the complete neutral pion structure for $G_{ij}^s(B)$ with a continuous decrease of its mass until there is no more solution for the neutral pion BSE around $eB \sim 1.3 \text{ GeV}^2$. Note that the complete neutral pion mass is not an average of the masses of states $\bar{u}u$ and $\bar{d}d$ because of the nonlinearity of the BSE but also due to the different up and down quark effective masses. It is interesting to emphasize that, whereas the current NJL predictions for the up and down quark condensates are improved with respect to the standard NJL method, the results for the neutral mesons masses need further physical input in their BSE.

The neutral kaon mass calculated either with G_0 or with G_{ij}^s provide decreasing values with eB although the magnetic field dependent coupling constants provide stronger decrease. By $eB_0 \sim 1.0 \text{ GeV}^2$, the difference between the two estimates is of the order of $M_{K^0}(G_0) - M_{K^0}(G_{ij}(B)) \sim 10 \text{ MeV}$, and larger for stronger magnetic fields. Finally estimates for the magnetic field dependence of the $\eta - \eta'$ mixing angle were provided by considering the mixing type interaction $G_{08}(B)$ according to Refs. [35,36]. As shown in Appendix B, $G_{08}(B) \sim G_{uu} + G_{dd} - 2G_{ss}$ is proportional to the up/down-strange quark effective mass nondegeneracy. For the $\eta - \eta'$ mixing angle, different behaviors of the magnetic field dependence of the mass difference $M_{\eta} - M_{\eta'}(B)$ were considered.

These results suggest that the present magnetic field corrections for the NJL coupling constant from quark polarization might be enough to describe results for the neutral pion mass from lattice QCD for not strong magnetic fields, i.e., $eB \lesssim 0.2, 0.4, \text{ or } 0.6 \text{ GeV}^2$, depending on the definition of the pion structure according to Figs. 7–9. Neutral kaon masses are also well reproduced for still weaker magnetic fields. The higher order polarization corrections should not provide large contributions because they are suppressed by $1/M^{*n}$ ($n \geq 2$). Therefore, further magnetic field dependencies might be needed for realistic predictions of the NJL model. Further comparisons of NJL predictions with first principles lattice QCD results will make possible to understand better, and eventually to improve, the predictive power of the model under finite magnetic fields. For that it is also important to provide further lattice calculations. Nevertheless, with calculations presented in this work, it is possible to identify how the NJL degrees of freedom exclusively come into play for the corresponding hadron observables under finite magnetic

fields. This procedure should help to disentangle somewhat both the understanding of hadron dynamics in terms of the fundamental degrees of freedom and in terms of hadron effective (and observable) degrees of freedom by trying to relate both levels of the description. Maybe this type of comparisons also might eventually help to conclude further which “sector” of QCD dynamics is at work for each observable under these external conditions.

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APPENDIX A: QUARK PROPAGATOR IN A CONSTANT MAGNETIC FIELD

By considering the proper time, representation for the quark propagator with the minimal coupling to the photon field is given by

$$S_0(x, y) = \Phi(x, y)S_0(x - y), \quad (\text{A1})$$

where

$$\Phi(x, y) \equiv \exp \left\{ iq \int_y^x d\xi^\mu \left[A_\mu(\xi) + \frac{1}{2} F_{\mu\nu}(\xi - y)^\nu \right] \right\} \quad (\text{A2})$$

is the Schwinger phase factor, which is explicitly gauge dependent and breaks the translation invariance of the propagator, and

$$\begin{aligned} S_0(x - y) &\equiv -(4\pi)^{-2} \int_0^\infty \frac{ds}{s^2} \left[m + \frac{1}{2} \gamma \cdot [q\mathbf{F} \coth(q\mathbf{F}s) + q\mathbf{F}](\mathbf{x} - \mathbf{y}) \right] \\ &\times \exp \left\{ -im^2 s - \frac{1}{2} \text{tr} \ln [(q\mathbf{F}s)^{-1} \sinh(q\mathbf{F}s)] \right\} \\ &\times \exp \left[-\frac{i}{4} (\mathbf{x} - \mathbf{y})^T q\mathbf{F} \coth(q\mathbf{F}s)(\mathbf{x} - \mathbf{y}) + \frac{i}{2} q\sigma_{\mu\nu} F^{\mu\nu} s \right] \end{aligned} \quad (\text{A3})$$

is the translational invariant term. Here the quark electric charge is denoted by q while m stands for its mass. The photon field strength tensor is denoted by $F^{\mu\nu}$ and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$.

Now we consider the case in which the photon field correspond to a constant magnetic field along the \hat{z} direction, $\vec{B} = B\hat{e}_z$, such that $F_{12} = B$. In this case, the translational invariant propagator becomes

$$\begin{aligned} S_0(x - y) &= -(4\pi)^{-2} \int_0^\infty \frac{ds}{s^2} \frac{|qB|s}{\sin(|qB|s)} \exp(-im^2 s + i\text{sign}(qB)|qB|s\sigma_3) \\ &\times \exp \left\{ -\frac{i}{4s} [(x - y)_\parallel^2 - |qB|s \cot(|qB|s)(x - y)_\perp^2] \right\} \\ &\times \left\{ m + \frac{1}{2s} \left[\gamma \cdot (x - y)_\parallel - \frac{|qB|s}{\sin(|qB|s)} \gamma \cdot (x - y)_\perp e^{-i\text{sign}(qB)|qB|s\sigma_3} \right] \right\}, \end{aligned} \quad (\text{A4})$$

where $\text{sign}(x)$ is the sign function and, for two arbitrary four-vectors a^μ and b^μ , we are denoting

$$\begin{aligned} (a \cdot b)_\parallel &= a^0 b^0 - a^3 b^3, \\ (a \cdot b)_\perp &= a^1 b^1 + a^2 b^2. \end{aligned}$$

The Fourier transformation of Eq. (A4) is found to be given by

$$\begin{aligned} S_0(p) &= -i \int_0^\infty ds \exp \left\{ -is \left[m^2 - p_\parallel^2 + \frac{\tan(|qB|s)}{|qB|s} p_\perp^2 \right] \right\} \\ &\times \{ [1 - \text{sign}(qB)\gamma_1\gamma_2 \tan(|qB|s)] \\ &\times (m + \gamma \cdot p_\parallel) - \gamma \cdot p_\perp [1 + \tan^2(|qB|s)] \}. \end{aligned} \quad (\text{A5})$$

APPENDIX B: COUPLING CONSTANTS IN DIFFERENT FLAVOR BASIS AND INTEGRALS

The coupling constants of the NJL interaction in the adjoint representation relates to the ones in the fundamental representation by

$$G^{00} = \frac{1}{3} [G_{uu}(B) + G_{dd}(B) + G_{ss}(B)], \quad (\text{B1a})$$

$$G^{11}(B) = G^{22}(B) = G_{ud}(B), \quad (\text{B1b})$$

$$G^{33}(B) = \frac{1}{2} [G_{uu}(B) + G_{dd}(B)], \quad (\text{B1c})$$

$$G^{44}(B) = G^{55}(B) = G_{us}(B), \quad (\text{B1d})$$

$$G^{66}(B) = G^{77}(B) = G_{ds}(B), \quad (\text{B1e})$$

$$G^{88}(B) = \frac{1}{6}[G_{uu}(B) + G_{dd}(B) + 4G_{ss}(B)], \quad (\text{B1f})$$

$$G^{03}(B) = G^{30}(B) = \frac{1}{\sqrt{6}}[G_{uu}(B) - G_{dd}(B)], \quad (\text{B1g})$$

$$G^{08}(B) = G^{80}(B) = \frac{1}{3\sqrt{2}}[G_{uu}(B) + G_{dd}(B) - 2G_{ss}(B)], \quad (\text{B1h})$$

$$G^{38}(B) = G^{83}(B) = \frac{1}{2\sqrt{3}}[G_{uu}(B) - G_{dd}(B)], \quad (\text{B1i})$$

both for scalar and pseudoscalar interactions. All the other couplings G^{ij} vanish. Here we are denoting

$$G_{fg}^s(B) = g + g^2\Pi_{fg}^s(B), \quad (\text{B2a})$$

$$G_{fg}^{\text{ps}}(B) = g + g^2\Pi_{fg}^{\text{ps}}(B), \quad (\text{B2b})$$

where

$$\Pi_{fg}^s(B) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \text{tr}_D[S_f^B(p)S_g^B(p)], \quad (\text{B3a})$$

$$\Pi_{fg}^{\text{ps}}(B) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \text{tr}_D[S_f^B(p)i\gamma_5 S_g^B(p)i\gamma_5], \quad (\text{B3b})$$

with $S_f^B(p)$ representing the quark propagator in momentum space in the presence of the uniform magnetic field B and tr_D representing the trace over Dirac indices.

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