Solving formally the auxiliary system of $O(N)$ nonlinear sigma model

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We show that the integrability of the $SO(N)/SO(N-1)$ principal chiral model (PCM) originates from the Pohlmeyer reduction of the $O(N)$ nonlinear sigma model (NLSM). In particular, we show that the Lax pair of the PCM is related upon redefinitions and identification of parameters to the zero curvature condition, which is a consequence of the flatness of the enhanced space used in the Pohlmeyer reduction. This identification provides the solution of the auxiliary system that corresponds to an *arbitrary* NLSM/PCM solution.

I. INTRODUCTION

The classical integrability of the principal chiral model (PCM) is guaranteed by the formulation of its equations of motion and the zero curvature condition, which is satisfied automatically by the current, as a Lax pair [1]. For a recent review see [2]. Both equations can be derived as the compatibility condition of the auxiliary system:

$$\partial_\pm \Psi(\lambda) = \mathcal{L}_\pm \Psi(\lambda), \qquad \mathcal{L}_\pm = \frac{(\partial_\pm g)g^{-1}}{1 \pm \lambda}, \qquad (1)$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. In the literature, this system also goes by the name of fundamental linear problem. In our previous work [3] we solved these equations for $g \in SO(3)/SO(2)$. This was achieved in a brute force manner, which was inspired by the application of the dressing method on elliptic strings in $\mathbb{R} \times S^2$ [4], as well as on the elliptic minimal surfaces in H^3 [5]. We introduced spherical coordinates and bootstrapped the solution of the auxiliary system using all the available equations. In this work, we will solve these equations for an arbitrary $g \in SO(N)/SO(N-1)$ in a more elegant way.

A key element of our derivation will be the so-called Pohlmeyer reduction of the $O(N)$ NLSM [6,7]. In a nutshell, considering the target space S^{N-1} embedded in $\mathbb{R}^N$, the embedding equations of the NLSM solution in $\mathbb{R}^N$ are multicomponent generalizations of the sine-Gordon equations, known as symmetric space sine Gordon models, for a recent review see [8]. This process is the Pohlmeyer reduction. It is important to point out that the Pohlmeyer reduction is a many-to-one mapping. In particular, there is an infinite family of NLSM solutions, which correspond to the same Pohlmeyer counterpart. Finally, it worth noticing that for a given solution of the Pohlmeyer reduced theory, the equations of motion of the NLSM become linear.

The solution of the auxiliary system (1) is of great interest since it is related to the monodromy matrix, which provides an infinite tower of conserved charges. In particular, the monodromy matrix $T(\sigma_f, \sigma_i; \lambda)$ is defined as

$$T(\sigma_f, \sigma_i; \lambda) = \Psi(\tau, \sigma_f; \lambda)\Psi^{-1}(\tau, \sigma_i; \lambda). \qquad (2)$$

It is straightforward to show that its time derivative reads

$$\partial_\tau T(\sigma_f, \sigma_i; \lambda) = \mathcal{L}_\tau(\tau, \sigma_f)T(\sigma_f, \sigma_i; \lambda) - T(\sigma_f, \sigma_i; \lambda)\mathcal{L}_\tau(\tau, \sigma_i). \qquad (3)$$

The Lax connection $\mathcal{L}_\tau$ appears in the auxiliary system when it is expressed in terms of the world sheet coordinates σ and τ. In particular, the relevant equation reads $\partial_\tau \Psi(\tau, \sigma; \lambda) = \mathcal{L}_\tau \Psi(\tau, \sigma; \lambda)$. The values of σ_i and σ_f are related to the specific solution and its boundary conditions. The latter also specify the explicit form of the conserved quantity. Let us make this statement clear. If $\mathcal{L}_\tau$ vanished both at σ_i and σ_f, which for instance may happen at $\pm\infty$, then the monodromy matrix is conserved. On the contrary, when imposing periodic boundary conditions, it is its trace that is conserved. Finally, let us also mention that in the case of open string boundary conditions one can employ

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the formalism of Cherdnik and Sklyanin by defining a boundary monodromy matrix [9,10].

There is another use for the solution of the auxiliary system. It is the necessary input for the application of the dressing method [1,11,12]. The latter is a technique which enables the construction of new solutions of the NLSM, once a solution is already known. The known solution is referred to as the seed solution. The Pohlmeyer reduced theory has a counterpart of the dressing transformations, known as the Bäcklund transformations. Interestingly enough, a dressing transformation on the NLSM solution, automatically performs a Bäcklund transformation at the avatar of the solution in the Pohlmeyer reduced theory [13].

In [3] we claimed that the systematic solution of the auxiliary system corresponding to the $O(3)$ NLSM, points to a nonlinear superposition rule for NLSMs on symmetric spaces. In order to obtain a new NLSM solution one can combine a NLSM solution with a "virtual" one, i.e., one that obeys inadmissible Virasoro constraints when the model is embedded in string theory, both corresponding to the same Pohlmeyer counterpart. We verify this statement in the case of $O(N)$ NLSM.

II. THE POHLMEYER REDUCTION OF THE $O(N)$ NLSM

In this section we review the Pohlmeyer reduction of the $O(N)$ NLSM. The main reason for doing so is that we will use slightly unusual conventions, which will facilitate the rest of the paper. The equations of motion of the NLSM read

$$\partial_+\partial_-\vec{X} + (\partial_+\vec{X}\cdot\partial_-\vec{X})\vec{X} = 0, \quad (4)$$

where we used the coordinates $\xi^\pm = \xi^1 \pm \xi^0$. They are accompanied by the geometric constraint

$$|\vec{X}|^2 = 1, \quad (5)$$

as well as by the conservation of the stress-energy tensor T, which in appropriate coordinates implies

$$T_{\pm\pm} = |\partial_\pm\vec{X}|^2 = m_\pm^2. \quad (6)$$

We introduce a basis $\mathcal{V} = \{\vec{v}_1, ..., \vec{v}_N\}$ in $\mathbb{R}^N$, where $\vec{v}_{N-2} = \partial_-\vec{X}$, $\vec{v}_{N-1} = \partial_+\vec{X}$ and $\vec{v}_N = \vec{X}$ [14]. On the rest of the vectors we impose the orthonormality conditions

$$\vec{v}_i \cdot \vec{v}_j = \delta_{ij}, \qquad \vec{v}_i \cdot \vec{v}_{N-2} = \vec{v}_i \cdot \vec{v}_{N-1} = \vec{v}_i \cdot \vec{v}_N = 0. \quad (7)$$

The norms of the last three vectors are fixed by Eqs. (5) and (6). As a consequence, in the subspace spanned by the vectors $\vec{X}$ and $\partial_\pm\vec{X}$ the only unconstrained degree of freedom is the angle between $\partial_+\vec{X}$ and $\partial_-\vec{X}$. Thus, we define the primary Pohlmeyer field φ through

$$\partial_+\vec{X}\cdot\partial_-\vec{X} = m_+m_-\cos\varphi. \quad (8)$$

Let us expand the derivatives of the vectors, which form the basis, on the basis itself as

$$\partial_\pm\vec{v}_\alpha = (A_\pm)_{\alpha\beta}\vec{v}_\beta, \quad (9)$$

or in matrix form as

$$\partial_\pm V = A_\pm V, \qquad V_{ij} = (\vec{v}_i)_j. \quad (10)$$

By definition $\partial_+\vec{v}_N = \vec{v}_{N-1}$ and $\partial_-\vec{v}_N = \vec{v}_{N-2}$, implying that the elements $(A_+)_{N\alpha} = \delta_{\alpha(N-1)}$ and $(A_-)_{N\alpha} = \delta_{\alpha(N-2)}$. The equations of motion imply that $(A_+)_{(N-2)\alpha} = -m_+m_-\cos\varphi\delta_{\alpha N}$ and similarly $(A_-)_{(N-1)\alpha} = -m_+m_-\cos\varphi\delta_{\alpha N}$. Taking into account the following inner products

$$(\partial_\pm^2\vec{X})\cdot\vec{X} = -m_\pm^2, \quad (11)$$

$$(\partial_\pm^2\vec{X})\cdot\partial_\pm\vec{X} = 0, \quad (12)$$

$$(\partial_\pm^2\vec{X})\cdot\partial_\mp\vec{X} = -m_+m_-\sin\varphi\partial_\pm\varphi, \quad (13)$$

we obtain

$$\partial_\pm^2\vec{X} = -m_\pm^2\vec{X} + \cot\varphi\partial_\pm\varphi\partial_\pm\vec{X}$$
$$- \frac{m_\pm}{m_\mp}\frac{\partial_\pm\varphi}{\sin\varphi}\partial_\mp\vec{X} + m_\pm a_i^\pm\vec{v}_i, \quad (14)$$

where $a_i^\pm$ are additional Pohlmeyer fields, which are grouped as two $(N-3)\times 1$ column matrices $\mathbf{a}_\pm$ [15]. Finally, we define

$$\vec{v}_j \cdot \partial_\pm\vec{v}_i = (\mathcal{A}_\pm)_{ij}, \quad (15)$$

where $\mathcal{A}_\pm$ are $(N-3)\times(N-3)$ antisymmetric matrices. Thus, the matrices $A^\pm$ are given by:

$$A_+ = \begin{pmatrix} \mathcal{A}_+ & \frac{\cos\varphi\mathbf{a}_+}{m_-\sin^2\phi} & -\frac{\mathbf{a}_+}{m_+\sin^2\phi} & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 & -m_+m_-\cos\varphi \\ m_+\mathbf{a}_+^T & -\frac{m_+}{m_-}\frac{\partial_+\varphi}{\sin\varphi} & \cot\varphi\partial_+\varphi & -m_+^2 \\ \mathbf{0}^T & 0 & 1 & 0 \end{pmatrix} \quad (16)$$

and

$$A_- = \begin{pmatrix} \mathcal{A}_- & -\frac{\mathbf{a}_-}{m_-\sin^2\phi} & \frac{\cos\varphi\mathbf{a}_-}{m_+\sin^2\phi} & \mathbf{0} \\ m_-\mathbf{a}_-^T & \cot\varphi\partial_-\varphi & -\frac{m_-}{m_+}\frac{\partial_-\varphi}{\sin\varphi} & -m_-^2 \\ \mathbf{0}^T & 0 & 0 & -m_+m_-\cos\varphi \\ \mathbf{0}^T & 1 & 0 & 0 \end{pmatrix}. \quad (17)$$

The compatibility condition $\partial_+\partial_-\vec{v}_i = \partial_-\partial_+\vec{v}_i$, is equivalent to the zero curvature condition

$$\partial_- A_+ - \partial_+ A_- + [A_+, A_-] = 0. \qquad (18)$$

Explicitly, the equations of motion of the Pohlmeyer reduced theory read:

$$F_{+-} = \frac{\cos\varphi}{\sin^2\varphi}[\mathbf{a}_-\mathbf{a}_+^T - \mathbf{a}_+\mathbf{a}_-^T] \qquad (19)$$

$$\mathcal{D}_\pm \mathbf{a}_\mp = \frac{\partial_\mp\varphi}{\sin\varphi}\mathbf{a}_\pm \qquad (20)$$

$$\partial_+\partial_-\varphi + \frac{\mathbf{a}_+^T\mathbf{a}_-}{\sin\varphi} + m_+ m_- \sin\varphi = 0, \qquad (21)$$

where $\mathcal{D}_\pm = I_{N-3}\partial_\pm - \mathcal{A}_\pm$ and $F_{+-} = [\mathcal{D}_+, \mathcal{D}_-]$ [16]. Notice that the Pohlmeyer reduced theory depends only on the product $m_+ m_-$. This is crucial for the rest of the paper. It seems that we ended up facing a more complicated problem than the original NLSM. However, this is not the case since the physical degrees of freedom are actually only $N - 2$. Let us define the $(N - 2) \times (N - 2)$ matrices

$$\tilde{A}_+ = \begin{pmatrix} \mathcal{A}_+ & -\frac{\mathbf{a}_+}{\tan\varphi} \\ \frac{\mathbf{a}_+^T}{\tan\varphi} & 0 \end{pmatrix}, \qquad \tilde{A}_- = \begin{pmatrix} \mathcal{A}_- & \frac{\mathbf{a}_-}{\sin\varphi} \\ -\frac{\mathbf{a}_-^T}{\sin\varphi} & 0 \end{pmatrix}. \qquad (22)$$

Notice that $\tilde{A}_\pm^T = -\tilde{A}_\pm$, thus $\tilde{A}_\pm$ are valued in the Lie algebra $\mathfrak{so}(N-2)$. The equations of motion (19)–(21) imply that $\tilde{A}_\pm$ constitute a flat connection, i.e.,

$$\partial_-\tilde{A}_+ - \partial_+\tilde{A}_- + [\tilde{A}_+, \tilde{A}_-] = 0. \qquad (23)$$

Moreover, Eq. (20) assumes the form

$$\tilde{\mathcal{D}}_-\tilde{\mathbf{a}}_+ = \frac{\tilde{\mathbf{a}}_-^T\tilde{\mathbf{a}}_+}{\sin\varphi}\tilde{\mathbf{Z}} + \frac{\partial_+\varphi}{\sin\varphi}\tilde{\mathbf{a}}_-, \qquad (24)$$

$$\tilde{\mathcal{D}}_+\tilde{\mathbf{a}}_- = -\frac{\tilde{\mathbf{a}}_-^T\tilde{\mathbf{a}}_+}{\tan\varphi}\tilde{\mathbf{Z}} + \frac{\partial_-\varphi}{\sin\varphi}\tilde{\mathbf{a}}_+, \qquad \tilde{\mathbf{a}}_\pm = \begin{pmatrix} \mathbf{a}_\pm \\ 0 \end{pmatrix}, \qquad (25)$$

where $\tilde{\mathbf{Z}}_i = \delta_{i,(N-2)}$ and $\tilde{\mathcal{D}}_\pm = I_{N-2}\partial_\pm - \tilde{A}_\pm$. Finally, Eq. (21) assumes the form

$$\partial_+\partial_-\varphi + \frac{\tilde{\mathbf{a}}_+^T\tilde{\mathbf{a}}_-}{\sin\varphi} + m_+ m_- \sin\varphi = 0. \qquad (26)$$

The solution of Eq. (23) is $\tilde{A}_\pm = (\partial_\pm\tilde{O})\tilde{O}^T$, where $\tilde{O} \in SO(N-2)$, which implies that

$$\tilde{\mathbf{a}}_+ = \tan\varphi\,\tilde{O}\partial_+\mathbf{Z}, \qquad \tilde{\mathbf{a}}_- = -\sin\varphi\,\tilde{O}\partial_-\mathbf{Z}, \qquad (27)$$

where $\mathbf{Z} = \tilde{O}^T\tilde{\mathbf{Z}}$ is of unit norm. It is straightforward to show that Eqs. (24) and (25) are equivalent to

$$\partial_+\partial_-\mathbf{Z} + \frac{\partial_+\varphi}{\tan\varphi}\partial_-\mathbf{Z} + \frac{\partial_-\varphi}{\sin\varphi\cos\varphi}\partial_+\mathbf{Z} + (\partial_-\mathbf{Z}^T\partial_+\mathbf{Z})\mathbf{Z} = 0, \qquad (28)$$

while Eq. (21) becomes

$$\partial_+\partial_-\varphi - \tan\varphi(\partial_-\mathbf{Z}^T\partial_+\mathbf{Z}) + m_+ m_- \sin\varphi = 0. \qquad (29)$$

These two equations, describing the $N - 2$ physical degrees of freedom, appeared in [17]. One can obtain another set of equations by interchanging $\tan\varphi \leftrightarrow -\sin\varphi$ in (22). Actually, the equations of motion can be derived from a $SO(N - 1)/SO(N - 2)$ gauged Wess-Zumino-Witten model perturbed by a potential term implementing an appropriate gauge fixing [18], see also [8]. Remnants of symmetry are also present in the above construction. Had $\tilde{\mathbf{Z}}$ been a dynamical field, the theory described by Eqs. (23)–(26) would have the following $SO(N - 2)$ gauge redundancy

$$\tilde{\mathbf{a}}_\pm \to \tilde{O}\tilde{\mathbf{a}}_\pm, \qquad \tilde{\mathbf{Z}} \to \tilde{O}\tilde{\mathbf{Z}},$$
$$\mathcal{S}\mathcal{A}_\pm \to \tilde{O}\tilde{\mathcal{S}}\tilde{\mathcal{A}}_\pm\tilde{O}^T + (\partial_\pm\tilde{O})\tilde{O}^T. \qquad (30)$$

Since $\tilde{\mathbf{Z}}$ has a specific value, one may interpret this fact as a gauge fixing condition.

III. THE AUXILIARY SYSTEM

The solution of the NLSM, henceforth denoted as the column X, corresponds to an element g of the coset $SO(N)/SO(N - 1)$ through the mapping

$$g = \theta(I - 2XX^T), \qquad \theta = (I - 2X_0 X_0^T), \qquad (31)$$

where I is the $N \times N$ identity matrix, X_0 is a constant vector and both X and X_0 are of unit norm. By construction g obeys:

$$\bar{g} = g, \qquad g\theta g\theta = I, \qquad g^T = g^{-1}, \qquad (32)$$

implying that indeed $g \in SO(N)/SO(N - 1)$. Let us define $\hat{\Psi}$ via the equation [19]

$$\Psi = g\hat{\Psi}. \qquad (33)$$

It is straightforward to show that the auxiliary system assumes the form

$$\partial_\pm\hat{\Psi} = \pm\frac{2\lambda}{1 \pm \lambda}\hat{j}_\pm\hat{\Psi}, \qquad (34)$$

where $\hat{j}_\pm = -\frac{1}{2}g^{-1}\partial_\pm g$. Its explicit form is

$$\hat{j}_\pm = (\partial_\pm X) X^T - X \partial_\pm X^T. \qquad (35)$$

For reasons that will become apparent shortly, we let

$$\hat{\Psi} = V^{-1} \Delta^{-1} \tilde{\Psi}, \qquad \Delta = \begin{pmatrix} I_{N-3} & 0 & 0 & 0 \\ 0 & \frac{1+\lambda}{1-\lambda} & 0 & 0 \\ 0 & 0 & \frac{1-\lambda}{1+\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (36)$$

where V is the matrix introduced in Eq. (10). Taking Eq. (36) into account, as well as Eq. (10), the auxiliary system (34) assumes the form

$$\partial_\pm \tilde{\Psi} = \Delta \left[A_\pm \pm \frac{2\lambda}{1 \pm \lambda} V \hat{j}_\pm V^{-1} \right] \Delta^{-1} \tilde{\Psi}. \qquad (37)$$

Equation (10) implies that the matrix V is given by

$$V^T = (v_1 \quad \dots \quad v_{N-2} \quad v_{N-1} \quad v_N). \qquad (38)$$

The inverse of this matrix, i.e., V^{-1} is

$$V^{-1} = (v_1 \quad \dots \quad v'_{N-2} \quad v'_{N-1} \quad v_N), \qquad (39)$$

where v'_{N-2} and v'_{N-1} are given by

$$\vec{v}'_{N-2} = \frac{1}{m_-^2 \sin^2 \varphi} \left[\partial_- \vec{X} - \frac{m_-}{m_+} \cos \varphi \, \partial_+ \vec{X} \right], \qquad (40)$$

$$\vec{v}'_{N-1} = \frac{1}{m_+^2 \sin^2 \varphi} \left[\partial_+ \vec{X} - \frac{m_+}{m_-} \cos \varphi \, \partial_- \vec{X} \right]. \qquad (41)$$

Using Eqs. (38), (39), and (35), one obtains

$$V \hat{j}_+ V^{-1} = \begin{pmatrix} 0_{N-3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 & m_+ m_- \cos \varphi \\ \mathbf{0}^T & 0 & 0 & m_+^2 \\ \mathbf{0}^T & 0 & -1 & 0 \end{pmatrix} \qquad (42)$$

$$V \hat{j}_- V^{-1} = \begin{pmatrix} 0_{N-3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 & m_-^2 \\ \mathbf{0}^T & 0 & 0 & m_+ m_- \cos \varphi \\ \mathbf{0}^T & -1 & 0 & 0 \end{pmatrix} \qquad (43)$$

where 0_{N-3} is the $(N-3) \times (N-3)$ zero matrix. Thus, it is straightforward to show Eqs. (37) actually assume the form

$$\partial_\pm \tilde{\Psi} = [A_\pm|_{m_\pm \to \frac{1 \mp \lambda}{1 \pm \lambda} m_\pm}] \tilde{\Psi}, \qquad (44)$$

which in view of Eq. (10) implies that $\tilde{\Psi}$ is given by

$$\tilde{\Psi} = V|_{m_\pm \to \frac{1 \mp \lambda}{1 \pm \lambda} m_\pm}. \qquad (45)$$

Putting everything together, Ψ reads

$$\Psi(\lambda) = g V^{-1} \Delta^{-1} \left(V|_{m_\pm \to \frac{1 \mp \lambda}{1 \pm \lambda} m_\pm} \right). \qquad (46)$$

This calculation indicates that given the whole family of NLSM solutions, which correspond to the same solution of the Pohlmeyer reduced theory, one can trivially construct the solution of the auxiliary system. This is possible because the transformation $m_\pm \to \frac{1 \mp \lambda}{1 \pm \lambda} m_\pm$ leaves invariant the Pohlmeyer reduced theory. In a sense, the fact that the Pohlmeyer reduction is a many-to-one mapping, generates the spectral parameter λ.

One could multiply the solution by any constant matrix as $\Psi \to \Psi C$, we choose the normalization used for the application of the dressing method, namely

$$\Psi(0) = g. \qquad (47)$$

As the elements of the matrix V are real functions of the real parameters $m_\pm$, it is evident that $\Psi(\lambda)$ obeys by construction the reality condition

$$\bar{\Psi}(\bar{\lambda}) = \Psi(\lambda). \qquad (48)$$

It is straightforward to show that

$$\Psi^T \Psi = \Psi \Psi^T = I. \qquad (49)$$

Finally, the consistency of the auxiliary system (1) for $\lambda \to 1/\lambda$ requires that $g\theta\Psi(1/\lambda)\theta$ belong to the set of solutions of the equations, i.e., $g\theta\Psi(1/\lambda)\theta = \Psi(\lambda)M$ for some constant matrix M. By an elementary multiplication of matrices, it turns out that

$$\Delta(\lambda) V \theta g V^{-1} \Delta^{-1}(1/\lambda) = \mathcal{I}, \quad \mathcal{I} = \begin{pmatrix} I_{N-3} & 0 \\ 0 & -I_3 \end{pmatrix}. \qquad (50)$$

Thus, using Eqs. (33) and (36) it follows that $\tilde{\Psi}$ is subject to the constraint

$$\mathcal{I} \tilde{\Psi}(1/\lambda)\theta = \tilde{\Psi}(\lambda)M. \qquad (51)$$

Finally, taking Eq. (45) into account, this constraint is equivalent to

$$V^{-1} \mathcal{I} (V|_{m_\pm \to -m_\pm})\theta = M, \qquad (52)$$

which has to be satisfied by the vector of the basis.

IV. DISCUSSION

In this work we obtained the formal solution of the auxiliary system corresponding to $O(N)$ NLSM. As long as the whole family of NLSM solutions, which correspond to a given solution of the Pohlmeyer reduced theory, is known, the solution of the auxiliary system can be constructed systematically.

There are various implications of this construction. To begin with, one can calculate the values of the conserved charges directly, by expanding the monodromy matrix. Also, in view of our result, the dressing method is the implementation of a nonlinear superposition of solutions with "virtual" ones. There are no differential equations to be solved. The same is also true for the Bäcklund transformation of the Pohlmeyer reduced theory. Essentially, the nonlinear superposition is the NLSM counterpart of the insertion of solitons in the Pohlmeyer reduced theory. In this spirit, our results, combined with the addition formula for the on-shell action derived in [5] enable the calculation of instanton contributions over any classical configuration of the $O(N)$ NLSMs. It would be compelling to use this formalism in order to make contact with studies on the possible path integration contours for such models [20] or to discuss semiclassical quantization.

Regarding the monodromy matrix, the form of the solution of the auxiliary system (46) implies that in the case of periodic boundary conditions its eigenvalues coincide with the eigenvalues of the matrix

$$\mathcal{T} = V(\tau, \sigma_f) V^{-1}(\tau, \sigma_i) |_{m_\pm \to \frac{1\mp\lambda}{1\pm\lambda} m_\pm}. \tag{53}$$

It is very important to point out that the NLSM solution and correspondingly the coset element g and the matrix V satisfy periodic boundary conditions for specific $m_\pm$. The rescaling of these parameters spoils the periodic boundary conditions (at least for arbitrary λ). This is the mechanism behind the nontrivial monodromy matrix. One should

perceive the notation of (53) as rescaling $m_\pm$ first and then substituting the value of σ. More details and a worked-out example can be found in the follow up publication [21].

Of course, there are intriguing generalizations of these results. For instance, one could study other symmetric spaces or introduce supersymmetry. Besides the mathematical curiosity, such generalizations are also interesting for practical reasons. In the context of AdS/CFT correspondence [22–24] it is known that classical free IIB superstring theory on $AdS_5 \times S^5$ and the operators of nonperturbative planar $\mathcal{N} = 4$ super-Yang- Mills share a spectral curve [25]. Generalizing the presented construction for the Metsaev-Tseytlin action [26], it could be the case that string configuration can be related to specific dual operators. Of course this would require the analogous construction on the field theory side. Nevertheless, the presented construction concerns the $SO(6)$ sector in the vector representation, whose spectral curve was presented in [27].

Finally, this construction may be relevant for the study of the stability of classical strings propagating on spheres in the spirit of [28,29]. Classical strings are unstable whenever superluminal solitons can propagate on the background of the Pohlmeyer counterpart of these configurations. As the boundary conditions are crucial for such studies, it is unsure how feasible it is to study the stability of arbitrary string configurations, yet one could study specific string solutions if available.

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[14] Lower-case English letters will always run from 1 to $N - 3$, while lower-case Greek letters will run from 1 to N.

[15] Symbols in bold will denote column matrices.

[16] I_{N-3} denotes the $(N - 3) \times (N - 3)$ identity matrix.

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[19] This is a different redefinition than the one used in our previous works [3,4]. The redefinition used in these works results in a pair of matrices multiplying $\hat{\Psi}$, which depend on a sole world-sheet coordinate. This selection facilitates the solution of the auxiliary system using the explicit form of the elliptic string string solutions.

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