Quantum instability of gauge theories on κ-Minkowski space

Kilian Hersent,[1,*] Philippe Mathieu,[2,†] and Jean-Christophe Wallet[1,‡]
[1]*IJCLab, Université Paris-Saclay, CNRS/IN2P3, 91405 Orsay, France*
[2]*Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556, USA*

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We consider a gauge theory on the 5D κ-Minkowski which can be viewed as the noncommutative analog of a $U(1)$ gauge theory. We show that the Hermiticity condition obeyed by the gauge potential A_μ is necessarily twisted. Performing a Becchi-Rouet-Stora-Tyutin gauge-fixing with a Lorentz-type gauge, we carry out a first exploration of the one loop quantum properties of this gauge theory. We find that the gauge-fixed theory gives rise to a nonvanishing tadpole for the time component of the gauge potential, while there is no nonvanishing tadpole 1-point function for the spatial components of A_μ. This signals that the classical vacuum of the theory is not stable against quantum fluctuations. Possible consequences regarding the symmetries of the gauge model and the fate of the tadpole in other gauges of noncovariant type are discussed.

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I. INTRODUCTION

It is currently believed that noncommutative structures are likely to show up at the Planck scale [1,2], where quantum gravity effects become sizeable [3–5] hence rendering questionable the standard description of spacetime as a smooth manifold to reconcile quantum mechanics and gravity. Among the noncommutative (quantum) spaces considered so far, the κ-Minkowski space [6–8] has received a considerable attention for more than two decades [9] as it appears to be a good candidate for a quantum space underlying the description of quantum gravity at least in some regime [3–5]. Numerous related works focus on algebraic aspects [9] while more phenomenological investigations resulted in a huge literature in connection in particular with doubly special relativity [10–14], relative locality [15–17] or with the quest of possible detectable effects [3–5]. The simplest description of the (d-dimensional) κ-Minkowski space $\mathcal{M}_\kappa^d$ is to view it as the enveloping algebra of the Lie algebra of noncommutative coordinates defined by $[x_0, x_i] = \frac{i}{\kappa} x_i$, $[x_i, x_j] = 0$, for $i, j = 1, \ldots, (d-1)$, where κ is the deformation parameter with mass dimension 1, often identified with the Planck mass in four dimensions. The bicrossproduct structure of the κ-Poincaré Hopf algebra $\mathcal{P}_\kappa^d$ exhibits clearly its role as coding the quantum symmetries of $\mathcal{M}_\kappa^d$, as first pointed out in [8], this latter being the dual of the subalgebra of $\mathcal{P}_\kappa^d$ generated by the so-called deformed translations.

$\mathcal{M}_\kappa^d$ can be conveniently modeled [18,19], by an associative * algebra equipped with the following star product and involution[1]

$$(f \star g)(x) = \int \frac{dp^0}{2\pi} dy_0 e^{-iy_0 p^0} f(x_0 + y_0, \vec{x}) g(x_0, e^{-p^0/\kappa}\vec{x}), \tag{1.1}$$

$$f^\dagger(x) = \int \frac{dp^0}{2\pi} dy_0 e^{-iy_0 p^0} \bar{f}(x_0 + y_0, e^{-p^0/\kappa}\vec{x}), \tag{1.2}$$

for any functions f, g in a suitable multiplier algebra of the algebra of Schwartz functions in $\mathbb{R}^d$ ($\bar{f}$ is the complex conjugate of f). It will be also denoted hereafter by $\mathcal{M}_\kappa^d$.[2] The construction basically combines the essential features of the Weyl-Wigner quantization map with properties of the convolution algebra of the affine group $\mathbb{R} \ltimes \mathbb{R}^{(d-1)}$. It is nothing but the extension of the old construction related to the Moyal product which used the notion of twisted convolution related to the Heisenberg group [20] (instead of the affine group) aiming to rigorously formalize the

*kilian.hersent@universite-paris-saclay.fr
†pmathieu@nd.edu
‡jean-christophe.wallet@universite-paris-saclay.fr

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[1]Conventions: latin indices $i, j, \cdots = 1, 2, \ldots, (d-1)$ refer to space coordinates. Timelike quantities are indexed by 0. We set $x := (x_\mu) = (x_0, \vec{x})$, $x.y := x_\mu y^\mu = x_0 y_0 + \vec{x}\vec{y}$. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is defined by $(\mathcal{F}f)(p) := \int d^d x e^{-i(p_0 x_0 + \vec{p}.\vec{x})} f(x)$ with inverse $\mathcal{F}^{-1}$.

[2]This algebra involves the smooth functions with polynomial bounds as well as all their derivatives, such that their inverse Fourier transform has compact support in the x_0 direction [18].

correspondence between classical and quantum observables initially introduced by Weyl [21] throughout the (Weyl-Wigner) quantization map. Notice that a somewhat similar extension to the case of $\mathbb{R}^3_\lambda$, a deformation of $\mathbb{R}^3$, based however on the $SU(2)$ group algebra, underlies the studies of quantum properties of field theories on this latter quantum space [22–24].

The star product (1.1) is especially convenient for practical purpose, a fact exploited in recent explorations of quantum properties of κ-Poincaré invariant scalar field theories on $\mathcal{M}^4_\kappa$ [19,25,26]. The κ-deformed relativistic symmetries of $\mathcal{M}^d_\kappa$ can be actually viewed as coded in the κ-Poincaré algebra. Hence, requiring κ-Poincaré invariance is physically natural. It is known [18,19] that this invariance is achieved provided the action is of the form $\int d^dx\mathcal{L}$ where $\mathcal{L}$ is some Lagrangian density and $\int d^dx$ is the simple Lebesgue measure which however is no longer cyclic with respect to the star product (1.1), since one has for any $f, g \in \mathcal{M}^d_\kappa$

$$\int d^dx(f \star g)(x) = \int d^dx((\mathcal{E}^{d-1} \triangleright g) \star f)(x), \quad (1.3)$$

where $\mathcal{E}^{d-1}$, called the modular twist, is defined by

$$(\mathcal{E} \triangleright f)(x) = f\left(x_0 + \frac{i}{\kappa}, \vec{x}\right), \quad \mathcal{E} = e^{-P_0/\kappa}, \quad (1.4)$$

with $(P_\mu \triangleright f)(x) = -i\partial_\mu f(x), \mu = 0, \ldots, d-1$. Equation (1.3) defines a twisted trace with respect to (1.1) which, as pointed out in [19,27,28], promotes the action to the status of Kubo-Martin-Schwinger (KMS) weight [29,30] a new property which replaces the lost cyclicity.[3] The loss of cyclicity complicates *a priori* the construction of a suitable gauge-invariant action for a κ-Poincaré invariant gauge theory, starting from any standard (untwisted) noncommutative differential calculus. Indeed, the usual compensation between unitary gauge factors in the noncommutative gauge transformation of the curvature $F_{\mu\nu}$, which takes the generic form $F^g_{\mu\nu} = g^\dagger \star F_{\mu\nu} \star g$, $g^\dagger \star g = g \star g^\dagger = 1$, cannot occur [33], due to the twisted trace relation (1.3). This prevents the construction of a gauge invariant polynomial action depending on the curvature. For a review on earlier algebraic and field theoretic developments on gauge theories on κ-Minkowski space and related problems, see [34].

As shown in [33], the troublesome effect of the modular twist can be neutralized, leading to a κ-Poincaré invariant *and* gauge invariant action with physically suitable commutative limit. This happens thanks to the existence of a unique twisted noncommutative differential calculus based on a family of twisted derivations related to the deformed translations of the κ-Poincaré algebra [33]. This forces the gauge transformations of the curvature to be twisted thus allowing now the unitary gauge factors to balance each other, which actually occurs for a unique value of d, namely $d = 5$ [33]. This comes from the fact that the modular twist (1.4) depends on the dimension d. Otherwise stated, the main physical prediction is that the coexistence of κ-Poincaré invariance and gauge invariance implies the existence of one extra dimension. For a first exploration of phenomenological features of the four-dimensional theory obtained from compactification scenarios, see [35].

Notice that the analysis carried out in [33] used the notion of noncommutative connection on a right module $\mathbb{E}$. It was assumed to be one copy of the algebra $\mathcal{M}^5_\kappa$ in order to describe a noncommutative analog of a $U(1)$ Yang-Mills theory while the action of the algebra $\mathcal{M}^5_\kappa$ on $\mathbb{E}$ was chosen to be $m \triangleleft a = m \star a$, for any $m \in \mathbb{E}$, $a \in \mathcal{M}^5_\kappa$, owing to $\mathbb{E} \simeq \mathcal{M}^5_\kappa$. One may wonder if a suitable choice for this action would lead to another value for d, since the corresponding gauge transformations should be modified. Algebraic constraints from right module structure together with physical requirement lead to actions of the form $m \triangleleft a = m \star \phi(a)$ where ϕ is any regular automorphism of the algebra, i.e., $(\phi(a))^\dagger = \phi^{-1}(a^\dagger)$. Again, one finds [36] that $d = 5$ is the only allowed value for which the κ-Poincaré invariant gauge theories exist. Thus, occurrence of one extra dimension seems to be a rather robust feature of these gauge theories on κ-Minkowski, which motivates further investigation on their quantum properties.

This is the purpose of the present paper which will exploit the Becchi-Rouet-Stora-Tyutin (BRST) symmetry linked to the twisted gauge symmetry elaborated in [37]. As a first step in the investigation, we will compute the one-loop 1-point (tadpole) function for this five-dimensional noncommutative gauge theory without matter. We will also pay attention to the Hermiticity condition obeyed by the gauge potential. We find that this latter Hermiticity condition is twisted. Furthermore, we find that the gauge-fixed theory gives rise to a nonvanishing tadpole for the time component of the gauge potential, while there is no nonvanishing tadpole 1-point function for the spatial components of A_μ. This signals that the classical vacuum of the theory is not stable against quantum fluctuations. Some possible consequences from the viewpoint of symmetries, namely Lorentz symmetry and noncommutative gauge symmetry are then discussed.

The paper is organized as follows. In Sec. II A, we first collect the useful properties which will be needed in the course of the analysis, including essential features of the twisted differential calculi singled out in [33,35]. We then show that the Hermiticity condition affecting the (noncommutative analog of the) gauge potential A_μ becomes twisted. In Sec. II B, we present the BRST gauge-fixing of the classical action. Section III is centered on the one-loop

[3] KMS weights which are basically KMS states up to a normalization, are linked to the Tomita group of modular automorphisms [31] whose generator here is $\mathcal{E}^{d-1}$. For discussions on physical consequences of KMS property, see [32].

computation of the tadpole 1-point function for the gauge potential. We find that the ghost-gauge vertex and the trilinear vertex of the gauge-fixed action give nonvanishing contributions only to the 1-point function for A_0, the time component of A_μ. The tadpole function for the spatial components of A_μ are all zero. In Sec. IV, we discuss the results and conclude.

II. GAUGE THEORIES ON κ-MINKOWSKI SPACE

A. Basic features of the classical action

The classical gauge-invariant action is given by [33]

$$S_{\rm cl} = \frac{1}{4}\int d^5x (F_{\mu\nu} \star F^\dagger_{\mu\nu})(x), \tag{2.1}$$

where the curvature can be expressed as

$$F_{\mu\nu} = \mathcal{E}^{-2\gamma} \triangleright (X_\mu A_\nu - X_\nu A_\mu) + (\mathcal{E}^{1-\gamma} \triangleright A_\mu) \star (\mathcal{E}^\gamma \triangleright A_\nu) - (\mathcal{E}^{1-\gamma} \triangleright A_\nu) \star (\mathcal{E}^\gamma \triangleright A_\mu), \tag{2.2}$$

in which γ is a real parameter and A_μ denotes as usual (the noncommutative analog of) the gauge potential which is assumed in the following to be *real valued*. Notice that in five dimensions the action (2.1) should normally be rescaled by a dimensionful parameter $\frac{1}{g^2}$ where g^2 has mass dimension -1. As this parameter is not essential in the following discussion, we will omit it.

In (2.1), the X_μs are the generators of an Abelian Lie algebra of twisted derivations denoted by $\mathfrak{D}_\gamma$. These are built from the so-called deformed translations which generate a sub-Hopf algebra of the κ-Poincaré algebra. The twisted derivations are given by

$$X_0 = \kappa\mathcal{E}^\gamma(1-\mathcal{E}), \qquad X_i = \mathcal{E}^\gamma P_i, \quad i = 1,2,3,4, \tag{2.3}$$

and satisfy the following twisted Leibniz rule

$$X_\mu(a \star b) = X_\mu(a) \star (\mathcal{E}^\gamma \triangleright b) + (\mathcal{E}^{\gamma+1} \triangleright a) \star X_\mu(b), \tag{2.4}$$

for any elements a, b of $\mathcal{M}^5_\kappa$. The twisted derivations of $\mathfrak{D}_\gamma$ generate the twisted noncommutative differential calculus underlying the analysis. The corresponding relevant properties are collected in the Appendix. For general algebraic properties of (untwisted) derivation-based differential calculus, see e.g., [38] and references therein.

Note that interesting twisted differential calculi for κ-Minkowski spaces stemming from the use of twist deformation formalism applied to Abelian and Jordanian twists are considered and discussed in [34]. These are not relevant here. Besides, interesting bicovariant (untwisted) differential calculi on κ-Minkowski spaces [39,40] cannot give rise to κ-Poincaré invariant and gauge invariant action functionals as discussed in [33].

In this paper, we will use a twisted version of the notion of noncommutative connection on a right module over $\mathcal{M}^5_\kappa$, introduced in [33]. The main mathematical properties of the notion of untwisted connection on a right (or left) module are characterized in details in [38] and further developed in [41–43]). Notice that untwisted connections on the right module underly the pioneering works [44–46].

Recall that the gauge group $\mathcal{U}$ is defined as the set of automorphisms of the right module over $\mathcal{M}^5_\kappa$, denoted by $\mathbb{E}$, assumed here to be one copy of $\mathcal{M}^5_\kappa$, i.e., $\mathbb{E} \simeq \mathcal{M}^5_\kappa$, which are required to preserve the canonical Hermitian structure

$$h(m_1, m_2) = m_1^\dagger \star m_2 \tag{2.5}$$

for any $m_1, m_2 \in \mathcal{M}^5_\kappa$. It is a simple matter of algebra to find that

$$\mathcal{U} = \{g \in \mathcal{M}^5_\kappa, g \star g^\dagger = g^\dagger \star g = \mathbb{I}\}, \tag{2.6}$$

which can be viewed as the noncommutative analog of the $U(1)$ group.

Now, recall that the twisted connection is defined [33] as a map $\nabla_{X_\mu}: \mathbb{E} \to \mathbb{E}$, for any $X_\mu \in \mathfrak{D}_\gamma$, fulfilling the following properties:

$$\nabla_{X_\mu + X'_\mu}(m) = \nabla_{X_\mu}(m) + \nabla_{X'_\mu}(m), \tag{2.7}$$

$$\nabla_{z.X_\mu}(m) = \nabla_{X_\mu}(m) \star z, \tag{2.8}$$

$$\nabla_{X_\mu}(m \star f) = \nabla_{X_\mu}(m) \star (\mathcal{E}^\gamma \triangleright f) + (\mathcal{E}^{\gamma+1} \triangleright m) \star X_\mu(f), \tag{2.9}$$

for any $m \in \mathbb{E} \simeq \mathcal{M}^5_\kappa$, $X_\mu, X'_\mu \in \mathfrak{D}_\gamma$, $z \in \mathcal{Z}(\mathcal{M}^5_\kappa)$ (the center of $\mathcal{M}^5_\kappa$), $f \in \mathcal{M}^5_\kappa$. In (2.9) the factor $(\mathcal{E}^{\gamma+1} \triangleright m)$ in the second term must be understood as a morphism $\tilde{\beta}: \mathbb{E} \to \mathbb{E}$ acting on the module as $\tilde{\beta}(m) = \mathcal{E}^{\gamma+1} \triangleright m$ for any m in $\mathbb{E} \simeq \mathcal{M}^5_\kappa$. Set

$$A_\mu := \nabla_{X_\mu}(\mathbb{I}), \qquad \nabla_\mu := \nabla_{X_\mu}. \tag{2.10}$$

Then, observe that the noncommutative analog of the "gauge potential" A_μ defined above and verifying

$$\nabla_\mu(f) = A_\mu \star \mathcal{E}^\gamma \triangleright f + X_\mu(f) \tag{2.11}$$

obtained by setting $m = \mathbb{I}$ in (2.9), does not generally satisfy the usual relation $A_\mu^\dagger = A_\mu$. This is a mere consequence of the fact that the X_μs are twisted and are *not* real derivations. Indeed, one has

$$(X_\mu(f))^\dagger = -\mathcal{E}^{-2\gamma-1} \triangleright (X_\mu(f^\dagger)) \neq X_\mu(f^\dagger). \tag{2.12}$$

In fact, one finds after standard algebraic calculations that A_μ satisfies

$$A_\mu = \mathcal{E}^{2\gamma+1} \triangleright A_\mu^\dagger, \tag{2.13}$$

together with the following twisted Hermiticity condition for the connection

$$h(\mathcal{E}^{-2\gamma-1} \triangleright i\nabla_{X_\mu}(m_1), \mathcal{E}^\gamma \triangleright m_2) + h(\mathcal{E}^{-\gamma-1} \triangleright m_1, i\nabla_{X_\mu}(m_2)) = iX_\mu h(m_1, m_2), \tag{2.14}$$

which holds true for any $X_\mu \in \mathfrak{D}_\gamma$, $m_1, m_2 \in \mathcal{M}_\kappa^5$.

Note that somewhat similar deformed Hermiticity condition for noncommutative connections also appeared within the framework of ε derivations giving rise to the notion of ε connections [43].

It is known that Hermitian connections play a central role in the physics described by (commutative) Yang-Mills theories. The present situation obviously deals with a noncommutative analog of a $U(1)$ Yang-Mills theory. Accordingly, we assume from now on that the twisted Hermitian condition for the connection (2.13) holds true. To simplify the analysis, we will further *assume* that

$$\gamma = 0, \tag{2.15}$$

which will not alter the conclusions of this paper.

It will be useful in the sequel to introduce the 1-form connection A and its associated curvature 2-form F. The main properties of the corresponding noncommutative differential calculus and the related notations introduced in [33] are collected for convenience in the Appendix. This provides a convenient formalism to deal with the BRST symmetry.

We will introduce below the material used in the ensuing analysis. The curvature 2-form $F \in \Omega^2(\mathfrak{D}_0)$ is easily found to be given by

$$F = \mathbf{d}A + ((\mathcal{E} \triangleright A) \times A), \tag{2.16}$$

where $A \in \Omega^1(\mathfrak{D}_0)$, $\times$ denotes the associative product of forms and $\mathbf{d}$ is the twisted differential with

$$\mathbf{d}^2 = 0 \tag{2.17}$$

satisfying the twisted Leibniz rule

$$\mathbf{d}(\omega \times \eta) = \mathbf{d}\omega \times \eta + (-1)^{\delta(\omega)}(\mathcal{E} \triangleright \omega) \times \mathbf{d}\eta \tag{2.18}$$

for any $\omega, \eta \in \Omega^\bullet$, where $\delta(\omega)$ is the form degree of ω. Recall that one has $A(X_\mu) = A_\mu$ and $F(X_\mu, X_\nu) = F_{\mu\nu}$.

In (2.16), (2.18) and in the sequel of the discussion, the action of $\mathcal{E}$ on the forms must be understood as $(\mathcal{E} \triangleright \omega)(X_1, \ldots, X_p) = (\mathcal{E} \triangleright (\omega(X_1, \ldots, X_p)))$ for any $\omega \in \Omega^\bullet$, i.e., $\mathcal{E}$ acts on the "components of the forms."

One can verify that the classical action is invariant under the gauge transformations

$$A_\mu^g = (\mathcal{E} \triangleright g^\dagger) \star A_\mu \star g + (\mathcal{E} \triangleright g^\dagger) \star X_\mu(g), \tag{2.19}$$

$$F_{\mu\nu}^g = (\mathcal{E}^2 \triangleright g^\dagger) \star F_{\mu\nu} \star g, \tag{2.20}$$

or equivalently on the connection 1-form and curvature 2-form

$$A^g = (\mathcal{E} \triangleright g^\dagger) \times A \times g + (\mathcal{E} \triangleright g^\dagger) \times \mathbf{d}g, \tag{2.21}$$

$$F^g = (\mathcal{E}^2 \triangleright g^\dagger) \times F \times g, \tag{2.22}$$

which hold true for any g of the gauge group $\mathcal{U}$.

B. BRST symmetry and the gauge-fixed action

The BRST symmetry associated with (2.19)–(2.22) is defined by the following structure equations [37]

$$s_0 A = -\mathbf{d}C - (\mathcal{E} \triangleright C) \times A - A \times C, \tag{2.23}$$

$$s_0 C = -C \times C, \tag{2.24}$$

so that the BRST transformation of the curvature 2-form is

$$s_0 F = F \times C - (\mathcal{E}^2 \triangleright C) \times F, \tag{2.25}$$

and one has $s_0^2 = 0$. Here, C is the Fadeev-Popov ghost, a real-valued field with ghost number $+1$ while s_0 is the Slavnov operation associated with the gauge transformations (2.19), (2.20), whose action on any field increases its ghost number by $+1$. The relevant technical materials needed in the ensuing discussion are collected in the Appendix.

The transformations of the components are easily found to be given by

$$s_0 A_\mu = X_\mu(C) - (\mathcal{E} \triangleright C) \star A_\mu + A_\mu \star C, \tag{2.26}$$

$$s_0 C = -C \star C, \tag{2.27}$$

$$s_0 F_{\mu\nu} = F_{\mu\nu} \star C - (\mathcal{E}^2 \triangleright C) \star F_{\mu\nu} \tag{2.28}$$

upon using $(A \times C)(X_\mu) = -A_\mu \star C$, $(C \times A)(X_\mu) = C \star A_\mu$, $(dC)(X_\mu) = -X_\mu(C)$. Furthermore, one can verify that

$$s_0 S_{\mathrm{cl}} = 0. \tag{2.29}$$

Recall that a suitable framework encompassing the differential calculus and the BRST symmetry is obtained by introducing a bigraded differential calculus. For more

mathematical details relevant to the present situation, see [37].

In particular, s_0 acts as an *untwisted* graded derivation. Namely, it satisfies the following Leibniz rule

$$s_0(\omega \times \eta) = s_0\omega \times \eta + (-1)^{|\omega|}\omega \times s_0\eta \tag{2.30}$$

for any bigraded forms $\omega, \eta \in \hat{\Omega} = \bigoplus_{p,q}\Omega^{p,q}(\mathfrak{D}_0)$, where $|\omega|$ denotes the total degree of ω defined as the sum of the form degree $\delta(\omega)$ and the ghost number of ω. Accordingly, the Leibniz rule for $\mathbf{d}$ (2.18) still holds for any bigraded forms $\omega, \eta \in \hat{\Omega}$ with however $\delta(\omega)$ replaced by $|\omega|$.

Notice by the way that one should have $C = C^\dagger$, stemming from the fact that s_0 can be viewed as a Grassmann version of the infinitesimal gauge transformations. This, combined with the Hermiticity relation (2.13) together with (2.23) yields $s_0(A^\dagger) = (s_0(A))^\dagger$.

The BRST operation s_0 generates the functional Slavnov identity which serves to control the UV behavior of the action $S_{\rm cl}$ after its gauge fixing. This latter is obtained by adding a BRST-exact term. A convenient gauge fixing is given by

$$S = S_{\rm cl} + s_0 \int d^5x(\bar{C}^\dagger \star (\mathcal{E}^{-4} \triangleright X_\mu(A_\mu)), \tag{2.31}$$

supplementing the BRST structure equations (2.26)–(2.28) by

$$s_0\bar{C}^\dagger = b^\dagger, \tag{2.32}$$

$$s_0 b^\dagger = 0, \tag{2.33}$$

where $\bar{C}$ (b) is the antighost (Stückelberg) real-valued field with ghost number -1 (0). A simple calculation yields

$$S = S_{\rm cl} + \int d^5x(b.X_\mu(A_\mu) + \bar{C}.X^2(C) - \bar{C}.X_\mu((\mathcal{E} \triangleright C) \star A_\mu - A_\mu \star C)), \tag{2.34}$$

where we set $X^2 = X^\mu X_\mu$ and we used the twisted trace property (1.3) for $\int d^5x$ together with the useful identity

$$\int d^5x(f \star g^\dagger)(x) = \int d^5x f(x).\bar{g}(x). \tag{2.35}$$

The quadratic part of the action defines the kinetic terms for the gauge potential and the ghosts. It is given by

$$S_{\rm kin} = \int d^5x \frac{1}{2}A_\mu(X^2\delta_{\mu\nu} - X_\mu X_\nu)A_\nu + \bar{C}.X^2(C), \tag{2.36}$$

where the term involving A_μ is easily obtained upon using $\langle a, X_\mu(b)\rangle = \langle X_\mu(a), b\rangle$ for any $a, b \in \mathcal{M}^5_\kappa$ where

$$\langle a, b\rangle := \int d^5x a^\dagger \star b \tag{2.37}$$

is the Hermitian product introduced in [19].

Performing the functional integration over the b field in the generating functional of the Green functions enforces the constraint $X_\mu(A_\mu) = 0$ everywhere in the gauge-fixed action (2.34). Accordingly, the kinetic terms for the gauge potential and the ghosts are, respectively, given by

$$S_{\rm kin}(A) = \frac{1}{2}\int d^5x d^5y A_\mu(x)K(x-y)A_\mu(y), \tag{2.38}$$

$$S_{\rm kin}(\bar{C}, C) = \int d^5p d^5q \bar{C}(x)K(x-y)C(y), \tag{2.39}$$

with

$$K(x-y) = \int d^5p e^{ip(x-y)}T(p) \tag{2.40}$$

in which

$$T(p) = (\vec{p}^2 + \kappa^2(1 - e^{-p_0/\kappa})^2). \tag{2.41}$$

Notice that $K(x-y) \neq K(y-x)$ which simply comes from the exponential $e^{-p_0/\kappa}$ in the second term of (2.41) so that $T(p) \neq T(-p)$. It is convenient to reexpress (2.38) as

$$S_{\rm kin}(A) = \frac{1}{2}\int d^5x d^5y A_\mu(x)S(x-y)A_\mu(y), \tag{2.42}$$

with

$$S(x-y) = \frac{1}{2}\int d^5p e^{ip(x-y)}(T(p) + T(-p)). \tag{2.43}$$

The trilinear gauge potential interaction can be written as

$$S_{AAA} = \langle (X_\mu A_\nu - X_\nu A_\mu)^\dagger, ((\mathcal{E} \triangleright A_\mu) \star A_\nu - (\mathcal{E} \triangleright A_\nu) \star A_\mu)^\dagger\rangle + \text{c.c.} \tag{2.44}$$

The corresponding vertex function can be read off from [we use the obvious notation for the momenta: $p = (p_0, \vec{p})$]

$$S_{AAA} = \int d^5p d^5q d^5r A_\mu(p)A_\nu(q)A_\nu(r)V^A_\mu(p,q,r) \times \delta(p_0 + q_0 + r_0) \tag{2.45}$$

with

$$V^A_\mu(p,q,r) = Q_\mu(q_0, \vec{q})(e^{-3p_0/\kappa}\delta(\vec{p}e^{-p_0/\kappa} + \vec{q}e^{-p_0/\kappa} + \vec{r}) - e^{-3r_0/\kappa}\delta(\vec{p} + \vec{q}e^{-r_0/\kappa} + \vec{r}e^{-r_0/\kappa})), \tag{2.46}$$

where

$$Q_\mu(q_0,\vec{q}) = (\kappa(1-e^{-q_0/\kappa}), q_i). \tag{2.47}$$

The trilinear gauge-ghost interaction can be cast into the form

$$S_{AC\bar{C}} = \int d^5x \bar{C}(x).((\mathcal{E}\triangleright A_\mu)\star X_\mu(C) - (\mathcal{E}\triangleright X_\mu(C))\star A_\mu), \tag{2.48}$$

where use has been made of the gauge function $X_\mu A_\mu = 0$. After some algebra, it can be put into the form [again we use the obvious notation for the momenta: $p = (p_0, \vec{p})$]

$$S_{AC\bar{C}} = \int d^5p d^5q d^5r \bar{C}(p) A_\mu(q) C(r) V_\mu^{\phi\pi}(p,q,r) \times \delta(p_0+q_0+r_0), \tag{2.49}$$

with

$$V_\mu^{\phi\pi}(p,q,r) = Q_\mu(r_0,\vec{r})(e^{-q_0/\kappa}\delta(\vec{p}+\vec{q}+\vec{r}e^{-q_0/\kappa}) - e^{-r_0/\kappa}\delta(\vec{p}+\vec{r}+\vec{q}e^{-r_0/\kappa})), \tag{2.50}$$

where Q_μ is still defined by (2.47).

One easily verifies that both vertex functions vanish in the commutative limit, which is consistent with the usual electrodynamics for which ghosts decouple.

III. COMPUTATION OF THE 1-POINT FUNCTION

A. Perturbative setup

Introduce the generating functional of the connected Green functions $W(J,\bar{\eta},\eta)$ defined by

$$e^{W(J,\bar{\eta},\eta)} = \int dA d\bar{C} dC e^{-(S+S_{\text{sources}})}, \tag{3.1}$$

where S is given by (2.34) and $J, \bar{\eta}, \eta$ are as usual sources to be defined just below. The part of the action relevant for the ensuing calculation involves the kinetic part together with the cubic vertices. We write the source term as

$$S_{\text{sources}} = \int d^5x A_\mu(x) J_\mu(x) + \bar{\eta}(x) C(x) + \bar{C}(x)\eta(x), \tag{3.2}$$

where J, $\bar{\eta}$, η are sources associated, respectively, to A_μ, C, $\bar{C}$ with respective ghost numbers $0, -1, 1$.

Recall that functional derivatives obey obvious graded Leibniz rules with grading defined by the ghost number. Besides, Green functions involving ghost fields are generated by the action on (3.1) of functional derivatives $\frac{\delta}{\delta\bar{\eta}}$ and $\frac{\delta}{\delta\eta}$, acting, respectively, from left and right. Namely, one has

$$\frac{\delta}{\delta\bar{\eta}(x)} e^{\int d^5y \bar{\eta}(y)C(y)+\bar{C}(y)\eta(y)} = C(x) e^{\int d^5y \bar{\eta}(y)C(y)+\bar{C}(y)\eta(y)}, \tag{3.3}$$

$$\frac{\delta}{\delta\eta(x)} e^{\int d^5y \bar{\eta}(y)C(y)+\bar{C}(y)\eta(y)} = e^{\int d^5y \bar{\eta}(y)C(y)+\bar{C}(y)\eta(y)} \bar{C}(x). \tag{3.4}$$

The perturbative expansion is generated by the following functional relation

$$W(J,\bar{\eta},\eta) = W_0(J,\bar{\eta},\eta) + \ln(1+e^{-W_0(J,\bar{\eta},\eta)}[e^{-S_{\text{int}}}-1]e^{W_0(J,\bar{\eta},\eta)}) \tag{3.5}$$

up to an unessential additive constant, where $W_0(J,\bar{\eta},\eta)$ denotes the free generating functional of the connected Green functions. It is given by

$$W_0(J,\bar{\eta},\eta) = \int d^5x d^5y \left(\frac{1}{2} J_\mu(x) S^{-1}(x-y) J_\mu(y) + \bar{\eta}(x) K^{-1}(x-y)\eta(y)\right), \tag{3.6}$$

with

$$S^{-1}(x-y) = 2\int d^5p e^{ip(x-y)}(T(p)+T(-p))^{-1}, \tag{3.7}$$

$$K^{-1}(x-y) = \int d^5p e^{ip(x-y)} T^{-1}(p), \tag{3.8}$$

where $T(p)$ still given by (2.41).

In (3.5), one has

$$S_{\text{int}} = S_{\text{int}}\left(\frac{\delta}{\delta J}, \frac{\delta}{\delta\bar{\eta}}, \frac{\delta}{\delta\eta}\right) = S_{AAA}\left(\frac{\delta}{\delta J}\right) + S_{AC\bar{C}}\left(\frac{\delta}{\delta J}, \frac{\delta}{\delta\bar{\eta}}, \frac{\delta}{\delta\eta}\right), \tag{3.9}$$

which is obtained by replacing each field in S_{AAA} and $S_{AC\bar{C}}$, respectively, (2.46), (2.49), by its associated functional derivative. We did not explicitly write the quartic interaction since it will not generate contributions to the tadpole 1-point function.

B. The tadpole at one-loop

The one-loop contribution to the 1-point tadpole function can be extracted from

$$W(J,\bar{\eta},\eta) = W_0(J,\bar{\eta},\eta) - e^{-W_0(J,\bar{\eta},\eta)} S_{\rm int}\left(\frac{\delta}{\delta J}\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta\eta}\right) e^{W_0(J,\bar{\eta},\eta)}. \quad (3.10)$$

The relevant part of $W(J,\bar{\eta},\eta)$ corresponding to the one-loop ghost contribution to the 1-point Green function for A_μ, denoted hereafter by $W_1^{\phi\pi}(J)$, is obtained by simply combining (2.49), (2.50), (3.6), (3.9) with (3.10). A standard calculation yields

$$W_1^{\phi\pi}(J) = \int d^5p d^5q d^5r d^5x d^5y d^5z d^5w e^{-i(px+qy+rz)} \times K^{-1}(z-x)S^{-1}(y-w)J_\mu(w)\delta(p_0+q_0+r_0) \times V_\mu^{\phi\pi}(p,q,r), \quad (3.11)$$

where $V_\mu^{\phi\pi}(p,q,r)$ is still given by (2.50). By further making use of the Legendre transform $\frac{\delta W}{\delta J_\mu(x)} = A_\mu(x)$ which at the first order, relevant here, reduces to

$$A_\mu(x) = \int d^5y S^{-1}(x-y)J_\mu(y), \quad (3.12)$$

one readily derives from (3.11), combined with $W_1^{\phi\pi}(J)$, the corresponding ghost contribution to the one-loop effective action, denoted by $\Gamma_1^{\phi\pi}(A)$. By taking into account the various delta functions occurring in $W_1^{\phi\pi}(J)$, we find that the expression reduces to

$$\Gamma_1^{\phi\pi}(A) = \int d^5q\delta(q)A_\mu(q)\mathcal{I}_\mu, \quad (3.13)$$

in which

$$\mathcal{I}_\mu = \int d^5s \frac{1-e^{3s_0/\kappa}}{\vec{s}^2+\kappa^2(1-e^{-s_0/\kappa})^2} \times Q_\mu(s). \quad (3.14)$$

From (3.14), one immediately infers that the spatial components of $\mathcal{I}_\mu$ vanish since the following relation $\int d\vec{s}\frac{s_i}{\vec{s}^2+M^2} = 0$, $i = 1,\ldots,4$ (M^2 is some positive quantity) holds true. Hence, one concludes that the ghosts only contribute to the 1-point function for A_0. One obtains finally

$$\Gamma_1^{\phi\pi}(A) = \int d^5x A_0(x)\mathcal{I}_0. \quad (3.15)$$

Notice that one can easily verify that $\lim_{\kappa\to\infty}\mathcal{I}_0 = 0$ hence signaling the vanishing of $\Gamma_1^{\phi\pi}(A)$ at the commutative limit as expected.

The relevant part of $W(J,\bar{\eta},\eta)$ corresponding to the one-loop A_μ contribution to the 1-point Green function for A_μ, denoted hereafter by $W_1^A(J)$, can be computed in a similar way. We find

$$W_1^A(J) = \int d^5p d^5q d^5r d^5x d^5y d^5z d^5w e^{-i(px+qy+rz)} \times \delta(p_0+q_0+r_0)V_\mu^A(p,q,r) \times J_\mu(w)(5S^{-1}(x-w)S^{-1}(y-z) + S^{-1}(y-w)S^{-1}(z-x) + S^{-1}(z-w)S^{-1}(x-y)). \quad (3.16)$$

Upon using (3.12), the corresponding contribution of the gauge potential to the one-loop effective action, denoted by $\Gamma_1^{\phi\pi}(A)$ can be written as a sum of three terms

$$\Gamma_1^A(A) := \Gamma_{11}(A) + \Gamma_{12}(A) + \Gamma_{13}(A) \quad (3.17)$$

with

$$\Gamma_{11}^A(A) = \int d\theta_\mu S^{-1}(z-x)A_\mu(y), \quad (3.18)$$

$$\Gamma_{12}^A(A) = \int d\theta_\mu S^{-1}(x-y)A_\mu(z), \quad (3.19)$$

$$\Gamma_{13}^A(A) = 5\int d\theta_\mu S^{-1}(y-z)A_\mu(x), \quad (3.20)$$

where

$$d\theta_\mu := d^5p d^5q d^5r d^5x d^5y d^5z e^{-i(px+qy+rz)} \times \delta(p_0+q_0+r_0)V_\mu^A(p,q,r). \quad (3.21)$$

The first contribution $\Gamma_{11}^A(A)$ vanishes. Indeed, from simple manipulations of the various delta functions occurring in $\Gamma_{11}^A(A)$, one easily realizes that the contribution to A_0 vanishes, due to the fact basically that one factor $\sim\delta(q_0)Q_0(q)$ appears in the expression which upon integrating over q_0 gives 0. For the spatial contributions, one arrives at

$$\Gamma_{11}^A(A) \sim \int d^5q ds_0 d^5y A_i(y)q_i e^{-iqy}\delta(q_0) \times (S^{-1}(-\vec{q}\Sigma(s_0),q_0) - S^{-1}(\vec{q}\Sigma(s_0),q_0)) = 0, \quad (3.22)$$

where the last equality comes from the fact that $S^{-1}(p)$ is an even function of $\vec{p}$. Hence

$$\Gamma_{11}^A(A) = 0. \quad (3.23)$$

The second contribution $\Gamma_{12}^A(A)$ can be cast into the form

$$\Gamma_{12}^A(A) = \int d^5x A_\mu(x)\mathcal{J}_\mu \quad (3.24)$$

with

$$\mathcal{J}_\mu = \int d^5 s 2 Q_\mu(s) \frac{e^{3s_0/\kappa} - 1}{T(s) + T(-s)}. \tag{3.25}$$

By observing that $\mathcal{J}_i = 0$, $i = 1, \ldots, 4$, one concludes that

$$\Gamma^A_{12}(A) = \int d^5 x A_0 \mathcal{J}_0. \tag{3.26}$$

Finally, the last contribution $\Gamma^A_{13}(A)$ yields

$$\Gamma^A_{13}(A) = -5 \int d^5 x A_\mu(x) \mathcal{J}_\mu = -5 \int d^5 x A_0 \mathcal{J}_0. \tag{3.27}$$

From (3.23), (3.26), (3.27) and (3.15), one concludes that the contribution of the 1-point function for the gauge potential is given by

$$\Gamma_1(A) = \int d^5 x A_0(x)(\mathcal{I}_0 - 4\mathcal{J}_0), \tag{3.28}$$

which, in view of (3.14) and (3.25), is nonzero.

Hence, only the time component A_0 of the gauge potential has a nonzero 1-point function at the one-loop order.

IV. DISCUSSION

We have considered a gauge theory on the (necessarily 5D) κ-Minkowski space which can be viewed as the noncommutative analog of a $U(1)$ gauge theory. First, we have shown in Sec. II A that the Hermiticity condition obeyed by the gauge potential A_μ is twisted. Then, we have carried out a first exploration of the quantum properties of this gauge theory suitably gauge-fixed using the twisted BRST symmetry framework elaborated in [37]. Assuming that A_μ is real valued and working at the one-loop order, we have find that the gauge-fixed theory gives rise to a nonvanishing tadpole for the A_μ corresponding to the occurrence in the 1-loop effective action of a term linear in A_μ of the form

$$\Gamma_1(A) \sim \int d^5 x \sigma_\mu A_\mu(x) = \int d^5 x K(\kappa) A_0(x), \tag{4.1}$$

where the diverging quantity $K(\kappa)$ can be read of from (3.28) and must then be suitably regularized.

Had we have relaxed the assumption on A_μ to be real valued,[4] thus starting from a complex-valued A_μ, then we would have find again a nonvanishing tadpole. This can be verified from a computation similar to the one presented in Sec. III. Indeed, to compute the 1-point contribution for A_μ, simply replace everywhere $S(x-y)$ defined in (2.42), (2.43) by $K(x-y)$ (2.40), (2.41), with the relevant trilinear vertex now given by

$$\begin{aligned} S_{AAA} = \int & d^5 p d^5 q d^5 r \bar{A}_\mu(p) A_\nu(q) \bar{A}_\nu(r) Q_\mu(q) \\ & \times V^A_\mu(p,q,r) \delta(p_0 - q_0 + r_0) \\ & \times (e^{3p_0/\kappa} \delta(\vec{p} e^{p_0/\kappa} - \vec{q} e^{p_0/\kappa} + \vec{r}) \\ & - e^{3r_0/\kappa} \delta(\vec{p} - \vec{q} e^{r_0/\kappa} + \vec{r} e^{r_0/\kappa})), \end{aligned} \tag{4.2}$$

and the gauge-ghost vertex unchanged, leading to $\Gamma_1(A, \bar{A}) \sim \int d^5 x A_0(x) \mathcal{I}_0$. One proceeds similarly for the $\bar{A}_\mu$ contribution.

Notice that the inclusion of fermions,[5] obtained by supplementing the action with the following gauge invariant coupling

$$S_F = \int d^5 x (\psi^\dagger \gamma^0 \star \mathcal{E}^{-1} \gamma^\mu \nabla_\mu \psi)(x), \tag{4.3}$$

with $\nabla_\mu \psi = A_\mu \star \psi + X_\mu(\psi)$, does not change the conclusion. This simply stems from the fact that the corresponding fermionic contribution to the 1-point function identically vanishes as in the commutative case, being proportional to the trace of a single gamma matrix, which can be easily verified by using the fermion propagator whose expression is given by $K_F^{-1}(x-y) = \int d^5 x e^{ip(x-y)} \gamma^\mu Q_\mu(p) T^{-1}(p)$.

The appearance of a nonzero tadpole at the 1-loop order has already been evidenced in various classes of gauge theories on quantum spaces. For instance, this shows up in the massless gauge theory on $\mathbb{R}^3_\lambda$, a deformation of the 3D space [47].

In the same way, a tadpole appears in a family of gauge matrix models on the Moyal plane $\mathbb{R}^2_\theta$ [48]. Recall that in these gauge models, the relevant field variable is a tensor form, sometimes called the covariant coordinate, which is the difference of two form connections, one of them having a distinguished status.[6] The corresponding (classical) action functional is quadratic and quartic in the tensor form but must be expanded around a symmetric vacuum [50] in order to obtain a dynamically nontrivial model, which generates an additional cubic vertex upon expansion. This results in a nonzero tadpole so that a term linear in A_μ, albeit absent at the classical order, is induced by quantum fluctuations in the 1-loop effective action.

Note that in both cases, the relevant BRST symmetry is untwisted. Note also that the analysis in [47] is based on a

[4]The commutative limit would however involve two gauge potentials.

[5]Recall that the corresponding commutative limit would lead to a nonrenormalizable model.

[6]This latter is the canonical gauge invariant connection rigidly linked with the coordinates of the Moyal space. For more details, see e.g., [41,49].

temporal gauge, giving rise to a nonvanishing tadpole. We will discuss more closely the consequences implied by the use of such a temporal gauge at the end of this section.

The complicated vacuum structure [50] of the gauge matrix models on 4D Moyal spaces $\mathbb{R}^4_\theta$ [49,51] forbids, so far, their complete exploration at the quantum level. Note however that a class of intensively studied gauge theories on $\mathbb{R}^4_\theta$ [52–58], bearing a formal similarity with the (commutative) Yang-Mills theories, do not produce 1-loop tadpoles. This stems from the mere algebraic structure of the trilinear gauge potential vertices which produces automatically vanishing tadpole contributions. Unfortunately, these noncommutative gauge theories suffer from UV/IR mixing [59], which thus likely precludes the achievement of their perturbative renomalizability.[7]

At this stage, some comments are in order.

A nonvanishing tadpole is linked with a nonvanishing 1-point function, i.e. vacuum expectation value (VEV), for the gauge potential, say $\langle A_\mu \rangle \neq 0$, which may have some noteworthy consequences as gauge symmetry breaking as well as Lorentz symmetry breaking may occur. In the present situation, the occurrence of the extra term $\Gamma_1(A)$ in the 1-loop effective action Γ_{eff} signals that the classical vacuum of the theory is not stable against quantum fluctuations. This can be traced back to the fact that the term linear in A_μ induced by "radiative corrections" obviously prevents the classical vacuum configuration characterized by $A_\mu = 0$ to be an extremal point of Γ_{eff} (i.e., it no longer verifies the equations of motion related to Γ_{eff}). Note that $\Gamma_1(A)$ is not gauge (BRST) invariant and that it seems unlikely possible to balance its gauge variation by another variation of some other higher order terms involved in Γ_{eff} suggesting that the classical symmetry is broken.

Getting rid of the linear term in A_μ is achieved as usual by expanding Γ_{eff} around the new vacuum $\tilde{A}_\mu$,[8] i.e., setting $A_\mu = \tilde{A}_\mu + \alpha_\mu$ in Γ_{eff} where the new field variable is α_μ, while the resulting (background) symmetry of the expanded action should be presumably obtained from the BRST operation (2.26), (2.27) combined with the field expansion. Its full characterization would require to carry out the complete 1-loop renormalization which is beyond the scope of this paper.

Besides, we note that having $\langle A_\mu \rangle \neq 0$ from (3.28) is reminiscent of some instance in which a Lorentz symmetry breaking does occur. Among the numerous models describing possible Lorentz violations (for a review, see for instance [5] and references therein), some effective models leading to a spontaneous Lorentz symmetry breaking [62] have been considered for some time, see e.g., [63–66]. In these models, defined either in flat space or coupled to a gravitational field, a nonzero VEV of a vector field, mostly obtained through the introduction of a suitable potential at the classical level, triggers the Lorentz symmetry breaking which may possibly exhibit some interplay with a kind of generation of a gauge symmetry [67].

The gauge theory on κ-Minkowski space considered in this paper bears at a first sight somewhat similar gross features with the models mentioned above. However, it differs in some respects. Indeed, one starts with a nonzero VEV of a gauge potential, instead of a simple vector field. This nonzero value is induced by radiative corrections and is not present at the classical level. Besides, the gauge theory under consideration is non local while the models mentioned above are essentially local. Nevertheless, note that the relevant field variable α_μ obtained after achieving the expansion of the effective action around the new vacuum $\tilde{A}_\mu$ is a vector field as being equal to the difference of two connections.

It is instructive to notice that the expression (4.1) actually depends on the gauge choice. This gauge dependence can be made more apparent by choosing a one-parameter family of "noncovariant" gauges involving the temporal/Weyl gauge $A_0 = 0$ for a special value of the parameter. It is worth recalling that this latter gauge choice generates some difficulties when performed within commutative gauge theories, stemming, for instance in QED, from the peculiar momentum dependence of the gauge-fixed photon propagator. Other noncovariant gauges are as well not free from difficulties. However, some of these difficulties can be (partly) overcome or circumvented. For technical details on noncovariant gauges, see e.g., [68,69].

For our present purpose, it is convenient to start, instead of (2.31), from the following gauge-fixing action

$$\begin{aligned} S_{GF} &= s_0 \int d^5x \left(\bar{C}^\dagger \star \mathcal{E}^{-4} \triangleright \left(\frac{\lambda}{4} b - A_0 \right) \right) \\ &= \int d^5x \bar{C} \left(s_0 A_0 + b \left(\frac{\lambda}{4} b - A_0 \right) \right), \end{aligned} \tag{4.4}$$

where λ is a real parameter. The functional integration over the Stückelberg field b yields

$$S_{GF} = \int d^5x - \frac{1}{\lambda} A_0^2 + \bar{C} X_0 C + \bar{C}(A_0 \star C - \mathcal{E} \triangleright C \star A_0). \tag{4.5}$$

By carrying out a computation similar to the one presented in Sec. III B, one easily realizes that (4.5) gives rise to a nonzero contribution to a tadpole for A_0 of the form (to be suitably regularized) $\Gamma_1^{\phi\pi}(A_0) \sim \int d^5x A_0(x) \times \left(\int d^5k \frac{F(k_0)}{(1-e^{-k_0/\kappa})} \right)$ where $F(k_0)$ is a function whose exact

[7]Note that an interesting interpretation of the UV/IR mixing in term of an induced gravity action has been presented in e.g., [60,61]. It takes place in a matrix formulation of the gauge theories on Moyal space.

[8]Solving the equations of motion related to Γ_{eff}.

expression is not needed here. But this latter contribution cannot be balanced by the corresponding contribution from the cubic self-interaction for the A_μ which is easily found to be of the form $\Gamma_1^A(A_0) \sim \int d^5x A_0(x) \times (\int d^5k (Q_0(k)K_{00}(k) + Q_i(k)K_{0i}(k)))$, where $K_{00}(k)$ and $K_{0i}(k)$ are the components of the gauge-fixed propagator for A_μ.[9] Again, a suitable regularization is understood. In view of the expressions for $K_{00}(k)$ and $K_{0i}(k)$, one obtains a contribution proportional to the gauge parameter λ, namely $\Gamma_1^A(A_0) \sim \lambda \int d^5x A_0(x) J$, where J is some constant. Hence, there is a nonvanishing tadpole for A_0 in this gauge,

$$\Gamma_1(A_0) \sim \Gamma_1^{\phi\pi}(A_0) + \lambda \int d^5x A_0(x) J. \tag{4.6}$$

Besides, there is no tadpole for the spatial components of the gauge potential. To see that, one computes the corresponding 1-point function $\Gamma_1(A_I)$ for a given component I. Simple algebraic manipulations yield $\Gamma_1(A_I) \sim \int d^5x A_I(x) \times (\int d^5k K_{I0}(k) Q_0(k) + K_{Ij}(k) Q_j(k))$ in obvious notations. But $K_{I0}(k)Q_0(k)$ and $K_{Ij}(k)Q_j(k)$ are linear in the internal spatial momentum k_i so that the corresponding integrals in $\Gamma^1(A_I)$ vanish. Hence

$$\Gamma_1(A_I) = 0. \tag{4.7}$$

The usual temporal gauge $A_0 = 0$ is obtained by taking the limit $\lambda \to 0$ at which the ghosts decouple so that $\Gamma_1^{\phi\pi}(A_0) = 0$. One can verify that (4.7) still holds true while the remaining contribution from the cubic gauge interaction also vanishes, simply because it is proportional to the gauge parameter λ as it is apparent in (4.6).

Summarizing this last observation, no tadpole occurs when the temporal gauge is used for the gauge fixing. Note however that one can check by inspection that the temporal gauge does not fix the x_0-dependant $\mathcal{U}$ gauge transformations (2.6), as it is the case for the commutative case, so that the corresponding gauge-fixed theory should support a residual gauge symmetry.

In view of the above discussion dealing with different natural choices for gauge conditions, it would be worth analyzing the consequences of the (possibly) nonvanishing 1-point function for A_μ from the viewpoint of Lorentz and noncommutative $\mathcal{U}$ gauge symmetries (2.6). We will come back to these aspects in a forthcoming work.

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[9] $K_{0i}(p) = \frac{\lambda}{Q_0(p_0)} p_i$, $K_{00}(p) = \lambda$, $K_{ij}(p) = \frac{1}{\hat{p}^2}(\delta_{ij} + (\lambda \hat{p}^2 + 1)\frac{p_i p_j}{Q_0(p_0)^2})$, where $\hat{p}^2 = Q_0(p_0)^2 + \vec{p}^2$.

APPENDIX: TWISTED DIFFERENTIAL CALCULUS

Let $\mathfrak{D}_\gamma$ denotes the set of twisted derivation defined in Eq. (2.3) and satisfying the Leibniz rule (2.4). One can verify that $[X_\mu, X_\nu] = X_\mu X_\nu - X_\nu X_\mu = 0$ so that $\mathfrak{D}_\gamma$ is an Abelian Lie algebra. Let $\mathcal{Z}(\mathcal{M}_\kappa^d)$ be the center of $\mathcal{M}_\kappa^d$. We denote by $\Omega^n(\mathfrak{D}_\gamma)$ the linear space of n-linear antisymmetric forms. Note that linearity of forms holds with respect to $\mathcal{Z}(\mathcal{M}_\kappa^d)$. The twisted differential calculus based on $\mathfrak{D}_\gamma$ is an extension of the derivation-based differential calculus introduced a long time ago. See e.g., [38] and references therein.

In the present situation, n-forms are defined from $\Omega^n(\mathfrak{D}_\gamma)$. For any $\alpha \in \Omega^n(\mathfrak{D}_\gamma)$, one has $\alpha\colon \mathfrak{D}_\gamma \to \mathcal{M}_\kappa^d$ together with

$$\begin{aligned} &\alpha(X_1, X_2, \ldots, X_n) \in \mathcal{M}_\kappa^d, \\ &\alpha(X_1, X_2, \ldots, X_n.z) = \alpha(X_1, X_2, \ldots, X_n) \star z, \end{aligned} \tag{A1}$$

for any z in $\mathcal{Z}(\mathcal{M}_\kappa^d)$ and any $X_1, \ldots, X_n \in \mathfrak{D}_\gamma$.

Define now the linear space $\Omega^\bullet := \bigoplus_{n=0}^d \Omega^n(\mathfrak{D}_\gamma)$, with $\Omega^0(\mathfrak{D}_\gamma) = \mathcal{M}_\kappa^d$. Then $(\Omega^\bullet, \times)$ is an associative algebra where the product of forms is defined for any $\alpha \in \Omega^p(\mathfrak{D}_\gamma)$, $\beta \in \Omega^q(\mathfrak{D}_\gamma)$ by $\alpha \times \beta \in \Omega^{p+q}(\mathfrak{D}_\gamma)$ with

$$\begin{aligned} &(\alpha \times \beta)(X_1, \ldots, X_{p+q}) \\ &= \frac{1}{p!q!} \sum_{s \in \mathfrak{S}(p+q)} (-1)^{\mathrm{sign}(s)} \alpha(X_{s(1)}, \ldots, X_{s(p)}) \\ &\quad \star \beta(X_{s(p+1)}, \ldots, X_{s(q)}). \end{aligned} \tag{A2}$$

In (A2), $\mathfrak{S}(p+q)$ is the symmetric group of a set of $p+q$ elements, $\mathrm{sign}(s)$ is the signature of the permutation s. Notice that $\alpha \times \beta \neq (-1)^{\delta(\alpha)\delta(\beta)} \beta \times \alpha$, where $\delta(\alpha)$ is the degree of α. Now the triple $(\Omega^\bullet, \times, \mathbf{d})$ is a graded differential algebra where the differential $\mathbf{d}$ satisfies $\mathbf{d}\colon \Omega^p(\mathfrak{D}_\gamma) \to \Omega^{p+1}(\mathfrak{D}_\gamma)$, $p = 0, \ldots, (d-1)$ and

$$\begin{aligned} &(\mathbf{d}\alpha)(X_1, X_2, \ldots, X_{p+1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\alpha(X_1, \ldots, \vee_i, \ldots, X_{p+1})), \end{aligned} \tag{A3}$$

where the symbol $\vee_i$ indicates the omission of X_i. The differential satisfies $\mathbf{d}^2 = 0$ and the following twisted Leibniz rule

$$\mathbf{d}(\alpha \times \beta) = \mathbf{d}\alpha \times \mathcal{E}^{\gamma}(\beta) + (-1)^{\delta(\alpha)}\mathcal{E}^{1+\gamma}(\alpha) \times \mathbf{d}\beta, \tag{A4}$$

where $\mathcal{E}^x(\alpha)$ is defined for any $x \in \mathbb{R}$ and any $\alpha \in \Omega^n(\mathfrak{D}_\gamma)$ by $\mathcal{E}^x(\alpha) \in \Omega^n(\mathfrak{D}_\gamma)$ with $\mathcal{E}^x(\alpha)(X_1, \ldots, X_n) = \mathcal{E}^x \triangleright (\alpha(X_1, \ldots, X_n))$.

The inclusion of the BRST operation s_0, i.e., the Slavnov operation, in the above framework amounts to introduce bigraded forms which carry a ghost number in addition of the form degree. One has now $\mathbf{d}\colon\ \Omega^{p,g}(\mathfrak{D}_\gamma) \to \Omega^{p+1,g}(\mathfrak{D}_\gamma)$ and $s_0 : \Omega^{p,g}(\mathfrak{D}_\gamma) \to \Omega^{p,g+1}(\mathfrak{D}_\gamma)$ where $\Omega^{p,g}(\mathfrak{D}_\gamma)$ is the space of p forms with ghost number g. The Slavnov operation s_0 acts as a graded but *untwisted* derivation with the Leibniz rule $s_0(\rho \times \eta) = s_0(\rho) \times \eta + (-1)^{|\rho|}\rho \times s_0(\eta)$ for any $\rho, \eta \in \hat{\Omega}(\mathfrak{D}_\gamma)$, where $|\rho| = \delta(\rho) + g$ with $\hat{\Omega}(\mathfrak{D}_\gamma) = \bigoplus_{p,g}\Omega^{p,g}(\mathfrak{D}_\gamma)$, while (A4) still holds with however $\delta(\alpha)$ replaced by $|\alpha|$. For more mathematical details, see [37].

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