Holographic thermodynamics requires a chemical potential for color

Manus R. Visser[ORCID]

Department of Theoretical Physics, University of Geneva,
24 quai Ernest-Ansermet, 1211 Genève 4, Switzerland

The thermodynamic Euler equation for high-energy states of large-N gauge theories is derived from the dependence of the extensive quantities on the number of colors N. This Euler equation relates the energy of the state to the temperature, entropy, number of degrees of freedom and its chemical potential, but not to the volume or pressure. In the context of the gauge/gravity duality we show that the Euler equation is dual to the generalized Smarr formula for black holes in the presence of a negative cosmological constant. We also match the fundamental variational equation of thermodynamics to the first law of black hole mechanics, when extended to include variations of the cosmological constant and Newton's constant.

I. INTRODUCTION

The thermodynamics of black holes remains one of the most important theoretical advancements in gravitational physics of the past half-century. In semiclassical general relativity the energy E, entropy S, and temperature T of a black hole can be identified with its mass, horizon area A and surface gravity κ, respectively, [1–3] (setting $k_B = \hbar = c = 1$)

$$E = M \qquad S = \frac{A}{4G} \qquad T = \frac{\kappa}{2\pi}. \qquad (1)$$

For static, asymptotically flat black holes these thermodynamic quantities are simply related by the Smarr formula [4] $M = \frac{d-1}{d-2}\frac{\kappa A}{8\pi G}$ in $d+1$ spacetime dimensions. The dimension dependent factors are a peculiar property of the Smarr formula, which typically do not appear in the Euler equation of thermodynamics relating all extensive and intensive quantities. Furthermore, for black holes in the presence of a negative cosmological constant Λ there is an additional term in the generalized Smarr formula $M = \frac{d-1}{d-2}\frac{\kappa A}{8\pi G} - \frac{1}{d-2}\frac{\Theta\Lambda}{4\pi G}$ [5,6]. In an *extended* version of black hole thermodynamics the cosmological constant is interpreted as the pressure $P = -\Lambda/8\pi G$, and is treated as a thermodynamic state variable in its own right [6–9]. Its conjugate quantity Θ in the extended first law of black holes is regarded as (minus) the thermodynamic volume. An important, but slightly odd aspect of this interpretation is

that the mass of the black hole is identified with the enthalpy instead of the internal energy of the system (see [10] for a recent review).

From a holographic perspective the dimension dependent factors and the Λ term in the generalized Smarr formula remain somewhat elusive. In holography, or gauge/gravity duality, the thermodynamics of black holes in the "bulk" spacetime is equivalent to the thermodynamics of large-N strongly coupled gauge theories living on the asymptotic boundary of the bulk spacetime [11–13]. In particular, the thermodynamic variables (1) for black holes correspond with the energy E, entropy S, and temperature T of thermal states in the boundary theory [14,15]. The generalized Smarr formula relating these variables in the gravity theory should be dual to the Euler relation for the thermodynamic quantities in the field theory. But the pressure interpretation of the Λ term in the Smarr formula does not directly carry over to the field theory, since the bulk pressure P is not dual to the boundary pressure p and the bulk thermodynamic volume is not related to spatial volume V of the boundary [16].

In several works [6,16–20] it has been suggested that varying Λ is related to varying the number of colors N, or the number of degrees of freedom N^2, in the boundary field theory. For gauge theories arising from coincident D-branes, varying N corresponds to varying the number of branes. Further, in conformal field theories (CFTs) the number of degrees of freedom is given by the central charge C, whose variation takes us from one CFT to another. In holographic CFTs dual to Einstein gravity the central charge corresponds to $C \sim L^{d-1}/G$ [21–23], where L is the curvature radius of the bulk geometry, related to the cosmological constant via $\Lambda = -d(d-1)/2L^2$, and G is Newton's constant in $d+1$ dimensions. So varying C in the boundary CFT is dual to varying Λ and G in the bulk

theory. In addition, it was argued in [20] that varying Λ in the bulk does not only correspond to varying C (or N), but also to varying the volume V of the spatial boundary geometry. This is because the bulk curvature radius L is equal to the boundary curvature radius for a particular boundary metric. We show, however, that for a different boundary metric varying Λ only corresponds to varying C (and E) in the boundary theory, and not to varying V. Overall, by building on (and refining) the holographic dictionary in [20], we propose a precise boundary description of extended black hole thermodynamics.

In this paper we argue that the dual field theory description of black hole thermodynamics requires a chemical potential μ for the central charge (see also [17,18]). From the large-N scaling properties of the field theory we derive the holographic Euler equation

$$E = TS + \nu^i B_i + \mu C, \tag{2}$$

and show that it is holographically dual to the generalized Smarr formula. Here ν_i are additional chemical potentials for the conserved quantities B_i (such as charge and angular momentum). As expected, the dimension dependent factors do not feature in (2), and the Λ term is incorporated in μ. Moreover, E is the standard energy of the field theory and not the enthalpy. Strikingly though, there is no pV term in the large-N Euler equation. We explain why this is consistent with the fundamental equation of thermodynamics, $dE = TdS - pdV + \nu_i dB_i + \mu dC$, in which both V and C are varied. Finally, we match this boundary variational equation with the extended first law of black holes.

II. THERMODYNAMICS OF LARGE-N THEORIES

We first derive the Euler equation from the scaling properties of gauge theories at finite temperature in the large-N 't Hooft limit [24], $N \to \infty$ for fixed coupling $\lambda \equiv g^2 N$. Large-N $SU(N)$ gauge theories on compact spaces, with fields in the adjoint representation, exhibit a separation between low-energy states with energy of $\mathcal{O}(N^0)$, and high-energy states for which the energy scales as $E \sim N^2$ [15,25,26]. This is because the low-energy excitations consist of color singlets, whose energy is independent of N, and at high energies all the N^2 adjoint degrees of freedom contribute on the same footing. The low-energy states are in a confined phase and are characterized by a thermal entropy that grows with energy, whereas the high-energy states are in a deconfined phase and behave as a gas of free particles (at nonzero λ there could exist an intermediate phase [27]). Other gauge theories at finite temperature display a similar (de)confinement phase transition, but the energy in the deconfined phase may scale with a different power of N, e.g., as N^3 for an exotic theory in $d = 6$ with (0,2) supersymmetry [28].

In conformal theories the central charge C counts the number of field degrees of freedom. For $SU(N)$ gauge theories with conformal symmetry the central charge scales as $C \sim N^2$ at large N, so high-energy states satisfy $E \sim C$. Since holographic CFTs are the main examples of large-N theories we have in mind, we denote the number of degrees of freedom simply as C for all large-N theories.

High-energy states in large-N theories obey interesting large-N scaling laws and are dual to black holes in holographic field theories. By definition the internal energy of these equilibrium states depends on extensive quantities, such as entropy S, volume V, and conserved quantities B_i, and on the (intensive) central charge C, i.e., $E = E(S, V, B_i, C)$. Formally, we can vary the energy with respect to each of these quantities, while holding the others fixed. This leads to Gibbs' fundamental equation of thermodynamics,

$$dE = TdS - pdV + \nu^i dB_i + \mu dC, \tag{3}$$

where the temperature T, pressure p, chemical potentials ν^i, and the chemical potential μ conjugate to C are defined as

$$T \equiv \left(\frac{\partial E}{\partial S}\right)_{V,B_i,C}, \qquad p \equiv -\left(\frac{\partial E}{\partial V}\right)_{S,B_i,C},$$

$$\nu^i \equiv \left(\frac{\partial E}{\partial B_i}\right)_{S,V,C}, \qquad \mu \equiv \left(\frac{\partial E}{\partial C}\right)_{S,V,B_i}. \tag{4}$$

The variation of C in (3) moves one away from the original field theory content to a theory with a different number of degrees of freedom. On the other hand, for variations which only compare different thermodynamic states within the same theory, the variable C is kept fixed. Hence, depending on the ensemble, the central charge could be varied or fixed in the fundamental equation of thermodynamics. However, we observe next that the central charge necessarily has to appear in the large-N Euler relation.

The entropy and conserved quantities scale with the central charge for high-energy states, $S, B_i \sim C$, reflecting the contribution from all the degrees of freedom. Thus, the energy function obeys the following scaling relation:

$$E(\alpha S, V, \alpha B_i, \alpha C) = \alpha E(S, V, B_i, C), \tag{5}$$

with α being a dimensionless scaling parameter. This means that in the deconfined phase of large-N theories on compact spaces the energy is not an extensive function. Differentiating with respect to α and putting $\alpha = 1$ leads to the Euler equation

$$E = TS + \nu^i B_i + \mu C. \tag{6}$$

Notice that pressure and volume do not appear in this Euler equation, since the volume does not generically scale

with C. It does scale with C in the infinite-volume limit of CFTs, i.e., $pV = -\mu C$ as $V \to \infty$ (see Appendix C). In that limit the energy becomes an extensive function, satisfying $E(\alpha S, \alpha V, \alpha B_i) = \alpha E(S, V, B_i)$. By varying the Euler relation (6) and employing the fundamental variational equation (3), we find a slightly unusual Gibbs-Duhem equation

$$0 = SdT + pdV + B_i d\nu^i + Cd\mu. \qquad (7)$$

The variation of volume (instead of pressure) features in this equation, since the Euler relation does not involve a pV term.

Furthermore, in the grand canonical ensemble the thermodynamic potential or free energy is defined as

$$W \equiv E - TS - \nu^i B_i = \mu C. \qquad (8)$$

It follows from the fundamental equation (3) that the grand canonical free energy is stationary at fixed (T, V, ν^i, C). The proportionality of free energy with C (or N^2) is a signature of deconfinement; in contrast, the free energy of the confined phase is of order one [29]. In fact, the relation $W \sim C$ can be viewed as the definition of the dimensionless central charge C in this paper, which could hence be called the "thermal free energy charge." This charge is generically not identical to other definitions of the central charge, such as anomaly coefficients or the coefficient of the two-point function of the stress-energy tensor, except in the special case of $d = 2$ and in the large-N limit of $SU(N)$ conformal gauge theories (where all central charges scale as N^2).

The Euler equation (6), or equivalently $W = \mu C$, only holds in a regime where $1/C$ corrections can be neglected. For generic CFTs on compact spaces this is the case in the high-temperature or large-volume regime $TR \gg 1$, where R is the curvature radius, since the free energy satisfies $W \sim (TR)^{d-1}$ in that regime and the central charge C is defined as the dimensionless proportionality coefficient. On the other hand, for holographic and $2d$ sparse CFTs the free energy already scales with the central charge at low temperatures $TR \sim \mathcal{O}(1)$ (i.e., if $ER \sim C$ with $C \gg 1$). Further, the Euler equation is satisfied for any value of λ, at weak and strong coupling, and for any large-N field theory, including conformal and confining theories, and theories with unusual scaling behavior like Lifshitz theories. Each of these theories, though, satisfies a different equation of state, which is not encoded in the large-N Euler equation [20]. For instance, the equation of state for conformal theories is $E = (d - 1)pV$, and for Lifshitz scale invariant theories with dynamical scaling exponent z it is given by $zE = (d - 1)pV$ (see Appendix C). The fact that the Euler relation applies to both conformal and Lifshitz theories, means that it not only holds for relativistic, but also for nonrelativistic theories.

III. HOLOGRAPHIC BLACK HOLE THERMODYNAMICS

The large-N Euler equation applies in particular to strongly coupled large-N CFTs with a semiclassical, gravitational dual description. We now investigate the gravity dual of the Euler equation.

The best-established example of holography, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, states that the partition function of the CFT and of the gravitational theory in asymptotically AdS spacetime are equal $Z_{CFT} = Z_{AdS}$ [12,13]. For field theories at finite temperature the thermal partition function is related to the free energy via $W = -T \ln Z_{CFT}$. On the other hand, the gravitational partition function is given by the Euclidean path integral, which in the saddle-point approximation is computed by the on-shell Euclidean action, $I_E = -\ln Z_{AdS}$ [30]. Since thermal states in the CFT are dual to black holes in AdS, the on-shell action should be evaluated on the classical black hole saddle. Here, we consider rotating, charged black hole solutions [5] to the Einstein-Maxwell action with a negative cosmological constant, i.e., $I_E = -\frac{1}{16\pi G}\int d^{d+1}x \sqrt{g}(R - 2\Lambda - F^2)$. In the grand canonical ensemble (at fixed T and Φ) the free energy of the holographic field theory corresponds to [31–33]

$$W = TI_E = M - \frac{\kappa A}{8\pi G} - \Omega J - \Phi Q. \qquad (9)$$

The final equality follows from evaluating the action—including the Gibbons-Hawking boundary term [30] and a background subtraction term—on the black hole solution with angular momentum J and electric charge Q. The corresponding chemical potentials are the angular velocity Ω and the electric potential Φ of the horizon. We note it is straightforward to generalize this equation to black holes with multiple electric charges and angular momenta [34–36].

The thermodynamic Euler equation for these black holes follows from (9) by inserting the holographic dictionary (1) for energy, entropy, and temperature, and the dictionary for the charge $\tilde{Q} = QL$ and potential $\tilde{\Phi} = \Phi/L$ [20,31], and using the relation (8) between free energy and chemical potential,

$$E = TS + \Omega J + \tilde{\Phi}\tilde{Q} + \mu C. \qquad (10)$$

The thermodynamic variables in this equation are well-known black hole parameters, except for the chemical potential and central charge. What is their gravitational dual description? Essentially, μC is the on-shell Euclidean action (times T). In addition, a different expression for the chemical potential is obtained from the generalized Smarr formula for AdS black holes, which relates all the black hole parameters and is thus a gravitational reorganization of the Euler relation [5–8,32,37],

$$M = \frac{d-1}{d-2}\left(\frac{\kappa A}{8\pi G} + \Omega J\right) + \Phi Q - \frac{1}{d-2}\frac{\Theta \Lambda}{4\pi G}. \quad (11)$$

The Λ term is absent for asymptotically flat black holes, but is necessary for the consistency of the Smarr formula of asymptotically AdS black holes. The quantity Θ can be defined as $\int_{\Sigma_{bh}} |\xi| dV - \int_{\Sigma_{AdS}} |\xi| dV$ [38], where a subtraction with respect to the pure AdS background is implemented to cancel the divergence at infinity. In this definition the domain of integration Σ_{BH} extends from the horizon to infinity, while Σ_{AdS} in the pure AdS integral extends across the entire spacetime. Further, ξ is the timelike Killing field $\xi = \partial_t + \Omega \partial_\phi$, which (in the black hole geometry) generates the event horizon, and $|\xi| = \sqrt{-\xi \cdot \xi}$ is its norm. In the literature [7,8] (minus) Θ is often called the "thermodynamic volume," since it is the conjugate quantity to Λ (the bulk pressure) in the first law of black hole mechanics, see Eq. (13). For our purposes, however, a geometric name is probably more suitable, such as (background subtracted) "Killing volume," because we are interested in the field theory thermodynamics rather than the bulk thermodynamics.

Comparing the Euler equation and the Smarr formula we see that the chemical potential (times central charge) corresponds to three combinations of the black hole parameters

$$\mu C = M - \frac{\kappa A}{8\pi G} - \Omega J - \Phi Q = \frac{1}{d-1}\left(M - \Phi Q - \frac{\Theta \Lambda}{4\pi G}\right)$$

$$= \frac{1}{d-2}\left(\frac{\kappa A}{8\pi G} + \Omega J - \frac{\Theta \Lambda}{4\pi G}\right). \quad (12)$$

Note that the dimension dependent factors in the Smarr formula are absorbed in the chemical potential. The expression above for the chemical potential should also follow from its definition in (4), $\mu \equiv (\frac{\partial E}{\partial C})_{S,V,J,\tilde{Q}}$. We check this explicitly by rewriting the extended first law of AdS black hole mechanics as a thermodynamic variational identity, where μ plays the role of the conjugate quantity to the central charge variation dC. For CFTs dual to Einstein gravity the holographic dictionary for the central charge depends on both the cosmological constant Λ and Newton's constant G. In order to keep track of the central charge variation, we vary both coupling constants as "bookkeeping devices" in the bulk first law [18,20].

The mass of rotating, charged AdS black holes can be regarded as the function $M(A, J, Q, \Lambda, G)$. From a scaling argument [4,6] and from the generalized Smarr formula (11) it follows that the extended first law for these black holes is

$$dM = \frac{\kappa}{8\pi G} dA + \Omega dJ + \Phi dQ + \frac{\Theta}{8\pi G} d\Lambda$$

$$- (M - \Omega J - \Phi Q)\frac{dG}{G}. \quad (13)$$

Usually, in extended black hole thermodynamics only the variation of Λ is taken into account in the first law, but the variation of Newton's constant can be easily included by noting that $M, J, Q \sim G^{-1}$ [39,40]. Remarkably, the Λ and G variations in (13) cannot be combined into one single term proportional to $d(\Lambda/G)$, because there is a term remaining involving the variation of G. This seems to imply that the standard interpretation of the extended first law in terms of bulk pressure $P = -\Lambda/8\pi G$ is inconsistent, if Newton's constant is allowed to vary. On the other hand, we can find a consistent boundary interpretation by expressing the right-hand side of the first law in terms of variations of the entropy $S = A/4G$, electric charge $\tilde{Q} = QL$, central charge $C \sim L^{d-1}/G$, and spatial volume $V \sim L^{d-1}$ of the holographic field theory. To this end we rewrite the extended first law above as

$$dM = \frac{\kappa}{2\pi} d\left(\frac{A}{4G}\right) + \Omega dJ + \frac{\Phi}{L} d(QL) - \frac{M}{d-1}\frac{dL^{d-1}}{L^{d-1}}$$

$$+ \left(M - \frac{\kappa A}{8\pi G} - \Omega J - \frac{\Phi}{L} QL\right)\frac{d(L^{d-1}/G)}{L^{d-1}/G}. \quad (14)$$

Here we used again the Smarr relation and $d\Lambda/\Lambda = -2dL/L$, and observed that $(d-1)dL/L - dG/G$ is equal to the fraction in the final term. It is crucial that the L and G variations appear in this combination, otherwise the holographically dual first law would not involve a variation of the central charge. Moreover, by allowing for variations of G we can clearly distinguish the variation of the spatial volume V from that of the central charge C [20]. Consequently, from the holographic dictionary we deduce that the extended first law for AdS black holes is dual to the fundamental equation in thermodynamics,

$$dE = TdS + \Omega dJ + \tilde{\Phi} d\tilde{Q} - pdV + \mu dC. \quad (15)$$

By comparing (14) and (15) we see that the pressure p satisfies the CFT equation of state, $E = (d-1)pV$, and the chemical potential μ fulfils the Euler equation (10). This shows that our dictionary (12) for the chemical potential is consistent, since the same expression follows from equating bulk and boundary free energy and from matching the "first laws." Note that we only used the scaling properties $V \sim L^{d-1}$ and $C \sim L^{d-1}/G$ to arrive at (15), but did not need their proportionality constants, because the fractions dV/V and dC/C appear in the first law and hence the proportionality constants drop out.

As an aside, we mention that the precise match between the first laws can be generalized to charged Lifshitz black holes [41] by replacing the holographic dictionary with: $E = ML^{1-z}$, $T = (\kappa/2\pi)L^{1-z}$ and $\tilde{\Phi} = \Phi/L^z$ (see Appendices C and D). The extended first law (13) with $J = 0$ still holds for Lifshitz black holes [42], and is dual to the fundamental equation (15), if p satisfies the Lifshitz

equation of state $zE = (d-1)pV$ and μ satisfies the Euler relation [43].

IV. COMPARISON WITH PREVIOUS LITERATURE

In Refs. [17,44] the chemical potential associated to the central charge (or N^2) was defined as $\mu \equiv \left(\frac{\partial E}{\partial C}\right)_S$ for AdS-Schwarzschild black holes. Compared to our Eq. (4) the fixed volume requirement is lacking in this definition. From the extended first law it follows that this definition of the chemical potential is proportional to the Killing volume Θ, which is inconsistent with the Euler equation and our expressions for μ in (12). This definition is especially problematic since these references take the boundary curvature radius R to be equal to the bulk curvature radius L, which implies that the spatial volume and central charge are both proportional to L^{d-1}. Thus, in [17,44] they mix up the bulk duals to spatial volume and central charge.

In [20] this issue was resolved by allowing for variations of G, in addition to Λ, so that the central charge variation can be distinguished from the volume variation. Our matching above between the first laws is based on this approach, but is still novel since [20] focused on finding the boundary dual to the Smarr relation and not to the first law. In [20] the free energy relation $W \sim N^2$ at large-N, which is equivalent to our Euler equation (8), was identified as the holographic origin of the Smarr formula. In addition, the pressure and its equation of state played an important role in their holographic derivation of the Smarr formula. However, their proof only holds for a particular choice of CFT metric, $ds^2 = -dt^2 + L^2 d\Omega_{k,d-1}^2$, where L is the AdS radius (see Appendix E). The derivation can be extended to a more general CFT metric with $R \neq L$ and, remarkably, it does not depend on the boundary pressure in this case. Rescaling the CFT metric above with the Weyl factor $\lambda = R/L$ changes the CFT time into Rt/L and the boundary curvature radius into R, so that the spatial volume becomes $V \sim R^{d-1}$. The added benefit of this more general boundary metric is that V is clearly distinct from C (and $\tilde{Q}$). The (refined) holographic dictionary for this metric is [45]

$$E = M\frac{L}{R}, \qquad T = \frac{\kappa}{2\pi}\frac{L}{R}, \qquad \tilde{\Omega} = \Omega\frac{L}{R}, \qquad \tilde{\Phi} = \frac{\Phi}{R}. \qquad (16)$$

Importantly, with this dictionary the bulk and boundary variational equations (13) and (15) still agree and the chemical potential again satisfies the Euler relation. We can even keep G fixed and only vary Λ in the bulk, since $dV \sim dR^{d-1}$ and $dC \sim dL^{d-1}|_G$ are obviously distinguishable for $R \neq L$.

Now, we derive the bulk Smarr formula purely from the boundary Euler equation and the holographic dictionary for $R \neq L$. The Λ term in the Smarr formula can be expressed as

$$-\frac{\Theta\Lambda}{4\pi G} = L\left(\frac{\partial M}{\partial L}\right)_{A,J,Q,G} = R\left(\frac{\partial E}{\partial L}\right)_{A,J,Q,G} - E\frac{R}{L}. \qquad (17)$$

Note that the bulk quantities A, J, Q and G are kept fixed in the partial derivative. The boundary energy depends implicitly on them as: $E = E(S(A,G), J, \tilde{Q}(L,Q), V(R), C(L,G))$. This implies (see Appendix D for more details)

$$-\frac{\Theta\Lambda}{4\pi G} = \frac{R}{L}(\tilde{\Phi}\tilde{Q} + (d-1)\mu C - E), \qquad (18)$$

where we used the definitions of $\tilde{\Phi}$ and μ from (4). Finally, inserting the Euler equation and the holographic dictionary (16) precisely yields the Smarr formula (11). Note that this derivation hinges on the chemical potential and not on the pressure.

V. DISCUSSION

In gauge/gravity duality, black holes in the bulk correspond to thermal states in the boundary theory. We proposed a new dictionary between the bulk and boundary thermodynamics by introducing a chemical potential for the number of colors in the gauge theory. The chemical potential is crucial for the correspondence between the Euler equation for large-N theories and the Smarr formula relating the black hole parameters. Since the Euler relation determines the energy as a function of other variables, it contains the essential thermodynamic information about the field theory.

Our field theory interpretation of the extended thermodynamics of black holes stands in contrast to the common gravitational interpretation in terms of bulk pressure and volume. One notable difference is that the black hole mass is equivalent to the internal energy of the field theory, whereas in [6] it is identified with the enthalpy of the gravitational system. Moreover, we found that the extended first law of black holes cannot be solely written in terms of the variation of bulk pressure, $P = -\Lambda/8\pi G$, if both Λ and G are allowed to vary, but can be consistently interpreted as a field theory first law. Thus, as the thermal field theory has a natural thermodynamic description, the boundary interpretation seems unavoidable.

As for future work, we expect that the dictionary for the chemical potential can be generalized to a multitude of black holes in the presence of a cosmological constant, such as black holes in higher-curvature gravity, hyper-scaling violating solutions, black rings and de Sitter black holes. On the field theory side, an interesting problem is to extend the Euler equation beyond the large-N limit, by including $1/N$ corrections [20,46].

APPENDIX A: EULER EQUATION FOR TWO-DIMENSIONAL CFTs

Examples of large-N field theories are $2d$ modular invariant CFTs with large central charge c. The microcanonical entropy for these theories is given by the Cardy formula (setting $c_L = c_R = c$) [47]

$$S(E_L, E_R, c) = 2\pi\sqrt{\frac{c}{6}E_L} + 2\pi\sqrt{\frac{c}{6}E_R}, \qquad \text{(A1)}$$

with $E_{L,R}$ the left- and right-moving energies. On a circle of length $V = 2\pi R$, the total energy and angular momentum are, respectively, $E = (E_L + E_R)/R$ and $J = E_L - E_R$. The Cardy formula holds for CFTs with a sparse light spectrum in the regime $C \to \infty$ with $ER \geq C$ [48], where we normalized the central charge (conjugate to μ) as $C = c/12$. If we view the entropy (A1) as the function $S = S(E, V, J, C)$, then the fundamental variational equation of thermodynamics with $\nu^i dB_i = \Omega dJ$, follows by taking partial derivatives of the entropy function. Consequently, the products of thermodynamic quantities are

$$TS = \frac{4}{R}\sqrt{E_L E_R}, \qquad pV = E,$$
$$\Omega J = E - \frac{2}{R}\sqrt{E_L E_R}, \qquad \mu C = -\frac{2}{R}\sqrt{E_L E_R}, \qquad \text{(A2)}$$

where Ω is the angular potential. They satisfy the relation

$$E = TS + \Omega J + \mu C. \qquad \text{(A3)}$$

Hence, the large-N Euler equation indeed holds for $2d$ CFTs. In fact, the Euler relation splits up into two separate equations, $E = \Omega J - \mu C$ and $TS = -2\mu C$.

In AdS$_3$ gravity the Smarr formula for the outer horizon of a BTZ black hole is given by $0 = TS + \Omega J - \Theta\Lambda/4\pi G$ [49]. Comparing this to the Euler equation (A3) we find that the chemical potential must correspond to $\mu C = E - \Theta\Lambda/4\pi G$. Using the holographic dictionary for the central charge $c = 3L/2G$ [50,51], it can be shown that the chemical potential is dual to $\mu = -(r_+^2 - r_-^2)/(L^2 R)$, where $r_\pm$ are the outer and inner horizon radii of the rotating BTZ black hole. Notably, μ vanishes for extremal black holes, if $r_+ = r_-$ or $ER = |J|$, which correspond to CFT states with $E_L = 0$ or $E_R = 0$.

APPENDIX B: THE EXTENDED FIRST LAW OF ENTANGLEMENT

In this Appendix we compare our chemical potential for AdS black holes to the chemical potential in the extended first law for entanglement entropy of ball-shaped regions in the CFT vacuum [18,52]. This CFT first law takes the form

$$d\bar{E} = \bar{T}dS_{\text{ent}} + \bar{\mu}dC, \qquad \text{(B1)}$$

where $\bar{E}$ denotes the modular Hamiltonian expectation value, S_{ent} is the vacuum entanglement entropy of the ball-shaped region and C is the universal coefficient of the entanglement entropy (commonly denoted as a_d^*) [23,53,54]. The CFT first law is dual to the first law of static hyperbolic AdS black holes which are isometric to pure AdS space [55–57], a special case of the black holes considered in the main text, with $J = Q = 0$. The boundary first law follows from reformulating our fundamental variational equation in terms of dimensionless quantities $\bar{E} = ML$, $\bar{T} = \kappa L/2\pi$, and $\bar{\mu} = \mu L$. The volume variation drops out of the first law, since it is a dimensionful quantity. In the vacuum $\bar{E} = 0$, hence the chemical potential reduces to $\bar{\mu} = -\bar{T}S_{\text{ent}}/C$, which agrees with the results in [18] (where the temperature was normalized as $\bar{T} = 1$).

APPENDIX C: EULER EQUATION IN FLAT SPACE

In flat spacetime, static equilibrium states satisfy the standard thermodynamic Euler equation,

$$E = TS + \nu^i B_i - pV, \qquad \text{(C1)}$$

which is often formulated instead in terms of densities since V is infinite. Note that the energy is purely extensive in this formula, since it satisfies $E(\alpha S, \alpha V, \alpha B_i) = \alpha E(S, V, B_i)$. This Euler relation applies in particular to conformal and Lifshitz theories on the plane (see e.g., [58,59]). It is not immediately clear why this equation is consistent with the large-N Euler equation, therefore in this Appendix we explain the relation between the two for Lifshitz scale invariant theories.

Anisotropic scaling symmetry $\{t, x^i\} \to \{\zeta^z t, \zeta x^i\}$ with dynamical scaling exponent z implies that the product TR^z is Lifshitz scale invariant, where R is the curvature radius of the compact space, such as a sphere. Therefore, for Lifshitz theories with positive z the infinite-volume limit $R \to \infty$ is effectively the same as $T \to \infty$, so on the plane these theories are essentially always in the high-temperature deconfining phase. In this limit, the energy scales as $E \sim T^{\frac{d-1+z}{z}}$ and entropy and conserved quantities as $S, B_i \sim T^{\frac{d-1}{z}}$, so the scaling relation is $E(\alpha^{\frac{d-1}{z}}S, V, \alpha^{\frac{d-1}{z}}B_i, C) = \alpha^{\frac{d-1+z}{z}}E(S, V, B_i, C)$. This imposes the condition $(d - 1 + z)E = (d - 1)(TS + \nu^i B_i)$, which in combination

with the large-N Euler equation yields $zE = -(d-1)\mu C$. We can now compare this to the Lifshitz equation of state $zE = (d-1)pV$, which is a consequence of the anisotropic scaling relation $E(S, \alpha^{d-1}V, B_i, C) = \alpha^{-z}E(S, V, B_i, C)$. As a result, we find $\mu C = -pV$ as $V \to \infty$, turning the large-N Euler equation into the standard one. The same argument works for conformal theories (by setting $z = 1$), hyperscaling violating theories and possibly other large-N theories. Notably, the standard Euler equation only applies in the infinite-volume limit of large-N theories. The large-N Euler relation, on the other hand, also holds at finite temperature on compact spaces for holographic field theories and $2d$ sparse CFTs (but not for generic CFTs).

APPENDIX D: HOLOGRAPHIC DERIVATION OF THE SMARR FORMULA FOR LIFSHITZ BLACK HOLES

In this Appendix we derive the Smarr formula for charged Lifshitz black holes [41], with curvature radius L and scaling exponent z, from the holographic Euler equation and the dictionary for the thermodynamic quantities involved. We put the dual Lifshitz field theory on a spatial geometry of curvature radius R. Our derivation generalizes Sec. 2.3 of [20] to $R \neq L$ and $z \neq 1$. Our aim is to prove that even for Lifshitz black holes the boundary pressure and its equation of state are not necessary input to deduce the Smarr formula (although they are in the special case $R = L$ considered in [20]).

The holographic dictionary for Lifshitz black holes reads

$$E = M\frac{L}{R^z}, \quad T = \frac{\kappa}{2\pi}\frac{L}{R^z}, \quad \tilde{\Phi} = \frac{\Phi}{R^z}, \quad \tilde{Q} = QL. \quad \text{(D1)}$$

Note that the factors of R and L are chosen such that the products ER^z, TR^z and $\tilde{\Phi}R^z$ are Lifshitz scale invariant (see Appendix C). First, we express the Λ term in the Smarr formula in terms of the boundary energy E

$$-\frac{\Theta\Lambda}{4\pi G} = L\left(\frac{\partial M}{\partial L}\right)_{A,Q,G} = R^z\left(\frac{\partial E}{\partial L}\right)_{A,Q,G} - E\frac{R^z}{L}. \quad \text{(D2)}$$

The strategy is to show that the right-hand side satisfies the Smarr formula. Note that the bulk quantities A, Q and G are fixed in the partial derivative with respect to L. The boundary energy depends on these bulk quantities as follows

$$E = E(S(A, G), \tilde{Q}(L, Q), V(R), C(L, G)). \quad \text{(D3)}$$

Note that $J = 0$. The partial derivative is hence given by

$$\left(\frac{\partial E}{\partial L}\right)_{A,Q,G} = \left(\frac{\partial E}{\partial \tilde{Q}}\right)_{S,V,C}\left(\frac{\partial \tilde{Q}}{\partial L}\right)_Q + \left(\frac{\partial E}{\partial C}\right)_{S,V,\tilde{Q}}\left(\frac{\partial C}{\partial L}\right)_G$$

$$= \frac{1}{L}(\tilde{\Phi}\tilde{Q} + (d-1)\mu C). \quad \text{(D4)}$$

In the second line we used $\tilde{Q} = QL$ and $C \sim L^{d-1}/G$, and we recognized the definitions of the electric potential $\tilde{\Phi}$ and chemical potential μ (see Eq. (4) in the main text). Thus, we find

$$-\frac{\Theta\Lambda}{4\pi G} = \frac{R^z}{L}(\tilde{\Phi}\tilde{Q} + (d-1)\mu C - E)$$

$$= \frac{R^z}{L}((d-2)E - (d-1)TS - (d-2)\tilde{\Phi}\tilde{Q}). \quad \text{(D5)}$$

Finally, by inserting the holographic dictionary (D1) we recover the Smarr formula. Note that the Smarr formula for Lifshitz black holes does not involve z and is hence the same as for black holes in Einstein gravity [42]. Crucially, the holographic Euler equation was employed in the second line of (D5) and is therefore dual to the Smarr formula, as pointed out in [20]. We emphasize that the boundary pressure does not play a role in this derivation, whereas the chemical potential μ does. For $R = L$ the pressure does feature in the derivation, since in that case the boundary volume depends on the bulk radius, i.e., $V(L)$, which yields an extra term $-(d-1)pV/L$ in (D4). But ultimately the result in (D5) remains the same, since this pressure term cancels, due to the Lifshitz equation of state, against a new term zEL^{z-1} on the right side of (D2).

APPENDIX E: THE RENORMALIZED HOLOGRAPHIC EULER EQUATION

In the main text the energy was defined with respect to the ground state, so the vacuum energy was effectively set to zero. However, CFTs on a curved background exhibit the Casimir effect, which implies that the ground state could have nonvanishing energy. In AdS/CFT the ground-state energy can be computed with the method of holographic renormalization, by regularizing the gravitational action with local counterterms at the boundary [21,60]. In this Appendix we derive the renormalized holographic Euler equation for static vacuum AdS black holes, and find that the ground-state energy contributes a constant term to the chemical potential.

We consider static, vacuum asymptotically AdS black holes with hyperbolic, planar and spherical horizons [61]

$$ds^2 = -f_k(r)dt^2 + \frac{dr^2}{f_k(r)} + r^2 d\Omega_{k,d-1}^2, \quad \text{(E1)}$$

where

$$f_k(r) = k + \frac{r^2}{L^2} - \frac{16\pi GM}{(d-1)\Omega_{k,d-1}r^{d-1}}. \qquad \text{(E2)}$$

For $k = 1$ the unit metric $d\Omega_{k,d-1}^2$ is the metric on a unit S^{d-1} sphere, for $k = 0$ it is the dimensionless metric $\frac{1}{L^2}\sum_{i=1}^{d-1} dx_i^2$ on the plane $\mathbb{R}^{d-1}$, and for $k = -1$ the unit metric on hyperbolic space H^{d-1} is $du^2 + \sinh^2 u\, d\Omega_{k=1,d-2}^2$. The mass parameter M is related to the horizon radius r_+ via

$$M = \frac{(d-1)\Omega_{k,d-1}r_+^{d-2}}{16\pi G}\left(\frac{r_+^2}{L^2} + k\right). \qquad \text{(E3)}$$

According to the Gubser-Klebanov-Polyakov-Witten prescription in AdS/CFT [12,13], the CFT metric is identified with the boundary metric of the dual asymptotically AdS spacetime up to a Weyl rescaling, i.e., $g_{\text{CFT}} = \lim_{r\to\infty}\lambda^2(x)g_{\text{AdS}}$ where $\lambda(x)$ is a Weyl scale factor. As $r \to \infty$ the boundary metric approaches

$$ds^2 = \frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + r^2 d\Omega_{k,d-1}^2. \qquad \text{(E4)}$$

A common choice of Weyl factor is $\lambda = L/r$, so that the CFT metric becomes $-dt^2 + L^2 d\Omega_{k,d-1}^2$. The boundary curvature radius is then equal to the AdS radius and the volume is $V = \Omega_{k,d-1}L^{d-1}/(d-1)$. Moreover, the CFT time is the same as the global AdS time t, which implies that the CFT energy E can be identified (up to a constant) with the ADM mass M, the conserved charge associated to time t translations.

The temperature, entropy and energy of the black holes are

$$T = \frac{dr_+^2 + k(d-2)L^2}{4\pi L^2 r_+}, \qquad S = \frac{\Omega_{k,d-1}r_+^{d-1}}{4G},$$

$$E_{\text{ren}} = \frac{(d-1)\Omega_{k,d-1}L^{d-2}}{16\pi G}\left(\frac{r_+^d}{L^d} + k\frac{r_+^{d-2}}{L^{d-2}} + \frac{2\epsilon_k^0}{d-1}\right). \qquad \text{(E5)}$$

The energy was derived from the renormalized boundary stress-energy tensor in [60] and from the on-shell Euclidean gravitational action with counterterms in [62]. The resulting energy, $E_{\text{ren}} = M + E_k^0$, differs from the mass parameter by a constant term, the Casimir energy of the dual field theory

$$E_k^0 = \frac{\Omega_{k,d-1}L^{d-2}}{8\pi G}\epsilon_k^0, \qquad \text{(E6)}$$

with $\epsilon_k^0 = 0$ for odd d and equal to [62]

$$\epsilon_k^0 = (-k)^{d/2}\frac{(d-1)!!^2}{d!} \quad \text{for even } d. \qquad \text{(E7)}$$

For instance, $\epsilon_k^0 = -k/2$ for $d = 2$ and $\epsilon_k^0 = 3k^2/8$ for $d = 4$. The renormalized version of the Smarr formula reads

$$E_{\text{ren}} = \frac{d-1}{d-2}TS - \frac{1}{d-2}\frac{\Theta_{\text{ren}}\Lambda}{4\pi G}, \qquad \text{(E8)}$$

with a new (counterterm subtracted) Killing volume

$$\Theta_{\text{ren}} = -\frac{\Omega_{k,d-1}}{d}\left(r_+^d - \frac{d-2}{d-1}L^d\epsilon_k^0\right). \qquad \text{(E9)}$$

The holographic Euler equation still takes the form

$$E_{\text{ren}} = TS + \mu_{\text{ren}}C, \qquad \text{(E10)}$$

since the Casimir energy is also proportional to the central charge, which we normalize here as $C = \Omega_{k,d-1}L^{d-1}/16\pi G$. But the chemical potential is not given by Eq. (12) in the main text anymore, since it receives a constant contribution from the vacuum energy

$$\mu_{\text{ren}} = -\frac{r_+^{d-2}}{L^{d-1}}\left(\frac{r_+^2}{L^2} - k\right) + \frac{2}{L}\epsilon_k^0. \qquad \text{(E11)}$$

For $d = 2$ we find the chemical potential $\mu_{\text{ren}} = -r_+^2/L^3$, which agrees with the expression found in Appendix A (for $r_- = 0$ and $R = L$). For planar black holes ($k = 0$) or very large hyperbolic or spherical black holes (with $r_+ \gg L$), the Casimir energy is effectively zero and hence there is no distinction between the renormalized energy and the vacuum-subtracted energy. As can be seen from (E5) and (E11), there are additional thermodynamic relations for these black holes

$$E = -(d-1)\mu C \quad \text{and} \quad TS = -d\mu C, \qquad \text{(E12)}$$

consistent with the infinite-volume limit of Appendix C.

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