# Background independence of effective actions at critical dimension

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Recently, by explicit calculations at orders $\alpha', \alpha'^2, \alpha'^3$, it has been observed that the effective action of string theory at the critical dimension is independent of the background for the closed spacetime manifolds. In this paper we speculate that for the open spacetime manifolds, the effective action is even independent of the character of the boundary, i.e., the boundary couplings for timelike and spacelike boundaries are the same. We support this proposal by calculating the boundary couplings in the bosonic string theory at order $\alpha'$ for the spacelike boundary and show that they are the same as the couplings for the timelike boundary that have been recently found.

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## I. INTRODUCTION

The critical string theory is an extension of the Einstein theory of general relativity which is consistent with the rules of quantum mechanics. As in the Einstein theory, one expects the string theory at the critical dimension $D = 1 + d$ to be background independent. In the low energy effective action, by the background independence we mean the coefficients of the independent gauge invariant couplings at each order of $\alpha'$ should be independent of the background. The background independence in the double field theory formalism has been discussed in [1].

The independent couplings in the effective action at a given order of $\alpha'$, are all gauge invariant couplings modulo the field redefinitions, the total derivative terms and the Bianchi identities. For the closed spacetime manifolds, the field redefinitions at a given order of $\alpha'$ involve the most general gauge invariant terms at that order [2]. The numbers of independent couplings involving the metric, dilaton and the $B$-field at orders $\alpha', \alpha'^2, \alpha'^3$ are 8,60,872, respectively [2–4]. The background independence assumption indicates that the coefficients of these couplings are independent of the background. If one can fix them for a particular background, then they are valid for any other background as well. On the other hand, it has been proved in [5,6] that the dimensional reduction of the classical effective actions of the bosonic and heterotic string theories on a torus $T^d$ are invariant under global $O(d,d)$ transformations. Hence, if one considers a particular background which includes one circle, compactifies the effective action on this circle and ignores the Kaluza-Klein massive modes (dimensional reduction), then the lower dimensional action which includes all parameters of the original action, must have the $O(1,1)$ symmetry. This symmetry may fix the couplings in the original action. Imposing this symmetry on the effective action of the bosonic string theory at orders $\alpha', \alpha'^2$, the coefficients of all independent couplings have been found in [7,8] up to an overall factor. Imposing this symmetry on the NS-NS couplings of the type II superstring theory at order $\alpha'^3$, the coefficients of all independent couplings have been found in [9–11] up to an overall factor. The resulting couplings must be valid for any other background. For example, the same couplings must be valid for the background which includes the compact submanifold $T^d$. The lower-dimensional action in this case must have the symmetry $O(d,d)$ [5,6]. In fact, it has been shown in [12,13] that the resulting couplings have exactly such symmetry. For the background which has the compact submanifold $T^2$, the lower dimensional action must have the symmetry $O(2,2)$. It has been shown in [14] that the couplings at order $\alpha'$ have such symmetry.

For the open spacetime manifolds, it has been speculated in [15] that the field redefinitions at a given order of $\alpha'$ involve only the restricted gauge invariant terms which respect the boundary conditions in the least action principle. In the presence of boundary, the boundary conditions which are consistent with the above $O(1,1)$ or $O(d,d)$ symmetries, require the massless fields and their derivatives at order $m$ to be known on the boundary for the effective actions at order $\alpha'^m$ [16]. The minimum numbers of independent couplings at order $\alpha'$ for the bosonic string theory and for the heterotic string theory after truncating the

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Yang-Mills gauge fields, are 17 bulk couplings and 38 boundary couplings [15,16]. The background independence assumption in this case indicates that the coefficients of these parameters are independent of the background and are independent of the character of the boundary, i.e., the coefficients in the bulk and boundary couplings must be the same for both timelike and spacelike boundaries. Using the background independence assumption, these couplings in a particular minimal scheme have been recently found for the timelike boundary to be [15]

$$\begin{aligned}\mathbf{S}_1 = &-\frac{48a_1}{\kappa^2}\int_M d^Dx\sqrt{-G}e^{-2\Phi}\Big[R^2_{\rm GB} + \frac{1}{24}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^{\varepsilon}H_{\gamma\epsilon\varepsilon} - \frac{1}{8}H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_\gamma{}^{\epsilon\varepsilon}H_{\delta\epsilon\varepsilon}\\ &+ R^{\alpha\beta}H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta} - \frac{1}{12}RH_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - \frac{1}{2}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\gamma\delta\epsilon} + 4R\nabla_\alpha\Phi\nabla^\alpha\Phi - 16R^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi\Big]\end{aligned} \tag{1}$$

$$\begin{aligned}\partial\mathbf{S}_1^t = &-\frac{48a_1}{\kappa^2}\int_{\partial M} d^{D-1}\sigma\sqrt{-g}e^{-2\Phi}\Big[Q_2^t + \frac{4}{3}n^\alpha n^\beta\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - \frac{1}{6}H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta}\\ &+ H_\alpha{}^{\delta\epsilon}H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta - 2H_\beta{}^{\delta\epsilon}H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi + 8K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi\\ &- 16K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta\Phi - 16K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + \frac{32}{3}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi\Big]\end{aligned}$$

where $n^\mu$ is the unit vector orthogonal to the boundary, $R^2_{\rm GB}$ is the Gauss-Bonnet gravity couplings and $Q_2^t$ is the Chern-Simons couplings that for timelike boundary, i.e., $n^\mu n_\mu = 1$, is given as [16,18]

$$Q_2^t = 4\Big[K^\mu{}_\mu R - 2K^{\mu\nu}R_{\mu\nu} - 2K_\alpha{}^\alpha n^\mu n^\nu R_{\mu\nu} + 2K^{\mu\nu}n^\alpha n^\beta R_{\alpha\mu\beta\nu} - \frac{1}{3}(6K^\alpha{}_\alpha K_{\mu\nu}K^{\mu\nu} - 2K^\mu{}_\mu K^\nu{}_\nu K^\alpha{}_\alpha - 4K_\mu{}^\nu K_{\nu\alpha}K^{\alpha\mu})\Big] \tag{2}$$

In the above equations, $K_{\mu\nu}$ is the extrinsic curvature which is defined as $K_{\mu\nu} = P^\alpha{}_\mu P^\beta{}_\nu\nabla_{(\alpha}n_{\beta)}$ where $P^{\mu\nu}$ is the first fundamental form which projects the spacetime tensors tangent to the boundary. For the timelike boundary, the first fundamental form is defined as $P^{\mu\nu} = G^{\mu\nu} - n^\mu n^\nu$ for which the extrinsic curvature becomes $K_{\mu\nu} = \nabla_\mu n_\nu - n_\mu n_\alpha\nabla^\alpha n_\nu$. In the above equations, the metric in the covariant derivatives and in the curvatures is the bulk metric $G_{\mu\nu}$. If one chooses the overall factor $a_1$ to be $a_1 = 1/96$ ($a_1 = 1/192$), then the bulk action becomes the Meissner action of the bosonic (heterotic) string theory found in [19], up to a restricted field redefinition [15]. For the superstring theory $a_1 = 0$.

For the spacelike boundary in which $n^\mu n_\mu = -1$, the first fundamental form is defined as $P^{\mu\nu} = G^{\mu\nu} + n^\mu n^\nu$ for which the extrinsic curvature becomes $K_{\mu\nu} = \nabla_\mu n_\nu + n_\mu n_\alpha\nabla^\alpha n_\nu$. We expect this sign difference in the second term in the definition of the first fundamental form changes the sign of all terms which involve three unit vector $n^\mu$ or extrinsic curvature. Therefore, when the spacetime has spacelike boundary, the background independence assumption of the effective action predicts that the bulk action to be the same as the bulk action (1) for the spacetime which has timelike boundary, whereas the character independence predicts the following boundary couplings for the spacelike boundary:

$$\begin{aligned}\partial\mathbf{S}_1^s = &-\frac{48a_1}{\kappa^2}\int_{\partial M} d^{D-1}\sigma\sqrt{g}e^{-2\Phi}\Big[Q_2^s - \frac{4}{3}n^\alpha n^\beta\nabla_\gamma\nabla^\gamma K_{\alpha\beta} - \frac{1}{6}H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta}\\ &- H_\alpha{}^{\delta\epsilon}H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta + 2H_\beta{}^{\delta\epsilon}H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi + 8K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi\\ &+ 16K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta\Phi - 16K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi - \frac{32}{3}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi\Big]\end{aligned} \tag{3}$$

where the terms involving three unit vector $n^\mu$ or extrinsic curvature have different sign compare to the timelike boundary couplings in (1). In above equation, $Q_2^s$ is the Chern-Simons couplings for the spacelike boundary

$$Q_2^s = 4\Big[K^\mu{}_\mu R - 2K^{\mu\nu}R_{\mu\nu} + 2K_\alpha{}^\alpha n^\mu n^\nu R_{\mu\nu} - 2K^{\mu\nu}n^\alpha n^\beta R_{\alpha\mu\beta\nu} + \frac{1}{3}(6K^\alpha{}_\alpha K_{\mu\nu}K^{\mu\nu} - 2K^\mu{}_\mu K^\nu{}_\nu K^\alpha{}_\alpha - 4K_\mu{}^\nu K_{\nu\alpha}K^{\alpha\mu})\Big]. \tag{4}$$

Note that the terms with three extrinsic curvatures have different sign compare to the Chern-Simons couplings of the timelike boundary (2). In this paper, using the background independence method, we are going to calculate the bulk and boundary couplings for the spacetime manifold which has spacelike boundary and show that the resulting couplings are exactly the same as the above couplings which are predicted by the background/character independence assumption.

The outline of the paper is as follows: In Sec. II, we write the 17 independent bulk couplings and the 38 independent boundary couplings at order $\alpha'$ which have been found in [15,16]. In Sec. III, using the background independence assumption, we consider the background which has a spacelike boundary and one circle, and use the dimensional reduction to find the corresponding couplings in the base space. We then impose the $O(1,1)$ symmetry on the reduced actions to constrain the parameters in the actions. In Sec. IV, we consider the background which has a spacelike boundary and the torus $T^d$, and use the cosmological reduction to find the one-dimensional bulk action and the zero-dimensional boundary action. We then impose the $O(d,d)$ symmetry on the resulting actions to further constrain the remaining parameters. The above two constraints fix the bulk action to be the bulk action in (1), and fix the boundary action up to two parameters. By requiring the gravity couplings on the boundary action to be consistent with the Chern-Simons couplings, the two boundary parameters are also fixed. We find that the final boundary action is exactly the same as (3).

## II. INDEPENDENT COUPLINGS AT ORDER $\alpha'$

The effective actions of string theory on an open manifold has both bulk and boundary actions. At the sphere-level, these actions have the following $\alpha'$-expansion:

$$\begin{aligned}\mathbf{S}_{\rm eff} &= \sum_{m=0}^{\infty}\alpha'^m\mathbf{S}_m = \mathbf{S}_0+\alpha'\mathbf{S}_1+\alpha'^2\mathbf{S}_2+\alpha'^3\mathbf{S}_3+\cdots\\ \partial\mathbf{S}_{\rm eff} &= \sum_{m=0}^{\infty}\alpha'^m\partial\mathbf{S}_m\\ &= \partial\mathbf{S}_0+\alpha'\partial\mathbf{S}_1+\alpha'^2\partial\mathbf{S}_2+\alpha'^3\partial\mathbf{S}_3+\cdots \end{aligned}\tag{5}$$

The leading order actions in the universal sector which includes metric, dilaton, and B-field, in the string frame for both timelike and spacelike boundaries are

$$\begin{aligned}&\mathbf{S}_0+\partial\mathbf{S}_0\\ &\quad=-\frac{2}{\kappa^2}\Bigg[\int d^Dx\sqrt{-G}e^{-2\Phi}\left(R+4\nabla_\mu\Phi\nabla^\mu\Phi-\frac{1}{12}H^2\right)\\ &\qquad+2\int d^{D-1}\sigma\sqrt{|g|}e^{-2\Phi}K\Bigg]\end{aligned}\tag{6}$$

where $G$ is determinant of the bulk metric $G_{\mu\nu}$ and boundary is specified by the functions $x^\mu = x^\mu(\sigma^{\tilde\mu})$. In the boundary term, $g$ is determinant of the induced metric on the boundary

$$g_{\tilde\mu\tilde\nu}=\frac{\partial x^\mu}{\partial\sigma^{\tilde\mu}}\frac{\partial x^\nu}{\partial\sigma^{\tilde\nu}}G_{\mu\nu}\tag{7}$$

and $K$ is the trace of the extrinsic curvature. The normal vector to the boundary is $n^\mu$. It is outward-pointing (inward-pointing) if the boundary is spacelike (timelike). Using the double field theory formalism, it has been shown in [17] that the leading order effective action (6) can be written in $O(D,D)$-invariant form in terms of the generalized metric and dilaton.

At order $\alpha'$ these actions in terms of their Lagrangians are

$$\begin{aligned}\mathbf{S}_1&=-\frac{2}{\kappa^2}\int_M d^Dx\sqrt{-G}e^{-2\Phi}\mathcal{L}_1;\\ \partial\mathbf{S}_1&=-\frac{2}{\kappa^2}\int_{\partial M}d^{D-1}\sigma\sqrt{|g|}e^{-2\Phi}\partial\mathcal{L}_1\end{aligned}\tag{8}$$

In general there are 41 gauge invariant couplings in the bulk Lagrangian. Removing the total derivative terms from the bulk to the boundary by using the Stokes' theorem and using the Bianchi identities, one can reduce the 41 couplings to 20 couplings. The most general field redefinitions reduce these couplings to 8 couplings [2]. However, in the presence of boundary one is not allowed to use the most general field redefinitions because they ruin the boundary conditions required in the least action principle for the effective actions of string theory [16]. The allowed field redefinitions at order $\alpha'$ requires the metric does not change, and the dilaton and B-field change to include only the first derivative of the massless fields. This restricted field redefinition has only three parameters. Hence, there are only 17 independent couplings in the bulk. The couplings in a particular minimal scheme are [15]

$$\begin{aligned}\mathcal{L}_1 =\ & a_1 H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^{\varepsilon}H_{\gamma\epsilon\varepsilon}+a_2H_{\alpha\beta}{}^{\delta}H^{\alpha\beta\gamma}H_\gamma{}^{\epsilon\varepsilon}H_{\delta\epsilon\varepsilon}+a_3H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}R^{\alpha\beta}+a_4R_{\alpha\beta}R^{\alpha\beta}\\ &+a_5H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}R+a_6R^2+a_7R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}+a_8H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\gamma\delta\epsilon}+a_9R\nabla_\alpha\Phi\nabla^\alpha\Phi\\ &+a_{10}R^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi+a_{11}R_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi+a_{12}\nabla_\alpha\Phi\nabla^\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi+a_{13}\nabla^\alpha\Phi\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\Phi\\ &+a_{14}\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\nabla^\alpha\Phi+a_{15}\nabla_\alpha H^{\alpha\beta\gamma}\nabla_\delta H_{\beta\gamma}{}^{\delta}+a_{16}H_\alpha{}^{\beta\gamma}\nabla^\alpha\Phi\nabla_\delta H_{\beta\gamma}{}^{\delta}+a_{17}\nabla_\delta H_{\alpha\beta\gamma}\nabla^\delta H^{\alpha\beta\gamma}\end{aligned}\tag{9}$$

where $a_1, \ldots, a_{17}$ are 17 background independent parameters.

The boundary of the spacetime has a unit normal vector $n^\mu$, hence, the boundary Lagrangian $\partial\mathcal{L}_1$ should include this vector and its derivatives as well as the bulk tensors. Since the field redefinition freedom has been already used in the bulk action, one is not allowed to use any field redefinition in the boundary action. Removing the boundary total derivative terms from the most general gauge invariant boundary couplings, and using the Bianchi identities and the identities corresponding to the unit vector, it has been shown in [16] that there are 38 independent couplings in the boundary action. For both timelike and spacelike boundaries, the couplings in a particular scheme are [16]

$$
\begin{aligned}
\partial\mathcal{L}_1 = {}& b_1 H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + b_2 H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta} + b_3 K_\alpha{}^\gamma K^{\alpha\beta}K_{\beta\gamma} + b_4 K^\alpha{}_\alpha K_{\beta\gamma}K^{\beta\gamma} \\
& + b_5 K^\alpha{}_\alpha K^\beta{}_\beta K^\gamma{}_\gamma + b_6 H_\alpha{}^{\delta\epsilon}H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta + b_7 H_{\alpha\gamma}{}^\epsilon H_{\beta\delta\epsilon}K^{\gamma\delta}n^\alpha n^\beta + b_8 K^{\alpha\beta}R_{\alpha\beta} \\
& + b_9 K^\gamma{}_\gamma n^\alpha n^\beta R_{\alpha\beta} + b_{10}K^\alpha{}_\alpha R + b_{11}K^{\gamma\delta}n^\alpha n^\beta R_{\alpha\gamma\beta\delta} + b_{12}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha H_{\beta\gamma\delta} \\
& + b_{13}K^{\beta\gamma}n^\alpha\nabla_\alpha K_{\beta\gamma} + b_{14}K^\beta{}_\beta n^\alpha\nabla_\alpha K^\gamma{}_\gamma + b_{15}n^\alpha\nabla_\alpha R + b_{16}H_{\beta\gamma\delta}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha\Phi \\
& + b_{17}K_{\beta\gamma}K^{\beta\gamma}n^\alpha\nabla_\alpha\Phi + b_{18}K^\beta{}_\beta K^\gamma{}_\gamma n^\alpha\nabla_\alpha\Phi + b_{19}H_\beta{}^{\delta\epsilon}H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi \\
& + b_{20}n^\alpha n^\beta n^\gamma R_{\beta\gamma}\nabla_\alpha\Phi + b_{21}n^\alpha R\nabla_\alpha\Phi + b_{22}K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi + b_{23}n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta K^\gamma{}_\gamma \\
& + b_{24}K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta\Phi + b_{25}n^\alpha n^\beta\nabla_\beta\nabla_\alpha K^\gamma{}_\gamma + b_{26}K^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi \\
& + b_{27}K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\beta\nabla_\alpha\Phi + b_{28}H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}n^\alpha\nabla^\beta\Phi + b_{29}n^\alpha R_{\alpha\beta}\nabla^\beta\Phi + b_{30}K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi \\
& + b_{31}n^\alpha\nabla_\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi + b_{32}n^\alpha\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\Phi + b_{33}H_\alpha{}^{\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\gamma H_{\beta\delta\epsilon} \\
& + b_{34}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi + b_{35}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\gamma\nabla_\beta\Phi + b_{36}n^\alpha n^\beta n^\gamma\nabla_\gamma\nabla_\beta\nabla_\alpha\Phi \\
& + b_{37}n^\alpha n^\beta\nabla_\beta K_{\alpha\gamma}\nabla^\gamma\Phi + b_{38}n^\alpha n^\beta n^\gamma n^\delta\nabla_\delta\nabla_\gamma K_{\alpha\beta}
\end{aligned}
\tag{10}
$$

where $b_1, \ldots, b_{38}$ are 38 background independent parameters. These parameters, however, depend on the character of boundary. They have been found in [15] for the timelike boundary. In the following sections we consider the boundary to be spacelike.

## III. BACKGROUND WITH SUBMANIFOLD $S^{(1)}$

We have used the gauge symmetries corresponding to the massless fields to write the independent couplings in the bulk action (9) and in the boundary action (10). The parameters in these actions are independent of the backgrounds. In general, there is no global symmetry in the universal sector of string theory at the critical dimension to be used for fixing these parameters. However, for some specific backgrounds which have compact submanifolds, the compactified actions may have some global symmetries after ignoring the Kaluza-Klein massive modes (dimensional reduction). Since the parameters in the actions (9) and (10) appears also in the lower dimensional actions, one can use the symmetry of the lower dimensional actions to fix these parameters. In this section we consider the background with submanifold $S^{(1)}$. That is, we choose the open manifold to have the structure $M^{(D)} = M^{(D-1)} \times S^{(1)}$, $\partial M^{(D)} = \partial M^{(D-1)} \times S^{(1)}$. The manifold $M^{(D)}$ has coordinates $x^\mu = (x^a, y)$ and its boundary $\partial M^{(D)}$ has coordinates $\sigma^{\tilde{\mu}} = (\sigma^{\tilde{a}}, y)$ where $y$ is the coordinate of the circle $S^{(1)}$. The boundary in the base space is specified by the functions $x^a = x^a(\sigma^{\tilde{a}})$. The dimensionally reduced action then should have the $O(1,1)$ symmetry. To simplify the calculation, we consider the $Z_2$-subgroup of the $O(1,1)$-group.

The reduction of the effective actions on the circle $S^{(1)}$ should then be invariant under the $Z_2$-transformations, up to some total derivative terms on the boundary [20], i.e.,

$$
S_{\rm eff}(\psi) + \partial S_{\rm eff}(\psi) = S_{\rm eff}(\psi') + \partial S_{\rm eff}(\psi') \tag{11}
$$

where $S_{\rm eff}$ and $\partial S_{\rm eff}$ are the reductions of the bulk action $\mathbf{S}_{eff}$ and boundary action $\partial\mathbf{S}_{\rm eff}$, respectively. In above equation $\psi$ represents all the massless fields in the base space which are defined in the following Kaluza-Klein reductions [21]:

$$
\begin{aligned}
G_{\mu\nu} &= \begin{pmatrix} \bar{g}_{ab} + e^\varphi g_a g_b & e^\varphi g_a \\ e^\varphi g_b & e^\varphi \end{pmatrix}, \\
B_{\mu\nu} &= \begin{pmatrix} \bar{b}_{ab} + \frac{1}{2}b_a g_b - \frac{1}{2}b_b g_a & b_a \\ -b_b & 0 \end{pmatrix}, \\
\Phi &= \bar{\phi} + \varphi/4, \qquad n^\mu = (n^a, 0)
\end{aligned}
\tag{12}
$$

and $\psi'$ represents its transformation under the $Z_2$-transformations or the T-duality transformations. At the leading order of $\alpha'$, the T-duality transformations are the Buscher rules [22,23]. To order $\alpha'$, they are the Buscher rules and the corrections at order $\alpha'$ which do not ruin the boundary conditions of the least action principle in the base space. They are [15]

$$\begin{aligned} \varphi' &= -\varphi + \alpha' \Delta\varphi, \quad g'_a = b_a + \alpha' e^{\varphi/2}\Delta g_a, \\ b'_a &= g_a + \alpha' e^{-\varphi/2}\Delta b_a, \quad \bar{g}'_{ab} = \bar{g}_{ab}, \\ \bar{H}'_{abc} &= \bar{H}_{abc} + \alpha' \Delta\bar{H}_{abc}, \quad \bar{\phi}' = \bar{\phi} + \alpha'\Delta\bar{\phi}, \quad n'_a = n_a \end{aligned} \tag{13}$$

where the corrections $\Delta\varphi, \Delta b_a, \Delta g_a, \Delta\bar{\phi}$ contain all contractions of the massless fields in the base space at order $\alpha'$ which involve only the first derivative of the massless fields. The correction $\Delta\bar{H}_{abc}$ is related to the corrections $\Delta g_a$, $\Delta b_a$ through the following relation:

$$\Delta\bar{H}_{abc} = \tilde{H}_{abc} - 3e^{-\varphi/2}W_{[ab}\Delta b_{c]} - 3e^{\varphi/2}\Delta g_{[a}V_{bc]} \tag{14}$$

where $\tilde{H}_{abc}$ is a $U(1)\times U(1)$ gauge invariant closed 3-form at order $\alpha'$ which is odd under parity. It has the following terms:

$$\tilde{H}_{abc} = e_1\partial_{[a}W_b{}^d V_{c]d} + e_2\partial_{[a}\bar{H}_{bc]d}\nabla^d\varphi \tag{15}$$

where $e_1$, $e_2$ and the coefficients in the corrections $\Delta\varphi, \Delta b_a, \Delta g_a, \Delta\bar{\phi}$ are parameters that the $Z_2$-symmetry of the effective action should fix them. The above transformations should also form the $Z_2$-group [7]. In the above equation, $V_{ab}$ is field strength of the $U(1)$ gauge field $g_a$, i.e., $V_{ab} = \partial_a g_b - \partial_b g_a$, and $W_{\mu\nu}$ is field strength of the $U(1)$ gauge field $b_a$, i.e., $W_{ab} = \partial_a b_\nu - \partial_b b_a$. The three-form $\bar{H}$ is defined as $\bar{H}_{abc} = \hat{H}_{abc} - \frac{3}{2}g_{[a}W_{bc]} - \frac{3}{2}b_{[a}V_{bc]}$ where the three-form $\hat{H}$ is field strength of the two-form $\bar{b}_{ab}$ in (12).

In [16], it has been shown that the constraint (11) can be written as two separate constraints. One for the bulk couplings and the other one for the boundary couplings. These constraints for the couplings at order $\alpha'$ are [16]

$$\begin{aligned} &S_1(\psi) - S_1(\psi'_0) - \Delta S_0 - \frac{2}{\kappa^2}\int d^{D-1}x\sqrt{-\bar{g}}\nabla_a(A_1^a e^{-2\bar{\phi}}) = 0 \\ &\partial S_1(\psi) - \partial S_1(\psi'_0) - \Delta\partial S_0 + T_1(\psi) \\ &\quad + \frac{2}{\kappa^2}\int d^{D-2}\sigma\sqrt{\tilde{g}}n_a A_1^a e^{-2\bar{\phi}} = 0 \end{aligned} \tag{16}$$

where $\bar{g}$ is the determinant of the base space metric $\bar{g}_{ab}$ and $\tilde{g}$ is the determinant of the induced base space metric on its boundary, i.e.,

$$\tilde{g}_{\tilde{a}\tilde{b}} = \frac{\partial x^a}{\partial\sigma^{\tilde{a}}}\frac{\partial x^b}{\partial\sigma^{\tilde{b}}}\bar{g}_{ab} \tag{17}$$

In Eq. (16), $\psi'_0$ is the transformation of the base space field $\psi$ under the Buscher rules, $A_1^a$ is a vector made of the massless fields in the base space at order $\alpha'$ with arbitrary coefficients, and $T_1(\psi)$ is the most general total derivative terms in the boundary at order $\alpha'$, i.e.,

$$T_1(\psi) = -\frac{2}{\kappa^2}\int_{\partial M^{(D-1)}} d^{D-2}\sigma\sqrt{|\tilde{g}|}n_a\nabla_b(e^{-2\bar{\phi}}F_1^{ab}) \tag{18}$$

where $F_1^{ab}$ is an antisymmetric tensor constructed from the massless fields in the base space at order $\alpha'$ with arbitrary coefficients. In the Eq. (16), $\Delta S_0$, $\Delta\partial S_0$ are the Taylor expansions of the reduction of the leading order actions (6) at order $\alpha'$,

$$\begin{aligned} S_0(\psi'_0 + \alpha'\psi'_1) &= S_0(\psi'_0) + \alpha'\Delta S_0 + \cdots \\ \partial S_0(\psi'_0 + \alpha'\psi'_1) &= \partial S_0(\psi'_0) + \alpha'\Delta\partial S_0 + \cdots \end{aligned} \tag{19}$$

where dots represent some terms at higher orders of $\alpha'$ in which we are not interested in this paper.

The first constraint in (16) involves only the bulk fields that their reductions are given in [8]. The second constraint involves the bulk fields and the boundary extrinsic curvature. The reduction of the extrinsic curvature and its first and second derivatives for the timelike boundary are calculated in [24]. We have checked explicitly that they are valid for spacelike boundary as well. Using these reductions and following the same steps as those in [16], one finds that the $Z_2$-symmetry fixes the bulk Lagrangian (9) to be the same as the one has been found in [15], i.e.,

$$\begin{aligned} \mathcal{L}_1 = {} & a_1 H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^\epsilon H_{\gamma\epsilon\epsilon} + \left(3a_1 + \frac{1}{64}a_{10} + \frac{1}{64}a_{11}\right)H_{\alpha\beta}{}^\delta H^{\alpha\beta\gamma}H_\gamma{}^{\epsilon\epsilon}H_{\delta\epsilon\epsilon} - \frac{1}{16}a_{11}H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}R^{\alpha\beta} + \left(\frac{1}{4}a_{10} + \frac{1}{4}a_{11}\right)R_{\alpha\beta}R^{\alpha\beta} \\ & + \frac{1}{192}a_{11}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}R - \frac{1}{16}a_{11}R^2 + 24a_1R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \left(-36a_1 - \frac{1}{8}a_{10} - \frac{1}{16}a_{11}\right)H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\gamma\delta\epsilon} \\ & - \frac{1}{4}a_{11}R\nabla_\alpha\Phi\nabla^\alpha\Phi + a_{10}R^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi + a_{11}R_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + a_{10}\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\nabla^\alpha\Phi - \frac{1}{16}a_{10}\nabla_\alpha H^{\alpha\beta\gamma}\nabla_\delta H_{\beta\gamma}{}^\delta \\ & + \frac{1}{8}a_{10}H_\alpha{}^{\beta\gamma}\nabla^\alpha\Phi\nabla_\delta H_{\beta\gamma}{}^\delta + \left(8a_1 + \frac{1}{24}a_{10} + \frac{1}{48}a_{11}\right)\nabla_\delta H_{\alpha\beta\gamma}\nabla^\delta H^{\alpha\beta\gamma} \end{aligned} \tag{20}$$

and the boundary Lagrangian (10) for the spacelike boundary to be

$$\begin{aligned}
\partial\mathcal{L}_1 = {}& b_1 H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + \frac{1}{16}(-2a_{10}-a_{11})H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta} + b_{11}K_\alpha{}^\gamma K^{\alpha\beta}K_{\beta\gamma} \\
&+\frac{1}{4}(-a_{11}-2(48b_1+b_{17}))K^\alpha{}_\alpha K_{\beta\gamma}K^{\beta\gamma} + \frac{1}{12}(a_{11}+96b_1-2b_{18})K^\alpha{}_\alpha K^\beta{}_\beta K^\gamma{}_\gamma \\
&-\frac{1}{2}b_{19}H_\alpha{}^{\delta\epsilon}H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta + \left(a_{10}+\frac{1}{2}a_{11}+12b_{12}\right)K^{\alpha\beta}R_{\alpha\beta} \\
&+\frac{1}{2}(a_{10}-48b_1+24b_{12}-b_{17}+4b_{19})K^\gamma{}_\gamma n^\alpha n^\beta R_{\alpha\beta} + \left(-\frac{1}{8}a_{11}-12b_1\right)K^\alpha{}_\alpha R \\
&+b_{11}K^{\gamma\delta}n^\alpha n^\beta R_{\alpha\gamma\beta\delta} + b_{12}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha H_{\beta\gamma\delta} + \left(\frac{1}{48}a_{11}-2b_1\right)H_{\beta\gamma\delta}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha\Phi \\
&+b_{17}K_{\beta\gamma}K^{\beta\gamma}n^\alpha\nabla_\alpha\Phi + b_{18}K^\beta{}_\beta K^\gamma{}_\gamma n^\alpha\nabla_\alpha\Phi + b_{19}H_\beta{}^{\delta\epsilon}H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi \\
&+\left(-a_{10}-\frac{1}{2}a_{11}-24b_{12}+b_{17}-4b_{19}\right)n^\alpha n^\beta n^\gamma R_{\beta\gamma}\nabla_\alpha\Phi + \left(-\frac{1}{4}a_{11}+24b_1\right)n^\alpha R\nabla_\alpha\Phi \\
&-48b_1K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi - 2(48b_1+b_{18})K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta\Phi \\
&+\frac{1}{2}(4a_{10}+a_{11}+48(-2b_1+b_{12}))K^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi \\
&+\left(a_{10}+\frac{1}{2}a_{11}+24b_{12}-b_{17}+4b_{19}\right)K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\beta\nabla_\alpha\Phi + \frac{1}{8}a_{10}H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}n^\alpha\nabla^\beta\Phi \\
&+\frac{1}{2}(a_{11}-96b_1)n^\alpha R_{\alpha\beta}\nabla^\beta\Phi + a_{11}K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + (-a_{11}+96b_1)n^\alpha\nabla_\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi \\
&+(a_{11}-96b_1)n^\alpha\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\Phi + \frac{1}{8}(2a_{10}+a_{11}-2b_{11}+48b_{12})H_\alpha{}^{\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\gamma H_{\beta\delta\epsilon} \\
&-\frac{2}{3}(a_{11}-2(96b_1+b_{18}))n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi \\
&-2(a_{10}+48b_1+24b_{12}-b_{17}+4b_{19})n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\gamma\nabla_\beta\Phi + b_{38}n^\alpha n^\beta n^\gamma n^\delta\nabla_\delta\nabla_\gamma K_{\alpha\beta}
\end{aligned} \tag{21}$$

which is not the same as its corresponding timelike Lagrangian found in [15]. The sign of some of the parameters are changed compare to the timelike case. We have imposed the identities corresponding to the unit vector in the base space, by writing it as

$$n^a = -\frac{\partial_a f}{\sqrt{|\partial_b f\partial^b f|}} \tag{22}$$

where $f$ is the function that specifies the spacelike boundary, i.e., $\partial_a f\partial^a f = -|\partial_a f\partial^a f|$.

The bulk Lagrangian has three parameters $a_1, a_{10}, a_{11}$ and the boundary Lagrangian has two bulk parameters $a_{10}$, $a_{11}$ and 7 boundary parameters $b_1, b_{11}, b_{12}, b_{17}, b_{18}, b_{19}, b_{38}$. Since not all parameters are fixed up to an overall factor, in the next section we consider another background.

## IV. BACKGROUND WITH SUBMANIFOLD $T^{(d)}$

In this section, we consider the background which has the submanifold $T^{(d)}$. That is, the open manifold has the structure $M^{(D)} = M^{(1)}\times T^{(d)}$, $\partial M^{(D)} = \partial M^{(1)}\times T^{(d)}$. The base space manifold $M^{(1)}$ has time coordinate $t$, hence, its boundary is spacelike boundary. The compactification on this background has massless modes as well as infinite tower of massive Kaluza-Klein modes. If one ignores the massive Kaluza-Klein modes (cosmological reduction), and uses the appropriate one-dimensional field redefinitions, then the cosmological action should have the $O(d,d)$ symmetry, i.e.,

$$S^c_{\rm eff}(\psi) + \partial S^c_{\rm eff}(\psi) = S^O_{\rm eff}(\psi') + \partial S^O_{\rm eff}(\psi') \tag{23}$$

where $S^c_{\rm eff}$ and $\partial S^c_{\rm eff}$ are the cosmological reductions of the bulk action $\mathbf{S}_{eff}$ and boundary action $\partial\mathbf{S}_{\rm eff}$, respectively. In above equation $\psi$ represents all the massless fields in the base space, i.e.,

$$G_{\mu\nu} = \begin{pmatrix} -n^2(t) & 0 \\ 0 & G_{ij}(t)\end{pmatrix}, \qquad B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ij}(t)\end{pmatrix},$$

$$2\Phi = \phi + \frac{1}{2}\log\ \det(G_{ij}) \tag{24}$$

The lapse function $n(t)$ can also be fixed to $n = 1$. This function at the boundary is the unit vector orthogonal to the boundary. On the right-hand side of Eq. (23), $\psi'$ represents their appropriate higher-derivative field redefinitions. The effective actions on the right-hand side must be invariant under $O(d,d)$-transformations.

In the absence of boundary, it has been shown in [25,26] that there are field redefinitions, including the lapse function, in which the nonlocal cosmological action which involves higher time-derivatives become local action which involves only the first time-derivative of the generalized metric. In the presence of the boundary, one should not use the field redefinitions for the lapse function because this function at the boundary represents the unit normal vector on the boundary. Moreover, in the presence of the boundary, the field redefinitions should be restricted to those which do not ruin the boundary conditions in the least action principle in the base space [15]. In the presence of boundary, there might be the field redefinitions that left intact the laps function and do not ruin the boundary conditions, however, the action in that scheme may involve the first derivative of the generalized metric as well as the first derivative of the one-dimensional dilaton, i.e., the action may still become local. On the other hand, for the local action, one expects the usual boundary condition in the least action principle in which only the values of the massless fields are known on the boundary. Hence, the boundary action should not include the derivative of the massless fields, i.e., as it has been speculated in [16], the boundary action must be zero in that particular scheme. Hence, in that scheme, the constraint (23) becomes

$$S^c_{\rm eff}(\psi) + \partial S^c_{\rm eff}(\psi) = S^O_{\rm eff}(\psi') \qquad (25)$$

It has been shown in [15] that there is such scheme at order $\alpha'$.

The above constraint at each order of $\alpha'$ produces two constraints. One bulk and one boundary constraints. At the leading order of $\alpha'$, they are

$$\begin{aligned} S^c_0(\psi) - \frac{2}{\kappa^2}\int dt \frac{d}{dt}(\mathcal{I}_0 e^{-\phi}) &= S^O_0(\psi) \\ \partial S^c_0(\psi) + \frac{2}{\kappa^2}\mathcal{I}_0 e^{-\phi} &= 0 \end{aligned} \qquad (26)$$

The second terms in the first equation is a total derivative term at the two derivative order. For a particular $\mathcal{I}_0$, the bulk constraint produces the following $O(d,d)$-invariant action [25,27,28]:

$$S^O_0 = -\frac{2}{\kappa^2}\int dt e^{-\phi}\left[-\dot{\phi}^2 - \frac{1}{8}{\rm tr}(\dot{\mathcal{S}}^2)\right] \qquad (27)$$

where $\mathcal{S}$ is the generalized metric. Taking into the account the appropriate $\mathcal{I}_0$ from the bulk constraint, one finds the boundary constraint (26) satisfies automatically [16].

The constraint (25) at order $\alpha'$ produces the following two constraints:

$$\begin{aligned} S^c_1(\psi) - \Delta S^O_0(\psi) - \frac{2}{\kappa^2}\int dt \frac{d}{dt}(\mathcal{I}_1 e^{-\phi}) &= S^O_1(\psi) \\ \partial S^c_1(\psi) + \frac{2}{\kappa^2}\mathcal{I}_1 e^{-\phi} &= 0 \end{aligned} \qquad (28)$$

where $\Delta S^O_0(\psi)$ is the Taylor expansion of the leading order cosmological action (27) at order $\alpha'$, i.e.,

$$S^O_0(\psi + \alpha'\psi'_1) = S^O_0(\psi) + \alpha'\Delta S^O_0(\psi) + \cdots \qquad (29)$$

It has been shown in [15] that the bulk constraint in (28) is satisfies when there are the following relations between the bulk parameters $a_1$, $a_{10}$, $a_{11}$:

$$a_{11} = -384 a_1, \qquad a_{10} = 0 \qquad (30)$$

The corresponding $O(d,d)$-invariant action is the cosmological action that has been found in [19], i.e.,

$$\begin{aligned} S^O_1(\psi) = -\frac{2}{\kappa^2}24a_1\int dt e^{-\phi}\bigg[&\frac{1}{16}{\rm tr}(\dot{\mathcal{S}}^4) - \frac{1}{64}({\rm tr}(\dot{\mathcal{S}}^2))^2 \\ &+ \frac{1}{2}{\rm tr}(\dot{\mathcal{S}}^2)\dot{\phi}^2 - \frac{1}{3}\dot{\phi}^4\bigg] \end{aligned} \qquad (31)$$

The corresponding total derivative terms are the following:

$$\begin{aligned} \mathcal{I}_1 = {}& 24a_1\dot{B}_i{}^k\dot{B}^{ij}\dot{G}_{jk} + 12a_1\dot{G}^i{}_i\dot{G}_{jk}\dot{G}^{jk} - 6a_1\dot{B}_{ij}\dot{B}^{ij}\dot{G}^k{}_k \\ & - 6a_1\dot{G}^i{}_i\dot{G}^j{}_j\dot{G}^k{}_k - 24a_1\dot{B}_{ij}\dot{B}^{ij}\dot{\phi} + 24a_1\dot{G}^i{}_i\dot{\phi}^2 \\ & + 32a_1\dot{\phi}^3 \end{aligned} \qquad (32)$$

The corresponding field redefinitions that involve only the first derivative of the massless fields have been also found in [15]. However, since they do not appear in the boundary constraint in (28), we are not interested in them. Inserting the relations (30) into the bulk Lagrangian (20), one reproduces the Lagrangian (1), as expected.

The one-dimensional reduction of the timelike boundary couplings (21) is the following:

$$\partial S_1^c = -\frac{2}{\kappa^2} e^{-\phi}\left[\frac{1}{4}(-96a_1 - (b_{11} - 12b_{12}))\dot{B}_i{}^k\dot{B}^{ij}\dot{G}_{jk} + \frac{1}{4}(-192a_1 - (b_{11} - 12b_{12}))\dot{G}_i{}^k\dot{G}^{ij}\dot{G}_{jk}\right.$$
$$+ (36a_1 + 6b_1 - 3b_{12})\dot{G}^i{}_i\dot{G}_{jk}\dot{G}^{jk} - 6a_1\dot{B}_{ij}\dot{B}^{ij}\dot{G}^k{}_k - 6a_1\dot{G}^i{}_i\dot{G}^j{}_j\dot{G}^k{}_k$$
$$-\frac{1}{2}(24a_1 + 6b_1 + b_{19})\dot{B}_{ij}\dot{B}^{ij}\dot{\phi} + \left(60a_1 + 9b_1 - 6b_{12} - \frac{1}{2}b_{19}\right)\dot{G}_{ij}\dot{G}^{ij}\dot{\phi}$$
$$+ 24a_1\dot{G}^i{}_i\dot{\phi}^2 + \frac{1}{6}(96a_1 - 24b_1 - b_{18})\dot{\phi}^3 + \frac{1}{4}(192a_1 + b_{11} - 12b_{12})\dot{B}^{ij}\ddot{B}_{ij}$$
$$\left.+\frac{1}{4}(192a_1 + b_{11} - 12b_{12})\dot{G}^{ij}\ddot{G}_{ij} + \frac{1}{2}(-192a_1 + 24b_{12} - b_{17} + 4b_{19})\dot{\phi}\,\ddot{\phi}\right] \tag{33}$$

where we have also used the relations (30). The above action is not invariant under the $O(d,d)$ transformations. If one includes $\mathcal{I}_1$ which is given in (32), one can choose the boundary parameters such that the result becomes invariant. For the following relations between the parameters:

$$b_{11} = -96a_1 + 24b_1, \qquad b_{12} = 8a_1 + 2b_1,$$
$$b_{19} = -12b_1 + \frac{1}{4}b_{17} \tag{34}$$

The boundary action becomes $O(d,d)$-invariant which involves the first derivative of the dilaton, i.e.,

$$\partial S_1^c(\psi) + \frac{2}{\kappa^2}\mathcal{I}_1 e^{-\phi}$$
$$= -\frac{2}{\kappa^2}e^{-\phi}\left[\left(12a_1 + 3b_1 - \frac{1}{8}b_{17}\right)(\dot{B}_{ij}\dot{B}^{ij} + \dot{G}_{ij}\dot{G}^{ij})\dot{\phi} - \left(16a_1 + 4b_1 + \frac{1}{6}b_{18}\right)\dot{\phi}^3\right]$$

The boundary constraint in (28) then dictates the following relations:

$$b_{17} = 96a_1 + 24b_1, \qquad b_{18} = -96a_1 - 24b_1 \tag{35}$$

The above relations (34) and (35), then reduce the 7 boundary parameters in (21) to 2 parameters $b_1$, $b_{38}$. Note that the coupling with coefficient $b_{38}$ is invariant under the $O(1,1)$ and $O(d,d)$ transformations.

For the spacetime manifolds which have boundary, both the bulk and boundary actions should satisfy the least action principle, i.e., $\delta(\mathbf{S}_1 + \partial\mathbf{S}_1) = 0$ with the appropriate boundary condition on the massless fields. Since the bulk action has at most the term with two derivatives, the variation of the bulk action satisfies $\delta\mathbf{S}_1 = 0$ using the assumption that the values of the massless fields and their first derivatives are known on the boundary [16]. The variation of the boundary action produces variation of the second derivatives of the massless fields which are not zero on the boundary for the effective action at order $\alpha'$. However, the parameters $b_1$, $b_{38}$, cannot be fixed because the nonzero variations are total derivative terms on the boundary which are zero. In fact inserting the relations (30), (34), and (35) into the boundary action (21), one finds the variation of the resulting boundary action against the metric variation produces the following terms:

$$-24(4a_1 + b_1)\partial^\alpha\Phi\partial^\beta f P^{\gamma\delta}\nabla_\alpha\nabla_\beta\delta G_{\gamma\delta}$$
$$-24(4a_1 + b_1)\partial^\alpha\Phi\partial_\alpha f\partial^\beta f\partial^\gamma f P^{\delta\epsilon}\nabla_\beta\nabla_\gamma\delta G_{\delta\epsilon} \tag{36}$$

where we have used the assumption that the variation of metric and its first derivative, and their tangent derivatives are zero, i.e., $\delta G_{\alpha\beta} = \partial_\mu\delta G_{\alpha\beta} = 0$ and $P^{\mu\nu}\partial_\mu\partial_\gamma\delta G_{\alpha\beta} = 0$. On the other hand, if one considers the following anti-symmetric tensor:

$$\mathcal{F}_1^{\alpha\beta} = (96a_1 + 24b_1)n^\mu n^{[\alpha}\partial^{\beta]}\Phi(\nabla_\mu\delta G^\nu{}_\nu - \nabla_\nu\delta G_\mu{}^\nu) \tag{37}$$

Then its corresponding boundary total derivative term, i.e.,

$$\int_{\partial M^{(D)}} d^{D-1}\sigma\sqrt{g}n_\alpha\nabla_\beta(e^{-2\Phi}\mathcal{F}_1^{\alpha\beta}) \tag{38}$$

would cancel the variations (36). Similar cancellations happen for the variations of the boundary action against the dilaton and B-field.

We fix the remaining boundary parameters $b_1$, $b_{38}$ by noting that the boundary couplings include the structures as those in the Chern-Simons form. Hence, we fix the remaining parameters in the boundary action such that the gravity couplings in the boundary include the Chern-Simons form. The Chern-Simons form has the following gravity couplings for the spacelike boundary [18]:

$$Q_2^s = 4\left[K^\mu{}_\mu\tilde{R} - 2K^{\mu\nu}\tilde{R}_{\mu\nu} - \frac{1}{3}(3K^\alpha{}_\alpha K_{\mu\nu}K^{\mu\nu} - K^\mu{}_\mu K^\nu{}_\nu K^\alpha{}_\alpha - 2K_\mu{}^\nu K_{\nu\alpha}K^{\alpha\mu})\right] \tag{39}$$

where $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ are curvatures that are constructed from the induced metric (7). Using the following Gauss-Codazzi relations for the spacelike boundary:

$$\tilde{R}_{\alpha\beta} = P_{\alpha\mu}P_{\beta\nu}R^{\mu\nu} + n^\mu n^\nu R_{\alpha\mu\beta\nu} + K_{\alpha\mu}K_\beta{}^\mu - K_{\alpha\beta}K_\mu{}^\mu$$
$$\tilde{R} = R + 2n^\mu n^\nu R_{\mu\nu} + K_{\mu\nu}K^{\mu\nu} - K_\mu{}^\mu K_\nu{}^\nu \tag{40}$$

and the identity $n^\mu K_{\mu\nu} = 0$, one can rewrite $Q_2^s$ in terms of the spacetime curvatures, i.e., (4). For the spacelike boundary, there is also the following identity:

$$n^\alpha n^\beta n^\gamma n^\delta \nabla_\delta \nabla_\gamma K_{\alpha\beta} = 2K_\alpha{}^\gamma K^{\alpha\beta}K_{\beta\gamma} - n^\alpha n^\beta \nabla_\gamma \nabla^\gamma K_{\alpha\beta} \tag{41}$$

which can be verified by writing both sides in the local frame and in terms of the function $f$, i.e.,

$$n^\mu = -\frac{\partial_\mu f}{\sqrt{|\partial_\nu f \partial^\nu f|}} \tag{42}$$

The identity (41) is the same as the corresponding identity in the timelike boundary [16] in which the terms which have three $n^\alpha$ or extrinsic curvature, have different sign. However, the term which has five $n^\alpha$ or extrinsic curvature, has the same sign.

Using the identity (41), one finds the gravity couplings in the boundary action become the same as the couplings in $Q_2^s$ for the following relations:

$$b_1 = -4a_1, \qquad b_{38} = 32a_1 \tag{43}$$

In fact, inserting the relations (30), (34), (35) and (43) into the boundary action (21), one finds the boundary couplings (3) dictated by the background/character independence of the effective actions at the critical dimension.

The boundary action for the non-null boundaries can then be written as

$$\begin{aligned}\partial\mathbf{S}_1 = -\frac{48a_1}{\kappa^2}\int d^{D-1}\sigma\sqrt{|g|}e^{-2\Phi}\Big[&Q_2 + \frac{4}{3}n^2 n^\alpha n^\beta \nabla_\gamma\nabla^\gamma K_{\alpha\beta} - \frac{1}{6}H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta}\\ &+ n^2 H_\alpha{}^{\delta\epsilon}H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta - 2n^2 H_\beta{}^{\delta\epsilon}H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma \nabla_\alpha\Phi + 8K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi\\ &- 16n^2 K^\gamma{}_\gamma n^\alpha n^\beta \nabla_\alpha\Phi\nabla_\beta\Phi - 16K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + \frac{32}{3}n^2 n^\alpha n^\beta n^\gamma \nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi\Big]\end{aligned} \tag{44}$$

where $n^2 = n^\mu n_\mu$, and the Chern-Simons density is

$$Q_2 = 4\Big[K^\mu{}_\mu R - 2K^{\mu\nu}R_{\mu\nu} - 2n^2 K_\alpha{}^\alpha n^\mu n^\nu R_{\mu\nu} + 2n^2 K^{\mu\nu}n^\alpha n^\beta R_{\alpha\mu\beta\nu} - \frac{1}{3}n^2(6K^\alpha{}_\alpha K_{\mu\nu}K^{\mu\nu} - 2K^\mu{}_\mu K^\nu{}_\nu K^\alpha{}_\alpha - 4K_\mu{}^\nu K_{\nu\alpha}K^{\alpha\mu})\Big] \tag{45}$$

The boundary action has terms which have one and five $n^\alpha$ and/or $K_{\alpha\beta}$. At the higher orders of $\alpha'$, one expects the boundary action to have terms with $1,5,9,13,\cdots$ unit vector $n^\alpha$ and/or $K_{\alpha\beta}$. Hence, to find the boundary actions at higher orders of $\alpha'$, one may first find the couplings for the timelike boundary, and then inserts $n^2 = 1$ in the couplings which have $3,7,11,\cdots$ unit vector $n^\alpha$ and/or $K_{\alpha\beta}$, to produce terms with $1,5,9,13,\cdots$ unit vector $n^\alpha$ and/or $K_{\alpha\beta}$. The result then would be valid for the spacelike boundary as well. It would be interesting to find the boundary couplings at order $\alpha'^2$ to check this speculation.

The boundary action for the null boundary may be obtained by treating the null boundary as a limit of a sequence of non-null boundaries [29]. Using this method, the coupling on the null boundary at the leading order has been found in [29] by taking the appropriate limit of the non-null boundary coupling in (6). One may use this method to find the boundary couplings at order $\alpha'$ for the null boundary by taking the limit of the non-null boundary couplings (44).

Having found the boundary couplings (44) corresponding to the bulk action (1) at order $\alpha'$, one may try to write them in manifest $O(D,D)$-invariant form, as has been done in [17] for the leading order effective action. Even for the closed spacetime manifold, it is hard to write the effective action at order $\alpha'$ in terms of the generalized metric because the conventional 2D-dimensional Riemann curvature does not transform covariantly under the generalized diffeomorphisms [30–33]. However, this action has been written in $O(D,D)$-invariant form using the generalized frame [34,35]. It would be interesting to write the bulk and boundary actions (1), (44) in the duality manifest actions in terms of the generalized frame.

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