Coupling the precursor of the most general theory of electromagnetism invariant under duality and conformal invariance with scalars, and BIon-type solutions

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Recently, the most general theory of electromagnetism invariant under duality and conformal invariance was written, and dubbed ModMax. It arises from a generalization of Born-Infeld (BI) theory by taking the infinite tension limit, $T \to \infty$. In this paper we show that this generalization can be obtained from a branelike construction, just like BI, and can thus be coupled to scalars in the same way to obtain a Dirac-Born-Infeld (DBI)-like action. All the BIon and catenoid solutions, and their interpolations, are still solutions of the generalized DBI-like action, suggesting that an interpretation within string theory could be possible. We also show that Rañada's knotted solutions (with nonzero helicities) are still valid, both in the ModMax theory, and in its precursor.

I. INTRODUCTION

In a remarkable paper [1], the unique conformally invariant and duality invariant extension of Maxwell electromagnetism was found (both in the Lagrangian formalism for $\mathcal{L}(\vec{E}, \vec{B})$, and in the Hamiltonian formalism for $\mathcal{H}(\vec{D}, \vec{B})$), depending on a dimensionless (numerical) parameter γ, reducing to Maxwell's theory at $\gamma = 0$.[1] The Hamiltonian was found as a $T \to \infty$ limit of a generalization of the Born-Infeld theory [3] with the same parameter γ (such that $T \to \infty$ gives the Maxwell theory at $\gamma = 0$). P-form generalizations were given in [4], and a supersymmetrization was presented in [5] (see also [6]). A number of papers studied the solutions and properties of the ModMax theory (coupled to gravity), among them [7–18].

However, it is known that the Born-Infeld theory can be coupled to scalars, and the resulting Dirac-Born-Infeld (DBI) theory has the simple interpretation in string theory as the theory for the massless modes appearing on a D-brane. As such, a supersymmetrization is available, and in terms of the scalars representing brane fluctuations and the gauge fields, one has both an extremal (supersymmetric) solution, the Bogomol'nyi-Prasad-Sommerfield (BPS) BIon [19], together with the original BIon of Born and

Infeld (devised as a nonsingular alternative to the Dirac electron), and a "catenoid" solution. Both the BPS BIon and catenoid have interpretations in terms of brane geometries, and there are interpolating solutions as well [20–22].

A relevant question that we are trying to answer here is can one interpret the precursor of ModMax, the generalization of the Born-Infeld action, also as a brane-type action? And are the same kind of solutions still valid, and if so, are they modified? To the first question we will find a positive answer, with a certain caveat, whereas to the second question we will find a positive answer, with no modification for the solution.

We will also consider whether knotted solutions of electromagnetism found by Rañada [23,24] are still solutions of ModMax theory and its precursor, and we will find that they are.

The paper is organized as follows. In Sec. II we will describe the BI generalization and the ModMax limit, in Sec. III we will rewrite it in terms of a branelike description, which will allow us to show that the same solutions are still valid, including the Rañada solutions, and in Sec. IV we will conclude.

II. BORN-INFELD DUALITY-INVARIANT GENERALIZATION AND MODMAX ELECTROMAGNETISM

The generalization of the Born-Infeld theory has the Hamiltonian

$$\mathcal{H}(T, \gamma; \vec{D}, \vec{B})$$
$$= \sqrt{T^2 + 2T(s \cosh\gamma - \sinh\gamma\sqrt{s^2 - p^2}) + p^2} - T, \quad (2.1)$$

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[1]For an alternative derivation of this result see [2].

where T is a parameter of dimension 4, $\vec{D}$ is the electric induction, and

$$s = \frac{\vec{D}^2 + \vec{B}^2}{2}, \qquad p = |\vec{D} \times \vec{B}| = \sqrt{\vec{D}^2\vec{B}^2 - (\vec{D} \cdot \vec{B})^2}.$$

$$(2.2)$$

The Lagrangian is, as usual, the Legendre transform of the Hamiltonian with respect to $\vec{D}$,

$$\mathcal{L}(T, \gamma; \vec{E}, \vec{B}) = \vec{E} \cdot \vec{D} - \mathcal{H}(T, \gamma; \vec{D}, \vec{B}), \qquad (2.3)$$

though an explicit formula was not given in [1]. As usual, the Legendre transform implies the constitutive relations

$$\vec{E} = \frac{\partial \mathcal{H}}{\partial \vec{D}}, \qquad \vec{H} = \frac{\partial \mathcal{H}}{\partial \vec{B}},$$
$$\vec{D} = \frac{\partial \mathcal{L}}{\partial \vec{E}}, \qquad \vec{H} = -\frac{\partial \mathcal{L}}{\partial \vec{B}}, \qquad (2.4)$$

and, in terms of $\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$, the equations of motion take the form of the equations of motion inside a material (in the presence of sources, in $\vec{\nabla} \cdot \vec{D} = \rho_{\text{ext}}/\epsilon_0$ we obtain only the *external* charge, delta function sources, introduced as an extra term in the Lagrangian, whereas in $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ we also obtain charges due to the polarization of the material, or in this case, of the vacuum, so the charge density is spread out)

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}, \qquad \vec{\nabla} \cdot \vec{B} = 0,$$
$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\partial_t \vec{D}, \qquad \vec{\nabla} \cdot \vec{D} = 0. \qquad (2.5)$$

In [1], it is also shown that conditions for conformal invariance are

$$\vec{E} \times \vec{H} = \vec{D} \times \vec{B}, \qquad \vec{D} \cdot \vec{E} + \vec{B} \cdot \vec{H} = 2\mathcal{H}, \qquad (2.6)$$

while a condition for duality is

$$\vec{E} \cdot \vec{B} = \vec{D} \cdot \vec{H}. \qquad (2.7)$$

In the Appendix we quickly review the case of the BI action, that we will be generalizing here.

A. ModMax theory

The conformal and duality invariant ModMax theory is obtained from the low energy, or $T \to \infty$, limit of the general theory (2.1), so

$$\mathcal{H}(\gamma; \vec{D}, \vec{B}) = s \cosh\gamma - \sinh\gamma\sqrt{s^2 - p^2}. \qquad (2.8)$$

In this case also the Legendre transform can be made explicitly, as was done in [1], and the result is the Lorentz invariant Lagrangian

$$\mathcal{L}^{\text{ModMax}}(\gamma; \vec{E}, \vec{B}) = \cosh\gamma S + \sinh\gamma\sqrt{S^2 + P^2}, \qquad (2.9)$$

where we have defined the quantities S and P as in [1],

$$S = \frac{\vec{E}^2 - \vec{B}^2}{2} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{b^2 F}{2}$$
$$P = \vec{E} \cdot \vec{B} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = b^2 G, \qquad (2.10)$$

and we have also related to the quantities F and G defined for the BI electromagnetism (see the Appendix).

B. Lagrangian for the general theory

For the general theory (2.1), we calculate

$$\vec{E} = \frac{\partial \mathcal{H}}{\partial \vec{D}} = \frac{T\left[\cosh\gamma\vec{D} - \frac{\sinh\gamma}{\sqrt{s^2 - p^2}}\left(s\vec{D} - (\vec{D}\vec{B}^2 - \vec{B}(\vec{B} \cdot \vec{D}))\right)\right] + \vec{D}\vec{B}^2 - \vec{B}(\vec{B} \cdot \vec{D})}{\sqrt{T^2 + 2T\left(\cosh\gamma s - \sinh\gamma\sqrt{s^2 - p^2}\right) + p^2}}, \qquad (2.11)$$

and then

$$\mathcal{L}(T, \gamma; \vec{E}, \vec{B}) = \vec{D} \cdot \vec{E} - \mathcal{H}(T, \gamma; \vec{E}, \vec{B})$$
$$= T + \frac{-T^2 + T\left[-\cosh\gamma\vec{B}^2 - \sinh\gamma\frac{s\vec{B}^2 + p^2}{\sqrt{s^2 - p^2}}\right]}{\sqrt{T^2 + 2T\left(\cosh\gamma s - \sinh\gamma\sqrt{s^2 - p^2}\right) + p^2}}, \qquad (2.12)$$

but this has to be rewritten in terms of $\vec{E}$, $\vec{B}$.

A natural guess for that is

$$\mathcal{L} = T - \sqrt{T^2 - 2T\left(S\cosh\gamma + \sinh\gamma\sqrt{S^2 + P^2}\right) - P^2}$$

$$= T\left[1 - \sqrt{1 + F\cosh\gamma - \sinh\gamma\sqrt{F^2 + 4G^2} - G^2}\right]$$

$$= T\left[1 - \sqrt{1 + \frac{\vec{B}^2 - \vec{E}^2}{T}\cosh\gamma - \sinh\gamma\sqrt{\frac{(\vec{B}^2 - \vec{E}^2)^2}{T^2} + \frac{4(\vec{B}\cdot\vec{E})^2}{T^2} - \frac{(\vec{B}\cdot\vec{E})^2}{T^2}}}\right]. \tag{2.13}$$

In the expansion of the above, the T-independent terms are definitely correct, since the small-field (or $T \to \infty$) limit gives the ModMax theory, but the next-order terms could in principle not be. However, at least its $\gamma = 0$ limit is correct, since then we must obtain the BI theory. It is too complicated to check the correctness of the above ansatz directly, however.

Instead, we must use indirect methods to prove it, as we will shortly do.

Actually, the Lagrangian (2.13), corresponding to the Hamiltonian (2.1), was also found already in [4], via yet another indirect method.[2]

We can also calculate back the Hamiltonian of this Lagrangian (i.e., assuming the Lagrangian is correct, find the $\vec{D}$, $\vec{H}$ and $\mathcal{H}$ from it), via first obtaining from (2.13) that

$$\vec{H} = -\frac{\partial\mathcal{L}}{\partial\vec{B}} = \frac{T\left[\vec{B}\left(\cosh\gamma + \frac{S\sinh\gamma}{\sqrt{S^2+P^2}}\right) - \vec{E}P\left(\frac{1}{T} + \frac{\sinh\gamma}{\sqrt{S^2+P^2}}\right)\right]}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}},$$

$$\vec{D} = \frac{\partial\mathcal{L}}{\partial\vec{E}} = \frac{T\left[\vec{E}\left(\cosh\gamma + \frac{S\sinh\gamma}{\sqrt{S^2+P^2}}\right) + \vec{B}P\left(\frac{1}{T} + \frac{\sinh\gamma}{\sqrt{S^2+P^2}}\right)\right]}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}},$$

$$\tag{2.14}$$

and then

$$\mathcal{H} = \vec{E}\cdot\vec{D} - \mathcal{L} = -T$$

$$+ \frac{T\left\{1 + \frac{\vec{B}^2}{T}\cosh\gamma + \frac{\sinh\gamma}{T^2\sqrt{\frac{(\vec{B}^2-\vec{E}^2)^2}{T^2}+4\frac{(\vec{B}\cdot\vec{E})^2}{T^2}}}\left[-\vec{B}^2(\vec{B}^2 - \vec{E}^2) - 2(\vec{E}\cdot\vec{B})^2\right]\right\}}{\sqrt{1 + \frac{\vec{B}^2-\vec{E}^2}{T}\cosh\gamma - \sinh\gamma\sqrt{\frac{(\vec{B}^2-\vec{E}^2)^2}{T^2} + \frac{4(\vec{B}\cdot\vec{E})^2}{T^2} - \frac{(\vec{B}\cdot\vec{E})^2}{T^2}}}}$$

$$= -T + \frac{T\left[T + \cosh\gamma\vec{B}^2 + \frac{\sinh\gamma}{\sqrt{S^2+P^2}}(S\vec{B}^2 - P^2) - \frac{P^2}{T}\right]}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}}. \tag{2.15}$$

The original Hamiltonian (2.1) satisfies the duality condition

$$\vec{E}\cdot\vec{B} = \vec{D}\cdot\vec{H}, \tag{2.16}$$

and the condition (needed for conformal invariance of the model, so only in the ModMax $T \to \infty$ limit)

$$\vec{E}\times\vec{H} = \vec{D}\times\vec{B}, \tag{2.17}$$

while the condition for conformal invariance

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[2]A fact that I realized only recently, shortly before finishing this paper.

$$\vec{D} \cdot \vec{E} + \vec{B} \cdot \vec{H} = 2\mathcal{H}, \tag{2.18}$$

is satisfied only in the $T \to \infty$ limit.

Indeed, we can check that, in terms of the Hamiltonian variables, $\vec{B}$ and $\vec{D}$, we have

$$\vec{E} = \frac{\partial \mathcal{H}}{\partial \vec{D}} = \frac{T\left[\cosh\gamma \vec{D} - \frac{\sinh\gamma}{\sqrt{s^2-p^2}}\left(\vec{D}s - (\vec{D}\vec{B}^2 - \vec{B}(\vec{B}\cdot\vec{D}))\right)\right] + \vec{D}\vec{B}^2 - \vec{B}(\vec{B}\cdot\vec{D})}{\sqrt{T^2 + 2T(s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2}) + p^2}},$$

$$\vec{H} = \frac{\partial \mathcal{H}}{\partial \vec{B}} = \frac{T\left[\cosh\gamma \vec{B} - \frac{\sinh\gamma}{\sqrt{s^2-p^2}}\left(\vec{B}s - (\vec{B}\vec{D}^2 - \vec{D}(\vec{B}\cdot\vec{D}))\right)\right] + \vec{B}\vec{D}^2 - \vec{D}(\vec{B}\cdot\vec{D})}{\sqrt{T^2 + 2T(s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2}) + p^2}}, \tag{2.19}$$

so that

$$\vec{E}\cdot\vec{B} = \vec{D}\cdot\vec{H} = \vec{B}\cdot\vec{D}\,\frac{\cosh\gamma - \frac{\sinh\gamma s}{\sqrt{s^2-p^2}}}{\sqrt{T^2 + 2T(s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2}) + p^2}},$$

$$\vec{E}\times\vec{H} = \vec{D}\times\vec{B}, \tag{2.20}$$

and moreover

$$\vec{D}\cdot\vec{E} = \frac{T\left[\vec{D}^2\left(\cosh\gamma - \frac{s\sinh\gamma}{\sqrt{s^2-p^2}}\right) + p^2\left(\frac{1}{T} + \frac{\sinh\gamma}{\sqrt{s^2-p^2}}\right)\right]}{\sqrt{T^2 + 2T(s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2}) + p^2}},$$

$$\vec{B}\cdot\vec{H} = \frac{T\left[\vec{B}^2\left(\cosh\gamma - \frac{s\sinh\gamma}{\sqrt{s^2-p^2}}\right) + p^2\left(\frac{1}{T} + \frac{\sinh\gamma}{\sqrt{s^2-p^2}}\right)\right]}{\sqrt{T^2 + 2T(s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2}) + p^2}}, \tag{2.21}$$

so that

$$\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H} = \frac{2T\left[s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2} + \frac{p^2}{T}\right]}{\sqrt{T^2 + 2T(s\cosh\gamma - \sinh\gamma\sqrt{s^2-p^2}) + p^2}}. \tag{2.22}$$

We see that the right-hand side equals $2\mathcal{H}$ *only in the $T \to \infty$ limit*, i.e., only for the ModMax theory (this is expected, since only then the theory is conformal).

But then, the above gives us a way to check the correctness of the ansatz (2.13) for the Lagrangian, namely that it should satisfy the same relations (and then the uniqueness of the theory, as stated in the original paper, ensures that we have the right result).

From (2.14), we obtain

$$\vec{E}\times\vec{H} = \vec{D}\times\vec{B} = \vec{E}\times\vec{B}\,\frac{T\left(\cosh\gamma + \frac{S\sinh\gamma}{\sqrt{S^2+P^2}}\right)}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}},$$

$$\vec{D}\cdot\vec{H} = \vec{E}\cdot\vec{B}, \tag{2.23}$$

and moreover

$$\vec{E} \cdot \vec{D} = \frac{T\left[\vec{E}^2\left(\cosh\gamma + \frac{S\sinh\gamma}{\sqrt{S^2+P^2}}\right) + P^2\left(\frac{1}{T} + \frac{\sinh\gamma}{\sqrt{S^2+P^2}}\right)\right]}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}},$$

$$\vec{B} \cdot \vec{H} = \frac{T\left[\vec{B}^2\left(\cosh\gamma + \frac{S\sinh\gamma}{\sqrt{S^2+P^2}}\right) - P^2\left(\frac{1}{T} + \frac{\sinh\gamma}{\sqrt{S^2+P^2}}\right)\right]}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}},$$

$$(2.24)$$

so that

$$\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}$$

$$= 2\frac{TS\left(\cosh\gamma + \frac{S\sinh\gamma}{\sqrt{S^2+P^2}}\right)}{\sqrt{T^2 - 2T(S\cosh\gamma + \sinh\gamma\sqrt{S^2+P^2}) - P^2}}, \quad (2.25)$$

and the right-hand side is different than (2.15), except in the $T \to \infty$ case, or ModMax case, in which it matches, as it should.

III. LORENTZ INVARIANT THEORY: COUPLING TO A SCALAR AND BION AND CATENOID SOLUTIONS

We would like to couple the above general model, with the Hamiltonian (2.1) and the Lagrangian (2.13) to a scalar field, and find its nonmagnetic static solutions. As we have previewed, we will find that the same solutions at $\gamma = 0$ (for

the DBI case) are still valid, so we will review the way they are obtained first.

A. DBI Lagrangian and BIon and catenoid solutions

In the case of the BI Lagrangian, coupling to scalars was easy, and it led to the DBI Lagrangian. Noting that the Lagrangian can be rewritten as

$$\mathcal{L}_{\text{BI}}(b; \vec{E}, \vec{B}) = b^2\left[1 - \sqrt{\det\left(\eta_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)}\right], \quad (3.1)$$

the Lorentz invariant coupling is simple, and it leads to

$$\mathcal{L}_{\text{DBI}}(b; F_{\mu\nu}, X) = b^2\left[1 - \sqrt{\det\left(\eta_{\mu\nu} + \frac{\partial_\mu X \partial_\nu X}{b^2} + \frac{F_{\mu\nu}}{b}\right)}\right],$$

$$(3.2)$$

which moreover, was found to the action for a D3-brane in string theory moving in flat 5 dimensions, with X/b the transverse position of the brane, and A_μ the open string field on the brane.

Note that the above form of the DBI action is the unique, explicitly Lorentz invariant, form. Unlike the $X = 0$ case, we cannot reexpress it in terms of the invariants F and G, and the invariant $\tilde{X} = (\partial_\mu X)^2/b^2$.

The most one can say in the DBI case is that, for $\vec{B} = 0$, we can rewrite it as [19]

$$\mathcal{L}_{\text{DBI}}(b; \vec{E}, X) = b^2\left[1 - \sqrt{\left(1 - \frac{\vec{E}^2}{b^2}\right)\left(1 - \frac{(\vec{\nabla}X)^2}{b^2}\right) + \frac{(\vec{E} \cdot \vec{\nabla}X)^2}{b^4} - \frac{\dot{X}^2}{b^2}}\right]. \quad (3.3)$$

The momenta for $\vec{A}$ and X, respectively, in this case are found to be (in the $A_0 = 0$ gauge)

$$\vec{\Pi} = \frac{\partial\mathcal{L}}{\partial\vec{E}} = \frac{\vec{E}\left(1 + \frac{(\vec{\nabla}X)^2}{b^2}\right) - \vec{\nabla}X\frac{\vec{E}\cdot\vec{\nabla}X}{b^2}}{\sqrt{\left(1 - \frac{\vec{E}^2}{b^2}\right)\left(1 - \frac{(\vec{\nabla}X)^2}{b^2}\right) + \frac{(\vec{E}\cdot\vec{\nabla}X)^2}{b^4} - \frac{\dot{X}^2}{b^2}}} \equiv \text{``}\vec{D}(\vec{E}, X)\text{''},$$

$$P = \frac{\partial\mathcal{L}}{\partial\dot{X}} = \frac{\dot{X}}{\sqrt{\left(1 - \frac{\vec{E}^2}{b^2}\right)\left(1 - \frac{(\vec{\nabla}X)^2}{b^2}\right) + \frac{(\vec{E}\cdot\vec{\nabla}X)^2}{b^4} - \frac{\dot{X}^2}{b^2}}}, \quad (3.4)$$

(we see that $\vec{\Pi}$ is a generalization, in the presence of X, of $\vec{D}$) so the Hamiltonian becomes

$$\mathcal{H}_{\text{DBI}}(b; \vec{\Pi}, P, \vec{\nabla}X) = b^2\left[\sqrt{\left(1 + \frac{(\vec{\nabla}X)^2}{b^2}\right)\left(1 + \frac{P^2}{b^2}\right) + \frac{\vec{\Pi}^2}{b^2} + \frac{(\vec{\Pi}\cdot\vec{\nabla}X)^2}{b^4}} - 1\right], \quad (3.5)$$

subject to the Gauss constraint $\vec{\nabla} \cdot \vec{\Pi} = 0$, due to the absence of the A_0 canonical momentum, as usual.

The static equation for X, obtained by varying the above Hamiltonian with respect to X, after putting $P = 0$ and $\vec{\nabla} \cdot \vec{\Pi} = 0$, is

$$\vec{\nabla} \cdot \left[\frac{\vec{\nabla}X + \vec{\Pi}\frac{\vec{\nabla}X \cdot \vec{\Pi}}{b^2}}{\sqrt{1 + \frac{(\vec{\nabla}X)^2}{b^2} + \frac{\vec{\Pi}^2}{b^2} + \frac{(\vec{\Pi} \cdot \vec{\nabla}X)^2}{b^4}}} \right]$$

$$= \vec{\nabla} \cdot \left[\frac{\vec{\nabla}X\left(1 - \frac{\vec{E}^2}{b^2}\right) + \vec{E}\frac{\vec{E} \cdot \vec{\nabla}X}{b^2}}{\sqrt{\left(1 - \frac{\vec{E}^2}{b^2}\right)\left(1 + \frac{(\vec{\nabla}X)^2}{b^2}\right) + \frac{(\vec{E} \cdot \vec{\nabla}X)^2}{b^4}}} \right] = 0. \quad (3.6)$$

Static solutions without magnetic fields need only solve the above equation and the constraint $\vec{\nabla} \cdot \vec{\Pi} = 0$. We have (see [20,21] for a review of these solutions),

(1) The BPS (extremal) BIon, corresponding in the string picture to a fundamental string attached to the D-brane, sourced by an electric field charge on the D-brane. This is obtained by putting (writing the sum of extremal BIons, which is also a solution, due to the BPS nature of it)

$$\vec{\Pi} = \vec{\nabla}X = \vec{\nabla}\Lambda, \qquad \Lambda = \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|} \Rightarrow$$

$$X = \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|}, \qquad \vec{E} = \vec{\nabla}X. \quad (3.7)$$

It is easy to see that it is a solution to both the constraint and (3.6). Its energy linearizes, and is

$$E = \int d^3x (\vec{\nabla}\Lambda)^2, \quad (3.8)$$

and gives rise to a finite energy.

(2) The usual BIon solution of Born and Infeld, with $X = 0$, and $\vec{E} = -\vec{\nabla}\phi$, and

$$\phi = C \int_r^\infty \frac{dx}{\sqrt{C^2b^2 + x^4}}, \quad (3.9)$$

which becomes $\phi \to \frac{C}{r}$ at $r \to \infty$. This solution has also a finite energy, which was identified by Born and Infeld with the energy of a nondivergent electron.

(3) The "catenoid" solution, describing one half of a D-brane–anti-D-brane bridge, with $\vec{E} = 0$ and

$$X = \tilde{C} \int_r^\infty \frac{dx}{\sqrt{x^4 - b^2\tilde{C}^2}}, \quad (3.10)$$

and also becomes $X \to \frac{\tilde{C}}{r}$ at $r \to \infty$. This has a "horizon" (where we glue to another solution) at $r_0 = \sqrt{b\tilde{C}}$, where $\vec{F} \equiv \vec{\nabla}X$ diverges, though X does not, so the solution makes sense only for $r > r_0$.

The most general static spherically symmetric solution is obtained by writing in this case the constraint $\vec{\nabla} \cdot \vec{\Pi} = 0$ and the equation of motion (3.6), which become

$$\frac{d}{dr}\left[\frac{r^2\phi'}{\sqrt{1 - \frac{\phi'^2}{b^2} + \frac{X'^2}{b^2}}} \right] = 0, \qquad \frac{d}{dr}\left[\frac{r^2X'}{\sqrt{1 - \frac{\phi'^2}{b^2} + \frac{X'^2}{b^2}}} \right] = 0,$$

$$(3.11)$$

which can be integrated with integration constants C and $\tilde{C}$, respectively, leading to the most general solution

$$\phi = C \int_r^\infty \frac{dx}{\sqrt{x^4 + b^2(C^2 - \tilde{C}^2)}},$$

$$X = \tilde{C} \int_r^\infty \frac{dx}{\sqrt{x^4 + b^2(C^2 - \tilde{C}^2)}}. \quad (3.12)$$

We see that $C^2 < \tilde{C}^2$ gives a catenoidlike solution, with $C = 0$ giving the catenoid, $C^2 > \tilde{C}^2$ gives a BIon-like solution, with $\tilde{C} = 0$ being the BIon, and $C = \tilde{C}$ gives the BPS BIon solution corresponding to the fundamental string ending on the D-brane.

B. Lorentz invariant theory and coupling to a scalar

In order to couple to a scalar, we must write a Lorentz invariant form of the action, but as we saw in the DBI case, that is only possible if we have it in a determinant-type form. Otherwise, if we try to rewrite it in terms of the Lorentz invariants F and G, that fails once the scalar is introduced. Then, the thing we expect to have inside the square root would be of the form

$$\det\left(\eta_{\mu\nu} + \alpha(F, \tilde{F})\frac{F_{\mu\nu}}{b} + \beta(F, \tilde{F})\frac{\tilde{F}_{\mu\nu}}{b} \right)$$

$$= \det\left(\eta_{\mu\nu} + \alpha(F, \tilde{F})\frac{F_{\mu\nu}}{b} + \frac{\beta(F, \tilde{F})}{2}\epsilon_{\mu\nu\rho\sigma}\frac{F^{\rho\sigma}}{b} \right), \quad (3.13)$$

i.e., starting with the standard BI action, replace

$$F_{\mu\nu} \to \alpha(F, \tilde{F})F_{\mu\nu} + \beta(F, \tilde{F})\tilde{F}_{\mu\nu}, \quad (3.14)$$

which gives rise to the change [in terms of $A = F_{\mu\nu}F^{\mu\nu}$ and $B = F_{\mu\nu}\tilde{F}^{\mu\nu}$, or more precisely $F = A/(2b^2)$ and $G = -B/(4b^2)$]

$$F \to (\alpha^2 - \beta^2)F - 4\alpha\beta G, \qquad G \to (\alpha^2 - \beta^2)G + \alpha\beta F. \quad (3.15)$$

But given that the determinant in the BI action gives

$$1 + F - G^2, \quad (3.16)$$

and it is supposed to (we want it to) get modified into

$$1 + F \cosh \gamma - \sinh \gamma \sqrt{F^2 + 4G^2} - G^2, \qquad (3.17)$$

we obtain two equations for two unknowns, α and β,

$$(\alpha^2 - \beta^2)F - 4\alpha\beta G = \cosh \gamma F - \sinh \gamma \sqrt{F^2 + 4G^2}$$
$$(\alpha^2 - \beta^2)G + \alpha\beta F = \pm G. \qquad (3.18)$$

To solve them, define

$$a \equiv \alpha^2 - \beta^2, \qquad b \equiv \alpha\beta, \qquad (3.19)$$

and first write $a = 1 - bF/G$, then find

$$\frac{(\alpha + \beta)^2 - (\alpha - \beta)^2}{4} = \alpha\beta = b = \frac{FG}{F^2 + 4G^2}(1 - \cosh \gamma) + \sinh \gamma \frac{G}{\sqrt{F^2 + 4G^2}},$$
$$(\alpha + \beta)(\alpha - \beta) = \alpha^2 - \beta^2 = a = 1 - \frac{F^2}{F^2 + 4G^2}(1 - \cosh \gamma) - \sinh \gamma \frac{F}{\sqrt{F^2 + 4G^2}}. \qquad (3.20)$$

We want to avoid singularities in both the $G = 0$ case (for instance, for $\vec{B} = 0$), and $F = 0$ case (null, for instance electromagnetic wave).

Indeed, for $G = 0$ (and $F \neq 0$), we have $a = e^{-\gamma}, b = 0$, so

$$\alpha = e^{-2\gamma}, \qquad \beta = 0. \qquad (3.21)$$

Also, for $F = 0$ (and $G \neq 0$), we have $b = \frac{1}{2}\sinh \gamma$, $a = 1$, giving

$$\alpha = \cosh\frac{\gamma}{2}, \qquad \beta = \sinh\frac{\gamma}{2}. \qquad (3.22)$$

The only potential singularity is for the case $F = G = 0$, just like for the limit ModMax theory.

Finally, coupling to the scalar is done in the usual way for the DBI action, assuming that that the scalar X part is not modified, so

$$\mathcal{L}(X, F_{\mu\nu}) = b^2 \left[1 - \sqrt{-\det\left(\eta_{\mu\nu} + \frac{\partial_\mu X \partial_\nu X}{b^2} + \alpha(F, G)\frac{F_{\mu\nu}}{b} + \beta(F, G)\frac{\tilde{F}_{\mu\nu}}{b}\right)} \right]. \qquad (3.23)$$

Since inside the square root we obtain $1 + (\partial_\mu X)^2/b^2 + F$ terms, when taking the large $T = b^2$ limit, we obtain only a free scalar action, $-\frac{1}{2}(\partial_\mu X)^2$, added to the ModMax theory, so it is not so interesting.

It is much more interesting (meaning, we have interactions to between the scalar and gauge fields), if we keep T finite, so that is what we will consider next.

C. Static solutions with no magnetic field

Since we are interested in solutions at $\vec{B} = 0$, consider $\vec{B} = 0$ in the Lagrangian (2.13), giving

$$\mathcal{L} = T\left[1 - \sqrt{1 - \frac{e^\gamma}{T}\vec{E}^2}\right], \qquad (3.24)$$

which by the way is consistent with the fact that when $\vec{B} = 0$, the Hamiltonian (2.1) becomes (then $p = 0, 2s = \vec{D}^2$)

$$\mathcal{H} = T\left[\sqrt{1 + \frac{\vec{D}^2}{Te^\gamma}} - 1\right]. \qquad (3.25)$$

Then (since up to the e^γ term, this is the same as in the DBI case) it follows that we can use the same logic as in the DBI case, and say that, in the $A_0 = 0$ gauge, we have the equation of motion reducing to

$$\vec{\nabla} \cdot \vec{D} = 0, \qquad (3.26)$$

or rather delta function sources on the right-hand side, which is solved as usual by

$$\vec{D} = \vec{\nabla}\frac{q}{|\vec{x} - \vec{x}_0|}, \qquad (3.27)$$

except now there are some factors of e^γ in the relation between $\vec{E}$ and $\vec{D}$,

$$\vec{E} = \frac{\partial \mathcal{H}}{\partial \vec{D}} = \frac{\vec{D}}{e^\gamma \sqrt{1 + \frac{\vec{D}^2}{Te^\gamma}}} = e^{-\gamma} \frac{\vec{D}}{\sqrt{1 + \frac{\vec{D}^2}{\tilde{T}}}}. \qquad (3.28)$$

We can then add X directly in the Hamiltonian at $\vec{B} = 0$, and obtain ($\tilde{T} = e^\gamma T$)

$$\mathcal{H}(\vec{\Pi}, P, X)$$

$$= e^{-\gamma}\tilde{T}\left[\sqrt{\left(1 + \frac{(\vec{\nabla}X)^2}{\tilde{T}}\right)\left(1 + \frac{P^2}{\tilde{T}}\right) + \frac{\vec{\Pi}^2}{\tilde{T}} + \frac{(\vec{\Pi}\cdot\vec{\nabla}X)^2}{\tilde{T}^2}} - 1\right]. \qquad (3.29)$$

Then the static solutions at $\vec{B} = 0$ are the same ones as for the usual DBI case, just rescaled by $e^{-\gamma}$; namely, ($\tilde{b}^2 = \tilde{T}$) in the most general case

$$\phi = e^{-\gamma}C\int_r^\infty \frac{dx}{\sqrt{x^4 + \tilde{b}^2(C^2 - \tilde{C}^2)}},$$

$$X = e^{-\gamma}\tilde{C}\int_r^\infty \frac{dx}{\sqrt{x^4 + \tilde{b}^2(C^2 - \tilde{C}^2)}}, \qquad (3.30)$$

$\vec{E} = -\vec{\nabla}\phi$, with $C = 0$ giving the catenoid, $\tilde{C} = 0$ the BIon, and $C = \tilde{C}$ the extremal BIon.

D. Knotted solutions to precursor of ModMax

Maxwell's equations in vacuum have solutions with nonzero helicities

$$H_{mm} = \int d^3x \epsilon^{ijk}A_i\partial_j A_k,$$

$$H_{ee} = \int d^3x \epsilon^{ijk}C_i\partial_j C_k,$$

$$H_{em} = \int d^3x \epsilon^{ijk}C_i\partial_j A_k,$$

$$H_{me} = \int d^3x \epsilon^{ijk}A_i\partial_j C_k, \qquad (3.31)$$

where $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = \vec{\nabla}\times\vec{C}$, with nonzero Hopf index, and perhaps knotted (with nonzero linking numbers). The basic Hopfion (time-dependent) solution was found in Bateman's formalism [25], via the ansatz

$$\vec{F} \equiv \vec{E} + i\vec{B} = \vec{\nabla}\alpha \times \vec{\nabla}\beta, \qquad (3.32)$$

where α and β are specific complex functions of spacetime variables,

$$\alpha = \frac{A - 1 + iz}{A + it}, \qquad \beta = \frac{x - iy}{A + it},$$

$$A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1), \qquad (3.33)$$

and (p, q)-knotted solutions are obtained by the replacement $\alpha \to \alpha^p$, $\beta \to \beta^q$, or by using acting on a plane wave (unknotted) solution with conformal symmetry transformations with *complex* instead of real parameters [26]. The Hopfion and knotted Hopfion solution have $F = G = 0$ and, since

$$\partial_t H_{mm} = -2\int d^3x \vec{E}\cdot\vec{B} = \partial_t H_{ee}$$

$$\partial_t H_{me} = -\int d^3x(\vec{E}^2 - \vec{B}^2) = -\partial_t H_{em}, \qquad (3.34)$$

all the helicities are conserved on the solution.

More general solutions were found by Rañada [23,24]. One set of solutions have $G = 0$ ($\vec{E}\cdot\vec{B} = 0$), but F arbitrary ($\vec{E}^2 - \vec{B}^2 \neq 0$), so have only H_{ee} and H_{mm} conserved on the solution, and are given by

$$\vec{B}(\vec{r}, t) = g(\phi, \bar{\phi})\vec{\nabla}\phi \times \vec{\nabla}\bar{\phi},$$

$$\vec{E}(\vec{r}, t) = g'(\theta, \bar{\theta})\vec{\nabla}\theta \times \vec{\nabla}\bar{\theta},$$

$$g(\phi, \bar{\phi}) = \frac{\sqrt{a}}{2\pi i}\frac{1}{(1 + \bar{\phi}\phi)^2}, \qquad g'(\theta, \bar{\theta}) = \frac{\sqrt{a}}{2\pi i}\frac{1}{(1 + \bar{\theta}\theta)^2}, \qquad (3.35)$$

satisfy $\partial_{[\mu}F_{\nu\rho]} = 0$, and can be rewritten in terms of real functions p, q, u, v (defined only on patches in whole space) as

$$\vec{B} = \vec{\nabla}p \times \vec{\nabla}q, \quad p = \frac{1}{1 + |\phi|^2}, \quad q = \frac{\arg(\phi)}{2\pi},$$

$$\vec{E} = \vec{\nabla}u \times \vec{\nabla}v, \quad u = \frac{1}{1 + |\theta|^2}, \quad v = \frac{\arg(\theta)}{2\pi}. \qquad (3.36)$$

There is also another set of solutions, with both $F \neq 0$ and $G \neq 0$ ($\vec{E}^2 - \vec{B}^2 \neq 0$ and $\vec{E}\cdot\vec{B} \neq 0$) found implicitly (no explicit form is given, only a condition on the form at time $t = 0$) from the Rañada construction in [27], but we are not interested in them here. In [15] there was a direct construction for deformed Rañada type solutions for the ModMax theory, that reduce to the Rañada solutions at $\gamma = 0$.

However, in [28,29], the following observation was made. Since the equations of motion of the nonlinear electromagnetic theory written in terms of F and G can be put into the form of the Maxwell equations in a medium, (in terms of $\vec{E}$, $\vec{B}$, $\vec{D}$, $\vec{H}$) as in (2.5), all we need to do is ensure that

$$\vec{H} = -\frac{\partial \mathcal{L}}{\partial \vec{B}} = \vec{B} + \mathcal{O}(F, G), \qquad \vec{D} = \frac{\partial \mathcal{L}}{\partial \vec{E}} = \vec{E} + \mathcal{O}(F, G),$$

(3.37)

for the same Bateman solutions with $F = G = 0$ to be still valid. That is satisfied for a Lagrangian that admits a Taylor expansion around $F = G = 0$ and reduces to Maxwell in the small field limit, i.e.,

$$\mathcal{L}(F, G) = -\frac{Fb^2}{2}\cosh\gamma + \frac{\sinh\gamma}{2}f(F^2, G^2), \qquad \mathcal{L}(F, G = 0) = -\frac{Fb^2}{2}e^{\gamma} \Rightarrow$$
$$\vec{H}(G = 0) = e^{\gamma}\vec{B}, \qquad \vec{D}(G = 0) = e^{\gamma}\vec{E}.$$

(3.39)

The factors of e^{γ} in $\vec{H}$ and $\vec{D}$ can be reabsorbed in a trivial redefinition, in order to obtain the same equations of motion as in vacuum, since classically a multiplicative constant in front of the Lagrangian is irrelevant.

That means that the Rañada solutions (3.35) are still valid for the ModMax theory. When going from ModMax to its precursor (2.13), the same logic as [28,29], described above, for going from Maxwell to BI, still applies. Namely, the expanded Lagrangian is now

$$\mathcal{L}(F, G) = \text{ModMax}(F, G)$$
$$+ \sum_{n\geq 2}\sum_{m\geq 0} c_{n,m}[\text{ModMax}(F, G)]^n G^{2m}, \quad (3.40)$$

and so again, it leads to (3.39), hence the Rañada solutions with $G = 0$, $F \neq 0$, (3.35) are still valid.

IV. CONCLUSIONS

In this paper, I have shown how to couple the precursor to the ModMax model to scalars, by writing it in a square root of determinant form, similar to the DBI action. For static solutions in the absence of magnetic fields, the action is the same as in the DBI case, which led to the same solutions still being valid; the supersymmetric (BPS) BIon, the original BIon, and the catenoid, as well as all the general solutions in between. I have also shown that the Rañada knotted solutions with $F \neq 0$, $G = 0$, are still solutions of both the ModMax theory and its precursor (and, of course, the full action with scalars, of which that is a consistent truncation, as the action contains only terms which are quadratic in scalars).

An important question remains; can one still understand this precursor to the ModMax model somehow in string theory, as a modification of the original DBI brane action? The answer is not clear, since the form we wrote for the action had some nontrivial scalar coefficients, α and β, depending on the Lorentz invariants F and G (or A and B)

$$\mathcal{L}(F, G) = b^2\left[-\frac{F}{2} + \sum_{n\geq 2}\sum_{m\geq 0} c_{n,m}F^n G^m\right]. \quad (3.38)$$

In particular, this is applied to the BI theory, for which $b^2[1 - \sqrt{1 + F - G^2}]$ can be expanded as above.

In the case of the ModMax theory, however, the limit $(F, G) \to (0, 0)$ is singular because of the square root, so the above seems not to hold. However, in this case of the ModMax theory, we observe that we have

made from the electromagnetic field, and it is not clear under what conditions these could be obtained.

APPENDIX: QUICK REVIEW OF BI ELECTRODYNAMICS

The Born-Infeld theory of electromagnetism [3] is obtained for $\gamma = 0$ in (2.1), so it is worth reviewing what happens in this case, in terms of the formalism above, in order to generalize it.

The Lagrangian is

$$\mathcal{L}(b; \vec{E}, \vec{B}) = b^2\left[1 - \sqrt{1 + F - G^2}\right], \quad (A1)$$

where

$$F = \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2) = \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu},$$
$$G = \frac{1}{b^2}\vec{E}\cdot\vec{B} = -\frac{1}{4b^2}F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (A2)$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

The constitutive relations become

$$\vec{H} = -\frac{\partial\mathcal{L}}{\partial\vec{B}} = \frac{\vec{B} - G\vec{E}}{\sqrt{1 + F - G^2}}, \qquad \vec{D} = \frac{\partial\mathcal{L}}{\partial\vec{E}} = \frac{\vec{E} + G\vec{B}}{\sqrt{1 + F - G^2}}.$$

(A3)

Then the Hamiltonian is

$$\mathcal{H} = \vec{E}\vec{D} - \mathcal{L} = b^2 \left[\frac{1 + \frac{\vec{B}^2}{b^2}}{\sqrt{1 + \frac{\vec{B}^2 - \vec{E}^2}{b^2} - \left(\frac{\vec{B}\cdot\vec{E}}{b^2}\right)^2}} - 1 \right], \quad (A4)$$

but it needs to be reexpressed in terms of $\vec{D}, \vec{B}$, where $\vec{D}$ is now explicitly

$$\vec{D} = \frac{\vec{E} + \vec{B}\left(\frac{\vec{E}\cdot\vec{B}}{b^2}\right)}{\sqrt{1 + \frac{\vec{B}^2 - \vec{E}^2}{b^2} - \left(\frac{\vec{B}\cdot\vec{E}}{b^2}\right)^2}}, \quad (A5)$$

and we can thus calculate

$$2s = \vec{D}^2 + \vec{B}^2 = \frac{\vec{E}^2 + \vec{B}^2\left(1 + \frac{\vec{B}^2 - \vec{E}^2}{b^2}\right) + 2\frac{(\vec{E}\cdot\vec{B})^2}{b^2}}{1 + \frac{\vec{B}^2 - \vec{E}^2}{b^2} - \left(\frac{\vec{B}\cdot\vec{E}}{b^2}\right)^2},$$

$$p^2 = \vec{D}^2\vec{B}^2 - (\vec{B}\cdot\vec{D})^2 = \frac{\vec{E}^2\vec{B}^2 - \frac{(\vec{E}\cdot\vec{B})^2}{b^2}}{1 + \frac{\vec{B}^2 - \vec{E}^2}{b^2} - \left(\frac{\vec{B}\cdot\vec{E}}{b^2}\right)^2}, \quad (A6)$$

which allows us to check that

$$\mathcal{H}(b; \vec{D}, \vec{B}) = b^2 \left[\sqrt{1 + \frac{2s}{b^2} + \frac{p^2}{b^4}} - 1 \right]$$

$$= b^2 \left[\sqrt{1 + \frac{\vec{D}^2 + \vec{B}^2}{b^2} + \frac{\vec{D}^2\vec{B}^2 - (\vec{D}\cdot\vec{B})^2}{b^4}} - 1 \right]. \quad (A7)$$

We now note, by comparing with the $\gamma = 0$ limit of (2.1), that we have $T = b^2$.

We can also, reversely, calculate

$$\vec{E} = \frac{\partial\mathcal{H}(b; \vec{B}, \vec{D})}{\partial\vec{D}} = \frac{\vec{D} + \frac{\vec{D}\vec{B}^2 - \vec{B}(\vec{D}\cdot\vec{B})}{b^2}}{\sqrt{1 + \frac{\vec{D}^2 + \vec{B}^2}{b^2} + \frac{\vec{D}^2\vec{B}^2 - (\vec{B}\cdot\vec{D})^2}{b^4}}}, \quad (A8)$$

and then

$$\mathcal{L}(b; \vec{E}, \vec{B}) = \vec{D}\cdot\vec{E} - \mathcal{H}(b; \vec{D}, \vec{B})$$

$$= b^2 \left[1 - \frac{1 + \frac{\vec{B}^2}{b^2}}{\sqrt{1 + \frac{\vec{D}^2 + \vec{B}^2}{b^2} + \frac{\vec{D}^2\vec{B}^2 - (\vec{B}\cdot\vec{D})^2}{b^4}}} \right], \quad (A9)$$

and then we can check that we can rewrite it in terms of $\vec{E}$, $\vec{D}$ as in (A1).

[1] I. Bandos, K. Lechner, D. Sorokin, and P. K. Townsend, A non-linear duality-invariant conformal extension of Maxwell's equations, Phys. Rev. D **102**, 121703 (2020).

[2] B. P. Kosyakov, Nonlinear electrodynamics with the maximum allowable symmetries, Phys. Lett. B **810**, 135840 (2020).

[3] M. Born and L. Infeld, Foundations of the new field theory, Proc. R. Soc. A **144**, 425 (1934).

[4] I. Bandos, K. Lechner, D. Sorokin, and P. K. Townsend, On p-form gauge theories and their conformal limits, J. High Energy Phys. 03 (2021) 022.

[5] I. Bandos, K. Lechner, D. Sorokin, and P. K. Townsend, ModMax meets Susy, J. High Energy Phys. 10 (2021) 031.

[6] S. M. Kuzenko, Superconformal duality-invariant models and $\mathcal{N} = 4$ SYM effective action, J. High Energy Phys. 09 (2021) 180.

[7] D. Flores-Alfonso, B. A. González-Morales, R. Linares, and M. Maceda, Black holes and gravitational waves sourced by non-linear duality rotation-invariant conformal electromagnetic matter, Phys. Lett. B **812**, 136011 (2021).

[8] A. Ballon Bordo, D. Kubizňák, and T. R. Perche, Taub-NUT solutions in conformal electrodynamics, Phys. Lett. B **817**, 136312 (2021).

[9] D. Flores-Alfonso, R. Linares, and M. Maceda, Nonlinear extensions of gravitating dyons: From NUT wormholes to Taub-Bolt instantons, J. High Energy Phys. 09 (2021) 104.

[10] Z. Amirabi and S. Habib Mazharimousavi, Black-hole solution in nonlinear electrodynamics with the maximum allowable symmetries, Eur. Phys. J. C **81**, 207 (2021).

[11] H. Babaei-Aghbolagh, K. B. Velni, D. M. Yekta, and H. Mohammadzadeh, $T\bar{T}$-like flows in non-linear electrodynamic theories and S-duality, J. High Energy Phys. 04 (2021) 187.

[12] M. J. Neves, J. B. de Oliveira, L. P. R. Ospedal, and J. A. Helayël-Neto, Dispersion relations in nonlinear electrodynamics and the kinematics of the Compton effect in a magnetic background, Phys. Rev. D **104**, 015006 (2021).

[13] A. Bokulić, T. Jurić, and I. Smolić, Black hole thermodynamics in the presence of nonlinear electromagnetic fields, Phys. Rev. D **103**, 124059 (2021).

[14] P. A. Cano and A. Murcia, Duality-invariant extensions of Einstein-Maxwell theory, J. High Energy Phys. 08 (2021) 042.

[15] C. Dassy and J. Govaerts, Deformed Hopfion-Rañada Knots in ModMax electrodynamics, J. Phys. A **54**, 265201 (2021).

[16] Y.-S. An, L. Li, and F.-G. Yang, No Cauchy horizon theorem for nonlinear electrodynamics black holes with charged scalar hairs, Phys. Rev. D **104**, 024040 (2021).

[17] S. I. Kruglov, On generalized ModMax model of nonlinear electrodynamics, Phys. Lett. B **822**, 136633 (2021).

[18] M. Zhang and J. Jiang, Conformal scalar NUT-like dyons in conformal electrodynamics, Phys. Rev. D **104**, 084094 (2021).

[19] C. G. Callan and J. M. Maldacena, Brane death and dynamics from the Born-Infeld action, Nucl. Phys. **B513**, 198 (1998).

[20] G. W. Gibbons, Born-Infeld particles and Dirichlet p-branes, Nucl. Phys. **B514**, 603 (1998).

[21] G. W. Gibbons, Aspects of Born-Infeld theory and string / M theory, AIP Conf. Proc. **589**, 324 (2001).

[22] H. Nastase, DBI skyrmion, high energy (large s) scattering and fireball production, arXiv:hep-th/0512171.

[23] A. F. Ranada, A topological theory of the electromagnetic field, Lett. Math. Phys. **18**, 97 (1989).

[24] A. F. Ranada, Knotted solutions of the Maxwell equations in vacuum, J. Phys. A **23**, L815 (1990).

[25] H. Bateman, *The Mathematical Analysis of Electrical and Optical Wave-motion on the Basis of Maxwell's Equations* (Cambridge University Press, Cambridge, England, 1915).

[26] C. Hoyos, N. Sircar, and J. Sonnenschein, New knotted solutions of Maxwell's equations, J. Phys. A **48**, 255204 (2015).

[27] M. Arrayás and J. L. Trueba, A class of non-null toroidal electromagnetic fields and its relation to the model of electromagnetic knots, J. Phys. A **48**, 025203 (2015).

[28] D. W. F. Alves, C. Hoyos, H. Nastase, and J. Sonnenschein, Knotted solutions for linear and nonlinear theories: Electromagnetism and fluid dynamics, Phys. Lett. B **773**, 412 (2017).

[29] D. W. F. Alves, C. Hoyos, H. Nastase, and J. Sonnenschein, Knotted solutions, from electromagnetism to fluid dynamics, Int. J. Mod. Phys. A **32**, 1750200 (2017).