Induced Chern-Simons term by dimensional reduction

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We derive an induced Abelian Chern-Simons (CS) term in $2+1$ dimensions, by dimensional reduction from the finite-temperature theory of a Dirac field with both vector and axial-vector couplings to two Abelian gauge fields, in $3+1$ dimensions. In our construction, the CS term emerges for the lowest Matsubara mode of the vector Abelian field, by integrating the fermionic field, under the assumption that the axial-vector field is in a "vacuum" configuration. This configuration is characterized by a single number, which in turn determines the coefficient of the induced CS term for the Abelian vector field.

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I. INTRODUCTION

Quantum field theories in $2+1$ dimensions have some features that make them an important subject of research, with great relevance both in theoretical developments and phenomenological applications. Among the latter, besides the celebrated condensed matter models involving planar systems, we should also mention the dimensional reduction at high temperatures in some high energy physics systems, typically, Yang-Mills theories [1], in the context of hot QCD.

Among the most characteristic properties of these theories, one of them shows up when considering gauge invariant systems, since they allow for the construction of a local, topological, and gauge-invariant functional of the gauge field, which breaks parity: the Chern-Simons (CS) term.[1] Unlike what happens in $3+1$ dimensions, parity is understood, in the $2+1$ dimensional context, to correspond to the reflection of just one of the two spatial coordinates (changing both coordinates has unit determinant: it is a rotation in π).

It has been realized some time ago that the CS term may appear in a system as a relic of the integration of matter degrees of freedom which break parity explicitly; indeed, this was first realized when evaluating the effective action for a massive Dirac field coupled to a gauge field [3], since the mass term in $2+1$ dimensions breaks parity. One of the terms appearing in the effective action for the gauge field is parity breaking and becomes a CS term when the mass of the fermion tends to infinity. It has moreover been realized that an explicit breaking is not required, since a quantum breaking is unavoidable, leading to a properly called parity anomaly [4–6]. The induced CS term and related objects have been subsequently studied in many different contexts [7] and from novel standpoints [8–11].

As originally pointed out in a well-honored work by Deser, Jackiw, and Templeton [3], one of the motivations to study the $d=3$ dimensional Chern-Simons (CS) action is that it leads to a topological mass term for the gauge field, which could possibly be connected with the high-temperature limit of a $d=4$ quantum field theory. This could result in a mass generation for the resulting effective Halmiltonian, as analyzed by Weinberg [12].

A recent work Pisarski [13] comes back to the possibility that the topological CS mass term may in fact provide the correct infrared regulation at high temperatures exposing doubts concerning the possibility that a theta term $\theta\tilde{F}F$ in a $d=4$ high-temperature gauge theory could be at the origin of such phenomenon.

In the present work, we follow a different strategy to connect the Chern-Simons action in $2+1$ spacetime dimensions by dimensional reduction from a finite-temperature $3+1$ dimensional theory, namely, Dirac field theory with vector and axial-vector couplings to two external Abelian gauge fields, in $3+1$ dimensions.

The dimensional reduction is implemented here under two assumptions about the gauge fields: the axial field is assumed to be in a vacuum state, while the vector one belongs to the lowest, zero Matsubara frequency configuration.

The structure of this paper is as follows: in Sec. II, we introduce a $3+1$ dimensional theory and the assumptions

[1] See, for example, [2] for a review.

we make before evaluating its effective action. Then, in Sec. III, we evaluate the imaginary part: the induced CS term. The real part of the effective action is briefly discussed in Sec. IV. In Sec. V, we present our conclusions.

II. THE SYSTEM

We consider a massless Dirac field in $3+1$ dimensions, at a finite temperature T, endowed with vector and axial-vector couplings to external Abelian gauge fields A_μ and B_μ, respectively. These fields are assumed to belong to some specific classes below, but we first introduce them as if they were arbitrary, for the sake of clarity. With this in mind, the Euclidean action $\mathcal{S}$ of the system, in the Matsubara formalism, is given by the expression

$$\mathcal{S}(\bar{\psi},\psi;A,B) = \int_0^\beta d\tau \int d^3x \bar{\psi}(\tau,x)(\not{\partial} + i\not{A} + i\not{B}\gamma_5)\psi(\tau,x), \tag{1}$$

where τ is the Euclidean time, and we use conventions whereby the Boltzmann constant $k_B \equiv 1$, so that $\beta = \frac{1}{T}$. Spacetime coordinates are denoted by x_μ, $\mu = 0, 1, 2, 3$, such that $x_0 \equiv \tau$, and $x \equiv (x_1, x_2, x_3)$, where x_i $(i = 1, 2, 3)$ are the spatial Cartesian coordinates.[2] On the other hand, Dirac's matrices γ_μ are Hermitian and satisfy the relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, while γ_5 is given by $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5^\dagger$.

We are interested in extracting the parity-breaking part of the effective action $\Gamma(A,B)$ due to the Dirac field quantum fluctuations,

$$e^{-\Gamma(A,B)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-\mathcal{S}(\bar{\psi},\psi;A,B)}. \tag{2}$$

In the Matsubara formalism, the fermionic fields are antiperiodic in the imaginary time interval, namely,

$$\psi(\tau+\beta,x) = -\psi(\tau,x), \qquad \bar{\psi}(\tau+\beta,x) = -\bar{\psi}(\tau,x), \tag{3}$$

while bosonic ones, in particular, the gauge fields, are periodic,

$$A_\mu(\tau+\beta,x) = A_\mu(\tau,x), \qquad B_\mu(\tau+\beta,x) = B_\mu(\tau,x). \tag{4}$$

As a consequence, when considering the set of allowed vector and axial-vector gauge transformations,

$$\psi(\tau,x) \to e^{-i\Omega_A(\tau,x)}\psi(\tau,x), \qquad \bar{\psi}(\tau,x) \to e^{i\Omega_A(\tau,x)}\bar{\psi}(\tau,x),$$
$$A_\mu(\tau,x) \to A_\mu(\tau,x) + \partial_\mu\Omega_A(\tau,x) \tag{5}$$

and

$$\psi(\tau,x) \to e^{-i\Omega_B(\tau,x)\gamma_5}\psi(\tau,x), \quad \bar{\psi}(\tau,x) \to \bar{\psi}(\tau,x)e^{-i\Omega_B(\tau,x)\gamma_5}$$
$$B_\mu(\tau,x) \to B_\mu(\tau,x) + \partial_\mu\Omega_B(\tau,x), \tag{6}$$

respectively, the functions $\Omega_{A,B}$ must be required to satisfy

$$\Omega_{A,B}(\beta,x) = \Omega_{A,B}(0,x) + 2\pi n_{A,B}, \tag{7}$$

where n_A and n_B are integers, which label the respective winding numbers of the large gauge transformations.

To date, no restriction about the gauge-field configurations has been implemented; let us now make them more explicit. First, since we have in mind the high-temperature regime, and the field is periodic, we can invoke the usual decoupling of the lowest mode. Indeed, in the Matsubara Fourier expansion of A_μ,

$$A_\mu(\tau,x) = \beta^{-\frac{1}{2}} \sum_{n=-\infty}^{+\infty} e^{\frac{2n\pi i}{\beta}} A_\mu^{(n)}(x), \tag{8}$$

we keep just the $n = 0$ mode, since the remaining ones have masses which increase with temperature: $A_\mu(\tau,x) \sim A_\mu^{(n)}(x) \equiv A_\mu(x)$. Note that this produces four space-dependent components; we add the further constraint of having a space-independent A_0. An alternative way of characterizing this is to say that we only keep purely magnetic (static) field configurations: the simplest non-trivial one allowing for the existence of a reduced (non-trivial) effective action. Note that the time component of A_μ may be assumed to depend on τ, what is gauge equivalent to a constant field.

The axial field, on the other hand, is assumed to be a vacuum configuration (vanishing electric and magnetic fields), since we are using it just as a seed to produce parity breaking in the reduced theory. As mentioned in [14], a vacuum gauge field configuration corresponds to no spatial components and a time-dependent temporal component. This is consistent with assuming a Maxwell action for that field and looking for its lowest action configuration.

Therefore, the class of configurations that we consider may be characterized as follows:

$$\partial_j A_0 = 0, \qquad \partial_j B_0 = 0,$$
$$\partial_0 A_j = 0, \qquad B_j = 0 \quad (j = 1,2,3). \tag{9}$$

III. IMAGINARY PART OF THE EFFECTIVE ACTION

In this Section, we evaluate the imaginary part of Γ, under the previous assumptions about the gauge field configurations.

[2]Indices from the middle of the greek alphabet run over the same range as μ, while those from the middle of the roman one correspond to spatial coordinates and have the same range as i.

Furthermore, A_0 and B_0 (that can only depend on τ) may be rendered τ independent (constant) having, respectively, the values $\tilde{A}_0$ and $\tilde{B}_0$,

$$A_0(\tau) \to \tilde{A}_0 = \frac{1}{\beta}\int_0^\beta d\tau A_0(\tau),$$
$$B_0(\tau) \to \tilde{B}_0 = \frac{1}{\beta}\int_0^\beta d\tau B_0(\tau), \tag{10}$$

by means of a gauge transformation of the fermions,

$$\psi(\tau,x) \to e^{i\int_0^\tau d\tau'(A_0(\tau')-\tilde{A}_0)} e^{i\gamma_5\int_0^\tau d\tau'(B_0(\tau')-\tilde{B}_0)\gamma_5}\psi(\tau,x),$$
$$\bar{\psi}(\tau,x) \to \bar{\psi}(\tau,x)e^{-i\int_0^\tau d\tau'(A_0(\tau')-\tilde{A}_0)} e^{i\gamma_5\int_0^\tau d\tau'(B_0(\tau')-\tilde{B}_0)}. \tag{11}$$

Note that this gauge transformation is "small", i.e., connected to the identity (its winding number vanishes). Under this transformation, the axial part the gauge transformation above does not produce a nontrivial Jacobian. Indeed, denoting by $\Omega_B(\tau)$ the parameter of that transformation, $\Omega_B(\tau) \equiv \int_0^\tau d\tau'(B_0(\tau') - \tilde{B}_0)$, and

$$K_\mu = \frac{1}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta, \tag{12}$$

we note that the anomalous Jacobian $\mathcal{J}$ is

$$\mathcal{J} = e^{-i\int_0^\beta d\tau\int d^3x\Omega_B(\tau)\partial_\mu K_\mu(\tau,x)} = e^{-i\int_0^\beta d\tau\int d^3x\Omega_B(\tau)\partial_\tau K_0(x)}$$
$$= e^0 = 1, \tag{13}$$

where we used the property that, for the configurations we are dealing with, $K_j = 0$ ($j = 1, 2, 3$), plus the time independence of the spatial components of A_μ. The periodicity of Ω_B for a "small" transformation is implicitly assumed in the fact that the anomalous Jacobian is known for transformations which do not change the boundary conditions.

Therefore, we arrive to an equivalent (i.e., having identical effective action) expression for the action

$$\mathcal{S}(\bar{\psi},\psi;A,B) = \int_0^\beta d\tau\int d^3x\bar{\psi}(\tau,x)$$
$$\times[\partial\!\!\!/ + i\gamma_j A_j(x) + i\gamma_0(\tilde{A}_0 + \tilde{B}_0\gamma_5)]\psi(\tau,x). \tag{14}$$

In the expression above, the constant values of the temporal components of the gauge fields can also be shifted by an integer number of $\frac{2\pi}{\beta}$. On the other hand, had we wanted to completely decouple also the constant fields $\tilde{A}_0$ and $\tilde{B}_0$, we should have performed a gauge transformation which, in general, would have spoiled the boundary conditions, namely, because $\frac{1}{2\pi}[\Omega_{A,B}(\beta,x) - \Omega_{A,B}(0,x)] \notin \mathbb{Z}$.

Parity is *explicitly* broken by the presence of B, and since the imaginary part of Γ coincides with its parity breaking part, one can obtain the former as the odd part (under parity) of the effective action. Note that a nonexplicit (i.e., anomalous) breaking of parity cannot be obtained by this procedure, which is adamant to B_μ-independent contributions. To obtain the imaginary part of Γ, we begin from

$$\mathrm{Im}[\Gamma(A,B)] = \Gamma_{\mathrm{odd}}(A,B)$$
$$= \frac{1}{2}\int_{-\tilde{B}_0}^{+\tilde{B}_0} d\tilde{B}_0\frac{\partial}{\partial\tilde{B}_0}\Gamma(A,B), \tag{15}$$

where A and B are implicitly assumed to belong to the class we are considering here, namely, $A_0 = \tilde{A}_0$, $A_j = A_j(x)$, $B_0 = \tilde{B}_0$, $B_j = B_j(x)$.

To proceed, we introduce an expansion in Matsubara modes for the fermions,

$$\psi(\tau,x) = \beta^{-\frac{1}{2}}\sum_{n=-\infty}^{+\infty}\psi_n(x)e^{-i\omega_n\tau},$$
$$\bar{\psi}(\tau,x) = \beta^{-\frac{1}{2}}\sum_{n=-\infty}^{+\infty}\bar{\psi}_n(x)e^{i\omega_n\tau}, \tag{16}$$

obtaining an alternative form of the action with all the modes decoupled,

$$\mathcal{S}(\bar{\psi},\psi;A,B) = \sum_{n=-\infty}^{+\infty}\int d^3x\bar{\psi}_n(x)\mathcal{D}_n\psi_n(x),$$
$$\mathcal{D}_n \equiv d\!\!\!/ + i\gamma_0(\omega_n + \tilde{A}_0 + \tilde{B}_0\gamma_5), \tag{17}$$

and $d\!\!\!/$ denotes a Dirac operator in three Euclidean dimensions, $d\!\!\!/ \equiv \gamma_j(\partial_j + iA_j(x))$, but built with 4×4 Dirac matrices γ_j. We then see that

$$e^{-\Gamma(A,B)} = \det[\partial\!\!\!/ + i\gamma_j A_j(x) + i\gamma_0(\tilde{A}_0 + \tilde{B}_0\gamma_5)]$$
$$= \prod_{n=-\infty}^{+\infty}\det[d\!\!\!/ + i\gamma_0(\omega_n + \tilde{A}_0 + \tilde{B}_0\gamma_5)] \tag{18}$$

and

$$\Gamma(A,B) = -\sum_{n=-\infty}^{+\infty}\mathrm{Tr}\log[d\!\!\!/ + i\gamma_0(\omega_n + \tilde{A}_0 + \tilde{B}_0\gamma_5)]. \tag{19}$$

Therefore,

$$\Gamma_{\rm odd}(A,B) = -\frac{1}{2}\int_{-\tilde{B}_0}^{+\tilde{B}_0} d\tilde{B}'_0 \times \sum_{n=-\infty}^{+\infty} {\rm Tr}\left[i\gamma_0\gamma_5 \frac{1}{\not{d} + i\gamma_0(\omega_n + \tilde{A}_0 + \tilde{B}'_0\gamma_5)}\right], \tag{20}$$

where "Tr" denotes trace over both spacetime arguments and Dirac matrices' indices (the latter be denoted by "tr").

We can produce a more explicit expression,

$$\Gamma_{\rm odd}(A,B) = \frac{1}{2}\int_{-\tilde{B}_0}^{+\tilde{B}_0} d\tilde{B}'_0 \sum_{n=-\infty}^{+\infty} \mathcal{Q}_n(A,B'), \tag{21}$$

where

$$\mathcal{Q}_n(A,B) = -i\int d^3x\, {\rm tr}\left[\gamma_0\gamma_5\langle x|\frac{1}{\not{d} + i\gamma_0(\omega_n + \tilde{A}_0 + \tilde{B}_0\gamma_5)}|x\rangle\right], \tag{22}$$

and we have adopted Dirac's bra-ket notation to denote operator kernels.

Let us note that, up to this point, we have not made any assumption about the magnitude of the temperature. From now on, we assume that $T \gg |A_j|$, the spatial components of the gauge field. The temporal components of A and B, on the other hand, are gauge equivalent to the constants $\tilde{A}_0$ and $\tilde{B}_0$ and therefore cannot be regarded as small just by invoking a similar argument to the one used for the spatial components. However, the fact that they are constants allows us to treat them exactly. Indeed, we expand in powers of A_j, since $\omega_n \gg A_j,\ \forall\, n$. The lowest nonvanishing contribution to $\mathcal{Q}_n$ is of the second order in A_j, as it may be seen from the vanishing of the Dirac traces for the previous two orders.

Keeping just the second-order contribution, we see that

$$\mathcal{Q}_n(A,B) = i\int d^3x\, {\rm tr}[\gamma_0\gamma_5\langle x|G_n\gamma_j A_j G_n \gamma_k A_k G_n|x\rangle], \tag{23}$$

where we have introduced the operator

$$G_n = \frac{1}{\not{d}_0 + i\gamma_0(\omega_n + \tilde{A}_0 + \tilde{B}_0\gamma_5)}, \qquad \not{d}_0 \equiv \gamma_j\partial_j. \tag{24}$$

Although one could use any representation for the Dirac's matrices, it is rather convenient, in this calculation, to use the chiral representation, built in terms of $\sigma_0 \equiv \mathbb{I}_{2\times 2}$ and the standard Pauli's matrices σ_j,

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu^\dagger \\ \sigma_\mu & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & -\mathbb{I}_{2\times 2} \end{pmatrix}. \tag{25}$$

When used in (26), this leads to an equation which is naturally decomposed into two terms, one for each chirality component, which in turn involve traces of 2×2 matrices,

$$\mathcal{Q}_n(A,B) = \mathcal{Q}_n^L(A,B) + \mathcal{Q}_n^R(A,B), \tag{26}$$

$$\mathcal{Q}_n^L(A,B) = -\int d^3x\, {\rm tr}[\langle x|(\not{\nabla} - \omega_n - \tilde{A}_0 - \tilde{B}_0)^{-2}\gamma_j A_j \times (\not{\nabla} - \omega_n - \tilde{A}_0 - \tilde{B}_0)^{-1}\gamma_k A_k|x\rangle],$$
$$\mathcal{Q}_n^R(A,B) = -\int d^3x\, {\rm tr}[\langle x|(\not{\nabla} + \omega_n + \tilde{A}_0 - \tilde{B}_0)^{-2}\gamma_j A_j \times (\not{\nabla} + \omega_n + \tilde{A}_0 - \tilde{B}_0)^{-1}\gamma_k A_k|x\rangle], \tag{27}$$

where $\not{\nabla} \equiv \sigma_j\partial_j$.

After some algebra, we find that $\Gamma_{\rm odd}$ may be expressed as

$$\begin{aligned}\Gamma_{\rm odd}(A,B) &= -\frac{1}{4}\sum_{n=-\infty}^{+\infty}\int d^3x\{{\rm tr}[\langle x|((\not{\nabla} - \omega_n - \tilde{A}_0 - \tilde{B}_0)^{-1}\gamma_j A_j)^2|x\rangle] \\ &+ {\rm tr}[\langle x|((\not{\nabla} + \omega_n + \tilde{A}_0 - \tilde{B}_0)^{-1}\gamma_j A_j)^2|x\rangle] \\ &- {\rm tr}[\langle x|((\not{\nabla} - \omega_n - \tilde{A}_0 + \tilde{B}_0)^{-1}\gamma_j A_j)^2|x\rangle] \\ &- {\rm tr}[\langle x|((\not{\nabla} + \omega_n + \tilde{A}_0 + \tilde{B}_0)^{-1}\gamma_j A_j)^2|x\rangle]\}.\end{aligned} \tag{28}$$

The structure of each term of the four terms inside the sum over n is identical, except for a global factor, to the one of the effective action for a massive Dirac field in $2+1$ dimensions. The difference being the values one should use for the respective masses, which depend on n, $\tilde{A}_0$, and $\tilde{B}_0$. One sees that only the odd part in the fermion mass (of each mode) is needed, after inserting the known $2+1$ dimensional result into (28).

Keeping just the leading terms in the corresponding "masses" (we discuss the next to leading terms below), one gets

$$\Gamma_{\rm odd}(A,B) = \frac{i}{8\pi}\xi(\tilde{A}_0,\tilde{B}_0)\int d^3x\,\epsilon_{jkl}A_j(x)\partial_k A_l(x), \tag{29}$$

where

$$\xi(\tilde{A}_0,\tilde{B}_0) = \sum_{n=-\infty}^{\infty}\left(\frac{\omega_n + \tilde{A}_0 - \tilde{B}_0}{|\omega_n + \tilde{A}_0 - \tilde{B}_0|} - \frac{\omega_n + \tilde{A}_0 + \tilde{B}_0}{|\omega_n + \tilde{A}_0 + \tilde{B}_0|}\right). \tag{30}$$

Thus, the odd part of the $3+1$ dimensional effective action looks like a Chern-Simons action with an effective coefficient: the function ξ. This function is expressed as a series in Matsubara frequencies space. A convenient way to render it in a more appealing form without spoiling its gauge transformation properties is by using Poisson summation or, in this context, Selberg's trace formula [15].

In this case, it amounts to replacing the series over frequencies by another one where each term is (anti)-Fourier transformed,

$$\xi(\tilde{A}_0, \tilde{B}_0) = -\frac{2}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \mathcal{P}\left(\frac{1}{k}\right) \sin(k\beta\tilde{B}_0) \cos(k\beta\tilde{A}_0), \tag{31}$$

where $\mathcal{P}$ denotes Cauchy's principal value. In the present context, its effect would be to get rid of a possible contribution from the $k = 0$ term; however, that term vanishes by itself. Besides, in the remaining terms, the principal value prescription is irrelevant and can be removed. Then,

$$\begin{aligned}\xi(\tilde{A}_0, \tilde{B}_0) &= \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(k\beta\tilde{B}_0) \cos(k\beta\tilde{A}_0)}{k} \\ &= \frac{2}{\pi} \{\mathrm{Im}[\log(1 + e^{i\beta(\tilde{B}_0 - \tilde{A}_0)})] \\ &\quad + \mathrm{Im}[\log(1 + e^{i\beta(\tilde{B}_0 + \tilde{A}_0)})]\} \\ &= \frac{2}{\pi} \left\{ \arctan\left[\tan\left(\frac{\beta\tilde{B}_0 - \beta\tilde{A}_0}{2}\right)\right] \right. \\ &\quad \left. + \arctan\left[\tan\left(\frac{\beta\tilde{B}_0 + \beta\tilde{A}_0}{2}\right)\right] \right\},\end{aligned} \tag{32}$$

i.e.,

$$\xi = \frac{2}{\pi} \int_0^\beta d\tau B_0(\tau). \tag{33}$$

Recalling the origin of this result, from the imaginary part of logarithmic functions, we see that under large gauge transformations with winding number equal to, say, n, then $\int_0^\beta d\tau B_0(\tau)$ will follow that winding; it is an angular function.

Finally, we have, for Γ_{odd},

$$\Gamma_{\mathrm{odd}}(A,B) = \frac{i}{4\pi^2} \int_0^\beta d\tau B_0(\tau) \int d^3x \epsilon_{jkl} A_j(x) \partial_k A_l(x), \tag{34}$$

which, we recall, has been obtained as the leading term in a high-temperature expansion.

Note that the previous result may be written in terms of a Chern-Simons (CS) action $\mathcal{S}_{\mathrm{CS}}$, defined (in our conventions) by

$$\mathcal{S}_{\mathrm{CS}}(A) \equiv \frac{1}{8\pi} \int d^3x \epsilon_{jkl} A_j(x) \partial_k A_l(x), \tag{35}$$

as

$$\Gamma_{\mathrm{odd}}(A,B) = i\frac{2}{\pi} \int_0^\beta d\tau B_0(\tau) \mathcal{S}_{\mathrm{CS}}(A). \tag{36}$$

This is the main result of this paper, namely, that an induced CS term emerges for the lowest (i.e., massless) Matsubara mode of A, in the high-temperature limit. The coefficient of that term is determined by a parameter which labels the vacuum configurations of the B field. In the next section, we propose a possible reason whereby a nontrivial value for such a parameter may naturally emerge.

The previous equation relates the content of the B_0-field configuration to the coefficient multiplying the CS action. We note that $\int_0^\beta d\tau B_0(\tau)$ may be identified with $\frac{\pi}{2} N$, with N being the number of fermionic flavors, had the induced CS proceeded from a $2+1$ dimensional calculation.

We note that the next-to-leading term contribution to the odd part of the effective action does contain an extra derivative of the spatial components of the gauge field, so that it has the structure,

$$\Gamma_{\mathrm{odd}}^{\mathrm{sub}} = i\chi(\tilde{A}_0, \tilde{B}_0) \mathcal{S}_{PC}(A), \tag{37}$$

where $\mathcal{S}_{PC}(A) = \frac{1}{4} \int d^3x F_{jk}^2$, where $\chi(\tilde{A}_0, \tilde{B}_0)$ is an odd function under the a reflection in $\tilde{B}_0$. Thus, this subleading contribution is indeed odd (and imaginary), although it does not contribute to the induced CS term, as it should be.

IV. REAL PART OF THE EFFECTIVE ACTION

The axial gauge field configuration has been assumed to be in a vacuum configuration, from the point of view of its corresponding action. Note, however, that the real part of its effective action, for the same kind of configuration as before, will receive quantum corrections. Besides, in the same limit as the one used for the imaginary part, the space-dependent part of A_j is suppressed. Thus, the real part of the effective action may also be conveniently obtained by first introducing the Matsubara modes for the fermions, as we did for the imaginary part. Besides, it will be extensive, so it is convenient to introduce the (real part of the) free energy per unit volume, $f(A,B)$,

$$\begin{aligned}\mathrm{Re}[\Gamma(A,B)] &= -\frac{1}{2} \sum_{n=-\infty}^{+\infty} \mathrm{Tr}\, \log(\mathcal{D}_n^\dagger \mathcal{D}_n) \\ &= -V\beta f(A,B),\end{aligned} \tag{38}$$

with

$$\begin{aligned}f(A,B) = \beta^{-1} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \{&\log[k^2 + (\omega_n + \tilde{A}_0 + \tilde{B}_0)^2] \\ &+ \log[k^2 + (\omega_n + \tilde{A}_0 - \tilde{B}_0)^2]\}.\end{aligned} \tag{39}$$

The sum over n can be performed using standard complex variable techniques, leading to

$$f(A,B) = -\frac{1}{\beta}\int \frac{d^3k}{(2\pi)^3}\{\log[\cosh(\beta k) + \cos(\beta\tilde{A}_0 + \beta\tilde{B}_0)] + \log[\cosh(\beta k) + \cos(\beta\tilde{A}_0 - \beta\tilde{B}_0)]\}, \tag{40}$$

from which one can subtract its zero-temperature (i.e., vacuum) part in order to render it finite. Besides, looking for extrema with respect to $\tilde{B}_0$, we find the necessary condition

$$0 = \int \frac{d^3k}{(2\pi)^3}\left[\left[\frac{\sin(\beta\tilde{A}_0 + \beta\tilde{B}_0)}{\cosh(\beta k) + \cos(\beta\tilde{A}_0 + \beta\tilde{B}_0)}\right] - \frac{\sin(\beta\tilde{A}_0 - \beta\tilde{B}_0)}{\cosh(\beta k) + \cos(\beta\tilde{A}_0 - \beta\tilde{B}_0)}\right]\right], \tag{41}$$

which can be satisfied for $\int_0^\beta d\tau B_0(\tau) = \pi\,(\mathrm{mod}\,\pi)$. Coming back to the imaginary part in (36), this implies

$$\Gamma_{\rm odd}(A,B) = 2i\mathcal{S}_{\rm CS}(A). \tag{42}$$

We see that corresponds to the standard result for the induced CS term in $2+1$ dimensions for two two-component fields.

V. CONCLUSIONS

We have evaluated the dimensionally reduced effective action due to a Dirac field in $3+1$ dimensions, in the presence of vector and axial-vector gauge fields. Under some assumptions abut the system, namely, a vacuum configuration for the axial one,and a purely magnetic one for the vector field, an induced CS term appears for the latter. The mechanism whereby this happens may be understood as due to an unbalance, due to the axial field, between the number of fermionic Matsubara modes having positive and negative masses. Note that a related reduction mechanism has been used in [16] to formulate the overlap prescription for the Dirac operator for lattice fermions in an odd number of dimensions, although without an explicit breaking of parity by an external axial field. The parity anomalous contribution, which as explained we do not study here, was shown in [16] to correspond to a specific prescription for the phase of the Dirac operator.

As a direction for future work, we note that it might be possible for alternative physical mechanisms to produce an induced CS term. Even at zero temperature, perfect conductor boundary conditions on the boundaries of a compact spatial coordinate [17] lead to an interesting $2+1$ dimensional structure, while for the Dirac field one could try using bag model [18] (or related) conditions.

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