Inferring the nature of active neutrinos: Dirac or Majorana?

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The nature of a neutrino, whether it is a Dirac type or Majorana type, may be comprehensively probed using their quantum statistical properties. If the neutrino is a Majorana fermion, then by definition it is identical and indistinguishable from the corresponding antineutrino. When a Majorana neutrino and antineutrino are pair produced, the corresponding state has to obey the Pauli principle unlike in the Dirac case. We use this property to distinguish between the two cases using the process $B^0 \to \mu^- \mu^+ \nu_\mu \bar{\nu}_\mu$. We show that the two cases differ dramatically in a special kinematic scenario where, in the rest frame of the parent B meson, the muons fly away back-to-back (i.e., fly with 3-momenta of equal magnitudes but opposite directions), and so do the neutrino and antineutrino. Unlike any other scenario, we know the energies and magnitudes of 3-momenta of both the neutrino and the antineutrino in this back-to-back configuration without even directly measuring them. This provides a way of avoiding the constraint imposed by the "practical Dirac-Majorana confusion theorem," as one need not fully integrate over neutrino and antineutrino in this case. As a true signature of the universal principle of quantum statistics which does not depend on the size of the mass of the particle but its spin, the difference between Dirac and Majorana cases in this special kinematic configuration does survive independent of the neutrino mass as long as neutrino mass is nonzero. The analysis presented here is applicable immediately to several other processes with the same final state as in the case of B^0 decay without any major change.

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I. INTRODUCTION

Neutrinos are the most ubiquitous elementary particles after the photon in the universe. Nevertheless they are also one of the least understood in terms of their properties. We know that the active neutrinos in the standard model (SM) come in three flavors: electron neutrino, muon neutrino, and tau neutrino each associated with a corresponding charged lepton. From neutrino oscillation experiments [1,2] it has been established that the neutrinos ν_ℓ with $\ell = e, \mu, \tau$ can oscillate from one flavor to another. This is usually explained by considering the flavor neutrinos as linear combinations of three different neutrino mass eigenstates. The oscillation experiments suggest that at least two of these mass eigenstates must have tiny but nonzero masses, whereas in the SM neutrinos are regarded as massless. As neutrinos are charge neutral and have nonzero mass, they could in principle be their own antiparticles. In that case they are called Majorana fermions [3–5]. Since a Majorana neutrino is quantum mechanically identical to its antiparticle, any state having a Majorana neutrino antineutrino pair must obey the Fermi-Dirac statistics, a fact that is independent of the magnitude of the neutrino mass. This means that the probability amplitude must be totally antisymmetric under exchange. There is no such requirement if neutrinos are of Dirac type. Thus the main difference between the Dirac or Majorana nature of the neutrino arises from its quantum statistical properties. We exploit this connection to construct a novel way of probing the nature of neutrinos.

Such a connection between the statistics and the nature of neutrino and antineutrino has been studied previously by using antisymmetrization of amplitude for final states having Majorana neutrino antineutrino pair. However, the effect of antisymmetrization gets lost when the unobservable neutrino and antineutrino get fully integrated out. This then leads to the "practical Dirac-Majorana confusion theorem" (DMCT) [6,7]. The theorem, which still lacks a rigorous, process

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independent, general proof, as far as we are aware, states that the difference between Dirac and Majorana neutrinos is proportional to some power of the neutrino mass. This poses a challenge since the neutrino masses are not known precisely, except that at least two of them must have nonzero masses as indicated by neutrino oscillation experiments and the masses are very small (< 1 eV) compared to other mass scales in the SM [1,2]. Thus, any proposal conforming to DMCT depends on this tiny neutrino mass and as a result carries this mass uncertainty apart from the probability being small. It is therefore necessary and important to explore whether there are any SM allowed processes that can directly probe the quantum statistics of Majorana neutrinos avoiding this DMCT constraint.

Both experimentally as well as theoretically, an important proposal to probe the Majorana nature of the neutrino is through the neutrinoless double beta decay ($0\nu\beta\beta$) [8–32]. Since the proposal looks at a lepton number violating (LNV) process, it is beyond SM. While there are many ongoing experiments [33–48], there is no conclusive evidence experimentally as yet or from any other LNV decays. Another LNV process, the neutrinoless double-electron capture [49–62] has also been studied experimentally and is yet to be observed. Both these processes involve a single Majorana neutrino as a propagator. One can also consider processes mediated by exchange of a pair of virtual neutrinos, as done in Ref. [63–65], as a way to distinguish Dirac and Majorana neutrinos by observing the resulting potential. In this method DMCT also holds except for distances that are of the same order or larger than the inverse of the unknown neutrino mass. Also the process of coherent scattering of neutrino on nucleus with bremsstrahlung radiation has been explored and shown to be consistent with DMCT [66]. Therefore it is worthwhile exploring other possibilities, especially those that do not involve LNV, or include Majorana neutrino(s) as propagator(s).

In Refs. [67–74] SM allowed process of radiative emission of neutrino pair was considered as a probe of Dirac or Majorana neutrino. Since the final state involves $\nu\bar{\nu}$, statistics was accounted in the Majorana case by explicit antisymmetrization. Earlier, Nieves and Pal [75] analysed the decay $K^+ \to \pi^+\nu\bar{\nu}$ as a test of Majorana neutrinos. They pointed out that while the Dirac case involves both vector and axial vector contribution, the Majorana case involves pure axial vector current. This is due to the explicit antisymmetrization of the final state of two identical particles. In all these cases the difference between Dirac and Majorana type appeared in the different event rates. Crucially, in all the above analyses, the neutrino and antineutrino variables were integrated out since they are not observable. The results for the rates was found to be directly proportional to some power of the neutrino mass as required by DMCT. As far as we know it is only in the analysis by Chhabra and Babu [76] that an effect independent of m_ν has been found between Dirac and Majorana type neutrinos. Chabra and Babu considered the process $e^+e^- \to \nu\bar{\nu}\gamma$. Their result is in conformity with DMCT when they integrate over all the neutrino variables. However, they also point out that the difference between Dirac and Majorana nature can be ascertained independent of the mass of the neutrinos provided their momenta are *not* integrated out.

It is important to note that when one considers massless neutrinos, i.e., $m_\nu = 0$, both the Dirac and Majorana neutrinos can be described as Weyl fermions. The reduction of neutrino degrees of freedom from 4 to 2 for $m_\nu = 0$ is a discrete jump, and not a continuous change. So the massless neutrino is an entirely different species than the massive one even with extremely tiny mass. Therefore, the presumed smooth transitional difference between Majorana and Dirac neutrinos at $m_\nu \to 0$ is only a misperception.

In this paper, we show that the difference between Dirac and Majorana neutrino persists independent of the magnitude of neutrino mass provided the neutrino/antineutrino momenta are either measured directly or indirectly fixed. As shown by Chabra and Babu, this is not in violation of DMCT. We elaborate on this theme in this paper. In particular we consider the decay $B^0(\text{or}\,\bar{B}^0) \to \mu^+\mu^-\nu\bar{\nu}$, for example, and discuss the rates and branching ratios in a chosen kinematic scenario in which we may indirectly discern the $\nu\bar{\nu}$ variables without the need of any explicit observation of the neutrinos which is extremely difficult any way at present. The method may be adopted to many other such processes simply by replacing the appropriate parameters like mass etc. We discuss in detail the dramatic differences between Dirac and Majorana scenarios in differential distributions in such SM allowed processes. *Most importantly this difference mainly involves well-known and measured quantities and is independent of the unknown neutrino mass as long as it is nonzero.* We do not consider massless neutrinos in this paper. Moreover, we would like to emphasize that our work is not dependent on specific details of any neutrino mass generation mechanism.

The paper is organized as follows. In Sec. II, we provide a brief overview of the previous studies using SM allowed processes to put things in perspective. In Sec. II we also lay down the basic issues that we address in this paper. This is followed by Sec. III in which we provide a broad outline of our approach showing the main differences between Dirac and Majorana neutrinos. In Sec. IV, we look at the decay of $B^0 \to \mu^+\mu^-\nu\bar{\nu}$ in detail. In this section we make a case study with its experimental feasibility and future prospects. This is followed by a discussion of other possible decay modes in Sec. V. Finally we conclude in Sec. VI emphasizing the salient features of our approach.

II. A BRIEF OVERVIEW OF PREVIOUS STUDIES

First a note about the convention here: In general a neutrino flavor is denoted by $\nu_\ell = \nu$ with $\ell = e, \mu, \tau$, where we drop the subscript which is already implicit in the process. Same for antineutrinos. When we explicitly denote the Majorana neutrinos in the Feynman diagrams or elsewhere, we use the convention $\nu \equiv \nu_\ell = \bar{\nu} \equiv \bar{\nu}_\ell \equiv \nu^M$.

A. Processes with 2-body final states

As noted earlier the practical Dirac-Majorana confusion theorem (DMCT) states that any difference between Dirac and Majorana neutrinos must vanish in the limit of neutrino mass going to zero. The DMCT was first discussed by Kayser in Ref. [6]. The loop induced process $\gamma^* \to \nu\bar{\nu}$ was discussed. Angular momentum analysis shows that $\nu\bar{\nu}$ final state can exist in any one of the four possible $J = 1$ states: ${}^3S_1$, ${}^3P_1$, ${}^3D_1$ and ${}^1P_1$. In the case of Dirac neutrinos all the four states are possible where as for Majorana neutrinos only the ${}^3P_1$ state is allowed, since this is the only antisymmetric state. This also fixes the parity of the Majorana neutrino relative to the photon while leaving it undetermined in the Dirac case. Using this information it was proposed that the angular distribution of neutrinos in the decay $\psi(J^P = 2^+) \to \nu\bar{\nu}$ could be different for Dirac and Majorana neutrinos. While this has not been realized experimentally, this remains the first application of the quantum statistics apart from proposing DMCT.

A more direct application, instead of a loop induced process, is the tree-level decay $Z^0 \to \nu\bar{\nu}$. This was discussed in Ref. [77]. In the Dirac case both vector and axial-vector currents contribute where as in the Majorana case it is a pure axial vector, due to antisymmetrization taking into account the statistics. The decay width, more appropriately called the missing width, is given by

$$\Gamma(Z^0 \to \nu\bar{\nu}) = \frac{G_F m_Z^3}{12\pi\sqrt{2}} \times \begin{cases} (1-r)(1-4r)^{1/2}, & \text{Dirac} \\ (1-4r)^{3/2}, & \text{Majorana} \end{cases} \tag{1}$$

where $r = (m_\nu/m_Z)^2$ with m_ν, m_Z being the masses of neutrino and Z boson respectively, and G_F denotes the Fermi coupling constant. Thus the difference between Dirac and Majorana cases is directly proportional to r or m_ν^2 as expected from DMCT. Alternatively, one could also study the process $e^+e^- \to \nu\bar{\nu}$ [78]. While spin dependent and spin-independent cross sections for Dirac and Majorana cases show substantial difference near threshold, the results are consistent with DMCT once the spins are summed over. This example comes close to the conclusions of this paper as we shall see later.

B. Processes with 3-body final states

The main difficulty with just $\nu\bar{\nu}$ in the final state is that it cannot be observed; in the case of Z decay this corresponds to the invisible width of the Z boson as the final state cannot be directly observed. One way to improve upon this situation is to look at 3- and 4- body finals states which contain the $\nu\bar{\nu}$ pair. Nieves and Pal [75] analyzed the decay $K^+ \to \pi^+\nu\bar{\nu}$. Because the final state pion is a pseudoscalar, the process still involves only the axial vector current in the Majorana case as in the two body decays. However, the presence of the pion allows for a differential distribution, even after integrating over the $\nu, \bar{\nu}$ variables. Once again, while the rates are different for Dirac and Majorana scenarios, the difference in pion energy distributions is proportional to the neutrino mass in accordance with DMCT. On the other hand, Chabra and Babu in Ref. [76] analysed in detail the scattering process $e^+e^- \to \nu\bar{\nu}\gamma$. Because of the presence of γ in the final state this process has a richer spin structure. Most importantly, it is shown that when there is no integration over $\nu, \bar{\nu}$ variables, the difference between Dirac and Majorana cases does not vanish even if the neutrino mass is set to zero. However, upon integration, the result is proportional to the neutrino mass in accordance with DMCT. This is a clear demonstration of both conformity and an exception to DMCT but suffers from the fact that it is still not possible to observe any neutrino related variables experimentally.

More recently, radiative emission of neutrino pair has been attracting some attention [67–73]. In this proposal, one looks at atomic transition from an excited state to a ground state as in $|es\rangle \to |gs\rangle + \gamma + \nu\bar{\nu}$. The photon energy spectrum is sensitive to the absolute masses of the neutrino mass eigenstates. The Dirac or Majorana cases may be probed by looking at the decay rate near the threshold for neutrino pair production. Since the momenta of neutrino (antineutrino) are integrated out, the difference between the two cases is always proportional to the neutrino mass, again in agreement with DMCT. Complimentary to the studies on radiative emission of neutrino pair in atomic experiments, the authors of Ref. [74] studied the stimulated emission of neutrino pair via the process $e^-\gamma \to e^-\nu\bar{\nu}$. Here also they consider the difference between Dirac and Majorana neutrinos close to the kinematic threshold of pair production and compare the decay rates for the two cases which is extremely small.

C. Summary of results from previous studies

The common features in all of the above studies are the following:

(1) All the processes considered, have a neutrino and an antineutrino in the final state, are SM allowed and do not violate lepton number.
(2) The amplitude is antisymmetrized in the case of Majorana neutrinos as required by statistics.
(3) The "observable" difference between Dirac and Majorana neutrinos is a direct consequence of the antisymmetrization in the Majorana case. It is proportional to the neutrino mass when the neutrino and antineutrino momenta are integrated out.
(4) However, exceptions to DMCT constraint occur under some special conditions, e.g., near kinematic threshold of pair production, when the spin sum is not done, or when the neutrino and antineutrino momenta are not integrated out.

We continue along the theme considered in many of the references cited above and show that the difference between Dirac and Majorana cases may be seen more clearly under certain kinematical conditions especially with 4-body fermion final states in an SM allowed process without lepton number violation.

In particular we choose processes in which we have a final state given by $\mu^+\mu^-\nu_\mu\bar{\nu}_\mu$. Of course, we could have chosen either e^+e^- or $\tau^+\tau^-$ instead of the muon pair. The analysis remains the same though experimentally muon pair production is preferred.

The initial state could be either a symmetric collision of e^-e^+ or decay of some resonance such as neutral B or D mesons or even the SM Higgs, the main criteria being which initial state offers the best ability to measure the total missing 4-momentum of the escaping neutrino and antineutrino pair. In this work we specifically focus on the decay $B^0(\bar{B}^0) \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. Even though ν_μ is strictly not a mass eigenstate, for simplicity we denote its effective mass by m_ν.

III. GENERAL FORMALISM

Consider the SM allowed decay,

$$B^0(p_B) \to \mu^-(p_-)\mu^+(p_+)\bar{\nu}_\mu(p_1)\nu_\mu(p_2),$$

where the corresponding 4-momenta are shown in parentheses. There are various other allowed initial states one could also consider, such as $\bar{B}^0$, D^0, $\bar{D}^0$, or neutral kaons, or even Higgs. The following analysis holds for all such decays with appropriate changes in the form factors or vertex factors as well as the allowed phase space due to the mass of the parent particle. The amplitude for Dirac case is denoted as

$$\mathscr{M}^D = \mathscr{M}(p_1, p_2), \tag{2}$$

where for brevity we have not shown any other dependencies in the amplitude. For Majorana case the amplitude is antisymmetrized with respect to the exchange of p_1, p_2 and is given by,

$$\mathscr{M}^M = \frac{1}{\sqrt{2}}(\mathscr{M}(p_1, p_2) - \mathscr{M}(p_2, p_1)). \tag{3}$$

The difference between amplitude squares for the two cases after summing over final spins is given by

$$|\mathscr{M}^D|^2 - |\mathscr{M}^M|^2 = \frac{1}{2}(\underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} - \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}}) + \underbrace{\mathrm{Re}(\mathscr{M}(p_1, p_2)^*\mathscr{M}(p_2, p_1))}_{\text{Interference term}}. \tag{4}$$

Consistent with the prior studies in the literature as mentioned in Sec. II, we observe the following.

(1) The antisymmetrization in Majorana amplitude gives rise to the three terms: direct, exchange, and interference terms, which are identified in Eq. (4). The Dirac case involves only the direct term.

(2) The interference term is *always* (except for $p_1 = p_2$) directly proportional to m_ν^2 as it involves helicity flips,

$$\mathrm{Re}(\mathscr{M}(p_1, p_2)^*\mathscr{M}(p_2, p_1)) \propto m_\nu^2. \tag{5}$$

(3) Neither the direct nor the exchange terms is proportional to m_ν. In general,

$$\underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} \neq \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}}. \tag{6}$$

The difference between direct and exchange terms is, in general, not proportional to m_ν. However, this difference vanishes after integration over the neutrino momenta, i.e.,

$$\iint \underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} \mathrm{d}^4p_1\mathrm{d}^4p_2 = \iint \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}} \mathrm{d}^4p_1\mathrm{d}^4p_2, \tag{7}$$

since the amplitude squared is symmetric under exchange of p_1, p_2 even though the amplitude is antisymmetric. Therefore,

$$\iint (|\mathscr{M}^D|^2 - |\mathscr{M}^M|^2)\mathrm{d}^4p_1\mathrm{d}^4p_2 = 2\iint \underbrace{\mathrm{Re}(\mathscr{M}(p_1, p_2)^*\mathscr{M}(p_2, p_1))}_{\text{Interference term}}\mathrm{d}^4p_1\mathrm{d}^4p_2 \propto m_\nu^2. \tag{8}$$

This is consistent with DMCT once the integration over neutrino and antineutrino momenta are done.

A. A thought experiment highlighting an exception to DMCT

In order to show that there exist exceptions to DMCT we consider a simple thought experiment for illustration only. Let us assume, for arguments sake, that the 4-momenta of both neutrino and anti-neutrino are individually measured. Consider the special case when neutrino and antineutrino are collinear, i.e., their 4-momenta are equal, $p_1 = p_2$. Due to antisymmetrization the amplitude for Majorana case in Eq. (3) vanishes for such collinear events ($\mathscr{M}^M_{\text{collinear}} = 0$). However, the amplitude for the Dirac case is nonzero, $\mathscr{M}^D_{\text{collinear}} \neq 0$. This is a dramatic illustration of the difference between the Dirac and Majorana cases. Furthermore, as we show later in the specific example of the B decay in Sec. IV, the $\mathscr{M}^D_{\text{collinear}}$ is in fact not proportional to m_ν^2. Hence, the difference between the Dirac and the Majorana cases does not vanish when we neglect terms proportional to m_ν. This starkly contradicts the DMCT. The kinematics chosen here is only for the purpose of illustration. The collinear $\nu\bar{\nu}$ scenario has never been probed experimentally. On the contrary, there exists another kinematic scenario, the back-to-back $\nu\bar{\nu}$ scenario, using which the exception to the DMCT may be easily explored. As we will show, this scenario is experimentally accessible. Unless otherwise mentioned, we focus on this new specific back-to-back kinematic scenario in our discussions ahead.

B. Back-to-back neutrino antineutrino configuration: an experimentally observable exception to DMCT

Before we discuss the detailed structure of the amplitudes, we can make certain statements based on angular momentum analysis and quantum statistics. In a frame where the neutrino and antineutrino are back-to-back, i.e., flying with 3-momenta of equal magnitude but opposite direction, this reduces to the helicity analysis.[1] This is the kinematic situation that we are interested. The transition from a left-handed neutrino to the right-handed antineutrino is achieved by the combined transformation of charge conjugation (C) and parity (P). Thus,

$$\mathrm{C\,P}|\nu_\ell(\vec{s},E_\nu,\vec{p}_\nu)\rangle = \eta_P|\bar{\nu}_\ell(\vec{s},E_\nu,-\vec{p}_\nu)\rangle, \qquad (9)$$

where $\vec{s}, E_\nu, \vec{p}_\nu$ denote the spin, energy, and 3-momentum of the neutrino respectively, and η_P is the parity phase factor which is arbitrary for Dirac neutrinos but takes the values $\pm i$ for Majorana neutrinos [6]. Disregarding this phase factor for the time being, we can schematically express Eq. (9) as follows,

$$\big|\nu_\ell(\vec{s},E_\nu,\vec{p}_\nu)\big\rangle \quad \overset{\mathrm{C\,P}}{\Longleftrightarrow} \quad \big|\bar{\nu}_\ell(\vec{s},E_\nu,-\vec{p}_\nu)\big\rangle \qquad (10)$$

where the long thin arrows represent the 3-momenta of the neutrino and antineutrino, and the short thick arrows represent their spins. It is clear from Eqs. (9) and (10) that if the Majorana neutrino and antineutrino are back-to-back we can consider the consequence of their exchange as a proper signature of the quantum statistics.

C. Helicity considerations

This back-to-back configuration has one important consequence. If in the rest frame of the parent B^0 meson the neutrino antineutrino pair is found to fly away back-to-back, the muon pair must also fly away back-to-back since 3-momentum is conserved. This is a much simpler kinematic configuration than the general kinematics for any 4-body decay. Instead of the usual five independent variables one needs to describe any 4-body decay, we only need two independent variables to describe the back-to-back configuration. In this case, the energies of the two muons are the same and let us denote them by E_μ. Similarly, the energies of the back-to-back neutrino and antineutrino are the same and let us denote them by E_ν. Either E_μ or E_ν is independent, because from conservation of energy we get,

$$E_\nu = m_B/2 - E_\mu, \qquad (11)$$

where m_B is the mass of the B^0 meson. Let us choose E_μ as one independent variable. The other independent variable would then be the angle, say θ, between the muon direction and the neutrino direction.

Let us analyze the helicity configuration of this back-to-back muons (and back-to-back neutrino antineutrino) case as shown in Fig. 1, where the long arrows represent particle momenta and the short thick arrows represent their spins. Let us denote the decay amplitude describing the back-to-back configuration by $\mathscr{M}_{\leftrightarrow}^{D/M}$ for Dirac/Majorana neutrinos. In the case of Dirac neutrinos, it is clear from Fig. 1(a) that for $\theta = 0$ we have a net final spin $\neq 0$. This violates conservation of angular momentum, since the parent B^0 meson has spin-0. Therefore, for the Dirac case we have,

$$|\mathscr{M}_{\leftrightarrow}^{D}|^2 \propto \underbrace{(1-\cos\theta)^2}_{\text{Direct term}}. \qquad (12)$$

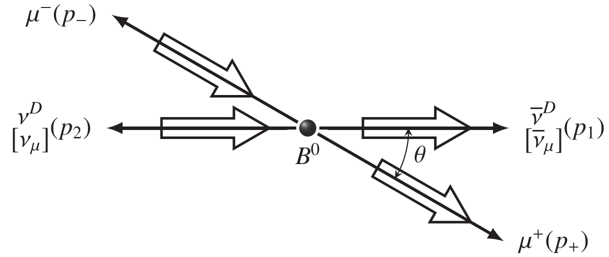
However for Majorana neutrinos, it is clear from Fig. 1(b) that both the θ and $\pi - \theta$ configurations are indistinguishable since ν_μ and $\bar{\nu}_\mu$ are quantum mechanically identical.[2] The interference term which is proportional to m_ν^2 can be neglected. Thus,

$$\begin{aligned} |\mathscr{M}_{\leftrightarrow}^{M}|^2 &\propto \frac{1}{2}\Big[\underbrace{(1-\cos\theta)^2}_{\text{Direct term}} + \underbrace{(1-\cos(\pi-\theta))^2}_{\text{Exchange term}} - \underbrace{\mathcal{O}(m_\nu^2)}_{\text{Interference term}}\Big] \\ &\simeq 1+\cos^2\theta. \end{aligned} \qquad (13)$$

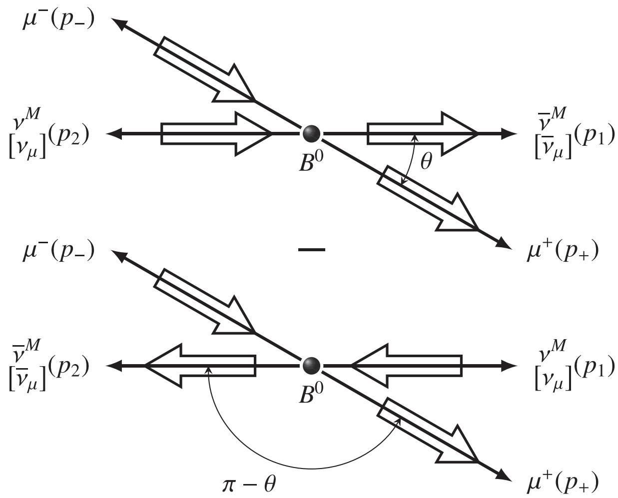
Thus, the Dirac and Majorana cases have completely different angular distributions in the back-to-back configuration, see Fig. 2. We would like to emphasize that this difference is simply a result of the antisymmetrization of the amplitude for Majorana neutrinos, and we have already neglected the interference term which is proportional to m_ν^2. Therefore, it can be considered as a proper test of the quantum statistics of the Majorana neutrinos.

[1] In this work we have $V - A$ interaction which fixes the helicity of all the particles involved. One could, in principle, consider mass dependent contributions, which are negligible for neutrino and antineutrino due to their tiny mass.

[2] The antisymmetrization for Majorana case gives the exchange term (via $p_1 \leftrightarrow p_2$ exchange) and is not associated with any helicity flip, as shown in Fig. 1 and Fig. 3. However, helicity flip is present in the interference term making it proportional to m_ν^2.



(a) Helicity configuration involving Dirac neutrinos, $\nu_\mu \equiv \nu^D$, $\bar{\nu}_\mu \equiv \bar{\nu}^D$.



(b) Helicity configuration involving Majorana neutrinos, $\nu_\mu = \bar{\nu}_\mu \equiv \nu^M$.

FIG. 1. The helicity configuration for back-to-back muons in the rest frame of B^0 in the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. Here the second diagram in Majorana case is a result of antisymmetrization and is not related to any helicity-flip. We show the $[\nu_\mu]$ and $[\bar{\nu}_\mu]$ labels just for bookkeeping.

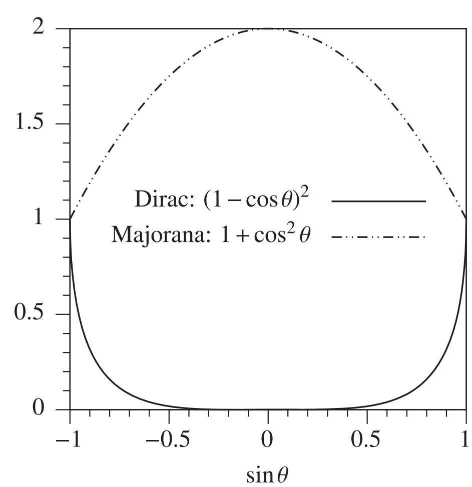


FIG. 2. Comparison of the angular distributions as given in Eqs. (12) and (13). The reason for taking $\sin\theta$ as the independent variable instead of $\cos\theta$ would be clear from the detailed discussion later.

The distinct signature between Dirac and Majorana cases as shown in Fig. 2 appears only in the restricted kinematic situation of back-to-back muons in the B^0 rest frame. The branching ratio in general is dominated by the non-back-to-back configurations which dominate the phase space and as we shall see later the branching ratio for back-to-back configuration is small but significant for distinguishing Dirac and Majorana cases. Therefore back-to-back configuration provides an exception to DMCT. Of course, once the full phase space integration over $\nu, \bar{\nu}$ variables is carried out, the difference between Dirac and Majorana cases is proportional to m_ν and we are back to DMCT domain.

In the next section a detailed analysis of the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ is presented covering all the nuances in the differences between Dirac and Majorana cases.

IV. A DETAILED STUDY OF THE DECAY $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$

In Fig. 3 the Feynman diagrams that contribute to the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ are shown. This is a doubly weak decay. The branching ratio of this mode will also have contributions from intermediate resonances such as π^- and D^- which tend to enhance the total branching ratio.

A. Structures in the decay amplitude

In order to present both the resonant and nonresonant contributions to the decay amplitudes in a uniform form, we note that the hadronic part will involve the following factors:

(i) product of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $V_{ub}^*V_{ud}$ or $V_{cb}^*V_{cd}$ or $V_{tb}^*V_{td}$ depending on whether u or c or t quark is being considered as the propagating quark,

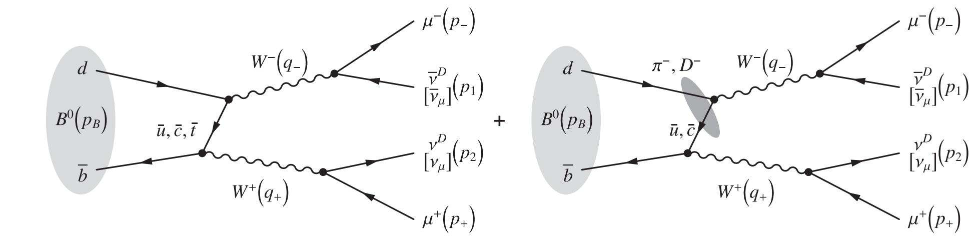
(ii) product of coupling constants and the virtual W propagators, which gives an overall factor of $\frac{g_w^4}{64m_W^4} = \frac{G_F^2}{2}$, as the Fermi constant (G_F) is related to the weak coupling constant (g_w) and the mass of W boson (m_W) by the relation $G_F = \frac{\sqrt{2}}{8}(\frac{g_w}{m_W})^2$,

(iii) the effective vertex factors for the contribution from $B^0(p_B) \to W^{+*}(q_+^{(\prime)})W^{-*}(q_-^{(\prime)})$, which are different for resonant and nonresonant channels (see Fig. 3 for the definitions of $q_\pm^{(\prime)}$ and more details about the vertex factors are given in Sec. IV B and IV C below).

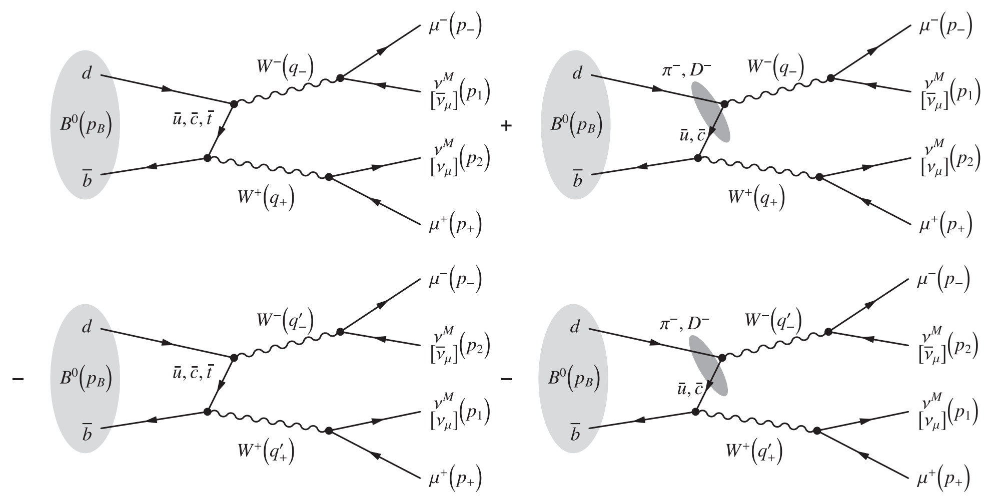
There are two combinations of product of leptonic currents in our case,

$$L_{\alpha\beta} = [\bar{u}(p_-)\gamma_\alpha(1-\gamma^5)v(p_1)][\bar{u}(p_2)\gamma_\beta(1-\gamma^5)v(p_+)], \tag{14a}$$

$$L'_{\alpha\beta} = [\bar{u}(p_-)\gamma_\alpha(1-\gamma^5)v(p_2)][\bar{u}(p_1)\gamma_\beta(1-\gamma^5)v(p_+)]. \tag{14b}$$



(a) For Dirac neutrinos: $\nu_\mu \equiv \nu^D$, $\bar{\nu}_\mu \equiv \bar{\nu}^D$.



(b) For Majorana neutrinos: $\nu_\mu = \bar{\nu}_\mu \equiv \nu^M$.

FIG. 3. The Feynman diagrams for $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ for both Dirac and Majorana cases. Here the internal 4-momenta are denoted by $q_- = p_- + p_1$, $q_+ = p_+ + p_2$, $q'_- = p_- + p_2$ and $q'_+ = p_+ + p_1$. Here one can consider $\pi^-(d\bar{u})$ and $D^-(d\bar{c})$ as the possible resonances. Due to the identical nature of Majorana neutrino and antineutrino, we have two probable resonant diagrams involving intermediate π^- or D^-. We show the $[\nu_\mu]$ and $[\bar{\nu}_\mu]$ labels just for bookkeeping.

It is easy to see that $L_{\alpha\beta}$ and $L'_{\alpha\beta}$ are related to one another by $p_1 \leftrightarrow p_2$ exchange.

The decay amplitudes for Dirac and Majorana neutrinos can be written as,

$$\mathscr{M}^D = \frac{G_F^2}{2} H^{\alpha\beta} L_{\alpha\beta} \equiv \mathscr{Q}_{12} + \mathscr{R}_{12}, \tag{15a}$$

$$\mathscr{M}^M = \frac{G_F^2}{2\sqrt{2}} (H^{\alpha\beta} L_{\alpha\beta} - H'^{\alpha\beta} L'_{\alpha\beta}) \equiv \frac{1}{\sqrt{2}} (\mathscr{Q}_{12} - \mathscr{Q}_{21} + \mathscr{R}_{12} - \mathscr{R}_{21}), \tag{15b}$$

where $H^{(\prime)\alpha\beta}$ denote the hadronic currents which contain the combination of products of CKM matrix elements and effective vertex factors, and we discuss about the structure of the hadronic currents below in detail leading to its final expression in Eq. (28), $\mathscr{Q}_{12}$ and $\mathscr{R}_{12}$ are respectively the nonresonant and resonant parts of the decay amplitude which are the sole contributors in case of Dirac neutrinos, and the nonresonant amplitude $\mathscr{Q}_{21}$, the resonant amplitude $\mathscr{R}_{21}$ which appear in Majorana case are obtained from $\mathscr{Q}_{12}$, $\mathscr{R}_{12}$ respectively by $p_1 \leftrightarrow p_2$ exchange. Below we look at the content of the hadronic currents, the resonant and nonresonant amplitudes in more detail.

B. Resonant amplitude and the hadronic current

For the resonant case, we can have both π^- and D^- as intermediate resonances depending on whether $q_-^{(\prime)2} = m_\pi^2$ or m_D^2 in the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. If we were to consider the conjugate process $\bar{B}^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$, then the resonances would be π^+ and D^+, both associated with the 4-momentum $q_+^{(\prime)}$ instead of $q_-^{(\prime)}$. Thus, knowing the flavor of the initial neutral B meson, i.e., whether it is B^0 or $\bar{B}^0$, the allowed resonances get fixed distinguishing the 4-momenta $q_+^{(\prime)}$ and $q_-^{(\prime)}$. This is in fact easily discernible from the expression for the effective vertex factors for $B^0 \to W^{+*}(q_+^{(\prime)}) W^{-*}(q_-^{(\prime)})$ from the resonant channel,

$$\mathbf{V}_R^{(\prime)\alpha\beta} = \frac{f_R}{q_-^{(\prime)2} - m_R^2 + i m_R \Gamma_R} q_-^{(\prime)\alpha} (F_{R+}^{(\prime)} q_+^{(\prime)\beta} + F_{R-}^{(\prime)} q_-^{(\prime)\beta}), \tag{16}$$

where $F_{R\pm}^{(\prime)} \equiv F_{R\pm}(q_+^{(\prime)2})$ are known form factors and f_R is the known decay constant of the resonance R ($R = \pi$, D) that has mass m_R and total decay rate Γ_R. Therefore, for the process $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$, the resonant part of the decay amplitude for Dirac case is given by

$$\mathscr{R}_{12} = \frac{G_F^2}{2}\mathbf{H}^{\alpha\beta}L_{\alpha\beta}, \tag{17}$$

where the resonant hadronic current is given by,

$$\mathbf{H}^{\alpha\beta} \equiv V_{ub}^* V_{ud}\mathbf{V}_\pi^{\alpha\beta} + V_{cb}^* V_{cd}\mathbf{V}_D^{\alpha\beta} = (\mathbf{F}_+ q_+^\beta + \mathbf{F}_- q_-^\beta)q_-^\alpha, \tag{18}$$

with the combined form factors $\mathbf{F}_\pm$ ("*resonant transition form factors*") being given by

$$\mathbf{F}_\pm \equiv \mathbf{F}_\pm(q_+^2, q_-^2) = \frac{V_{ub}^* V_{ud} f_\pi}{q_-^2 - m_\pi^2 + i m_\pi \Gamma_\pi} F_{\pi\pm}(q_+^2) + \frac{V_{cb}^* V_{cd} f_D}{q_-^2 - m_D^2 + i m_D \Gamma_D} F_{D\pm}(q_+^2). \tag{19}$$

It is important to reiterate that the vertex factor for resonant case as defined in Eq. (16) and the related form factors of Eq. (19) are specific to the decay mode $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$.

For Majorana case, in addition to $\mathscr{R}_{12}$ we have $\mathscr{R}_{21}$ which is given by

$$\mathscr{R}_{21} \equiv \frac{G_F^2}{2}\mathbf{H}'^{\alpha\beta}L'_{\alpha\beta}, \tag{20}$$

with the resonant hadronic current $\mathbf{H}'^{\alpha\beta}$ being given by,

$$\mathbf{H}'^{\alpha\beta} \equiv V_{ub}^* V_{ud}\mathbf{V}'^{\alpha\beta}_\pi + V_{cb}^* V_{cd}\mathbf{V}'^{\alpha\beta}_D = (\mathbf{F}'_+ q'^\beta_+ + \mathbf{F}'_- q'^\beta_-)q'^\alpha_-, \tag{21}$$

which includes the combined form factors $\mathbf{F}'_\pm$ that can be easily obtained by substituting $q_\pm^2$ by $q'^2_\pm$ in Eq. (19).

C. Non-resonant amplitude and the hadronic current

Unlike the resonant case, in the nonresonant case neither $q_+^{(\prime)}$ nor $q_-^{(\prime)}$ has any preferred role over the other. Hence, following Lorentz covariance, the effective vertex factors for $B^0(p_B) \to W^{+*}(q_+^{(\prime)})W^{-*}(q_-^{(\prime)})$ for nonresonant case (involving intermediate quark $Q = u$, c, t) can be written as

$$\mathbb{V}_Q^{(\prime)\alpha\beta} = F_a^{(\prime)Q} g^{\alpha\beta} + F_b^{(\prime)Q} p_B^\alpha p_B^\beta + i F_c^{(\prime)Q} \epsilon^{\alpha\beta\mu\nu} q_{+\mu}^{(\prime)} q_{-\nu}^{(\prime)}, \tag{22}$$

where $F_a^{(\prime)Q} \equiv F_a^{(\prime)Q}(q_+^{(\prime)2}, q_-^{(\prime)2})$, $F_b^{(\prime)Q} \equiv F_b^{(\prime)Q}(q_+^{(\prime)2}, q_-^{(\prime)2})$, $F_c^{(\prime)Q} \equiv F_c^{(\prime)Q}(q_+^{(\prime)2}, q_-^{(\prime)2})$ are the relevant "*nonresonant transition form factors*[3]" and $p_B = q_+^{(\prime)} + q_-^{(\prime)}$. Currently, the exact expressions for the form factors $F_a^{(\prime)Q}, F_b^{(\prime)Q}$ and $F_c^{(\prime)Q}$ are unknown and we consider them to be complex, in general. Thus, the nonresonant decay amplitude for Dirac case neutrinos is given by,

$$\mathscr{Q}_{12} = \frac{G_F^2}{2}\left(\sum_{Q=u,c,t} V_{Qb}^* V_{Qd}\mathbb{V}_Q^{\alpha\beta}\right)L_{\alpha\beta} = \frac{G_F^2}{2}\mathbb{H}^{\alpha\beta}L_{\alpha\beta}, \tag{23}$$

where the nonresonant hadronic current is given by

$$\mathbb{H}^{\alpha\beta} = \mathbb{F}_a g^{\alpha\beta} + \mathbb{F}_b p_B^\alpha p_B^\beta + i\mathbb{F}_c \epsilon^{\alpha\beta\mu\nu} q_{+\mu} q_{-\nu}, \tag{24}$$

with the combined form factors being,

$$\mathbb{F}_i \equiv \mathbb{F}_i(q_+^2, q_-^2) = \sum_{Q=u,c,t} V_{Qb}^* V_{Qd} F_i^Q(q_+^2, q_-^2), \tag{25}$$

with $i = a$, b, c. For Majorana case, in addition to $\mathscr{Q}_{12}$ we have $\mathscr{Q}_{21}$ which is given by,

$$\mathscr{Q}_{21} = \frac{G_F^2}{2}\mathbb{H}'^{\alpha\beta}L'_{\alpha\beta} \tag{26}$$

with

$$\mathbb{H}'^{\alpha\beta} = \mathbb{F}'_a g^{\alpha\beta} + \mathbb{F}'_b p_B^\alpha p_B^\beta + i\mathbb{F}'_c \epsilon^{\alpha\beta\mu\nu} q'_{+\mu} q'_{-\nu}, \tag{27}$$

and the combined form factors $\mathbb{F}'_i$ with $i = a$, b, c can be easily obtained by substituting $q_\pm^2$ by $q'^2_\pm$ in Eq. (25).

D. Complete expressions for the hadronic currents

Taking both resonant and nonresonant contributions, the decay amplitudes for Dirac and Majorana cases for the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ are given by Eq. (15) with the hadronic currents having both resonant and nonresonant components,

$$H^{(\prime)\alpha\beta} = \mathbf{H}^{(\prime)\alpha\beta} + \mathbb{H}^{(\prime)\alpha\beta}, \tag{28}$$

where the expressions for $\mathbf{H}^{\alpha\beta}$, $\mathbf{H}'^{\alpha\beta}$, $\mathbb{H}^{\alpha\beta}$ and $\mathbb{H}'^{\alpha\beta}$ are shown in Eqs. (18), (21), (24), and (27) respectively. For brevity the primed and unprimed hadronic currents are written in the same equation above.

It should be noted that the form factors $\mathbf{F}_\pm$ and $\mathbf{F}'_\pm$, as well as $\mathbb{F}_{a,b,c}$ and $\mathbb{F}'_{a,b,c}$ are the same functions with different arguments since the hadronic structure is independent of the process. They are simply related by the exchange $p_1 \leftrightarrow p_2$.

[3]These *transition form factors* are functions of two different q^2. An example where similar situation occurs is while considering the pion transition form factors for $\gamma^*\gamma^*\pi^0$ vertex, see Ref. [79].

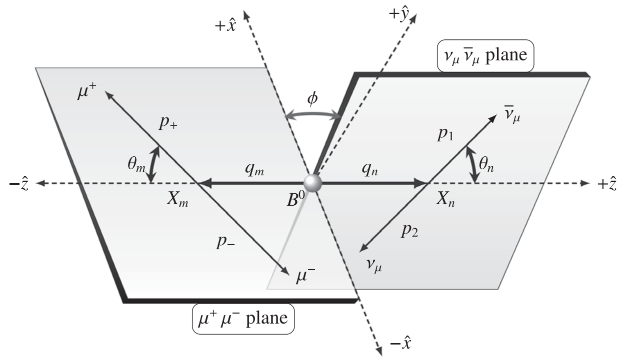


FIG. 4. The kinematics of $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ in the rest frame of B, showing the polar angles θ_m and θ_n, as well as the azimuthal angle ϕ. Here X_m and X_n denote the muon pair and the neutrino pair.

Furthermore, currently we do not know the exact functional forms of the various nonresonant transition form factors. On the other hand the individual resonant form factors for any given resonance are known, but there could be relative phase difference between resonant and nonresonant form factors. Though the resonance contribution is substantial for the total branching ratio, as we show later they are not important for the back-to-back kinematic configuration which is the focus here.

E. General kinematics and differential decay rates

It is convenient to visualize the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$, in the rest frame of the B^0 meson, as a two-body decay into a "dimuon" X_m of mass $m_{\mu\mu}$ and a "dineutrino" X_n of mass $m_{\nu\nu}$. The subsequent decay of each of these two subsystems is considered in its own center-of-momentum frame as shown in Fig. 4. The 4-momentum of the dimuon is denoted by q_m and that of the dineutrino is denoted by q_n. The process is then described by the following five variables:

(1) $m_{\mu\mu}^2 \equiv q_m^2 = (p_+ + p_-)^2$, the invariant effective mass squared of the dimuon system,
(2) $m_{\nu\nu}^2 \equiv q_n^2 = (p_1 + p_2)^2$, the invariant effective mass squared of the dineutrino system,
(3) θ_m, the angle between the direction of flight of the μ^+ in the center-of-momentum frame of the dimuon and the direction of flight of the dimuon in the B^0 rest frame,
(4) θ_n, the angle between the direction of flight of the $\bar{\nu}_\mu$ in the center-of-momentum frame of the dineutrino and the direction of flight of the dineutrino in the B^0 rest frame, and
(5) ϕ, the angle between the plane formed by the muons in the B^0 rest frame and the corresponding plane formed by the neutrino and antineutrino.

The angles θ_m and θ_n are polar; ϕ is azimuthal.

The differential decay rate for the decay $B \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ is given by,

$$\frac{\mathrm{d}^5\Gamma^{D/M}}{\mathrm{d}m_{\mu\mu}^2\mathrm{d}m_{\nu\nu}^2\mathrm{d}\cos\theta_m\mathrm{d}\cos\theta_n\mathrm{d}\phi} = \frac{YY_mY_n\langle|\mathscr{M}^{D/M}|^2\rangle}{(4\pi)^6 m_B^2 m_{\mu\mu}m_{\nu\nu}}, \quad (29)$$

where m_B is the mass of the B meson, the magnitude of 3-momentum of X_m or X_n in the B rest frame is Y, the magnitude of 3-momentum of μ^- or μ^+ in the rest frame of the dimuon is Y_m, the magnitude of 3-momentum of ν_μ or $\bar{\nu}_\mu$ in the rest frame of the dineutrino is Y_n and these are given by

$$Y = \frac{\sqrt{\lambda(m_B^2, m_{\mu\mu}^2, m_{\nu\nu}^2)}}{2m_B}, \quad (30a)$$

$$Y_m = \frac{\sqrt{m_{\mu\mu}^2 - 4m_\mu^2}}{2}, \quad (30b)$$

$$Y_n = \frac{\sqrt{m_{\nu\nu}^2 - 4m_\nu^2}}{2}, \quad (30c)$$

with m_μ, m_ν being the masses of muon and neutrino respectively, and $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$ is the Källén function. The expression for the square of the modulus of the decay amplitude with average over initial spins and sum over final spins, $\langle|\mathscr{M}^{D/M}|^2\rangle$, is a complicated function of θ_m, θ_n, ϕ, $m_{\mu\mu}$ and $m_{\nu\nu}$ for both Dirac and Majorana cases, and can be written as,

$$\begin{aligned}\langle|\mathscr{M}^D|^2\rangle = G_F^4(&|\mathbb{F}_a|^2S_{aa}^D + |\mathbb{F}_b|^2S_{bb}^D + |\mathbb{F}_c^2|S_{cc}^D + |\mathbf{F}_+|^2S_{pp}^D + |\mathbf{F}_-|^2S_{mm}^D + \mathrm{Re}(\mathbb{F}_a\mathbb{F}_b^*)R_{ab}^D + \mathrm{Re}(\mathbb{F}_a\mathbb{F}_c^*)R_{ac}^D + \mathrm{Re}(\mathbb{F}_a\mathbf{F}_+^*)R_{ap}^D \\ &+ \mathrm{Re}(\mathbb{F}_a\mathbf{F}_-^*)R_{am}^D + \mathrm{Im}(\mathbb{F}_a\mathbb{F}_b^*)I_{ab}^D + \mathrm{Im}(\mathbb{F}_a\mathbb{F}_c^*)I_{ac}^D + \mathrm{Im}(\mathbb{F}_a\mathbf{F}_+^*)I_{ap}^D + \mathrm{Im}(\mathbb{F}_a\mathbf{F}_-^*)I_{am}^D + \mathrm{Re}(\mathbb{F}_b\mathbb{F}_c^*)R_{bc}^D \\ &+ \mathrm{Re}(\mathbb{F}_b\mathbf{F}_+^*)R_{bp}^D + \mathrm{Re}(\mathbb{F}_b\mathbf{F}_-^*)R_{bm}^D + \mathrm{Re}(\mathbb{F}_c\mathbf{F}_+^*)R_{cp}^D + \mathrm{Im}(\mathbb{F}_b\mathbb{F}_c^*)I_{bc}^D + \mathrm{Im}(\mathbb{F}_b\mathbf{F}_+^*)I_{bp}^D + \mathrm{Im}(\mathbb{F}_b\mathbf{F}_-^*)I_{bm}^D \\ &+ \mathrm{Im}(\mathbb{F}_c\mathbf{F}_+^*)I_{cp}^D + \mathrm{Re}(\mathbb{F}_c\mathbf{F}_-^*)R_{cm}^D + \mathrm{Re}(\mathbf{F}_+\mathbf{F}_-^*)R_{pm}^D + \mathrm{Im}(\mathbb{F}_c\mathbf{F}_-^*)I_{cm}^D + \mathrm{Im}(\mathbf{F}_+\mathbf{F}_-^*)I_{pm}^D),\end{aligned} \quad (31)$$

$$
\begin{aligned}
\langle|\mathscr{M}^M|^2\rangle = \frac{G_F^4}{2}\big(&|\mathbb{F}_a|^2 S^M_{aa} + |\mathbb{F}_b|^2 S^M_{bb} + |\mathbb{F}_c^2| S^M_{cc} + |\mathbf{F}_+|^2 S^M_{pp} + |\mathbf{F}_-|^2 S^M_{mm} + |\mathbb{F}'_a|^2 S^M_{a'a'} + |\mathbb{F}'_b|^2 S^M_{b'b'} + |\mathbb{F}'_c|^2 S^M_{c'c'} + |\mathbf{F}'_+|^2 S^M_{p'p'} \\
&+ |\mathbf{F}'_-|^2 S^M_{m'm'} + \mathrm{Re}(\mathbb{F}_a\mathbb{F}^*_b)R^M_{ab} + \mathrm{Re}(\mathbb{F}_a\mathbb{F}^*_c)R^M_{ac} + \mathrm{Re}(\mathbb{F}_a\mathbf{F}^*_+)R^M_{ap} + \mathrm{Re}(\mathbb{F}_a\mathbf{F}^*_-)R^M_{am} + \mathrm{Re}(\mathbb{F}_b\mathbb{F}^*_c)R^M_{bc} \\
&+ \mathrm{Re}(\mathbb{F}_b\mathbf{F}^*_+)R^M_{bp} + \mathrm{Re}(\mathbb{F}_b\mathbf{F}^*_-)R^M_{bm} + \mathrm{Re}(\mathbb{F}_c\mathbf{F}^*_+)R^M_{cp} + \mathrm{Re}(\mathbb{F}_c\mathbf{F}^*_-)R^M_{cm} + \mathrm{Re}(\mathbb{F}'_a\mathbb{F}'^*_b)R^M_{a'b'} + \mathrm{Re}(\mathbb{F}'_a\mathbb{F}'^*_c)R^M_{a'c'} \\
&+ \mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_+)R^M_{a'p'} + \mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_-)R^M_{a'm'} + \mathrm{Re}(\mathbb{F}'_b\mathbb{F}'^*_c)R^M_{b'c'} + \mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_+)R^M_{b'p'} + \mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_-)R^M_{b'm'} + \mathrm{Re}(\mathbb{F}'_c\mathbf{F}'^*_+)R^M_{c'p'} \\
&+ \mathrm{Re}(\mathbb{F}'_c\mathbf{F}'^*_-)R^M_{c'm'} + \mathrm{Re}(\mathbf{F}_+\mathbf{F}^*_-)R^M_{pm} + \mathrm{Re}(\mathbf{F}'_+\mathbf{F}'^*_-)R^M_{p'm'} + \mathrm{Im}(\mathbb{F}_a\mathbb{F}^*_b)I^M_{ab} + \mathrm{Im}(\mathbb{F}_a\mathbb{F}^*_c)I^M_{ac} + \mathrm{Im}(\mathbb{F}_a\mathbf{F}^*_+)I^M_{ap} \\
&+ \mathrm{Im}(\mathbb{F}_a\mathbf{F}^*_-)I^M_{am} + \mathrm{Im}(\mathbb{F}_b\mathbb{F}^*_c)I^M_{bc} + \mathrm{Im}(\mathbb{F}_b\mathbf{F}^*_+)I^M_{bp} + \mathrm{Im}(\mathbb{F}_b\mathbf{F}^*_-)I^M_{bm} + \mathrm{Im}(\mathbb{F}_c\mathbf{F}^*_+)I^M_{cp} + \mathrm{Im}(\mathbb{F}_c\mathbf{F}^*_-)I^M_{cm} \\
&+ \mathrm{Im}(\mathbb{F}'_a\mathbb{F}'^*_b)I^M_{a'b'} + \mathrm{Im}(\mathbb{F}'_a\mathbb{F}'^*_c)I^M_{a'c'} + \mathrm{Im}(\mathbb{F}'_a\mathbf{F}'^*_+)I^M_{a'p'} + \mathrm{Im}(\mathbb{F}'_a\mathbf{F}'^*_-)I^M_{a'm'} + \mathrm{Im}(\mathbb{F}'_b\mathbb{F}'^*_c)I^M_{b'c'} + \mathrm{Im}(\mathbb{F}'_b\mathbf{F}'^*_+)I^M_{b'p'} \\
&+ \mathrm{Im}(\mathbb{F}'_b\mathbf{F}'^*_-)I^M_{b'm'} + \mathrm{Im}(\mathbb{F}'_c\mathbf{F}'^*_+)I^M_{c'p'} + \mathrm{Im}(\mathbb{F}'_c\mathbf{F}'^*_-)I^M_{c'm'} + \mathrm{Im}(\mathbf{F}_+\mathbf{F}^*_-)I^M_{pm} + \mathrm{Im}(\mathbf{F}'_+\mathbf{F}'^*_-)I^M_{p'm'} \\
&+ m_\nu^2\big(\mathrm{Re}(\mathbb{F}_a\mathbb{F}'^*_a)R^M_{aa'} + \mathrm{Re}(\mathbb{F}_a\mathbb{F}'^*_b)R^M_{ab'} + \mathrm{Re}(\mathbb{F}_a\mathbb{F}'^*_c)R^M_{ac'} + \mathrm{Re}(\mathbb{F}_a\mathbf{F}'^*_+)R^M_{ap'} + \mathrm{Re}(\mathbb{F}_a\mathbf{F}'^*_-)R^M_{am'} + \mathrm{Re}(\mathbb{F}_b\mathbb{F}'^*_a)R^M_{ba'} \\
&+ \mathrm{Re}(\mathbb{F}_b\mathbb{F}'^*_b)R^M_{bb'} + \mathrm{Re}(\mathbb{F}_b\mathbb{F}'^*_c)R^M_{bc'} + \mathrm{Re}(\mathbb{F}_b\mathbf{F}'^*_+)R^M_{bp'} + \mathrm{Re}(\mathbb{F}_b\mathbf{F}'^*_-)R^M_{bm'} + \mathrm{Re}(\mathbb{F}_c\mathbb{F}'^*_a)R^M_{ca'} + \mathrm{Re}(\mathbb{F}_c\mathbb{F}'^*_b)R^M_{cb'} \\
&+ \mathrm{Re}(\mathbb{F}_c\mathbb{F}'^*_c)R^M_{cc'} + \mathrm{Re}(\mathbb{F}_c\mathbf{F}'^*_+)R^M_{cp'} + \mathrm{Re}(\mathbb{F}_c\mathbf{F}'^*_-)R^M_{cm'} + \mathrm{Re}(\mathbb{F}'_a\mathbf{F}^*_+)R^M_{a'p} + \mathrm{Re}(\mathbb{F}'_a\mathbf{F}^*_-)R^M_{a'm} + \mathrm{Re}(\mathbb{F}'_b\mathbf{F}^*_+)R^M_{b'p} \\
&+ \mathrm{Re}(\mathbb{F}'_b\mathbf{F}^*_-)R^M_{b'm} + \mathrm{Re}(\mathbb{F}'_c\mathbf{F}^*_+)R^M_{c'p} + \mathrm{Re}(\mathbb{F}'_c\mathbf{F}^*_-)R^M_{c'm} + \mathrm{Im}(\mathbb{F}_a\mathbb{F}'^*_a)I^M_{aa'} + \mathrm{Im}(\mathbb{F}_a\mathbb{F}'^*_b)I^M_{ab'} + \mathrm{Im}(\mathbb{F}_a\mathbb{F}'^*_c)I^M_{ac'} \\
&+ \mathrm{Im}(\mathbb{F}_a\mathbf{F}'^*_+)I^M_{ap'} + \mathrm{Im}(\mathbb{F}_a\mathbf{F}'^*_-)I^M_{am'} + \mathrm{Im}(\mathbb{F}_b\mathbb{F}'^*_a)I^M_{ba'} + \mathrm{Im}(\mathbb{F}_b\mathbb{F}'^*_b)I^M_{bb'} + \mathrm{Im}(\mathbb{F}_b\mathbb{F}'^*_c)I^M_{bc'} + \mathrm{Im}(\mathbb{F}_b\mathbf{F}'^*_+)I^M_{bp'} \\
&+ \mathrm{Im}(\mathbb{F}_b\mathbf{F}'^*_-)I^M_{bm'} + \mathrm{Im}(\mathbb{F}_c\mathbb{F}'^*_a)I^M_{ca'} + \mathrm{Im}(\mathbb{F}_c\mathbb{F}'^*_b)I^M_{cb'} + \mathrm{Im}(\mathbb{F}_c\mathbb{F}'^*_c)I^M_{cc'} + \mathrm{Im}(\mathbb{F}_c\mathbf{F}'^*_+)I^M_{cp'} + \mathrm{Im}(\mathbb{F}_c\mathbf{F}'^*_-)I^M_{cm'} \\
&+ \mathrm{Im}(\mathbb{F}'_a\mathbf{F}^*_+)I^M_{a'p} + \mathrm{Im}(\mathbb{F}'_a\mathbf{F}^*_-)I^M_{a'm} + \mathrm{Im}(\mathbb{F}'_b\mathbf{F}^*_+)I^M_{b'p} + \mathrm{Im}(\mathbb{F}'_b\mathbf{F}^*_-)I^M_{b'm} + \mathrm{Im}(\mathbb{F}'_c\mathbf{F}^*_+)I^M_{c'p} + \mathrm{Im}(\mathbb{F}'_c\mathbf{F}^*_-)I^M_{c'm}\big)\big),
\end{aligned}
\tag{32}
$$

where the terms $S^{D/M}_{i^{(\prime)}i^{(\prime)}}$ are associated with the squares of the form factors $|\mathbb{F}^{(\prime)}_i|^2$ or $|\mathbf{F}^{(\prime)}_i|^2$, and the terms $R^{D/M}_{i^{(\prime)}j^{(\prime)}}$ (or $I^{D/M}_{i^{(\prime)}j^{(\prime)}}$) are associated with the real (or imaginary) part of the products of form factors $\mathbb{F}^{(\prime)}_i\mathbb{F}^{(\prime)*}_j$ or $\mathbb{F}^{(\prime)}_i\mathbf{F}^{(\prime)*}_j$ or $\mathbf{F}^{(\prime)}_i\mathbb{F}^{(\prime)*}_j$ or $\mathbf{F}^{(\prime)}_i\mathbf{F}^{(\prime)*}_j$ with $i \neq j$ and $i, j \in \{a, b, c, p \equiv +, m \equiv -\}$. The total number of possible terms for Dirac and Majorana cases are 25 and 92 respectively. We have 25 direct terms, 25 exchange terms and 42 interference terms in the Majorana case which are directly proportional to m_ν^2 as shown in Eq. (32). Some of these terms, shown in Eq. (A1) of Appendix A, are zero. The detailed expressions for the 70 nonvanishing terms (with Majorana and Dirac cases sharing 20 terms) are given explicitly in Appendix A from Eqs. (A7)–(A76).

Note: To briefly illustrate that the difference between Dirac and Majorana neutrinos need not necessarily be proportional to some power of m_ν, we reconsider the simple example of collinear neutrino and antineutrino ($p_1 = p_2$) that was mentioned in Sec. III A. We note that the Majorana amplitude for this collinear case vanishes exactly, while the Dirac amplitude does not. Considering the leading contribution that comes from the form factor $\mathbb{F}_a$ alone, and substituting $p_1 = p_2 \equiv p_\nu(\text{say})$ we obtain

$$\langle|\mathscr{M}^D_{\text{collinear}}|^2\rangle = 64 G_F^4 |\mathbb{F}_a|^2 (p_\nu \cdot p_+)(p_\nu \cdot p_-),$$

which is not proportional to any power of m_ν. This proves that the collinear $\nu\bar{\nu}$ scenario is indeed another exception to DMCT, albeit being experimentally inaccessible as mentioned in Sec. III A.

To analyze the experimentally accessible back-to-back kinematic configuration which is also capable of distinguishing Dirac and Majorana neutrinos, we need to first study the differential decay distribution in detail.

F. Differential decay distribution

The angles θ_n and ϕ (see Fig. 4) are indeed inaccessible, as the neutrino pair goes missing. Therefore, for a physically useful differential decay rate we must integrate over both θ_n and ϕ in Eq. (29), i.e.,

$$
\begin{aligned}
&\frac{\mathrm{d}^3\Gamma^{D/M}}{\mathrm{d}m^2_{\mu\mu}\mathrm{d}m^2_{\nu\nu}\mathrm{d}\cos\theta_m} \\
&= \frac{YY_mY_n}{(4\pi)^6 m_B^2 m_{\mu\mu} m_{\nu\nu}} \int_{-1}^{1}\int_0^{2\pi} \langle|\mathscr{M}^{D/M}|^2\rangle \mathrm{d}\cos\theta_n \mathrm{d}\phi.
\end{aligned}
\tag{33}
$$

It is straightforward to show that the difference between Dirac and Majorana cases is given by

$$
\begin{aligned}
&\frac{\mathrm{d}^3\Gamma^M}{\mathrm{d}m_{\mu\mu}^2\mathrm{d}m_{\nu\nu}^2\mathrm{d}\cos\theta_m}-\frac{\mathrm{d}^3\Gamma^D}{\mathrm{d}m_{\mu\mu}^2\mathrm{d}m_{\nu\nu}^2\mathrm{d}\cos\theta_m}=\frac{G_F^4 Y Y_m Y_n}{2(4\pi)^6 m_B^2 m_{\mu\mu} m_{\nu\nu}}\int_{-1}^{1}\int_0^{2\pi}\mathrm{d}\cos\theta_n\mathrm{d}\phi \\
&\quad\times(-|\mathbb{F}_a|^2 S_{aa}^M-|\mathbb{F}_b|^2 S_{bb}^M-|\mathbb{F}_c^2|S_{cc}^M-|\mathbf{F}_+|^2 S_{pp}^M-|\mathbf{F}_-|^2 S_{mm}^M+|\mathbb{F}'_a|^2 S_{a'a'}^M+|\mathbb{F}'_b|^2 S_{b'b'}^M+|\mathbb{F}'_c|^2 S_{c'c'}^M+|\mathbf{F}'_+|^2 S_{p'p'}^M+|\mathbf{F}'_-|^2 S_{m'm'}^M \\
&\quad-\mathrm{Re}(\mathbb{F}_a\mathbb{F}_b^*)R_{ab}^M-\mathrm{Re}(\mathbb{F}_a\mathbb{F}_c^*)R_{ac}^M-\mathrm{Re}(\mathbb{F}_a\mathbf{F}_+^*)R_{ap}^M-\mathrm{Re}(\mathbb{F}_a\mathbf{F}_-^*)R_{am}^M-\mathrm{Re}(\mathbb{F}_b\mathbb{F}_c^*)R_{bc}^M-\mathrm{Re}(\mathbb{F}_b\mathbf{F}_+^*)R_{bp}^M-\mathrm{Re}(\mathbb{F}_b\mathbf{F}_-^*)R_{bm}^M \\
&\quad-\mathrm{Re}(\mathbb{F}_c\mathbf{F}_-^*)R_{cm}^M+\mathrm{Re}(\mathbb{F}'_a\mathbb{F}'^*_b)R_{a'b'}^M+\mathrm{Re}(\mathbb{F}'_a\mathbb{F}'^*_c)R_{a'c'}^M+\mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_+)R_{a'p'}^M+\mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_-)R_{a'm'}^M+\mathrm{Re}(\mathbb{F}'_b\mathbb{F}'^*_c)R_{b'c'}^M \\
&\quad+\mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_+)R_{b'p'}^M+\mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_-)R_{b'm'}^M+\mathrm{Re}(\mathbb{F}'_c\mathbf{F}'^*_-)R_{c'm'}^M-\mathrm{Re}(\mathbf{F}_+\mathbf{F}_-^*)R_{pm}^M+\mathrm{Re}(\mathbf{F}'_+\mathbf{F}'^*_-)R_{p'm'}^M \\
&\quad-\mathrm{Im}(\mathbb{F}_a\mathbb{F}_b^*)I_{ab}^M-\mathrm{Im}(\mathbb{F}_a\mathbb{F}_c^*)I_{ac}^M-\mathrm{Im}(\mathbb{F}_a\mathbf{F}_-^*)I_{am}^M-\mathrm{Im}(\mathbb{F}_b\mathbb{F}_c^*)I_{bc}^M-\mathrm{Im}(\mathbb{F}_c\mathbf{F}_+^*)I_{cp}^M-\mathrm{Im}(\mathbb{F}_c\mathbf{F}_-^*)I_{cm}^M \\
&\quad+\mathrm{Im}(\mathbb{F}'_a\mathbb{F}'^*_b)I_{a'b'}^M+\mathrm{Im}(\mathbb{F}'_a\mathbb{F}'^*_c)I_{a'c'}^M+\mathrm{Im}(\mathbb{F}'_a\mathbf{F}'^*_-)I_{a'm'}^M+\mathrm{Im}(\mathbb{F}'_b\mathbb{F}'^*_c)I_{b'c'}^M+\mathrm{Im}(\mathbb{F}'_c\mathbf{F}'^*_+)I_{c'p'}^M+\mathrm{Im}(\mathbb{F}'_c\mathbf{F}'^*_-)I_{c'm'}^M \\
&\quad+m_\nu^2(\mathrm{Re}(\mathbb{F}_a\mathbb{F}'^*_a)R_{aa'}^M+\mathrm{Re}(\mathbb{F}_a\mathbb{F}'^*_b)R_{ab'}^M+\mathrm{Re}(\mathbb{F}_a\mathbf{F}'^*_+)R_{ap'}^M+\mathrm{Re}(\mathbb{F}_b\mathbb{F}'^*_a)R_{ba'}^M+\mathrm{Re}(\mathbb{F}_b\mathbb{F}'^*_b)R_{bb'}^M+\mathrm{Re}(\mathbb{F}_b\mathbb{F}'^*_c)R_{bc'}^M \\
&\quad+\mathrm{Re}(\mathbb{F}_b\mathbf{F}'^*_+)R_{bp'}^M+\mathrm{Re}(\mathbb{F}_b\mathbf{F}'^*_-)R_{bm'}^M+\mathrm{Re}(\mathbb{F}_c\mathbb{F}'^*_b)R_{cb'}^M+\mathrm{Re}(\mathbb{F}_c\mathbb{F}'^*_c)R_{cc'}^M+\mathrm{Re}(\mathbb{F}_c\mathbf{F}'^*_+)R_{cp'}^M+\mathrm{Re}(\mathbb{F}_c\mathbf{F}'^*_-)R_{cm'}^M \\
&\quad+\mathrm{Re}(\mathbb{F}'_a\mathbf{F}_+^*)R_{a'p}^M+\mathrm{Re}(\mathbb{F}'_a\mathbf{F}_-^*)R_{a'm}^M+\mathrm{Re}(\mathbb{F}'_b\mathbf{F}_+^*)R_{b'p}^M+\mathrm{Re}(\mathbb{F}'_b\mathbf{F}_-^*)R_{b'm}^M+\mathrm{Re}(\mathbb{F}'_c\mathbf{F}_+^*)R_{c'p}^M+\mathrm{Re}(\mathbb{F}'_c\mathbf{F}_-^*)R_{c'm}^M \\
&\quad+\mathrm{Im}(\mathbb{F}_a\mathbb{F}'^*_c)I_{ac'}^M+\mathrm{Im}(\mathbb{F}_a\mathbf{F}'^*_-)I_{am'}^M+\mathrm{Im}(\mathbb{F}_b\mathbf{F}'^*_+)I_{bp'}^M+\mathrm{Im}(\mathbb{F}_b\mathbf{F}'^*_-)I_{bm'}^M+\mathrm{Im}(\mathbb{F}_c\mathbb{F}'^*_a)I_{ca'}^M+\mathrm{Im}(\mathbb{F}_c\mathbb{F}'^*_c)I_{cc'}^M \\
&\quad+\mathrm{Im}(\mathbb{F}_c\mathbf{F}'^*_+)I_{cp'}^M+\mathrm{Im}(\mathbb{F}_c\mathbf{F}'^*_-)I_{cm'}^M+\mathrm{Im}(\mathbb{F}'_b\mathbf{F}_+^*)I_{b'p}^M+\mathrm{Im}(\mathbb{F}'_b\mathbf{F}_-^*)I_{b'm}^M+\mathrm{Im}(\mathbb{F}'_c\mathbf{F}_+^*)I_{c'p}^M+\mathrm{Im}(\mathbb{F}'_c\mathbf{F}_-^*)I_{c'm}^M)).
\end{aligned}
\tag{34}
$$

In absence of analytical expressions for all the form factors, we note the following important features that can be easily observed in Eq. (34).

(1) There are direct and exchange terms which are related to one another by $p_1 \leftrightarrow p_2$ exchange. In Eq. (34) all the direct terms appear with negative sign. And the corresponding exchange terms have positive sign. Therefore, these terms after integration over $\cos\theta_n$ and ϕ should vanish, as they would have equal and opposite contributions.

(2) There are interference terms which are invariant under the $p_1 \leftrightarrow p_2$ exchange. All these interference terms are explicitly found to be proportional to m_ν^2. These terms would survive after integration over $\cos\theta_n$ and ϕ, simply because there is no way to cancel them, unless the integral itself vanishes. For example, if one were to assume the form factors to be constants, then all the $I^M_{j^{(\prime)}k^{(\prime)}}$ terms (for $j, k = a, b, c, p, m$) would vanish.

(3) The integration has been carried out over $\cos\theta_n$ and ϕ, the variables necessary to describe the individual ν and $\bar{\nu}$. The variable $m_{\nu\nu}^2$, also associated with the $\nu, \bar{\nu}$ pair, is however unaffected by the $p_1 \leftrightarrow p_2$ exchange. Essentially the integration over $\cos\theta_n$ and ϕ wipes out the difference between the direct and the exchange terms.

Therefore, the difference between the Dirac and Majorana cases, as shown in Eq. (34) after integration over neutrino pair variables, is now proportional to m_ν^2:

$$
\frac{\mathrm{d}^3\Gamma^M}{\mathrm{d}m_{\mu\mu}^2\mathrm{d}m_{\nu\nu}^2\mathrm{d}\cos\theta_m}-\frac{\mathrm{d}^3\Gamma^D}{\mathrm{d}m_{\mu\mu}^2\mathrm{d}m_{\nu\nu}^2\mathrm{d}\cos\theta_m}\propto m_\nu^2, \tag{35}
$$

which proves DMCT in the present case. This is not our main point since DMCT is a well-known result. It would be interesting to see if we may avoid the constraint imposed by the DMCT. We do this next, and as a bonus we find the difference between Dirac and Majorana scenarios is not just substantial, but *it eliminates the dependence on the unknown neutrino mass* to a very good approximation. The corrections coming from nonzero neutrino mass is negligible.

G. Change of variables for back-to-back configuration

As shown in Sec. III it is the back-to-back configuration which holds the promise to probe the quantum statistics of Majorana neutrinos most effectively. For this case, our choice of kinematic variables is not helpful. We need to make change of variables. Let us assume that the angle between the neutrino and antineutrino in the rest frame of B^0 be Θ. Then, in terms of the neutrino energies E_1 and E_2 we have

$$
\cos\Theta=\frac{Y^2-E_1^2-E_2^2+2m_\nu^2}{2\sqrt{E_1^2-m_\nu^2}\sqrt{E_2^2-m_\nu^2}}. \tag{36}
$$

When the neutrino and antineutrino are back-to-back in the B^0 rest frame, we have $Y=0$ and $E_1=E_2=E_\nu$ (say). This implies that, $\cos\Theta=-1$, as it should be for $\Theta=\pi$. It is easy to show that,

$$
\begin{aligned}
\mathrm{d}m_{\mu\mu}^2\mathrm{d}m_{\nu\nu}^2\mathrm{d}\cos\theta_n &= -\frac{4m_B m_{\nu\nu}}{Y Y_n}\sqrt{(E_1^2-m_\nu^2)(E_2^2-m_\nu^2)} \\
&\quad\times\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}\cos\Theta,
\end{aligned}
\tag{37}
$$

where

$$m_{\nu\nu}^2 = 2m_\nu^2 + 2E_1E_2 - 2\sqrt{(E_1^2 - m_\nu^2)(E_2^2 - m_\nu^2)}\cos\Theta, \tag{38a}$$

$$m_{\mu\mu}^2 = m_B^2 + 2m_\nu^2 - 2m_B(E_1 + E_2) + 2E_1E_2 - 2\sqrt{(E_1^2 - m_\nu^2)(E_2^2 - m_\nu^2)}\cos\Theta. \tag{38b}$$

Therefore,

$$\frac{\mathrm{d}^5\Gamma^{D/M}}{\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}\cos\Theta\mathrm{d}\cos\theta_m\mathrm{d}\phi} = -\frac{Y_m\langle|\mathscr{M}^{D/M}|^2\rangle}{4^5\pi^6 m_B m_{\mu\mu}} \times\sqrt{(E_1^2 - m_\nu^2)(E_2^2 - m_\nu^2)} \tag{39}$$

It should be noted that the differential decay rate for back-to-back configuration is obtained from the full five variable differential decay rate as shown in Eq. (29) without any integration and after making the suitable change of variables mentioned above.

H. Addressing the back-to-back case

For back-to-back case, with $E_1 = E_2 = E_\nu$ (say) and $\Theta = \pi$, we get the following from Eqs. (38a) and (38b),

$$m_{\nu\nu}^2 = 4E_\nu^2, \tag{40a}$$

$$m_{\mu\mu}^2 = (m_B - 2E_\nu)^2, \tag{40b}$$

which correctly implies $Y = 0$, meaning that the dimuon and dineutrino systems are at rest in the B^0 rest frame, as they should be. Moreover, for the back-to-back case we have

$$Y_m = \sqrt{\left(\frac{m_B}{2} - E_\nu\right)^2 - m_\mu^2}, \tag{41a}$$

$$Y_n = \sqrt{E_\nu^2 - m_\nu^2}. \tag{41b}$$

It can be shown that, in general,

$$\cos\theta_n = \frac{m_{\nu\nu}(E_1 - E_2)}{2YY_n}. \tag{42}$$

Whenever $E_1 = E_2$ for any value of the angle Θ between the neutrino and antineutrino we get $\cos\theta_n = 0$. This would therefore hold true for the back-to-back case. Moreover, in the back-to-back case we have both the back-to-back muons and the back-to-back neutrino antineutrino pair, in one single plane. This implies that for the back-to-back case we have $\phi = 0$. These choices put the orientation of the coordinate axes in such a way that the back-to-back neutrino and antineutrino fly away defining the x-axis. The xz-plane in Fig. 4 is the one in which the 3-momenta of muons lie, and now the back-to-back neutrino antineutrino define the x-direction. The direction perpendicular to the neutrino direction is the z-direction. If we define the angle between the neutrino and muon directions to be θ, then $\theta_m = \pi/2 - \theta$. This implies that

$$\cos\theta_m = \sin\theta. \tag{43}$$

Finally, we note that the energy of neutrino E_ν in the back-to-back case can be easily known from the experimentally measured energy of either of the back-to-back muons E_μ via Eq. (11). The muon energy E_μ, in the back-to-back case, can vary in the range $[m_\mu, m_B/2 - m_\nu]$. It is easy to show that for the back-to-back configuration,

$$p_1\cdot p_\pm = E_\mu\left(\frac{m_B}{2} - E_\mu\right) \mp Y_mY_n\cos\theta, \tag{44a}$$

$$p_2\cdot p_\pm = E_\mu\left(\frac{m_B}{2} - E_\mu\right) \pm Y_mY_n\cos\theta, \tag{44b}$$

with $Y_m = \sqrt{E_\mu^2 - m_\mu^2}$ and $Y_n = \sqrt{(m_B/2 - E_\mu)^2 - m_\nu^2}$.

The differential decay rate in the back-to-back case is therefore given by,

$$\frac{\mathrm{d}^3\Gamma_{\leftrightarrow}^{D/M}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta} = \frac{2\sqrt{E_\mu^2 - m_\mu^2}}{(4\pi)^6 m_B E_\mu}\left(\left(\frac{m_B}{2} - E_\mu\right)^2 - m_\nu^2\right)\langle|\mathscr{M}_{\leftrightarrow}^{D/M}|^2\rangle, \tag{45}$$

where $\langle|\mathscr{M}_{\leftrightarrow}^{D/M}|^2\rangle$ is same as $\langle|\mathscr{M}^{D/M}|^2\rangle$ with the necessary dot product substitutions as shown in Eq. (44). In the expression for $\langle|\mathscr{M}_{\leftrightarrow}^{D/M}|^2\rangle$ we have form factors which are functions of $q_\pm^2$ or $q_\pm'^2$ and it is easy to show that,

$$q_\pm^2 = m_\mu^2 + m_\nu^2 + E_\mu(m_B - 2E_\mu) + 2Y_mY_n\cos\theta, \tag{46a}$$

$$q_\pm'^2 = m_\mu^2 + m_\nu^2 + E_\mu(m_B - 2E_\mu) - 2Y_mY_n\cos\theta. \tag{46b}$$

I. The difference between Dirac and Majorana cases in back-to-back configuration

We are interested in whether there is any difference between Dirac and Majorana cases in the back-to-back configuration which would be independent of the mass m_ν which can be practically neglected in comparison with other masses and the energy E_μ. The difference between the decay rates for Dirac and Majorana cases can be obtained using Eq. (45). We find it convenient to express the difference in differential decay rates for back-to-back case, after neglecting the neutrino mass in comparison with other masses, as follows,

$$\begin{aligned}\frac{\mathrm{d}^3\Gamma^D_{\leftrightarrow}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta}-\frac{\mathrm{d}^3\Gamma^M_{\leftrightarrow}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta}=&\frac{G_F^4\sqrt{E_\mu^2-m_\mu^2}}{(4\pi)^6 m_B E_\mu}\left(\frac{m_B}{2}-E_\mu\right)^2\\
&\times((|\mathbb{F}_a|^2-|\mathbb{F}'_a|^2)\Delta_{aa}+(|\mathbb{F}_b|^2-|\mathbb{F}'_b|^2)\Delta_{bb}+(|\mathbb{F}_c|^2-|\mathbb{F}'_c|^2)\Delta_{cc}\\
&+(|\mathbf{F}_+|^2-|\mathbf{F}'_+|^2)\Delta_{pp}+(|\mathbf{F}_-|^2-|\mathbf{F}'_-|^2)\Delta_{mm}+(\mathrm{Re}(\mathbb{F}_a\mathbb{F}_b^*)-\mathrm{Re}(\mathbb{F}'_a\mathbb{F}'^*_b))\Delta_{ab}\\
&+(\mathrm{Re}(\mathbb{F}_a\mathbf{F}_+^*)-\mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_+))\Delta_{ap}+(\mathrm{Re}(\mathbb{F}_b\mathbf{F}_+^*)-\mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_+))\Delta_{bp}+(\mathrm{Re}(\mathbb{F}_a\mathbf{F}_-^*)-\mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_-))\Delta_{am}\\
&+(\mathrm{Re}(\mathbb{F}_b\mathbf{F}_-^*)-\mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_-))\Delta_{bm}+(\mathrm{Re}(\mathbb{F}_c\mathbf{F}_-^*)-\mathrm{Re}(\mathbb{F}'_c\mathbf{F}'^*_-))\Delta_{cm}+(\mathrm{Re}(\mathbf{F}_+\mathbf{F}_-^*)-\mathrm{Re}(\mathbf{F}'_+\mathbf{F}'^*_-))\Delta_{pm}\\
&+\cos\theta((|\mathbb{F}_a|^2+|\mathbb{F}'_a|^2)\Sigma_{aa}+(|\mathbb{F}_b|^2+|\mathbb{F}'_b|^2)\Sigma_{bb}+(|\mathbf{F}_+|^2+|\mathbf{F}'_+|^2)\Sigma_{pp}+(|\mathbf{F}_-|^2+|\mathbf{F}'_-|^2)\Sigma_{mm}\\
&+(\mathrm{Re}(\mathbb{F}_a\mathbb{F}_b^*)+\mathrm{Re}(\mathbb{F}'_a\mathbb{F}'^*_b))\Sigma_{ab}+(\mathrm{Re}(\mathbb{F}_a\mathbf{F}_-^*)+\mathrm{Re}(\mathbb{F}'_a\mathbf{F}'^*_-))\Sigma_{am}\\
&+(\mathrm{Re}(\mathbb{F}_b\mathbf{F}_-^*)+\mathrm{Re}(\mathbb{F}'_b\mathbf{F}'^*_-))\Sigma_{bm}+(\mathrm{Re}(\mathbf{F}_+\mathbf{F}_-^*)+\mathrm{Re}(\mathbf{F}'_+\mathbf{F}'^*_-))\Sigma_{pm})),\end{aligned}\tag{47}$$

where the various nonvanishing Δ_{ij} and Σ_{ij} terms, with $i,j\in\{a,b,c,p\equiv+,m\equiv-\}$, are given in Appendix B. It is interesting to note that, all the Σ_{ij} terms are directly proportional to $\cos\theta$, and therefore do not contribute when $\theta=\pi/2$, i.e., when the back-to-back neutrino antineutrino pair is perpendicular to the back-to-back muons. It is also true that for this very special case of $\theta=\pi/2$, we have $q_\pm^2=q_\pm'^2$ which implies that both the primed and unprimed form factors are equal in this case. This implies that when the muons fly perpendicular to the neutrino antineutrino pair in the back-to-back case, there is no difference between the Dirac and Majorana cases (when the neutrino mass is neglected in comparison with other masses). For other values of θ, the difference between Dirac and Majorana cases is nonzero, in general.

J. A simple case study

If we consider the helicity arguments of Sec. III at this stage, we have to neglect the masses of muons and neutrinos in comparison with the mass of B^0 as well as the energies. We also consider only the nonresonant contributions in the first approximation (only the Σ_{aa}, Σ_{bb} and Σ_{ab} terms survive when the muon and neutrino mass dependencies are neglected in comparison with other terms, see Appendix B). As an example, let us consider the dominant contribution that arises from the form factors $\mathbb{F}_a^{(\prime)}$ alone. For simplicity we also assume it to be a constant form factor. The full differential back-to-back decay rates are then given by,

$$\frac{\mathrm{d}^3\Gamma^D_{\leftrightarrow}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta}=\frac{G_F^4|\mathbb{F}_a|^2(m_B-2E_\mu)^4K_\mu}{512\pi^6 m_B E_\mu}(E_\mu-K_\mu\cos\theta)^2,\tag{48a}$$

$$\frac{\mathrm{d}^3\Gamma^M_{\leftrightarrow}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta}=\frac{G_F^4|\mathbb{F}_a|^2(m_B-2E_\mu)^4K_\mu}{512\pi^6 m_B E_\mu}(E_\mu^2+K_\mu^2\cos^2\theta),\tag{48b}$$

where $K_\mu=\sqrt{E_\mu^2-m_\mu^2}$ is the magnitude of the 3-momentum of the back-to-back muons. There are no m_ν dependent terms here. These distributions are shown in Fig. 5. In Figs. 5(a) and 5(b) the full distributions are shown. The one dimensional muon energy distribution obtained after integrating over $\sin\theta$ is shown in 5(c) and the angular distribution with respect to $\sin\theta$ alone is shown in Fig. 2. If we neglect m_μ as well, it is easy to see from Eqs. (48) that

$$\frac{\mathrm{d}^3\Gamma^D_{\leftrightarrow}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta}=\frac{G_F^4|\mathbb{F}_a|^2(m_B-2E_\mu)^4E_\mu^2}{512\pi^6 m_B}(1-\cos\theta)^2,\tag{49a}$$

$$\frac{\mathrm{d}^3\Gamma^M_{\leftrightarrow}}{\mathrm{d}E_\mu^2\mathrm{d}\sin\theta}=\frac{G_F^4|\mathbb{F}_a|^2(m_B-2E_\mu)^4E_\mu^2}{512\pi^6 m_B}(1+\cos^2\theta),\tag{49b}$$

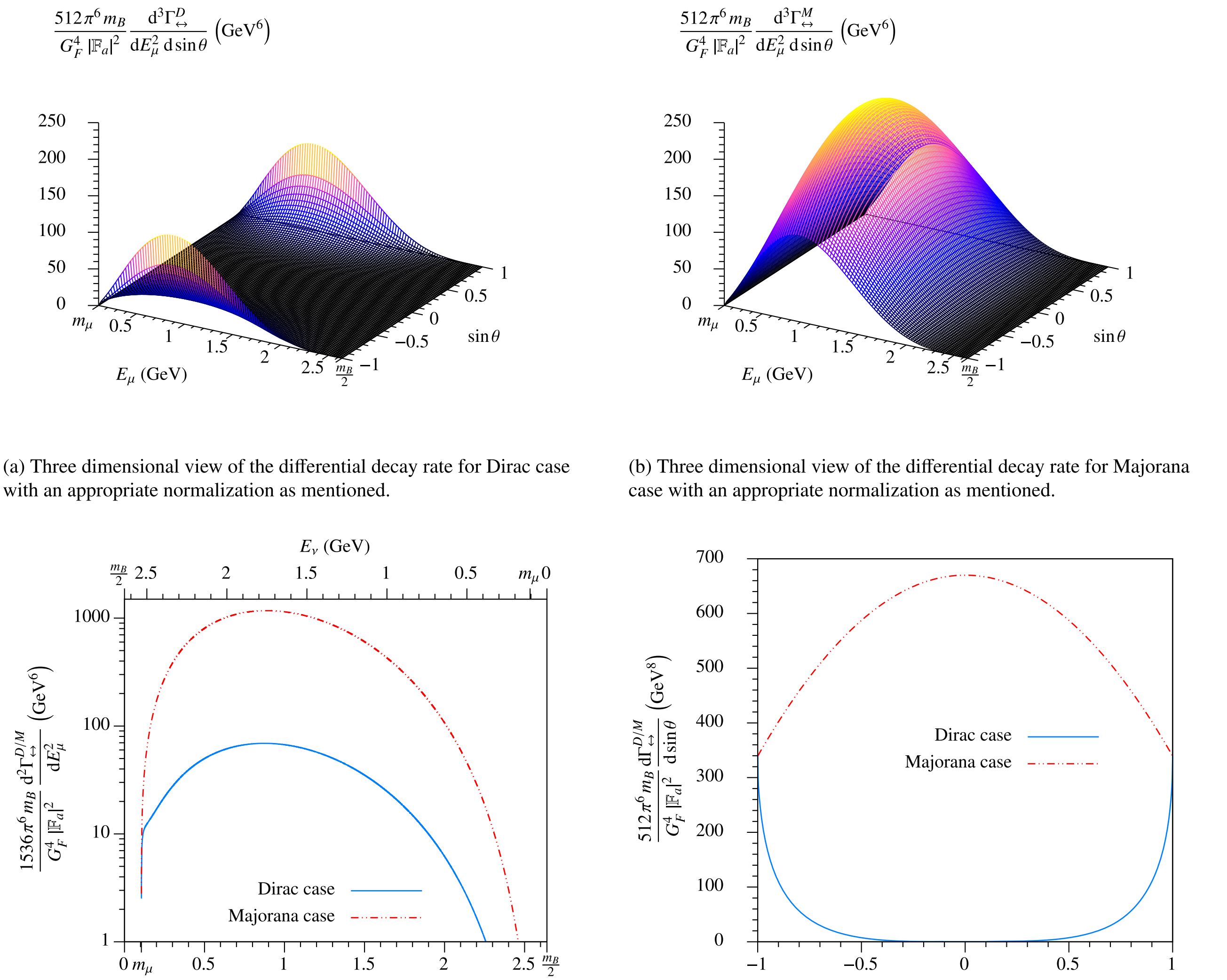
confirming our expectation in Eqs. (12), (13). The similarity between Figs. 2 and 5(d) is unmissable.

Integrating over the currently unobservable angle θ in Eq. (48) we get the muon energy distributions for Dirac and Majorana cases,

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma^D_{\leftrightarrow}}{\mathrm{d}E_\mu^2}=&\frac{G_F^4|\mathbb{F}_a|^2}{1536\pi^6 m_B E_\mu}(m_B-2E_\mu)^4K_\mu\\&\times(10E_\mu^2-3\pi E_\mu K_\mu-4m_\mu^2),\end{aligned}\tag{50a}$$

$$\frac{\mathrm{d}^2\Gamma^M_{\leftrightarrow}}{\mathrm{d}E_\mu^2}=\frac{G_F^4|\mathbb{F}_a|^2}{1536\pi^6 m_B E_\mu}(m_B-2E_\mu)^4K_\mu(10E_\mu^2-4m_\mu^2),\tag{50b}$$

which are shown in Fig. 5(c). It is also clear that there is still nonzero difference between muon energy distributions for Dirac and Majorana cases. Thus, this back-to-back muon energy distribution can be explored to distinguish between Dirac and Majorana neutrinos, and this difference is a direct consequence of the antisymmetrization of the decay amplitude for Majorana neutrinos. Therefore, the back-to-back muon energy distribution also probes the quantum statistics of the Majorana neutrinos. While observing the neutrino



(a) Three dimensional view of the differential decay rate for Dirac case with an appropriate normalization as mentioned.

(b) Three dimensional view of the differential decay rate for Majorana case with an appropriate normalization as mentioned.

(c) Comparison of muon energy distributions between Dirac and Majorana cases in the back-to-back scenario.

(d) Comparison of $\sin\theta$ distribution alone between Dirac and Majorana cases. Compare with Fig. 2.

FIG. 5. Comparison of Dirac and Majorana cases via angular distribution as given in Eq. (48) after being normalized appropriately.

energy distribution we are not using Eq. (8), instead we are utilizing Eq. (6) directly to distinguish between Dirac and Majorana neutrinos. Also note that the available phase space, as shown in Fig. 5 is nonnegligible and the effect of m_ν on the phase space can be neglected.

(a) *Branching ratio and experimental feasibility:* The branching ratios of the back-to-back configuration for Dirac and Majorana cases are estimated to be,

$$\mathcal{B}^D_{\leftrightarrow} = \Gamma^D_{\leftrightarrow}/\Gamma_B \approx 1.1 \times 10^{-12}\ \text{GeV}^{-2} \times |\mathbb{F}_a|^2, \tag{51a}$$

$$\mathcal{B}^M_{\leftrightarrow} = \Gamma^M_{\leftrightarrow}/\Gamma_B \approx 1.8 \times 10^{-11}\ \text{GeV}^{-2} \times |\mathbb{F}_a|^2, \tag{51b}$$

where $\mathbb{F}_a$ has mass dimension 1 (expressed in GeV), and Γ_B is the total decay rate of the B^0 meson. Adding the $\bar{B}^0$ mode would double the statistics. The branching ratios in Eq. (51) are very small, and at present with about 4.8×10^8 fully reconstructable B decays at Belle II [80] it is not possible to observe these back-to-back events. If in addition to the muon mode $B^0(\bar{B}^0) \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ one also considers the electron mode $B^0(\bar{B}^0) \to e^+e^-\nu_e\bar{\nu}_e$, the statistics could be increased four fold, such that the next generation of B factories might start to investigate these. One would probably require a very high-luminosity B factory to experimentally probe this back-to-back configuration. Note that the B decay considered here is only one out of many possible modes that can be exploited which we discuss in more detail in Sec. V. Therefore, the apparent experimental difficulty of observing our example B decay must be considered in this context.

(b) *Background processes:* Since flavor changing neutral current is absent at tree-level in the SM, we do not have background events that are of the same order for the B^0 decay under our consideration. Nevertheless, there are two possible B^0 decays that can mimic the final state experimental signature of $\mu^-\mu^+ +$ "missing momentum":

(1) $B^0 \to \tau^+\nu_\tau\mu^-\bar{\nu}_\mu \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu\nu_\tau\bar{\nu}_\tau$, and

(2) $B^0 \to \tau^+\tau^- \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu\nu_\tau\bar{\nu}_\tau$.

Both these decays involve (i) additional weak vertices, and (ii) phase space suppression due to six final particles when compared with the signal that has four final particles. Thus these decays are further suppressed in comparison with the doubly weak signal decay mode. One can therefore safely neglect these background processes.

(c) *For the future:* Though at present, it is not possible to detect and measure the 4-momentum of neutrinos at their place of origin, one might consider a future where technological advancements could make this feasible. Such futuristic detectors dedicated to neutrino detection might as well follow the trend of additional detector setups such as FASER [81–85] or CODEX-b [86] at the LHC, or the proposed MATHUSLA [87–91] and SHiP [92–94] detectors at the high luminosity LHC, or the proposed GAZELLE [95] detector at Belle II. Such futuristic detectors could enable us to directly probe the angular distribution of Fig. 5(d), which dramatically shows the difference between Dirac and Majorana neutrinos. Finally we note that, if the angle θ could be measured, then in addition to the difference in back-to-back branching ratios as well as the muon energy distributions of Fig. 5(c) for Dirac and Majorana cases, one can also probe whether the number of events increases away from $\theta = 0$ or not. From Fig. 5(d) the angular distribution for Majorana (or Dirac) case exhibits a down-ward (or up-ward) trend while going away from $\theta = 0$.

V. DISCUSSION ON OTHER POSSIBLE DECAY MODES

From the discussion following Eq. (51) it is clear that the study of the back-to-back kinematics of the decay $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ will have to wait for future experimental advancement (along with additional theoretical knowledge about the various form factors). Thus, it might be helpful to identify some other potential decay modes which could exhibit similar signatures as what we have found in the B^0 decay here.

One fine possibility would be to consider the Higgs decay $H \to W^{(*)}W^* \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. This could help us avoid the consideration of the unknown form factors all together. Nevertheless, a careful consideration of the Higgs decay mode presents the following challenges.

(1) *Initial 4-momentum of the Higgs boson:* The initial 4-momentum of the Higgs must be known before. This is probably achievable in an e^-e^+ collider tuned to produce the Higgs boson at rest. Therefore, the study of the Higgs decay under consideration is not feasible in any ongoing experiment such as the LHC where the initial 4-momentum of the Higgs bosons varies.

(2) *Background processes:* The final state $\mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ can also arise from other Higgs decays, such as the sequential decay involving two Z bosons, $H \to Z^{(*)}Z^* \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. Moreover, since the two neutrinos are missing, one indeed needs to distinguish the signal events from the dominant background coming from decays of τ arising from Higgs decay, $H \to \tau^-\tau^+ \to \mu^-\mu^+\nu_\tau\bar{\nu}_\tau\nu_\mu\bar{\nu}_\mu$ which also has the final signature of $\mu^-\mu^+ +$ "missing." It is easy to throw away on-shell Z contributions by studying the invariant mass square of the final muon pair or neutrino pair. However, there is no such strategy to throw away the $H \to \tau^-\tau^+$ mediated events, though such background is expected to be low in comparison with the signal decay. The major background events from two off-shell Z bosons would imply that additional Feynman diagrams must be taken into consideration, and it is not be possible to obtain analytical results for the Higgs decay by making simple substitutions in Eq. (34). It must be noted that the B decay we have considered before is free from such background processes in the SM due to absence of flavor changing neutral currents at tree level.

From these considerations, we therefore conclude that it is not straightforward to apply our results from Sec. IV in the case of the Higgs decay $H \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. Since the Higgs decay can probe a much larger set of heavy neutrino scenarios than the B decay of Sec. IV, it would be interesting to study the Higgs decay. However, a detailed study of the Higgs decay mode to differentiate Majorana neutrinos from Dirac neutrinos is beyond the scope of this paper and is thus reserved for a future work.

In addition to the B meson and Higgs (H) decays to $\mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ final state, one can also consider some other decay modes, such as $D \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ (with dominant K pole contribution), $J/\psi \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ (involving the $WW\gamma$ vertex), and $\psi(2S) \to \pi^+\pi^-\nu_\tau\bar{\nu}_\tau$ (with dominant τ pole contributions). The D decay can be analyzed in exactly the same fashion as the B decay we have considered in Sec. IV. Analogous calculations can be undertaken for the decays of J/ψ and $\psi(2S)$ with the later probing the Majorana nature of tau-neutrino.

A very interesting possibility from the experimental perspective could be the kaon decay $K^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$. If we were to consider the special case of contribution from the form factors $\mathbb{F}_a^{(\prime)}$ alone, then we would get the following branching ratios for the decays of K_S^0 and K_L^0 for the back-to-back muons configuration in the kaon rest frame:

$$\mathcal{B}^{D}_{\leftrightarrow}(K^0_S \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu) = 1.6\times 10^{-18}\ \mathrm{GeV}^{-2}\times|\mathbb{F}_a|^2, \quad (52a)$$

$$\mathcal{B}^{M}_{\leftrightarrow}(K^0_S \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu) = 7.0\times 10^{-18}\ \mathrm{GeV}^{-2}\times|\mathbb{F}_a|^2, \quad (52b)$$

$$\mathcal{B}^{D}_{\leftrightarrow}(K^0_L \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu) = 9.2\times 10^{-16}\ \mathrm{GeV}^{-2}\times|\mathbb{F}_a|^2, \quad (52c)$$

$$\mathcal{B}^{M}_{\leftrightarrow}(K^0_L \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu) = 4.0\times 10^{-15}\ \mathrm{GeV}^{-2}\times|\mathbb{F}_a|^2. \quad (52d)$$

The branching ratios given in Eq. (52) are much smaller than those for the B^0 decay as given in Eq. (51) because of the much reduced phase-space for the kaon decay. Nevertheless, kaons are both relatively much easier to produce in experiments and with extremely larger numbers than the B^0 mesons. This might make the kaon decays experimentally more accessible in near future.

Thus, we have found that one can think of many other decay modes which can potentially be tapped in a manner similar to the B^0 mode we have studied in this paper to probe the Dirac or Majorana nature of the neutrinos, light as well as heavy, if they exist. One could, in principle, extend our formalism to explore the Majorana nature of heavy neutrinos, supersymmetric neutralinos, or any other exotic electrically neutral fermions.

VI. CONCLUSION

In this paper, we have presented a technique, which is complimentary to lepton number violating processes, to probe the Majorana nature of neutrino. It is based on the idea of implementing the Fermi-Dirac statistics and hence requires presence of a neutrino antineutrino pair in the final state. We consider specifically the B meson decay $B\to\mu^-\mu^+\nu_\mu\bar{\nu}_\mu$, taking both resonant and nonresonant contributions simultaneously, in a very generalized manner. We consider the most general vertex factor for the $B\to W^*W^*$ vertex, involving three presently unknown, complex, transition form factors. The differential decay rates for Dirac and Majorana cases are expressed in terms of five independent variables: two mass squares and three angles. If we integrate over the neutrino and antineutrino momenta completely, the difference between the differential decay rates for Dirac and Majorana cases is *nonzero* albeit being directly proportional to the square of the neutrino mass. The smallness of neutrino mass could be compensated by the resonant enhancements in the decay under consideration. Nevertheless, this difference is in agreement with the "practical Dirac Majorana confusion theorem" which states that the difference between Dirac and Majorana cases would vanish when the mass of neutrino goes to zero. This mass dependence would, nevertheless, favor heavy neutrino scenarios more than the active neutrinos, if they exist in the kinematically allowed mass range.

We have demonstrated that it is possible that there can exist striking difference between Dirac and Majorana cases which do not depend on the mass of the neutrino, if we consider the special kinematic configuration of back-to-back muons in the B^0 rest frame. The branching ratio for this special kinematic situation is going to be very small. The unknown nonresonant transition form factors imply that a proper numerical study of the B decay process under our consideration, including a reliable estimate of the branching ratios, is currently not possible. Finally we note that the study of a similar decay of the Higgs $H\to\mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ is much more complicated with contributions from W mediated and Z mediated channels as well as background contributions from τ decays that arise from $H\to\tau^-\tau^+$ mode. This puts meson decays such as the decays of $B, D, J/\psi, \psi(2S)$ at a unique position that these are free from such background processes in the SM. Our approach proposed in this paper is important from the point of view that our methodology probes the Majorana nature of neutrinos by exploiting their quantum statistics which is a fundamental property.

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APPENDIX A: THE VARIOUS TERMS OF EQS. (31) AND (32)

In Eqs. (31) and (32) the following terms vanish

$$\begin{aligned} R^M_{ca'} &= R^M_{ac'} = R^{D/M}_{cp} = R^M_{c'p'} = I^M_{aa'} = I^M_{ba'} = I^M_{ab'} = I^M_{bb'} \\ &= I^M_{cb'} = I^M_{bc'} = I^{D/M}_{ap} = I^{D/M}_{bp} = I^M_{a'p} = I^{D/M}_{bm} = I^M_{a'm} \\ &= I^{D/M}_{pm} = I^M_{ap'} = I^M_{a'p'} = I^M_{b'p'} = I^M_{p'm'} = I^M_{am'} = I^M_{b'm'} = 0. \end{aligned} \quad (A1)$$

The remaining 20 nonvanishing terms for Dirac case as well as the 70 nonvanishing terms for Majorana case can be expressed by using the following expressions involving the 4-momenta of the final particles in the rest frame of the parent B meson and in terms of $m^2_{\mu\mu}$, $m^2_{\nu\nu}$, θ_m, θ_n and ϕ,

$$p_1 \cdot p_+ = -Y_m Y_n \sin\theta_m \sin\theta_n \cos\phi + \frac{Y_m Y_n (m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2)}{2m_{\nu\nu} m_{\mu\mu}} \cos\theta_m \cos\theta_n + \frac{Y Y_n m_B}{2m_{\nu\nu}} \cos\theta_n + \frac{Y Y_m m_B}{2m_{\mu\mu}} \cos\theta_m + \frac{1}{8}(m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2), \tag{A2}$$

$$p_1 \cdot p_- = Y_m Y_n \sin\theta_m \sin\theta_n \cos\phi - \frac{Y_m Y_n (m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2)}{2m_{\nu\nu} m_{\mu\mu}} \cos\theta_m \cos\theta_n + \frac{Y Y_n m_B}{2m_{\nu\nu}} \cos\theta_n - \frac{Y Y_m m_B}{2m_{\mu\mu}} \cos\theta_m + \frac{1}{8}(m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2), \tag{A3}$$

$$p_2 \cdot p_+ = Y_m Y_n \sin\theta_m \sin\theta_n \cos\phi - \frac{Y_m Y_n (m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2)}{2m_{\nu\nu} m_{\mu\mu}} \cos\theta_m \cos\theta_n - \frac{Y Y_n m_B}{2m_{\nu\nu}} \cos\theta_n + \frac{Y Y_m m_B}{2m_{\mu\mu}} \cos\theta_m + \frac{1}{8}(m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2), \tag{A4}$$

$$p_2 \cdot p_- = -Y_m Y_n \sin\theta_m \sin\theta_n \cos\phi + \frac{Y_m Y_n (m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2)}{2m_{\nu\nu} m_{\mu\mu}} \cos\theta_m \cos\theta_n - \frac{Y Y_n m_B}{2m_{\nu\nu}} \cos\theta_n - \frac{Y Y_m m_B}{2m_{\mu\mu}} \cos\theta_m + \frac{1}{8}(m_B^2 - m_{\mu\mu}^2 - m_{\nu\nu}^2), \tag{A5}$$

$$\mathbb{O} = -Y Y_m Y_n m_B \sin\theta_m \sin\theta_n \sin\phi. \tag{A6}$$

Using these, the 20 nonvanishing terms common to both Eqs. (31) and (32) are given by,

$$S_{aa}^{D/M} = 64(p_1 \cdot p_+)(p_2 \cdot p_-), \tag{A7}$$

$$\begin{aligned} S_{bb}^{D/M} = {} & 4(2(2(p_1 \cdot p_+)(p_2 \cdot p_-) + 2(p_1 \cdot p_-)(p_2 \cdot p_-) + 2(p_1 \cdot p_-)(p_1 \cdot p_+) + 2(p_1 \cdot p_-)^2 - m_B^2(p_1 \cdot p_-)) \\ & + 2m_{\nu\nu}^2((p_2 \cdot p_-) + (p_1 \cdot p_-)) + 2m_{\mu\mu}^2((p_1 \cdot p_+) + (p_1 \cdot p_-)) + m_{\nu\nu}^2 m_{\mu\mu}^2) \\ & \times (2(2(p_2 \cdot p_+)^2 + 2(p_2 \cdot p_-)(p_2 \cdot p_+) + 2(p_1 \cdot p_+)(p_2 \cdot p_+) - m_B^2(p_2 \cdot p_+) + 2(p_1 \cdot p_+)(p_2 \cdot p_-)) \\ & + 2m_{\mu\mu}^2((p_2 \cdot p_+) + (p_2 \cdot p_-)) + 2m_{\nu\nu}^2((p_2 \cdot p_+) + (p_1 \cdot p_+)) + m_{\nu\nu}^2 m_{\mu\mu}^2), \end{aligned} \tag{A8}$$

$$\begin{aligned} S_{cc}^{D/M} = {} & -4(8m_\nu^2((p_1 \cdot p_-)(p_2 \cdot p_+)^2 - 2(p_2 \cdot p_-)^2(p_2 \cdot p_+) + (p_1 \cdot p_+)(p_2 \cdot p_-)(p_2 \cdot p_+) \\ & + (p_1 \cdot p_-)^2(p_2 \cdot p_+) + (p_1 \cdot p_-)(p_1 \cdot p_+)(p_2 \cdot p_-) - 2(p_1 \cdot p_-)(p_1 \cdot p_+)^2) \\ & + 8m_\mu^2((p_1 \cdot p_-)(p_2 \cdot p_+)^2 + (p_1 \cdot p_+)(p_2 \cdot p_-)(p_2 \cdot p_+) - 2(p_1 \cdot p_+)^2(p_2 \cdot p_+) \\ & + (p_1 \cdot p_-)^2(p_2 \cdot p_+) - 2(p_1 \cdot p_-)(p_2 \cdot p_-)^2 + (p_1 \cdot p_-)(p_1 \cdot p_+)(p_2 \cdot p_-)) \\ & - 8m_\mu^2 m_\nu^2(3(p_2 \cdot p_-)(p_2 \cdot p_+) + 3(p_1 \cdot p_+)(p_2 \cdot p_+) + 2(p_1 \cdot p_-)(p_2 \cdot p_+) + 3(p_2 \cdot p_-)^2 \\ & + 4(p_1 \cdot p_+)(p_2 \cdot p_-) + 3(p_1 \cdot p_-)(p_2 \cdot p_-) + 3(p_1 \cdot p_+)^2 + 3(p_1 \cdot p_-)(p_1 \cdot p_+)) \\ & + 4m_{\mu\mu}^2 m_\nu^2(2(p_2 \cdot p_-)(p_2 \cdot p_+) + (p_1 \cdot p_+)(p_2 \cdot p_+) + 4(p_1 \cdot p_-)(p_2 \cdot p_+) \\ & + 4(p_1 \cdot p_+)(p_2 \cdot p_-) + (p_1 \cdot p_-)(p_2 \cdot p_-) + 2(p_1 \cdot p_-)(p_1 \cdot p_+)) \\ & + 4m_{\nu\nu}^2 m_\mu^2((p_2 \cdot p_-)(p_2 \cdot p_+) + 2(p_1 \cdot p_+)(p_2 \cdot p_+) + 4(p_1 \cdot p_-)(p_2 \cdot p_+) \\ & + 4(p_1 \cdot p_+)(p_2 \cdot p_-) + 2(p_1 \cdot p_-)(p_2 \cdot p_-) + (p_1 \cdot p_-)(p_1 \cdot p_+)) \\ & - 4m_{\nu\nu}^2 m_\nu^2((p_2 \cdot p_-)(p_2 \cdot p_+) + (p_1 \cdot p_-)(p_1 \cdot p_+)) \\ & - 4m_{\mu\mu}^2 m_\mu^2((p_1 \cdot p_+)(p_2 \cdot p_+) + (p_1 \cdot p_-)(p_2 \cdot p_-)) \\ & + 16((p_1 \cdot p_-)(p_2 \cdot p_+) - (p_1 \cdot p_+)^2)((p_1 \cdot p_-)(p_2 \cdot p_+) - (p_2 \cdot p_-)^2) \end{aligned}$$

$$
\begin{aligned}
&- 8m_{\nu\nu}^2 m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-))\\
&+ 4m_{\mu\mu}^2 m_\nu^4((p_2\cdot p_+) + (p_2\cdot p_-) + (p_1\cdot p_+) + (p_1\cdot p_-))\\
&+ 4m_{\nu\nu}^2 m_\mu^4((p_2\cdot p_+) + (p_2\cdot p_-) + (p_1\cdot p_+) + (p_1\cdot p_-))\\
&+ 8m_\mu^4((p_1\cdot p_+) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_2\cdot p_-))\\
&+ 4m_{\mu\mu}^2 m_\mu^2 m_\nu^2((p_2\cdot p_+) - (p_2\cdot p_-) - (p_1\cdot p_+) + (p_1\cdot p_-))\\
&+ 4m_{\nu\nu}^2 m_\mu^2 m_\nu^2((p_2\cdot p_+) - (p_2\cdot p_-) - (p_1\cdot p_+) + (p_1\cdot p_-))\\
&+ 8m_\nu^4((p_2\cdot p_-) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_+))\\
&- 2m_{\nu\nu}^2 m_{\mu\mu}^2 m_\nu^2((p_2\cdot p_+) + (p_1\cdot p_-)) - 2m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2((p_2\cdot p_+) + (p_1\cdot p_-))\\
&+ 2m_{\mu\mu}^4 m_\nu^4 + 12m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2 m_\nu^2 - 4m_{\nu\nu}^2 m_{\mu\mu}^4 m_\nu^2 + 2m_{\nu\nu}^4 m_\mu^4 - 4m_{\nu\nu}^4 m_{\mu\mu}^2 m_\mu^2 + m_{\nu\nu}^4 m_{\mu\mu}^4),
\end{aligned}
\tag{A9}
$$

$$
S_{pp}^{D/M} = 16(m_\nu^2(p_1\cdot p_-) + m_\mu^2(p_1\cdot p_-) + 2m_\mu^2 m_\nu^2)(m_\nu^2(p_2\cdot p_+) + m_\mu^2(p_2\cdot p_+) + 2m_\mu^2 m_\nu^2),
\tag{A10}
$$

$$
\begin{aligned}
S_{mm}^{D/M} = &-8(m_\nu^2(p_1\cdot p_-) + m_\mu^2(p_1\cdot p_-) + 2m_\mu^2 m_\nu^2)(4((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-))\\
&+ 2m_\mu^2((p_2\cdot p_+) + 2(p_2\cdot p_-)) + 2m_\nu^2((p_2\cdot p_+) + 2(p_1\cdot p_+)) - 2m_{\mu\mu}^2(p_2\cdot p_-) - 2m_{\nu\nu}^2(p_1\cdot p_+)\\
&- 4m_\mu^2 m_\nu^2 + 2m_{\mu\mu}^2 m_\nu^2 + 2m_{\nu\nu}^2 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^2),
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
R_{ab}^{D/M} = &-8(4(2(p_1\cdot p_-)(p_2\cdot p_+)^2 + 2(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) + 4(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)\\
&+ 4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+) + 2(p_1\cdot p_-)^2(p_2\cdot p_+) - m_B^2(p_1\cdot p_-)(p_2\cdot p_+)\\
&+ 2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) - m_B^2(p_1\cdot p_+)(p_2\cdot p_-))\\
&+ 4m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-)) + 4m_{\nu\nu}^2((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-))\\
&- m_{\nu\nu}^2 m_{\mu\mu}^2(2(p_2\cdot p_+) + 4(p_2\cdot p_-) + 4(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&+ 2m_{\mu\mu}^2 m_\nu^2(2(p_2\cdot p_+) + 2(p_2\cdot p_-) + 2(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&+ 2m_{\nu\nu}^2 m_\mu^2(2(p_2\cdot p_+) + 2(p_2\cdot p_-) + 2(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&+ 8m_\mu^2((p_1\cdot p_+) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_2\cdot p_-))\\
&+ 8m_\nu^2((p_2\cdot p_-) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_+))\\
&+ 4m_\mu^2 m_\nu^2 m_B^2 + 2m_{\mu\mu}^4 m_\nu^2 + 2m_{\nu\nu}^4 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^4 - m_{\nu\nu}^4 m_{\mu\mu}^2),
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
R_{ac}^{D/M} = &-16(4m_\nu^2((p_2\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_-)(p_1\cdot p_+)) - 4m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+) - (p_1\cdot p_-)(p_2\cdot p_-))\\
&+ 4((p_2\cdot p_-) - (p_1\cdot p_+))((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-))\\
&+ 2m_{\mu\mu}^2 m_\nu^2((p_2\cdot p_-) - (p_1\cdot p_+)) + 2m_{\nu\nu}^2 m_\mu^2((p_2\cdot p_-) - (p_1\cdot p_+))\\
&- m_{\nu\nu}^2 m_{\mu\mu}^2((p_2\cdot p_-) - (p_1\cdot p_+))),
\end{aligned}
\tag{A13}
$$

$$
\begin{aligned}
R_{bc}^{D/M} = &\,8(4m_\nu^2(2(p_2\cdot p_-)(p_2\cdot p_+)^2 + 4(p_2\cdot p_-)^2(p_2\cdot p_+) + 4(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)\\
&- m_B^2(p_2\cdot p_-)(p_2\cdot p_+) - 4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+) + 2(p_1\cdot p_+)(p_2\cdot p_-)^2\\
&- 2(p_1\cdot p_+)^2(p_2\cdot p_-) - 4(p_1\cdot p_-)(p_1\cdot p_+)^2 - 2(p_1\cdot p_-)^2(p_1\cdot p_+) + m_B^2(p_1\cdot p_-)(p_1\cdot p_+))\\
&- 4m_\mu^2(2(p_1\cdot p_+)(p_2\cdot p_+)^2 - 4(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+) + 4(p_1\cdot p_+)^2(p_2\cdot p_+)\\
&+ 4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+) - m_B^2(p_1\cdot p_+)(p_2\cdot p_+) - 2(p_1\cdot p_+)(p_2\cdot p_-)^2\\
&- 4(p_1\cdot p_-)(p_2\cdot p_-)^2 + 2(p_1\cdot p_+)^2(p_2\cdot p_-) - 2(p_1\cdot p_-)^2(p_2\cdot p_-) + m_B^2(p_1\cdot p_-)(p_2\cdot p_-))
\end{aligned}
$$

$$
\begin{aligned}
&+8((p_2\cdot p_-)-(p_1\cdot p_+))(2(p_1\cdot p_-)(p_2\cdot p_+)^2+3(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)\\
&+3(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+)+2(p_1\cdot p_-)^2(p_2\cdot p_+)-m_B^2(p_1\cdot p_-)(p_2\cdot p_+)\\
&-(p_1\cdot p_+)(p_2\cdot p_-)^2-(p_1\cdot p_+)^2(p_2\cdot p_-))\\
&-2m_{\nu\nu}^2m_\mu^2(2(p_2\cdot p_+)^2+4(p_2\cdot p_-)(p_2\cdot p_+)+2(p_1\cdot p_+)(p_2\cdot p_+)-m_B^2(p_2\cdot p_+)+2(p_2\cdot p_-)^2\\
&-2(p_1\cdot p_-)(p_2\cdot p_-)-2(p_1\cdot p_+)^2-4(p_1\cdot p_-)(p_1\cdot p_+)-2(p_1\cdot p_-)^2+m_B^2(p_1\cdot p_-))\\
&+2m_{\mu\mu}^2m_\nu^2(2(p_2\cdot p_+)^2+2(p_2\cdot p_-)(p_2\cdot p_+)+4(p_1\cdot p_+)(p_2\cdot p_+)-m_B^2(p_2\cdot p_+)-2(p_2\cdot p_-)^2\\
&-4(p_1\cdot p_-)(p_2\cdot p_-)+2(p_1\cdot p_+)^2-2(p_1\cdot p_-)(p_1\cdot p_+)-2(p_1\cdot p_-)^2+m_B^2(p_1\cdot p_-))\\
&+4m_{\nu\nu}^2m_\nu^2((p_2\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_-)(p_1\cdot p_+))\\
&-4m_{\mu\mu}^2m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+)-(p_1\cdot p_-)(p_2\cdot p_-))\\
&+4(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_2\cdot p_-)-(p_1\cdot p_+))((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))\\
&-2m_{\nu\nu}^2m_{\mu\mu}^2m_\mu^2((p_2\cdot p_+)+2(p_2\cdot p_-)-2(p_1\cdot p_+)-(p_1\cdot p_-))\\
&+16m_\mu^2m_\nu^2((p_2\cdot p_-)-(p_1\cdot p_+))((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))\\
&-2m_{\nu\nu}^4m_\mu^2((p_2\cdot p_+)+(p_2\cdot p_-)-(p_1\cdot p_+)-(p_1\cdot p_-))\\
&+2m_{\mu\mu}^4m_\nu^2((p_2\cdot p_+)-(p_2\cdot p_-)+(p_1\cdot p_+)-(p_1\cdot p_-))\\
&+2m_{\nu\nu}^2m_{\mu\mu}^2m_\nu^2((p_2\cdot p_+)-2(p_2\cdot p_-)+2(p_1\cdot p_+)-(p_1\cdot p_-))\\
&+2m_{\nu\nu}^2m_{\mu\mu}^2((p_2\cdot p_-)-(p_1\cdot p_+))((p_2\cdot p_-)+(p_1\cdot p_+))\\
&+(8m_\mu^2m_\nu^2+m_{\nu\nu}^2m_{\mu\mu}^2)(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_2\cdot p_-)-(p_1\cdot p_+))),
\end{aligned}\tag{A14}
$$

$$
R_{ap}^{D/M}=16(2m_\mu^2m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_+))-2m_\mu^2m_\nu^4+m_{\mu\mu}^2m_\nu^4-2m_\mu^4m_\nu^2+m_{\nu\nu}^2m_\mu^4),\tag{A15}
$$

$$
\begin{aligned}
R_{bp}^{D/M}=&\,8(2m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))+2m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2)\\
&\times(2m_\mu^2((p_2\cdot p_+)+(p_2\cdot p_-))+2m_\nu^2((p_2\cdot p_+)+(p_1\cdot p_+))+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2),
\end{aligned}\tag{A16}
$$

$$
\begin{aligned}
R_{am}^{D/M}=&-8(4m_\nu^2((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))+4m_\mu^2((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))\\
&+8m_\mu^2m_\nu^2((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+))-4m_{\mu\mu}^2m_\nu^2(p_2\cdot p_-)-4m_{\nu\nu}^2m_\mu^2(p_1\cdot p_+)-4m_\mu^2m_\nu^4\\
&+2m_{\mu\mu}^2m_\nu^4-4m_\mu^4m_\nu^2+2m_{\mu\mu}^2m_\mu^2m_\nu^2+2m_{\nu\nu}^2m_\mu^2m_\nu^2-m_{\nu\nu}^2m_{\mu\mu}^2m_\nu^2+2m_{\nu\nu}^2m_\mu^4-m_{\nu\nu}^2m_{\mu\mu}^2m_\mu^2),
\end{aligned}\tag{A17}
$$

$$
\begin{aligned}
R_{bm}^{D/M}=&-8(2m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))+2m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2)\\
&\times(4((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))+2m_\mu^2((p_2\cdot p_+)+(p_2\cdot p_-))\\
&+2m_\nu^2((p_2\cdot p_+)+(p_1\cdot p_+))-2m_{\mu\mu}^2(p_2\cdot p_-)-2m_{\nu\nu}^2(p_1\cdot p_+)+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2-m_{\nu\nu}^2m_{\mu\mu}^2),
\end{aligned}\tag{A18}
$$

$$
\begin{aligned}
R_{cm}^{D/M}=&\,4(m_\mu^2-m_\nu^2)(-4(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_1\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-))\\
&+8((p_2\cdot p_-)+(p_1\cdot p_+))((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))\\
&+16m_\mu^2m_\nu^2((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))\\
&-4(m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2)((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))\\
&+8m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_2\cdot p_-))\\
&+8m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_1\cdot p_+))\\
&+2m_{\nu\nu}^2m_{\mu\mu}^2((p_2\cdot p_-)+(p_1\cdot p_+))+8m_{\mu\mu}^2m_\mu^2m_\nu^2+8m_{\nu\nu}^2m_\mu^2m_\nu^2-2m_{\mu\mu}^4m_\nu^2\\
&-4m_{\nu\nu}^2m_{\mu\mu}^2m_\nu^2-4m_{\nu\nu}^2m_{\mu\mu}^2m_\mu^2-2m_{\nu\nu}^4m_\mu^2+m_{\nu\nu}^2m_{\mu\mu}^4+m_{\nu\nu}^4m_{\mu\mu}^2),
\end{aligned}\tag{A19}
$$

$$R_{pm}^{D/M} = 16(m_\nu^2(p_1 \cdot p_-) + m_\mu^2(p_1 \cdot p_-) + 2m_\mu^2 m_\nu^2)(2m_\mu^2(p_2 \cdot p_-) + 2m_\nu^2(p_1 \cdot p_+) - 4m_\mu^2 m_\nu^2 + m_{\mu\mu}^2 m_\nu^2 + m_{\nu\nu}^2 m_\mu^2), \quad \text{(A20)}$$

$$I_{ab}^{D/M} = 32\mathbb{O}(2(p_2 \cdot p_+) + 4(p_1 \cdot p_+) + 2(p_1 \cdot p_-) - m_B^2 + m_{\mu\mu}^2 + m_{\nu\nu}^2), \quad \text{(A21)}$$

$$I_{ac}^{D/M} = -64\mathbb{O}((p_2 \cdot p_-) + (p_1 \cdot p_+)), \quad \text{(A22)}$$

$$I_{bc}^{D/M} = 32\mathbb{O}(2(2(p_1 \cdot p_-)(p_2 \cdot p_+) - (p_2 \cdot p_-)^2 - (p_1 \cdot p_+)^2) + 2(m_\mu^2 + m_\nu^2)((p_2 \cdot p_+) + (p_2 \cdot p_-) + (p_1 \cdot p_+) + (p_1 \cdot p_-)) - (m_{\mu\mu}^2 + m_{\nu\nu}^2)((p_2 \cdot p_-) + (p_1 \cdot p_+)) + (m_{\mu\mu}^2 + m_{\nu\nu}^2)(m_\mu^2 + m_\nu^2) - m_{\nu\nu}^2 m_{\mu\mu}^2), \quad \text{(A23)}$$

$$I_{cp}^{D/M} = -32\mathbb{O}(m_\mu^2 - m_\nu^2)^2, \quad \text{(A24)}$$

$$I_{am}^{D/M} = -32\mathbb{O}(m_\mu^2 - m_\nu^2), \quad \text{(A25)}$$

$$I_{cm}^{D/M} = 16\mathbb{O}(m_\mu^2 - m_\nu^2)(2((p_2 \cdot p_-) - (p_1 \cdot p_+)) - 2m_\nu^2 + 2m_\mu^2 - m_{\mu\mu}^2 + m_{\nu\nu}^2). \quad \text{(A26)}$$

The rest of the 50 nonvanishing terms exclusive to Majorana case and appearing in Eq. (32) are given by,

$$S_{a'a'}^M = 64(p_1 \cdot p_-)(p_2 \cdot p_+), \quad \text{(A27)}$$

$$S_{b'b'}^M = 4(2(2(p_1 \cdot p_+)(p_2 \cdot p_+) + 2(p_1 \cdot p_-)(p_2 \cdot p_+) + 2(p_1 \cdot p_+)^2 + 2(p_1 \cdot p_-)(p_1 \cdot p_+) - m_B^2(p_1 \cdot p_+)) + 2m_{\nu\nu}^2((p_2 \cdot p_+) + (p_1 \cdot p_+)) + 2m_{\mu\mu}^2((p_1 \cdot p_+) + (p_1 \cdot p_-)) + m_{\nu\nu}^2 m_{\mu\mu}^2) \times (2(2(p_2 \cdot p_-)(p_2 \cdot p_+) + 2(p_1 \cdot p_-)(p_2 \cdot p_+) + 2(p_2 \cdot p_-)^2 + 2(p_1 \cdot p_-)(p_2 \cdot p_-) - m_B^2(p_2 \cdot p_-)) + 2m_{\mu\mu}^2((p_2 \cdot p_+) + (p_2 \cdot p_-)) + 2m_{\nu\nu}^2((p_2 \cdot p_-) + (p_1 \cdot p_-)) + m_{\nu\nu}^2 m_{\mu\mu}^2), \quad \text{(A28)}$$

$$\begin{aligned}
S_{c'c'}^M = {} & 4(8m_\nu^2(2(p_2 \cdot p_-)(p_2 \cdot p_+)^2 - (p_1 \cdot p_-)(p_2 \cdot p_-)(p_2 \cdot p_+) - (p_1 \cdot p_-)(p_1 \cdot p_+)(p_2 \cdot p_+) \\
& - (p_1 \cdot p_+)(p_2 \cdot p_-)^2 - (p_1 \cdot p_+)^2(p_2 \cdot p_-) + 2(p_1 \cdot p_-)^2(p_1 \cdot p_+)) \\
& + 8m_\mu^2(2(p_1 \cdot p_+)(p_2 \cdot p_+)^2 - (p_1 \cdot p_-)(p_2 \cdot p_-)(p_2 \cdot p_+) - (p_1 \cdot p_-)(p_1 \cdot p_+)(p_2 \cdot p_+) \\
& - (p_1 \cdot p_+)(p_2 \cdot p_-)^2 - (p_1 \cdot p_+)^2(p_2 \cdot p_-) + 2(p_1 \cdot p_-)^2(p_2 \cdot p_-)) \\
& + 8m_\mu^2 m_\nu^2(3(p_2 \cdot p_+)^2 + 3(p_2 \cdot p_-)(p_2 \cdot p_+) + 3(p_1 \cdot p_+)(p_2 \cdot p_+) + 4(p_1 \cdot p_-)(p_2 \cdot p_+) \\
& + 2(p_1 \cdot p_+)(p_2 \cdot p_-) + 3(p_1 \cdot p_-)(p_2 \cdot p_-) + 3(p_1 \cdot p_-)(p_1 \cdot p_+) + 3(p_1 \cdot p_-)^2) \\
& + 16((p_1 \cdot p_+)(p_2 \cdot p_-) - (p_1 \cdot p_-)^2)((p_2 \cdot p_+)^2 - (p_1 \cdot p_+)(p_2 \cdot p_-)) \\
& - 4m_{\mu\mu}^2 m_\nu^2(2(p_2 \cdot p_-)(p_2 \cdot p_+) + (p_1 \cdot p_+)(p_2 \cdot p_+) + 4(p_1 \cdot p_-)(p_2 \cdot p_+) \\
& + 4(p_1 \cdot p_+)(p_2 \cdot p_-) + (p_1 \cdot p_-)(p_2 \cdot p_-) + 2(p_1 \cdot p_-)(p_1 \cdot p_+)) \\
& - 4m_{\nu\nu}^2 m_\mu^2((p_2 \cdot p_-)(p_2 \cdot p_+) + 2(p_1 \cdot p_+)(p_2 \cdot p_+) + 4(p_1 \cdot p_-)(p_2 \cdot p_+) \\
& + 4(p_1 \cdot p_+)(p_2 \cdot p_-) + 2(p_1 \cdot p_-)(p_2 \cdot p_-) + (p_1 \cdot p_-)(p_1 \cdot p_+)) \\
& + 4m_{\nu\nu}^2 m_\nu^2((p_2 \cdot p_-)(p_2 \cdot p_+) + (p_1 \cdot p_-)(p_1 \cdot p_+)) \\
& + 4m_{\mu\mu}^2 m_\mu^2((p_1 \cdot p_+)(p_2 \cdot p_+) + (p_1 \cdot p_-)(p_2 \cdot p_-)) \\
& + 8m_{\nu\nu}^2 m_{\mu\mu}^2((p_1 \cdot p_-)(p_2 \cdot p_+) + (p_1 \cdot p_+)(p_2 \cdot p_-)) \\
& - 4(m_{\mu\mu}^2 + m_{\nu\nu}^2)m_\nu^4((p_2 \cdot p_+) + (p_2 \cdot p_-) + (p_1 \cdot p_+) + (p_1 \cdot p_-)) \\
& - 8m_\mu^4((p_1 \cdot p_+) + (p_1 \cdot p_-))((p_2 \cdot p_+) + (p_2 \cdot p_-))
\end{aligned}$$

$$
\begin{aligned}
&+4(m_{\mu\mu}^2+m_{\nu\nu}^2)m_\mu^2 m_\nu^2((p_2\cdot p_+)-(p_2\cdot p_-)-(p_1\cdot p_+)+(p_1\cdot p_-))\\
&-8m_\nu^4((p_2\cdot p_-)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_1\cdot p_+))\\
&+2m_{\nu\nu}^2 m_{\mu\mu}^2(m_\mu^2+m_\nu^2)((p_2\cdot p_-)+(p_1\cdot p_+))-2m_{\mu\mu}^4 m_\nu^4-12m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2 m_\nu^2+4m_{\nu\nu}^2 m_{\mu\mu}^4 m_\nu^2\\
&-2m_{\nu\nu}^4 m_\mu^4+4m_{\nu\nu}^4 m_{\mu\mu}^2 m_\mu^2-m_{\nu\nu}^4 m_{\mu\mu}^4),
\end{aligned}
\tag{A29}
$$

$$
S^M_{p'p'}=16((m_\mu^2+m_\nu^2)(p_1\cdot p_+)+2m_\mu^2 m_\nu^2)((m_\mu^2+m_\nu^2)(p_2\cdot p_-)+2m_\mu^2 m_\nu^2),
\tag{A30}
$$

$$
\begin{aligned}
S^M_{m'm'}=\;&8(m_\nu^2(p_2\cdot p_-)+m_\mu^2(p_2\cdot p_-)+2m_\mu^2 m_\nu^2)\\
&\times(4((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))-2m_\nu^2(2(p_2\cdot p_+)+(p_1\cdot p_+))\\
&+2m_{\nu\nu}^2(p_2\cdot p_+)-2m_\mu^2((p_1\cdot p_+)+2(p_1\cdot p_-))+2m_{\mu\mu}^2(p_1\cdot p_-)\\
&+4m_\mu^2 m_\nu^2-2m_{\mu\mu}^2 m_\nu^2-2m_{\nu\nu}^2 m_\mu^2+m_{\nu\nu}^2 m_{\mu\mu}^2),
\end{aligned}
\tag{A31}
$$

$$
R^M_{aa'}=32(2m_\mu^2-m_{\mu\mu}^2),
\tag{A32}
$$

$$
R^M_{ba'}=-16(2((p_2\cdot p_-)+(p_1\cdot p_-))+m_{\mu\mu}^2)(2((p_2\cdot p_+)+(p_1\cdot p_+))+m_{\mu\mu}^2),
\tag{A33}
$$

$$
R^M_{ab'}=-16(2((p_2\cdot p_-)+(p_1\cdot p_-))+m_{\mu\mu}^2)(2((p_2\cdot p_+)+(p_1\cdot p_+))+m_{\mu\mu}^2),
\tag{A34}
$$

$$
R^M_{bb'}=8m_B^4(2m_\mu^2-m_{\mu\mu}^2),
\tag{A35}
$$

$$
\begin{aligned}
R^M_{cb'}=\;&-16(4((p_1\cdot p_-)(p_2\cdot p_+)^2-(p_2\cdot p_-)^2(p_2\cdot p_+)-2(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)\\
&+2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+)-(p_1\cdot p_-)^2(p_2\cdot p_+)+(p_1\cdot p_-)(p_1\cdot p_+)^2)\\
&-2(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_2\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_-)(p_1\cdot p_+))\\
&+4m_\mu^2(((p_2\cdot p_+)+(p_1\cdot p_+))^2-((p_1\cdot p_-)+(p_2\cdot p_-))^2)\\
&+2(m_{\mu\mu}^2+m_{\nu\nu}^2)m_\mu^2((p_2\cdot p_+)-(p_2\cdot p_-)+(p_1\cdot p_+)-(p_1\cdot p_-))\\
&-m_{\nu\nu}^2 m_{\mu\mu}^2((p_2\cdot p_+)-(p_1\cdot p_-))),
\end{aligned}
\tag{A36}
$$

$$
\begin{aligned}
R^M_{a'b'}=\;&-8(4(4(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+)+2(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)+2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+)\\
&-m_B^2(p_1\cdot p_-)(p_2\cdot p_+)+2(p_1\cdot p_+)(p_2\cdot p_-)^2+2(p_1\cdot p_+)^2(p_2\cdot p_-)\\
&+4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-)-m_B^2(p_1\cdot p_+)(p_2\cdot p_-))\\
&-4(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))\\
&-m_{\nu\nu}^2 m_{\mu\mu}^2(4(p_2\cdot p_+)+2(p_2\cdot p_-)+2(p_1\cdot p_+)+4(p_1\cdot p_-)-m_B^2)\\
&+2(m_{\mu\mu}^2 m_\nu^2+m_{\nu\nu}^2 m_\mu^2)(2(p_2\cdot p_+)+2(p_2\cdot p_-)+2(p_1\cdot p_+)+2(p_1\cdot p_-)-m_B^2)\\
&+8m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_2\cdot p_-))\\
&+8m_\nu^2((p_1\cdot p_+)+(p_2\cdot p_+))((p_1\cdot p_-)+(p_2\cdot p_-))\\
&+4m_\mu^2 m_\nu^2 m_B^2+2m_{\mu\mu}^4 m_\nu^2+2m_{\nu\nu}^4 m_\mu^2-m_{\nu\nu}^2 m_{\mu\mu}^4-m_{\nu\nu}^4 m_{\mu\mu}^2),
\end{aligned}
\tag{A37}
$$

$$
\begin{aligned}
R^M_{bc'}=\;&-16(4((p_2\cdot p_-)(p_2\cdot p_+)^2+2(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-)^2\\
&+(p_1\cdot p_+)^2(p_2\cdot p_-)-2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-)-(p_1\cdot p_-)^2(p_1\cdot p_+))\\
&+2(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_2\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_-)(p_1\cdot p_+))\\
&+4m_\mu^2(((p_2\cdot p_+)+(p_1\cdot p_+))^2-((p_1\cdot p_-)+(p_2\cdot p_-))^2)\\
&+2(m_{\mu\mu}^2+m_{\nu\nu}^2)m_\mu^2((p_2\cdot p_+)-(p_2\cdot p_-)+(p_1\cdot p_+)-(p_1\cdot p_-))\\
&+m_{\nu\nu}^2 m_{\mu\mu}^2((p_2\cdot p_-)-(p_1\cdot p_+))),
\end{aligned}
\tag{A38}
$$

$$
\begin{aligned}
R^M_{cc'} = {} & 8(4m_\mu^2((p_2\cdot p_+)^2 + 2(p_2\cdot p_-)(p_2\cdot p_+) + (p_2\cdot p_-)^2 \\
& + (p_1\cdot p_+)^2 + 2(p_1\cdot p_-)(p_1\cdot p_+) + (p_1\cdot p_-)^2) \\
& - 4m_{\mu\mu}^2((p_2\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_-)(p_1\cdot p_+)) \\
& + 4m_{\nu\nu}^2((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-)) \\
& + 4((p_2\cdot p_+) - (p_2\cdot p_-) - (p_1\cdot p_+) + (p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-)) \\
& - 16m_\mu^2 m_\nu^2((p_2\cdot p_+) + (p_2\cdot p_-) + (p_1\cdot p_+) + (p_1\cdot p_-)) \\
& + 4(m_{\mu\mu}^2 m_\nu^2 + m_{\nu\nu}^2 m_\mu^2)((p_2\cdot p_+) + (p_2\cdot p_-) + (p_1\cdot p_+) + (p_1\cdot p_-)) \\
& - m_{\nu\nu}^2 m_{\mu\mu}^2((p_2\cdot p_+) + (p_2\cdot p_-) + (p_1\cdot p_+) + (p_1\cdot p_-)) \\
& - 8m_\nu^2((p_2\cdot p_-) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_+)) \\
& - 8m_{\mu\mu}^2 m_\mu^2 m_\nu^2 - 8m_{\nu\nu}^2 m_\mu^2 m_\nu^2 + 2m_{\mu\mu}^4 m_\nu^2 + 4m_{\nu\nu}^2 m_{\mu\mu}^2 m_\nu^2 + 2m_{\nu\nu}^4 m_\mu^2 - m_{\nu\nu}^4 m_{\mu\mu}^2),
\end{aligned} \tag{A39}
$$

$$
\begin{aligned}
R^M_{a'c'} = {} & 16(4m_\nu^2((p_2\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_-)(p_1\cdot p_+)) + 4m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+) - (p_1\cdot p_-)(p_2\cdot p_-)) \\
& + 4((p_2\cdot p_+) - (p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-)) \\
& + (2(m_{\mu\mu}^2 m_\nu^2 + m_{\nu\nu}^2 m_\mu^2) - m_{\nu\nu}^2 m_{\mu\mu}^2)((p_2\cdot p_+) - (p_1\cdot p_-))),
\end{aligned} \tag{A40}
$$

$$
\begin{aligned}
R^M_{b'c'} = {} & 8(-4m_\nu^2(4(p_2\cdot p_-)(p_2\cdot p_+)^2 + 2(p_1\cdot p_-)(p_2\cdot p_+)^2 + 2(p_2\cdot p_-)^2(p_2\cdot p_+) \\
& + 4(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) - m_B^2(p_2\cdot p_-)(p_2\cdot p_+) - 2(p_1\cdot p_-)^2(p_2\cdot p_+) \\
& - 4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) - 2(p_1\cdot p_-)(p_1\cdot p_+)^2 - 4(p_1\cdot p_-)^2(p_1\cdot p_+) \\
& + m_B^2(p_1\cdot p_-)(p_1\cdot p_+)) \\
& - 4m_\mu^2(4(p_1\cdot p_+)(p_2\cdot p_+)^2 + 2(p_1\cdot p_-)(p_2\cdot p_+)^2 + 4(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) \\
& + 2(p_1\cdot p_+)^2(p_2\cdot p_+) - m_B^2(p_1\cdot p_+)(p_2\cdot p_+) - 2(p_1\cdot p_-)^2(p_2\cdot p_+) \\
& - 2(p_1\cdot p_-)(p_2\cdot p_-)^2 - 4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) \\
& - 4(p_1\cdot p_-)^2(p_2\cdot p_-) + m_B^2(p_1\cdot p_-)(p_2\cdot p_-)) \\
& + 8((p_2\cdot p_+) - (p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+)^2 - 3(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) \\
& + (p_1\cdot p_-)^2(p_2\cdot p_+) - 2(p_1\cdot p_+)(p_2\cdot p_-)^2 \\
& - 2(p_1\cdot p_+)^2(p_2\cdot p_-) - 3(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) \\
& + m_B^2(p_1\cdot p_+)(p_2\cdot p_-)) \\
& + 2m_{\nu\nu}^2 m_\mu^2(2(p_2\cdot p_+)^2 + 4(p_2\cdot p_-)(p_2\cdot p_+) - 2(p_1\cdot p_+)(p_2\cdot p_+) + 2(p_2\cdot p_-)^2 \\
& + 2(p_1\cdot p_-)(p_2\cdot p_-) - m_B^2(p_2\cdot p_-) - 2(p_1\cdot p_+)^2 - 4(p_1\cdot p_-)(p_1\cdot p_+) \\
& + m_B^2(p_1\cdot p_+) - 2(p_1\cdot p_-)^2) \\
& + 2m_{\mu\mu}^2 m_\nu^2(2(p_2\cdot p_+)^2 - 2(p_2\cdot p_-)(p_2\cdot p_+) + 4(p_1\cdot p_+)(p_2\cdot p_+) - 2(p_2\cdot p_-)^2 \\
& - 4(p_1\cdot p_-)(p_2\cdot p_-) + m_B^2(p_2\cdot p_-) + 2(p_1\cdot p_+)^2 + 2(p_1\cdot p_-)(p_1\cdot p_+) \\
& - m_B^2(p_1\cdot p_+) - 2(p_1\cdot p_-)^2) \\
& - 4m_{\nu\nu}^2 m_\nu^2((p_2\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_-)(p_1\cdot p_+)) \\
& - 4m_{\mu\mu}^2 m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+) - (p_1\cdot p_-)(p_2\cdot p_-)) \\
& + 4m_{\mu\mu}^2((p_2\cdot p_+) - (p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-)) \\
& + 4m_{\nu\nu}^2((p_2\cdot p_+) - (p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-))
\end{aligned}
$$

$$
\begin{aligned}
&+2m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2 (2(p_2\cdot p_+) + (p_2\cdot p_-) - (p_1\cdot p_+) - 2(p_1\cdot p_-))\\
&+2m_{\nu\nu}^2 m_{\mu\mu}^2 m_\nu^2 (2(p_2\cdot p_+) - (p_2\cdot p_-) + (p_1\cdot p_+) - 2(p_1\cdot p_-))\\
&-16m_\mu^2 m_\nu^2 ((p_2\cdot p_+) - (p_1\cdot p_-))((p_2\cdot p_+) + (p_2\cdot p_-) + (p_1\cdot p_+) + (p_1\cdot p_-))\\
&+2m_{\nu\nu}^4 m_\mu^2 ((p_2\cdot p_+) + (p_2\cdot p_-) - (p_1\cdot p_+) - (p_1\cdot p_-))\\
&+2m_{\mu\mu}^4 m_\nu^2 ((p_2\cdot p_+) - (p_2\cdot p_-) + (p_1\cdot p_+) - (p_1\cdot p_-))\\
&-2m_{\nu\nu}^2 m_{\mu\mu}^2 ((p_2\cdot p_+) - (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_-))\\
&-(m_{\mu\mu}^2 + m_{\nu\nu}^2)(8m_\mu^2 m_\nu^2 + m_{\mu\mu}^2 m_{\nu\nu}^2)((p_2\cdot p_+) - (p_1\cdot p_-))),
\end{aligned}
\tag{A41}
$$

$$
R_{a'p}^M = -64((p_1\cdot p_-) + m_\mu^2)((p_2\cdot p_+) + m_\mu^2),
\tag{A42}
$$

$$
\begin{aligned}
R_{b'p}^M = {}& 4(4(4(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) + 2(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+) + 2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+)\\
&- m_B^2(p_1\cdot p_-)(p_2\cdot p_+) + 2(p_1\cdot p_+)(p_2\cdot p_-)^2 + 2(p_1\cdot p_+)^2(p_2\cdot p_-)\\
&+4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) - m_B^2(p_1\cdot p_+)(p_2\cdot p_-))\\
&+4m_\mu^2(2(p_1\cdot p_+)(p_2\cdot p_+) + 2(p_1\cdot p_-)(p_2\cdot p_+) - m_B^2(p_2\cdot p_+) + 2(p_1\cdot p_+)(p_2\cdot p_-)\\
&+2(p_1\cdot p_-)(p_2\cdot p_-) - m_B^2(p_1\cdot p_-))\\
&-4(m_{\mu\mu}^2 + m_{\nu\nu}^2)((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-))\\
&-m_{\nu\nu}^2 m_{\mu\mu}^2(4(p_2\cdot p_+) + 2(p_2\cdot p_-) + 2(p_1\cdot p_+) + 4(p_1\cdot p_-) - m_B^2)\\
&+2(m_{\mu\mu}^2 m_\nu^2 + m_{\nu\nu}^2 m_\mu^2)(2(p_2\cdot p_+) + 2(p_2\cdot p_-) + 2(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&+8m_\nu^2((p_2\cdot p_-) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_+))\\
&+4m_\mu^2 m_\nu^2 m_B^2 - 4m_\mu^4 m_B^2 + 2m_{\mu\mu}^4 m_\nu^2 + 2m_{\nu\nu}^4 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^4 - m_{\nu\nu}^4 m_{\mu\mu}^2),
\end{aligned}
\tag{A43}
$$

$$
\begin{aligned}
R_{c'p}^M = {}& -8(4m_\mu^2((p_2\cdot p_+)^2 + (p_1\cdot p_+)(p_2\cdot p_+) - (p_1\cdot p_-)(p_2\cdot p_-) - (p_1\cdot p_-)^2)\\
&+4m_\nu^2((p_2\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_-)(p_1\cdot p_+))\\
&+4((p_2\cdot p_+) - (p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-))\\
&-4m_\mu^4((p_2\cdot p_+) + (p_2\cdot p_-) - (p_1\cdot p_+) - (p_1\cdot p_-))\\
&-4m_\mu^2 m_\nu^2((p_2\cdot p_+) - (p_2\cdot p_-) + (p_1\cdot p_+) - (p_1\cdot p_-))\\
&+(2m_{\mu\mu}^2 m_\nu^2 + 2m_{\mu\mu}^2 m_\mu^2 + 4m_{\nu\nu}^2 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^2)((p_2\cdot p_+) - (p_1\cdot p_-))),
\end{aligned}
\tag{A44}
$$

$$
R_{a'm}^M = -32((p_1\cdot p_-) + m_\mu^2)(2(p_1\cdot p_+) - 2m_\mu^2 + m_{\mu\mu}^2),
\tag{A45}
$$

$$
\begin{aligned}
R_{b'm}^M = {}& -4(8((p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-)^2\\
&-(p_1\cdot p_+)^2(p_2\cdot p_-) - 2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) - 2(p_1\cdot p_-)(p_1\cdot p_+)^2\\
&-2(p_1\cdot p_-)^2(p_1\cdot p_+) + m_B^2(p_1\cdot p_-)(p_1\cdot p_+))\\
&-8m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+) + (p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-) + (p_1\cdot p_-)(p_2\cdot p_-)\\
&+2(p_1\cdot p_+)^2 + 4(p_1\cdot p_-)(p_1\cdot p_+) - m_B^2(p_1\cdot p_+) + 2(p_1\cdot p_-)^2)\\
&-4m_{\nu\nu}^2((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-) + 2(p_1\cdot p_-)(p_1\cdot p_+))\\
&+4m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-) + 2(p_1\cdot p_-)(p_2\cdot p_-) + 2(p_1\cdot p_-)^2)\\
&+2m_{\mu\mu}^2 m_\nu^2(2(p_2\cdot p_+) + 2(p_2\cdot p_-) + 2(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&-4m_{\nu\nu}^2 m_\mu^2((p_2\cdot p_+) + (p_2\cdot p_-) + 3(p_1\cdot p_+) + 3(p_1\cdot p_-))
\end{aligned}
$$

$$+ 8m_\nu^2((p_2\cdot p_-) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_+))$$
$$+ 2m_{\nu\nu}^2 m_{\mu\mu}^2((p_2\cdot p_-) + (p_1\cdot p_+) + 2(p_1\cdot p_-))$$
$$- 2m_{\mu\mu}^2 m_\mu^2(4(p_1\cdot p_+) + 4(p_1\cdot p_-) - m_B^2) + 4m_{\mu\mu}^4(p_1\cdot p_-) + 4m_\mu^2 m_\nu^2 m_B^2 - 4m_\mu^4 m_B^2$$
$$+ 2m_{\mu\mu}^4 m_\nu^2 - 4m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2 - 2m_{\nu\nu}^4 m_\mu^2 + m_{\nu\nu}^2 m_{\mu\mu}^4 + m_{\nu\nu}^4 m_{\mu\mu}^2), \tag{A46}$$

$$\begin{aligned}R_{c'm}^M = -4(&8((p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+) + (p_1\cdot p_-)^2(p_2\cdot p_+) + (p_1\cdot p_+)^2(p_2\cdot p_-)\\
&- (p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) - 2(p_1\cdot p_-)^2(p_1\cdot p_+))\\
&+ 8m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+) - 2(p_1\cdot p_+)(p_2\cdot p_-)\\
&- (p_1\cdot p_-)(p_2\cdot p_-) + (p_1\cdot p_+)^2 - 2(p_1\cdot p_-)(p_1\cdot p_+) - (p_1\cdot p_-)^2)\\
&+ 8m_\nu^2((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-) + 2(p_1\cdot p_-)(p_1\cdot p_+))\\
&+ 4m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-))\\
&- 8m_\mu^4((p_2\cdot p_+) + (p_2\cdot p_-) + 3(p_1\cdot p_+) + 3(p_1\cdot p_-))\\
&+ 8m_\mu^2 m_\nu^2((p_2\cdot p_+) - (p_2\cdot p_-) + (p_1\cdot p_+) - (p_1\cdot p_-))\\
&+ 4m_{\mu\mu}^2 m_\mu^2((p_2\cdot p_+) + 2(p_1\cdot p_+) + (p_1\cdot p_-))\\
&+ 4m_{\mu\mu}^2 m_\nu^2((p_2\cdot p_-) + (p_1\cdot p_+) + 2(p_1\cdot p_-))\\
&- 2m_{\nu\nu}^2 m_{\mu\mu}^2((p_1\cdot p_+) + (p_1\cdot p_-)) - 8m_{\nu\nu}^2(p_1\cdot p_-)(p_1\cdot p_+)\\
&+ 2m_{\mu\mu}^4 m_\nu^2 - 8m_{\mu\mu}^2 m_\mu^4 - 8m_{\nu\nu}^2 m_\mu^4 + 2m_{\mu\mu}^4 m_\mu^2 + 4m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^4),\end{aligned} \tag{A47}$$

$$R_{ap'}^M = -64((p_1\cdot p_+) + m_\mu^2)((p_2\cdot p_-) + m_\mu^2), \tag{A48}$$

$$\begin{aligned}R_{bp'}^M = 4(&4(2(p_1\cdot p_-)(p_2\cdot p_+)^2 + 2(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) + 4(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)\\
&+ 4(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_+) + 2(p_1\cdot p_-)^2(p_2\cdot p_+) - m_B^2(p_1\cdot p_-)(p_2\cdot p_+)\\
&+ 2(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-) - m_B^2(p_1\cdot p_+)(p_2\cdot p_-))\\
&+ 4m_\mu^2(2(p_1\cdot p_+)(p_2\cdot p_+) + 2(p_1\cdot p_-)(p_2\cdot p_+)\\
&+ 2(p_1\cdot p_+)(p_2\cdot p_-) + 2(p_1\cdot p_-)(p_2\cdot p_-) - m_B^2(p_2\cdot p_-) - m_B^2(p_1\cdot p_+))\\
&+ 4(m_{\mu\mu}^2 + m_{\nu\nu}^2)((p_1\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_+)(p_2\cdot p_-))\\
&- m_{\nu\nu}^2 m_{\mu\mu}^2(2(p_2\cdot p_+) + 4(p_2\cdot p_-) + 4(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&+ 2(m_{\mu\mu}^2 m_\nu^2 + m_{\nu\nu}^2 m_\mu^2)(2(p_2\cdot p_+) + 2(p_2\cdot p_-) + 2(p_1\cdot p_+) + 2(p_1\cdot p_-) - m_B^2)\\
&+ 8m_\nu^2((p_2\cdot p_-) + (p_1\cdot p_-))((p_2\cdot p_+) + (p_1\cdot p_+))\\
&+ 4m_\mu^2 m_\nu^2 m_B^2 - 4m_\mu^4 m_B^2 + 2m_{\mu\mu}^4 m_\nu^2 + 2m_{\nu\nu}^4 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^4 - m_{\nu\nu}^4 m_{\mu\mu}^2),\end{aligned} \tag{A49}$$

$$\begin{aligned}R_{cp'}^M = 8(&4m_\nu^2((p_2\cdot p_-)(p_2\cdot p_+) - (p_1\cdot p_-)(p_1\cdot p_+))\\
&- 4m_\mu^2((p_1\cdot p_+)(p_2\cdot p_+) - (p_2\cdot p_-)^2 - (p_1\cdot p_-)(p_2\cdot p_-) + (p_1\cdot p_+)^2)\\
&+ 4((p_2\cdot p_-) - (p_1\cdot p_+))((p_1\cdot p_-)(p_2\cdot p_+) + (p_1\cdot p_+)(p_2\cdot p_-))\\
&- 4m_\mu^4((p_2\cdot p_+) + (p_2\cdot p_-) - (p_1\cdot p_+) - (p_1\cdot p_-))\\
&+ 4m_\mu^2 m_\nu^2((p_2\cdot p_+) - (p_2\cdot p_-) + (p_1\cdot p_+) - (p_1\cdot p_-))\\
&+ (2m_{\mu\mu}^2(m_\mu^2 + m_\nu^2) + 4m_{\nu\nu}^2 m_\mu^2 - m_{\nu\nu}^2 m_{\mu\mu}^2)((p_2\cdot p_-) - (p_1\cdot p_+))),\end{aligned} \tag{A50}$$

$$R_{a'p'}^M = 16(2m_\mu^2 m_\nu^2((p_2\cdot p_+) + (p_1\cdot p_-)) - 2m_\mu^2 m_\nu^4 + m_{\mu\mu}^2 m_\nu^4 - 2m_\mu^4 m_\nu^2 + m_{\nu\nu}^2 m_\mu^4), \tag{A51}$$

$$\begin{aligned} R^M_{b'p'} &= 8(2m_\nu^2((p_2\cdot p_+)+(p_1\cdot p_+))+2m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2) \\ &\quad\times(2m_\mu^2((p_2\cdot p_+)+(p_2\cdot p_-))+2m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2), \end{aligned} \tag{A52}$$

$$\begin{aligned} R^M_{p'm'} &= 16(m_\nu^2(p_2\cdot p_-)+m_\mu^2(p_2\cdot p_-)+2m_\mu^2m_\nu^2) \\ &\quad\times(2m_\nu^2(p_2\cdot p_+)+2m_\mu^2(p_1\cdot p_-)-4m_\mu^2m_\nu^2+m_{\mu\mu}^2m_\nu^2+m_{\nu\nu}^2m_\mu^2), \end{aligned} \tag{A53}$$

$$R^M_{am'} = -32((p_2\cdot p_-)+m_\mu^2)(2(p_2\cdot p_+)-2m_\mu^2+m_{\mu\mu}^2), \tag{A54}$$

$$\begin{aligned} R^M_{bm'} &= 4(8(2(p_2\cdot p_-)(p_2\cdot p_+)^2+(p_1\cdot p_-)(p_2\cdot p_+)^2 \\ &\quad+2(p_2\cdot p_-)^2(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+)+2(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+) \\ &\quad-m_B^2(p_2\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_-)^2(p_2\cdot p_+)-(p_1\cdot p_-)(p_1\cdot p_+)(p_2\cdot p_-)) \\ &\quad+8m_\mu^2(2(p_2\cdot p_+)^2+4(p_2\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_+)+(p_1\cdot p_-)(p_2\cdot p_+) \\ &\quad-m_B^2(p_2\cdot p_+)+2(p_2\cdot p_-)^2+(p_1\cdot p_+)(p_2\cdot p_-)+(p_1\cdot p_-)(p_2\cdot p_-)) \\ &\quad+4m_{\nu\nu}^2(2(p_2\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-)) \\ &\quad+4m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+)-2(p_2\cdot p_-)^2-(p_1\cdot p_+)(p_2\cdot p_-)-2(p_1\cdot p_-)(p_2\cdot p_-)) \\ &\quad+2m_{\mu\mu}^2m_\mu^2(4(p_2\cdot p_+)+4(p_2\cdot p_-)-m_B^2) \\ &\quad+4m_{\nu\nu}^2m_\mu^2(3(p_2\cdot p_+)+3(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-)) \\ &\quad-2m_{\mu\mu}^2m_\nu^2(2(p_2\cdot p_+)+2(p_2\cdot p_-)+2(p_1\cdot p_+)+2(p_1\cdot p_-)-m_B^2) \\ &\quad-2m_{\nu\nu}^2m_{\mu\mu}^2((p_2\cdot p_+)+2(p_2\cdot p_-)+(p_1\cdot p_-)) \\ &\quad-8m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_1\cdot p_+))-4m_{\mu\mu}^4(p_2\cdot p_-) \\ &\quad-4m_\mu^2m_\nu^2m_B^2+4m_\mu^4m_B^2-2m_{\mu\mu}^4m_\nu^2+4m_{\nu\nu}^2m_{\mu\mu}^2m_\mu^2+2m_{\nu\nu}^4m_\mu^2-m_{\nu\nu}^2m_{\mu\mu}^4-m_{\nu\nu}^4m_{\mu\mu}^2), \end{aligned} \tag{A55}$$

$$\begin{aligned} R^M_{cm'} &= -4(8((p_1\cdot p_-)(p_2\cdot p_+)^2-2(p_2\cdot p_-)^2(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-)(p_2\cdot p_+) \\ &\quad-(p_1\cdot p_-)(p_2\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-)^2) \\ &\quad+8m_\mu^2((p_2\cdot p_+)^2-2(p_2\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_+)-2(p_1\cdot p_-)(p_2\cdot p_+) \\ &\quad-(p_2\cdot p_-)^2-(p_1\cdot p_-)(p_2\cdot p_-)) \\ &\quad+8m_\nu^2(2(p_2\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-)) \\ &\quad+4m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-)) \\ &\quad-8m_\mu^4(3(p_2\cdot p_+)+3(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-)) \\ &\quad+4m_{\mu\mu}^2m_\mu^2(2(p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)) \\ &\quad+4m_{\mu\mu}^2m_\nu^2((p_2\cdot p_+)+2(p_2\cdot p_-)+(p_1\cdot p_-)) \\ &\quad-2m_{\nu\nu}^2m_{\mu\mu}^2((p_2\cdot p_+)+(p_2\cdot p_-)) \\ &\quad+8m_\mu^2m_\nu^2((p_2\cdot p_+)-(p_2\cdot p_-)+(p_1\cdot p_+)-(p_1\cdot p_-))-8m_{\nu\nu}^2(p_2\cdot p_-)(p_2\cdot p_+) \\ &\quad+2m_{\mu\mu}^4m_\nu^2-8m_{\mu\mu}^2m_\mu^4-8m_{\nu\nu}^2m_\mu^4+2m_{\mu\mu}^4m_\mu^2+4m_{\nu\nu}^2m_{\mu\mu}^2m_\mu^2-m_{\nu\nu}^2m_{\mu\mu}^4), \end{aligned} \tag{A56}$$

$$\begin{aligned} R^M_{a'm'} &= 8(4m_\nu^2((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))+4m_\mu^2((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-)) \\ &\quad-8m_\mu^2m_\nu^2((p_2\cdot p_+)+(p_1\cdot p_+)+(p_1\cdot p_-))+4m_{\nu\nu}^2m_\mu^2(p_2\cdot p_+)+4m_{\mu\mu}^2m_\nu^2(p_1\cdot p_-)+4m_\mu^2m_\nu^4 \\ &\quad-2m_{\mu\mu}^2m_\nu^4+4m_\mu^4m_\nu^2-2m_{\mu\mu}^2m_\mu^2m_\nu^2-2m_{\nu\nu}^2m_\mu^2m_\nu^2+m_{\nu\nu}^2m_{\mu\mu}^2m_\nu^2-2m_{\nu\nu}^2m_\mu^4+m_{\nu\nu}^2m_{\mu\mu}^2m_\mu^2), \end{aligned} \tag{A57}$$

$$R^M_{b'm'} = 8(2m_\mu^2((p_2\cdot p_+)+(p_2\cdot p_-))+2m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))+m_{\mu\mu}^2 m_\nu^2+m_{\nu\nu}^2 m_\mu^2)$$
$$\times(4((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))-2m_\nu^2((p_2\cdot p_+)+(p_1\cdot p_+))+2m_{\nu\nu}^2(p_2\cdot p_+)$$
$$-2m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))+2m_{\mu\mu}^2(p_1\cdot p_-)-m_{\mu\mu}^2 m_\nu^2-m_{\nu\nu}^2 m_\mu^2+m_{\nu\nu}^2 m_{\mu\mu}^2), \tag{A58}$$

$$R^M_{c'm'} = 4(m_\nu^2-m_\mu^2)(4m_{\mu\mu}^2((p_1\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-))$$
$$+4m_{\nu\nu}^2((p_1\cdot p_-)(p_2\cdot p_+)+(p_1\cdot p_+)(p_2\cdot p_-))$$
$$+8((p_2\cdot p_+)+(p_1\cdot p_-))((p_1\cdot p_-)(p_2\cdot p_+)-(p_1\cdot p_+)(p_2\cdot p_-))$$
$$-16m_\mu^2 m_\nu^2((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))$$
$$+4m_{\mu\mu}^2 m_\nu^2((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))$$
$$+4m_{\nu\nu}^2 m_\mu^2((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))$$
$$-8m_\mu^2((p_1\cdot p_+)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_2\cdot p_-))$$
$$-8m_\nu^2((p_2\cdot p_-)+(p_1\cdot p_-))((p_2\cdot p_+)+(p_1\cdot p_+))$$
$$-2m_{\nu\nu}^2 m_{\mu\mu}^2((p_2\cdot p_+)+(p_1\cdot p_-))-8m_{\mu\mu}^2 m_\mu^2 m_\nu^2-8m_{\nu\nu}^2 m_\mu^2 m_\nu^2+2m_{\mu\mu}^4 m_\nu^2$$
$$+4m_{\nu\nu}^2 m_{\mu\mu}^2 m_\nu^2+4m_{\nu\nu}^2 m_{\mu\mu}^2 m_\mu^2+2m_{\nu\nu}^4 m_\mu^2-m_{\nu\nu}^2 m_{\mu\mu}^4-m_{\nu\nu}^4 m_{\mu\mu}^2), \tag{A59}$$

$$I^M_{ca'} = 64\mathbb{O}, \tag{A60}$$

$$I^M_{a'b'} = -32\mathbb{O}(4(p_2\cdot p_+)+2(p_2\cdot p_-)+2(p_1\cdot p_+)-m_B^2+m_{\mu\mu}^2+m_{\nu\nu}^2), \tag{A61}$$

$$I^M_{ac'} = 64\mathbb{O}, \tag{A62}$$

$$I^M_{cc'} = 32\mathbb{O}((p_2\cdot p_+)-(p_2\cdot p_-)+(p_1\cdot p_+)-(p_1\cdot p_-)), \tag{A63}$$

$$I^M_{a'c'} = 64\mathbb{O}((p_2\cdot p_+)+(p_1\cdot p_-)), \tag{A64}$$

$$I^M_{b'c'} = 32\mathbb{O}(2((p_2\cdot p_+)^2-2(p_1\cdot p_+)(p_2\cdot p_-)+(p_1\cdot p_-)^2)+(m_{\mu\mu}^2+m_{\nu\nu}^2)((p_2\cdot p_+)+(p_1\cdot p_-))$$
$$-2(m_\mu^2+m_\nu^2)((p_2\cdot p_+)+(p_2\cdot p_-)+(p_1\cdot p_+)+(p_1\cdot p_-))$$
$$-m_{\mu\mu}^2 m_\nu^2-m_{\nu\nu}^2 m_\nu^2-m_{\mu\mu}^2 m_\mu^2-m_{\nu\nu}^2 m_\mu^2+m_{\nu\nu}^2 m_{\mu\mu}^2), \tag{A65}$$

$$I^M_{b'p} = -16\mathbb{O}(4(p_2\cdot p_+)+2(p_2\cdot p_-)+2(p_1\cdot p_+)-m_B^2+m_{\mu\mu}^2+m_{\nu\nu}^2), \tag{A66}$$

$$I^M_{c'p} = 32\mathbb{O}((p_2\cdot p_+)+(p_1\cdot p_-)+2m_\mu^2), \tag{A67}$$

$$I^M_{b'm} = -16\mathbb{O}(2((p_2\cdot p_-)+(p_1\cdot p_+)+2(p_1\cdot p_-))+m_{\mu\mu}^2+m_{\nu\nu}^2), \tag{A68}$$

$$I^M_{c'm} = 16\mathbb{O}(2((p_1\cdot p_+)-(p_1\cdot p_-))-4m_\mu^2+m_{\mu\mu}^2), \tag{A69}$$

$$I^M_{bp'} = 16\mathbb{O}(2(p_2\cdot p_+)+4(p_1\cdot p_+)+2(p_1\cdot p_-)-m_B^2+m_{\mu\mu}^2+m_{\nu\nu}^2), \tag{A70}$$

$$I^M_{cp'} = -32\mathbb{O}((p_2\cdot p_-)+(p_1\cdot p_+)+2m_\mu^2), \tag{A71}$$

$$I^M_{c'p'} = 32\mathbb{O}(m_\nu^2-m_\mu^2)^2, \tag{A72}$$

$$I^M_{bm'} = 16\mathbb{O}(2((p_2\cdot p_+)+2(p_2\cdot p_-)+(p_1\cdot p_-))+m_{\mu\mu}^2+m_{\nu\nu}^2), \tag{A73}$$

$$I^M_{cm'} = -16\mathbb{O}(2((p_2\cdot p_+)-(p_2\cdot p_-))-4m_\mu^2+m_{\mu\mu}^2), \tag{A74}$$

$$I^M_{a'm'} = 32\mathbb{O}(m_\mu^2 - m_\nu^2), \tag{A75}$$

$$I^M_{c'm'} = 16\mathbb{O}(m_\mu^2 - m_\nu^2)(2((p_2 \cdot p_+) - (p_1 \cdot p_-)) + 2m_\nu^2 - 2m_\mu^2 + m_{\mu\mu}^2 - m_{\nu\nu}^2). \tag{A76}$$

APPENDIX B: EXPRESSIONS FOR THE VARIOUS Σ_{ij} AND Δ_{ij} TERMS

The Δ_{ij} terms appearing in Eq. (47) are given by

$$\Delta_{aa} = -16(m_B - 2E_\mu)^2((m_\mu^2 - E_\mu^2)\cos^2\theta - E_\mu^2), \tag{B1}$$

$$\Delta_{bb} = -4m_B^4(m_B - 2E_\mu)^2((m_\mu^2 - E_\mu^2)\cos^2\theta - E_\mu^2), \tag{B2}$$

$$\Delta_{cc} = -8m_\mu^2(m_\mu^2 - E_\mu^2)m_B^2(m_B - 2E_\mu)^2\sin^2\theta, \tag{B3}$$

$$\Delta_{pp} = -4m_\mu^4(m_B - 2E_\mu)^2((m_\mu^2 - E_\mu^2)\cos^2\theta - E_\mu^2), \tag{B4}$$

$$\Delta_{mm} = 4m_\mu^2(m_B - 2E_\mu)^2((m_\mu^2 - E_\mu^2)(m_B^2 - m_\mu^2)\cos^2\theta + E_\mu(E_\mu m_B^2 - 2m_\mu^2 m_B + E_\mu m_\mu^2)), \tag{B5}$$

$$\Delta_{ab} = -16m_B^2(m_B - 2E_\mu)^2 \times ((m_\mu^2 - E_\mu^2)\cos^2\theta - E_\mu^2), \tag{B6}$$

$$\Delta_{ap} = 16m_\mu^4(m_B - 2E_\mu)^2, \tag{B7}$$

$$\Delta_{bp} = 8m_\mu^4 m_B^2(m_B - 2E_\mu)^2, \tag{B8}$$

$$\Delta_{am} = 16m_\mu^2(m_B - 2E_\mu)^2(E_\mu m_B - m_\mu^2), \tag{B9}$$

$$\Delta_{bm} = 8m_\mu^2 m_B^2(m_B - 2E_\mu)^2(E_\mu m_B - m_\mu^2), \tag{B10}$$

$$\Delta_{cm} = 8m_\mu^2 m_B^2(E_\mu^2 - m_\mu^2)(m_B - 2E_\mu)^2\sin^2\theta, \tag{B11}$$

$$\Delta_{pm} = 8m_\mu^4(m_B - 2E_\mu)^2 \times ((m_\mu^2 - E_\mu^2)\cos^2\theta + E_\mu(m_B - E_\mu)), \tag{B12}$$

and the Σ_{ij} terms are given by,

$$\Sigma_{aa} = -32E_\mu\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2, \tag{B13}$$

$$\Sigma_{bb} = -8m_B^4 E_\mu\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2, \tag{B14}$$

$$\Sigma_{pp} = 8E_\mu m_\mu^4\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2, \tag{B15}$$

$$\Sigma_{mm} = -8m_\mu^4\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2(m_B - E_\mu), \tag{B16}$$

$$\Sigma_{ab} = -32m_B^2 E_\mu\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2, \tag{B17}$$

$$\Sigma_{am} = -16m_\mu^2 m_B\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2, \tag{B18}$$

$$\Sigma_{bm} = -8m_\mu^2 m_B^3\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^2, \tag{B19}$$

$$\Sigma_{pm} = 8m_\mu^4\sqrt{E_\mu^2 - m_\mu^2}(m_B - 2E_\mu)^3. \tag{B20}$$

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