Properties of the linearized functional renormalization group

Tim R. Morris *

STAG Research Centre & Department of Physics and Astronomy, University of Southampton, Highfield, Southampton SO17 1BJ, United Kingdom

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Interactions growing slower than a certain exponential of the square of a scalar field are well behaved when evolved under the functional renormalization group linearized around the Gaussian fixed point. They satisfy properties usually taken for granted, and reproduce standard perturbative quantization. However, the more challenging effects appear, the more interactions grow faster than this. We show explicitly that firstly, the flow no longer splits uniquely into operators of definite scaling dimension; secondly, (linearized) flows to the infrared can end prematurely in a singularity; and finally, new interactions can spontaneously appear at any scale.

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I. INTRODUCTION

In this paper, we will be mostly concerned with the functional (also known as exact) renormalization group (RG) linearized around the Gaussian fixed point. One might think that everything is known about such a simple situation. However, we show that new effects appear once interactions are allowed to grow as fast as an exponential of the square of the field (for large field). These effects challenge our expectations of the RG. As the speed of growth is increased, the first effect to appear is that the linearized flow no longer splits uniquely into a sum over eigenoperators (operators of definite scaling dimension). The next effect to appear is that the linearized flows toward the infrared (IR) can end prematurely in a singularity (after which the flow ceases to exist). These cases include linear combinations of the hypothesized Halpern-Huang interactions [1–23]. Finally, if interactions growing faster than any exponential of the field-squared are allowed, then the effective action at one point on the flow no longer determines its form at lower scales, even at the linearized level. New interactions can spontaneously appear at any lower scale.

For clarity and simplicity, we focus on the linearized effective potential $V(\varphi, \Lambda)$ for a single component scalar field φ (with standard kinetic term), where Λ is the effective cutoff, and choose four (Euclidean) dimensions. The effective action is then simply

$$\int d^4x \left\{ \frac{1}{2} (\partial_\mu \varphi)^2 + V(\varphi, \Lambda) \right\}. \tag{1.1}$$

However, it will be clear that the qualitative conclusions are the same in any dimension greater than two,[1] for more general field theories (e.g., multi-component scalar fields) and for general local interactions (thus also containing space-time derivatives). Although there are different versions of the flow equation [25–30] and different choices of the cutoff profile, at the linearized level they all collapse to the same thing. For the effective potential $V(\varphi, \Lambda)$, we have (e.g., see [23] or Sec. 2 of Ref. [31])

$$\Lambda \partial_\Lambda V(\varphi, \Lambda) = -\frac{\Lambda^2}{2a^2} V''(\varphi, \Lambda), \tag{1.2}$$

where prime stands for ∂_φ. At the linearized level, there is no anomalous dimension. The only quantum correction is the tadpole integral $\langle \varphi(x)\varphi(x) \rangle$ of the massless scalar field. The effective ultraviolet (UV) regularized version is $\Lambda^2/2a^2$ (independent of x), where the Λ^2 dependence is guaranteed by dimensions and the dimensionless parameter a captures the entire dependence on the regularization scheme in this situation.

The point of a flow equation is that it tells us how the effective action changes as we lower the cutoff Λ. Choosing some initial[2] potential $V(\varphi, \Lambda_0)$ at some high scale $\Lambda = \Lambda_0$, we can follow its evolution at the linearized level as we

*T.R.Morris@soton.ac.uk

[1]In two dimensions, the engineering dimension of a scalar field vanishes. The linearized RG leads instead to Sine-Gordan models. This was derived in [24], at $O(\partial^0)$ (which at the linearized level is exact).

[2]Basically, this is the bare potential. See Refs. [32,33] for the precise relationship.

integrate out all the modes by solving (1.2). As $\Lambda \to 0$, we recover the physical potential $V(\varphi, 0)$.

We stress that at the linearized level, the flow equation (1.2) is exact if we start with only potential interactions. No other approximation has been applied apart from linearization. In particular the result should not be confused with use of the so-called Local Potential Approximation [34]. The exact quantum correction arises from a term

$$\int d^4x d^4y \langle \varphi(x)\varphi(y)\rangle \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} \int d^4z\, V(\varphi(z), \Lambda) = \langle \varphi(x)\varphi(x)\rangle \int d^4x\, V''(\varphi(x), \Lambda), \tag{1.3}$$

so no terms are generated other than corrections to the effective potential.

Of course one should question if linearization is appropriate. We will come back to this in Sec. VI of the conclusion. For the moment, we just note that this is standard practice, being the first step in deriving the eigenoperators (e.g., see the reviews [25,35–44]). Our exposition follows [23,31]. We recast in dimensionless terms using the cutoff, $\varphi = \tilde{\varphi}\Lambda$, $V = \tilde{V}\Lambda^4$, and then separate variables, leading to a solution

$$\tilde{V}(\tilde{\varphi}, \Lambda) = \tilde{g}(\Lambda)\tilde{V}(\tilde{\varphi}), \tag{1.4}$$

where $\tilde{g}$ is the scaled coupling

$$\tilde{g}(\Lambda) = \frac{g}{\Lambda^\lambda}, \tag{1.5}$$

λ being the RG eigenvalue and g a constant of dimension λ. According to (1.5), this linearized coupling grows, stays constant, or shrinks, as we flow to the infrared, i.e., the coupling is relevant, marginal, or irrelevant, depending on whether λ is positive, zero, or negative, respectively. The function $\tilde{V}(\tilde{\varphi})$ satisfies the eigenoperator equation

$$\lambda\tilde{V}(\tilde{\varphi}) + \tilde{\varphi}\tilde{V}' - 4\tilde{V} = \frac{\tilde{V}''}{2a^2}, \tag{1.6}$$

where prime is now the differentiation with respect to $\tilde{\varphi}$.

The general nonsingular solution of (1.6) is a linear combination of the Kummer M functions [45–47]:

$$\omega^\lambda(\tilde{\varphi}) := M\left(\frac{\lambda}{2} - 2, \frac{1}{2}, a^2\tilde{\varphi}^2\right), \qquad \tilde{\varphi}M\left(\frac{\lambda}{2} - \frac{3}{2}, \frac{3}{2}, a^2\tilde{\varphi}^2\right). \tag{1.7}$$

These are in fact entire functions of $\tilde{\varphi}$, the first (second) being an even (odd) function of $\tilde{\varphi}$. For simplicity, we will mostly focus on the even eigenoperators and as in [23], we call them $\omega^\lambda(\tilde{\varphi})$.

For $\lambda = 4 - n$, with n being an even (odd) non-negative integer, the first (second) solution is proportional to a Hermite polynomial. We normalize these polynomials as[3]

$$\mathcal{O}_n(\tilde{\varphi}) = H_n(a\tilde{\varphi})/(2a)^n = \tilde{\varphi}^n - n(n-1)\tilde{\varphi}^{n-2}/4a^2 + \cdots. \tag{1.8}$$

The (scaling) dimension of the operator $\mathcal{O}_n$ is thus $4 - \lambda = n$, coinciding with the engineering dimension $[\varphi^n]$ of its top term. The lower powers appear due to the tadpole corrections. At the nonlinear level, expanding over these $\mathcal{O}_n$ operators reproduces perturbation theory [23,25,31,35–44].

The remaining eigenoperator solutions are the hypothesized Halpern-Huang (HH) interactions [1–23]. They attract interest especially because the $\lambda > 0$ operators appear to offer relevant interactions that would allow genuine interacting continuum limits for scalar fields (such as the Higgs field) in four space-time dimensions. They grow like $e^{a^2\tilde{\varphi}^2}$ for large field. More precisely, we have for the even operators that asymptotically

$$\omega^\lambda(\tilde{\varphi}) \sim \frac{\sqrt{\pi}}{\Gamma(\lambda/2 - 2)}|\tilde{\varphi}|^{\lambda-5}e^{a^2\tilde{\varphi}^2}, \quad \text{as } \tilde{\varphi} \to \pm\infty \tag{1.9}$$

($\lambda \neq 4 - 2n$, with n being a non-negative integer).

It will be useful for us to note that at $\lambda = 5 + n$ (n being a non-negative integer), the HH operators are up to normalization given by[4]

$$\mathcal{O}^n(\tilde{\varphi}) := e^{a^2\tilde{\varphi}^2}H_n(ia\tilde{\varphi})/(2ia)^n = e^{a^2\tilde{\varphi}^2}(\tilde{\varphi}^n + n(n-1)\tilde{\varphi}^{n-2}/4a^2 + \cdots), \quad (\lambda = 5 + n). \tag{1.10}$$

To prove this, note that after substituting $\tilde{V}(\tilde{\varphi}) \mapsto \tilde{V}(\tilde{\varphi})e^{a^2\tilde{\varphi}^2}$ into (1.6) and simplifying (1.6), one recovers the eigenoperator equation again but with λ replaced by $9 - \lambda$ and a replaced with ia [31]. This new equation is therefore solved by (1.8) with a replaced by ia, and $\lambda = 5 + n$.

The eigenoperator equation is of the Sturm-Liouville type. The corresponding Sturm-Liouville (SL) measure is

$$e^{-a^2\tilde{\varphi}^2}. \tag{1.11}$$

Perturbations $\tilde{V}(\tilde{\varphi})$ that are square integrable under this measure are particularly well behaved. We start by reviewing these properties. We then prove that there are solutions to the linearized flow equation (1.2) that start inside this space, stay inside this space, and have all the desired

[3] Kummer functions are normalized such that $M = 1$ at $\tilde{\varphi} = 0$.

[4] Note that these are indexed by a superscript in contrast to the (1.8). These are the analytic continuation of $\delta_n(\tilde{\varphi})$ operators [31] under $a \mapsto ia$. See Ref. [31] for more properties.

properties that are usually taken for granted. In particular, the linearized flow of the effective interaction is then unique and can be split uniquely into a convergent sum over eigenoperators. The point of proving this is to contrast it with solutions that lie outside of this space. As we will see, the further we move outside of this space, the less these properties can be taken for granted.

II. SL PERTURBATIONS

Let us refer to perturbations that are square integrable under the SL measure as "SL perturbations" and the space of such perturbations as the "SL space".[5] Mathematically this space is a Hilbert space (which should not, however, be confused with state space in quantum mechanics). The eigenoperator solutions in the SL space are the Hermite polynomials (1.8). In this space they are orthonormal

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\, \mathrm{e}^{-a^2\tilde{\varphi}^2} \mathcal{O}_n(\tilde{\varphi})\mathcal{O}_m(\tilde{\varphi}) = \frac{1}{a}\left(\frac{1}{2a^2}\right)^n n!\sqrt{\pi}\delta_{nm}, \tag{2.1}$$

and complete. To see what this implies, start with some initial perturbation $V = V(\varphi, \Lambda_0)$ at an initial scale $\Lambda = \Lambda_0$. If $V(\varphi, \Lambda_0)$ grows slower than

$$\frac{1}{\sqrt{|\varphi|}} \exp\left(\frac{a^2\varphi^2}{2\Lambda_0^2}\right) \tag{2.2}$$

as $\varphi \to \pm\infty$, then it is inside the SL space. Then completeness means we are guaranteed a convergent expansion over the operators $\mathcal{O}_n$ in the square-integrable sense (which means it converges in the usual point-wise sense almost everywhere). Explicitly, if we define the coefficients

$$\tilde{g}_n(\Lambda_0) = \frac{a}{\sqrt{\pi}} \frac{(2a^2)^n}{n!} \int_{-\infty}^{\infty} d\tilde{\varphi}\, \mathrm{e}^{-a^2\tilde{\varphi}^2} \mathcal{O}_n(\tilde{\varphi})\tilde{V}(\tilde{\varphi}, \Lambda_0), \tag{2.3}$$

then we are guaranteed that the norm-squared of the remainder

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\, \mathrm{e}^{-a^2\tilde{\varphi}^2} \left(\tilde{V}(\tilde{\varphi}, \Lambda_0) - \sum_{n=0}^{N} \tilde{g}_n(\Lambda_0)\mathcal{O}_n(\tilde{\varphi})\right)^2 \to 0 \quad \text{as } N \to \infty \tag{2.4}$$

vanishes as we send $N \to \infty$. Taking the limit, we have a well-defined expansion

$$\tilde{V}(\tilde{\varphi}, \Lambda_0) = \sum_{n=0}^{\infty} \tilde{g}_n(\Lambda_0)\mathcal{O}_n(\tilde{\varphi}). \tag{2.5}$$

Notice that these coefficients $\tilde{g}_n(\Lambda_0)$ are defined by (2.3) and are not yet expressed in terms of dimensionful couplings. However, if we now define dimensionful couplings g_n by writing

$$\tilde{g}_n(\Lambda) = g_n\Lambda^{n-4} \tag{2.6}$$

(in particular at $\Lambda = \Lambda_0$), then the result builds in the separation of the variables solution (1.4). Thus the expansion

$$\tilde{V}(\tilde{\varphi}, \Lambda) = \sum_{n=0}^{\infty} \tilde{g}_n(\Lambda)\mathcal{O}_n(\tilde{\varphi}) \tag{2.7}$$

provides the solution to the flow equation for the given initial condition $V = V(\varphi, \Lambda_0)$. Furthermore by orthonormality (2.1), these $\tilde{g}_n(\Lambda)$ are also given by

$$\tilde{g}_n(\Lambda) = \frac{a}{\sqrt{\pi}} \frac{(2a^2)^n}{n!} \int_{-\infty}^{\infty} d\tilde{\varphi}\, \mathrm{e}^{-a^2\tilde{\varphi}^2} \mathcal{O}_n(\tilde{\varphi})\tilde{V}(\tilde{\varphi}, \Lambda) \tag{2.8}$$

at these lower scales. Finally (2.7) still converges at these lower scales. To prove this, we use the orthonormality relation (2.1) together with (2.6) to compute the norm-squared

$$\int_{-\infty}^{\infty} d\tilde{\varphi}\, \mathrm{e}^{-a^2\tilde{\varphi}^2} \tilde{V}^2(\tilde{\varphi}, \Lambda) = \frac{\sqrt{\pi}}{\Lambda^8 a} \sum_{n=0}^{\infty} n! g_n^2 \left(\frac{\Lambda^2}{2a^2}\right)^n. \tag{2.9}$$

Since the right hand side (RHS) converges for $\Lambda = \Lambda_0$, the radius of convergence is greater than or equal to Λ_0, and therefore it continues to converge for all $0 < \Lambda \leq \Lambda_0$. Therefore, we have proven that $\tilde{V}(\tilde{\varphi}, \Lambda)$ remains in the SL space as Λ is lowered. This in turn implies that the equivalent bound to (2.2) remains satisfied at lower scales, i.e., for all $\Lambda \leq \Lambda_0$ we have proven that $V(\varphi, \Lambda)$ grows slower than

$$\frac{1}{\sqrt{|\varphi|}} \exp\left(\frac{a^2\varphi^2}{2\Lambda^2}\right) \tag{2.10}$$

as $\varphi \to \pm\infty$.

Writing the expansion over eigenoperators (2.7) instead in dimensionful terms

$$\begin{aligned} V(\varphi,\Lambda) &= \sum_{n=0}^{\infty} \Lambda^n g_n \mathcal{O}_n(\varphi/\Lambda) \\ &= \sum_{n=0}^{\infty} g_n(\varphi^n - n(n-1)\Lambda^2\varphi^{n-2}/4a^2 + \cdots), \end{aligned} \tag{2.11}$$

the expansion converges for all $0 \leq \Lambda \leq \Lambda_0$, i.e., also in the physical limit $\Lambda \to 0$. Notice that this means that the g_n

[5] It has been studied within model approximations (which, however, are exact at the linearized level) in Ref. [23], seen also [21,22], and exactly in Ref. [31]. In the following, we also use insights from the behavior for the negative kinetic term (see footnote 4). However, we keep the exposition self contained.

are just the Taylor expansion coefficients of the physical potential $V(\varphi, 0)$ (at the linearized level):

$$V(\varphi, 0) = \sum_{n=0}^{\infty} g_n \varphi^n. \tag{2.12}$$

In fact, $V(\varphi, 0)$ is an entire function, the RHS converging for all real φ. This follows from (2.9) since convergence of its RHS requires that the g_n vanish faster than $1/\sqrt{n!}$ for large n.

Substituting (2.8) into (2.7) for the case $\tilde{V}(\tilde{\varphi}, \Lambda_0) = \tilde{V}_0(\tilde{\varphi})$, we have

$$V(\varphi, \Lambda) = \int_{-\infty}^{\infty} d\varphi_0 \, G_{\Lambda,\Lambda_0}(\varphi - \varphi_0) V_0(\varphi_0), \tag{2.13}$$

where the Green's function G is given by its spectral expansion:

$$G_{\Lambda,\Lambda_0}(\varphi - \varphi_0) = \frac{a}{\Lambda_0 \sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2a^2\Lambda}{\Lambda_0}\right)^n \mathcal{O}_n\left(\frac{\varphi}{\Lambda}\right) \mathcal{O}_n\left(\frac{\varphi_0}{\Lambda_0}\right). \tag{2.14}$$

We can get it in closed form by recognizing that the flow equation (1.2) is the heat diffusion equation in disguise. Indeed, introducing the "time"

$$T = {\Lambda_0}^2 - \Lambda^2, \tag{2.15}$$

we get precisely the heat diffusion equation for diffusion coefficient $1/4a^2$,

$$\frac{\partial}{\partial T} V(\varphi, T) = \frac{1}{4a^2} V''(\varphi, T), \tag{2.16}$$

and thus from the well-known form of its Green's function (e.g., see [48]) we find

$$G_{\Lambda,\Lambda_0}(\varphi - \varphi_0) = \frac{a}{\sqrt{\pi(\Lambda_0^2 - \Lambda^2)}} \exp\left(-\frac{a^2(\varphi - \varphi_0)^2}{\Lambda_0^2 - \Lambda^2}\right), \quad (\Lambda < \Lambda_0). \tag{2.17}$$

Note that as a function of φ and Λ, $G_{\Lambda,\Lambda_0}(\varphi - \varphi_0)$ satisfies the flow equation (1.2) for $\Lambda < \Lambda_0$, as it must by (2.13), but which can also be checked explicitly. In fact, it evidently satisfies the flow equation for all $\Lambda \neq \Lambda_0$. However, since $G_{\Lambda,\Lambda_0}(\varphi - \varphi_0) \to \delta(\varphi - \varphi_0)$ as $\Lambda \to \Lambda_0$, and is pure imaginary for $\Lambda_0 > \Lambda$, this representation only makes physical sense for $\Lambda \leq \Lambda_0$, reflecting the fact that (2.16) is parabolic, so that the Cauchy initial value problem is only well defined in the positive T direction, i.e., for RG flows in the IR direction.

In fact, while the linearized RG flows can exist all the way to $\Lambda \to \infty$ [finite sums over the polynomials (2.7) such that $g_n = 0$ for $n > n_{\max}$ are examples], typically they extend only up to a finite range, failing at some higher critical scale. An example is provided by starting at $\Lambda = \Lambda_0$ with the bare potential

$$V_0(\varphi) = A_1 \exp\left(-\frac{\varphi^2}{\mu_1^2}\right) + A_2 \exp\left(-\frac{\varphi^2}{\mu_2^2}\right), \tag{2.18}$$

where the A_i are constants of dimension four and we set $0 < \mu_1 < \mu_2$. It is easy to see from (2.17) that this is $\sqrt{\pi}\mu_1 A_1 G_{\Lambda_0,\Lambda_1}(\varphi) + \sqrt{\pi}\mu_2 A_2 G_{\Lambda_0,\Lambda_2}(\varphi)$, where the $\Lambda_i = \sqrt{\Lambda_0^2 + a^2\mu_i^2}$. Therefore, the solution to the flow equation is

$$V(\varphi, \Lambda) = \sqrt{\pi}\mu_1 A_1 G_{\Lambda,\Lambda_1}(\varphi) + \sqrt{\pi}\mu_2 A_2 G_{\Lambda,\Lambda_2}(\varphi) \tag{2.19}$$

for all scales $\Lambda < \Lambda_1$. However, as Λ approaches Λ_1 from below,

$$V(\varphi, \Lambda) \to \sqrt{\pi}\mu_1 A \delta(\varphi) + \sqrt{\pi}\mu_2 A_2 G_{\Lambda,\Lambda_2}(\varphi), \tag{2.20}$$

after which the flow ceases to exist. If we persist in trying to use it for $\Lambda > \Lambda_1$, we find a potential that is now complex.

This is also a generic feature. To see this (for simplicity), assume a potential that is square integrable [such as is the case for (2.18)]. Then from the flow equation (1.2), we see that

$$\Lambda \frac{\partial}{\partial \Lambda} \int_{-\infty}^{\infty} d\varphi \, V^2(\varphi, \Lambda) = \frac{\Lambda^2}{a^2} \int_{-\infty}^{\infty} d\varphi \, \{V'(\varphi, \Lambda)\}^2, \tag{2.21}$$

by integration of parts. Since the right hand side is positive, the integral over V^2 can only increase as Λ increases, in turn increasing the right hand side even more. The integrals diverge at the singularity. The only way the integral over V^2 can be finite once Λ is above this is if the right hand side then contributes an infinitely negative amount. But since the right hand side is the integral of a square, this can only happen if V is no longer real.

III. ALTERNATIVE EXPANSIONS

We recall briefly the physical importance of being able to split the linearized flow uniquely into irrelevant, marginal, and relevant parts, as implied by (2.7). It leads to universality of the continuum limit since the latter is parameterized only by the few marginal and relevant operators [25] (e.g., see [35]). However, once we violate the bound (2.10) (and are thus outside the SL space), it is no longer possible to split the linearized flow uniquely into relevant and irrelevant parts, at least in a way that holds for all points on the flow. A perturbation can start outside the SL space, where it can be expanded in HH interactions

(1.7) and then, at lower scales, enter the SL space where instead it is expanded in the polynomial eigenoperators. These two expansions can disagree about what parts of the flow are relevant.

For example, consider the Green's function $G_{\Lambda,\Lambda_0}(\varphi)$ but where we replace Λ_0 with $i\mu$. Since $G_{\Lambda,\Lambda_0}(\varphi)$ solves the flow equation when $\Lambda \neq \Lambda_0$, it also solves it for any $\mu > 0$. Discarding $\sqrt{\pi}$ and reintroducing the dimension four constant, A, we thus have the solution

$$V(\varphi,\Lambda) = \frac{\mu a A}{\sqrt{\mu^2+\Lambda^2}} \exp\left(\frac{a^2\varphi^2}{\mu^2+\Lambda^2}\right). \tag{3.1}$$

Notice that while $\Lambda > \mu$, this solution violates the bound (2.10). Now we Taylor expand it in μ, yielding odd powers of μ, and thus by dimensions,

$$\tilde{V}(\tilde{\varphi},\Lambda) = aA\sum_{n=0}^{\infty} \frac{\mu^{2n+1}}{\Lambda^{2n+5}} \tilde{v}_n(\tilde{\varphi}). \tag{3.2}$$

Since the RHS solves the flow equation for all μ, it must be that the $\Lambda^{-2n-5}\tilde{v}_n(\tilde{\varphi})$ separately solve the flow equation. We recognize (from both the eigenvalue and the asymptotic behavior) that these are proportional to the HH interactions defined in (1.10), and therefore

$$V(\varphi,\Lambda) = A\sum_{n=0}^{\infty} c_n (a\mu/\Lambda)^{2n+1} \mathcal{O}^{2n}(\varphi/\Lambda), \tag{3.3}$$

where the c_n are numbers. In complex μ space, (3.1) is analytic except at $\mu = \pm i\Lambda$. Thus the series above converges for all φ and for all $\Lambda > \mu$.

On the other hand, when $\Lambda < \mu$, we see that the growth for large φ lies inside the bound (2.10), and thus that the solution lies inside the SL space. Therefore, here we can write the solution (3.1) uniquely as a series expansion over the polynomial eigenoperators $\mathcal{O}_n(\tilde{\varphi})$. Recalling (2.12), we can read off the conjugate couplings from the Taylor expansion of $V(\varphi, 0)$. Thus using (2.11), we have the expansion

$$V(\varphi,\Lambda) = aA\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a\Lambda}{\mu}\right)^{2n} \mathcal{O}_{2n}\left(\frac{\varphi}{\Lambda}\right), \tag{3.4}$$

which converges for all φ and for all $\Lambda < \mu$. [The expansion can also be derived directly from (2.14) on using properties of Hermite polynomials.]

We thus have two equivalent descriptions of the same flow, but with apparently contradictory RG behavior. In the former case (3.3), one would deduce that the flow involves only relevant couplings, since in the expansion all the eigenoperators are relevant. In the latter case (3.4), however, all the eigenoperators are irrelevant apart from the first three.

IV. SINGULAR FLOWS

We have already seen at the end of Sec. II that flows upwards can end in a singularity. This is not unexpected; there is no reason *a priori* why a freely chosen effective action $V(\varphi,\Lambda_0)$ should be the result of integrating out modes starting with some potential $V(\varphi,\Lambda_1)$ at a higher scale $\Lambda_1 > \Lambda_0$. However, once we are even further outside the SL space, it is also the case that linearized flows towards the IR may end in a singularity. This can happen if $V(\varphi,\Lambda)$ grows faster than the square of the bound (2.10) (as we explain in the next section). The solution

$$V(\varphi,\Lambda) = \frac{\mu a A}{\sqrt{\Lambda^2-\mu^2}} \exp\left(\frac{a^2\varphi^2}{\Lambda^2-\mu^2}\right), \quad (\Lambda > \mu) \tag{4.1}$$

will illustrate this behavior. It is just the previous solution (3.1) after replacing μ with $i\mu$, and dividing by i. Comparing to the bound (2.10), we see that (4.1) indeed grows faster than its square (for all $\Lambda > \mu$). Mapping from (3.3), we see that it has a convergent HH expansion in this domain:

$$V(\varphi,\Lambda) = \sum_{n=0}^{\infty} (-)^n c_n (a\mu/\Lambda)^{2n+1} \mathcal{O}^{2n}(\varphi/\Lambda) \tag{4.2}$$

(the c_n being the same numbers as before). However, we see from (4.1) that it ends in a singularity as $\Lambda \to \mu$, where it diverges for all values of φ. To be clear, we emphasize that since it diverges everywhere, there is no sense in which it can still be regarded as a solution once we reach $\Lambda = \mu$. If we nevertheless cavalierly attempt to continue the solution below this point, the solution becomes pure imaginary.

Notice that (4.1) is proportional to the Green's function $G_{\Lambda,\mu}(\varphi)$ continued above its domain of validity. A generic solution illustrating these properties would follow from the Green's function representation (2.13) if we choose $V_0(\varphi_0)$ to be integrable, imaginary, and of compact support. Then

$$V(\varphi,\Lambda) = \int_{-\infty}^{\infty} d\varphi_0\, G_{\Lambda,\mu}(\varphi-\varphi_0) V_0(\varphi_0) \tag{4.3}$$

is a solution that is real and well behaved for $\Lambda > \mu$, but which diverges everywhere as $\Lambda \to \mu$ from above.

V. NONUNIQUE FLOWS

In the Wilsonian RG literature, it is taken for granted that the solution to the flow equation is unique once the initial effective action is specified as $V(\varphi,\Lambda_0) = V_0(\varphi)$. In fact, this property is true for solutions only if they grow sufficiently slowly for large φ.

Of course by construction the solution expanded over the eigenoperators $\mathcal{O}_n(\tilde{\varphi})$ (2.7), (2.11), or written as the convolution (2.13), is unique. While the former makes sense only if the initial perturbation $V_0(\varphi)$ grows slower than the bound (2.2), the latter converges for a larger space of initial perturbations $V_0(\varphi)$. From the explicit form of the Green's function (2.17), the convolution form of the solution converges for all $\Lambda \leq \Lambda_0$, provided $V_0(\varphi)$ is integrable and grows slower than the square of the bound (2.2).

Note that these properties are consistent with our previous examples. Example (3.1) lies outside the bound (2.10) for $\Lambda > \mu$ and therefore does not have an expansion over the $\mathcal{O}_n$. However, it lies inside the square and the bound. Therefore, it has a convergent Green's function representation which gives a nonsingular flow to the IR. On the other hand, for any $\Lambda > \mu$, the example (4.1) violates both the bound (2.10) and its square. Therefore, it can neither be expanded over the $\mathcal{O}_n$ nor does it have a convergent Green's function representation. This is reflected in the fact that actually the solution becomes singular as $\Lambda \to \mu$ from above.

Even though the Green's function construction (when convergent) yields a unique solution, this does not mean it is the only solution. However, it is the only solution if we restrict the solution space to $V(\varphi, \Lambda)$ that grow slower than some exponential of φ^2, i.e., to solutions that, for all $0 \leq \Lambda \leq \Lambda_0$, grow slower than

$$\exp(B\varphi^2), \tag{5.1}$$

for some fixed (sufficiently large) positive constant B. The proof follows from the equivalence (2.15), (2.16) to the heat equation, since uniqueness of such bounded solutions is proven for the latter. E.g., see Theorem 7 of Sec. 2.3 in Ref. [49] where the proof is the result of applying the maximum principle together with some careful limits.

Now, we show that solutions are no longer unique if they are allowed to grow faster than any such exponential (5.1). In this case one can have two solutions, $V_1(\varphi, \Lambda)$ and $V_2(\varphi, \Lambda)$, to the flow equation (1.2) which agree for all scales $\Lambda \geq \mu$, but disagree once $\Lambda < \mu$. Since the flow equation is linear, this is equivalent to the statement that their difference, $V(\varphi, \Lambda) = V_1(\varphi, \Lambda) - V_2(\varphi, \Lambda)$, is a nontrivial solution that nevertheless vanishes identically at all scales $\Lambda \geq \mu$. Using the equivalence (2.15), (2.16) and following Tychonoff [50], we show that

$$V(\varphi, \Lambda) = v \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left(\frac{2a\varphi}{\mu}\right)^{2k} g^{(k)} \left(\frac{\mu^2 - \Lambda^2}{\mu^2}\right) \tag{5.2}$$

is such a solution (v being a proportionality constant of dimension four). It is constructed from the kth derivatives of the function

$$\begin{aligned} g(t) &= e^{-t^{-\alpha}} \quad \text{for } t > 0 \\ &= 0 \quad \text{for } t \leq 0 \end{aligned} \tag{5.3}$$

where one must choose the parameter $\alpha > 1$. Note that $g(t)$ and all its derivatives are continuous at $t = 0$, so in the series (5.2), each term vanishes smoothly as $\Lambda \to \mu$ from below and of course vanishes identically for all $\Lambda \geq \mu$. It is straightforward to verify by direct substitution that (5.2) is indeed a (formal) solution of the flow equation (1.2). It would only be a formal solution, however, unless one can show that the series (5.2) converges. In fact, the series is absolutely convergent, cf., Appendix [50,51]. From that analysis, one also sees that V stays within the envelopes

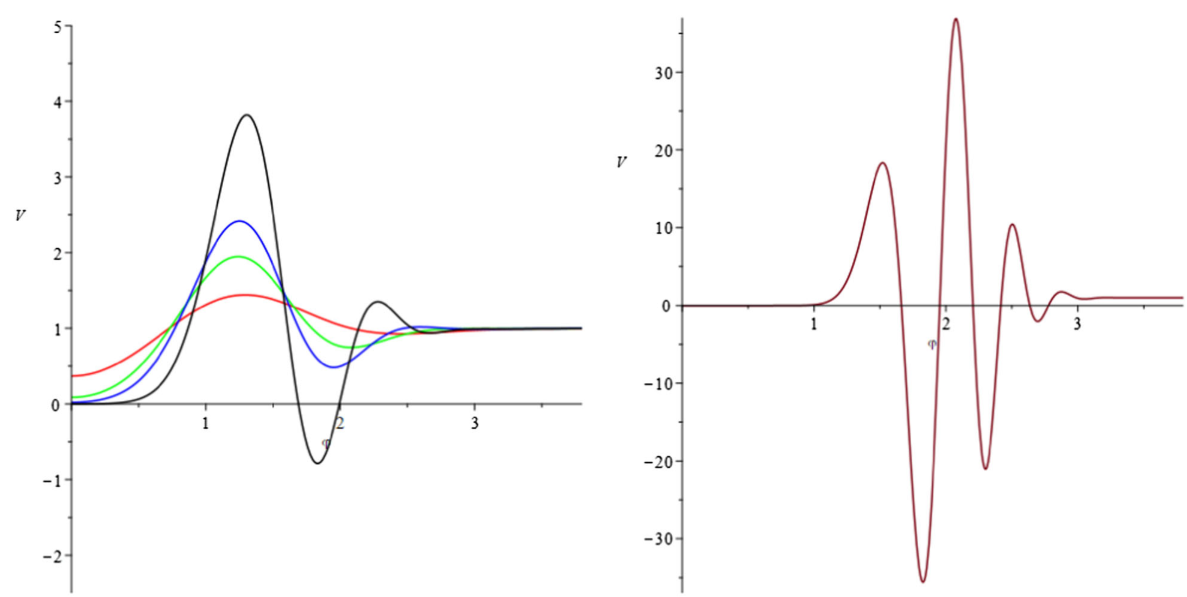

FIG. 1. The $\alpha = 2$ Tychonoff solution in units of v and μ/a. On the left, it is plotted for $\Lambda = 0$ (red), 0.6 μ (green), 0.7 μ (blue), and 0.8 μ (black), and on the right it is plotted for $\Lambda = 0.9$ μ.

$$|V(\varphi,\Lambda)| \le |v| \exp\left[\frac{4a^2\varphi^2}{(\mu^2-\Lambda^2)r} - F(r)\left(\frac{\mu^2}{\mu^2-\Lambda^2}\right)^{\alpha}\right] \quad \text{for} \quad \Lambda < \mu. \tag{5.4}$$

These envelopes are parameterized by an $0 < r < 1$ which additionally must be chosen so that

$$\mathrm{Re}(1 + re^{i\theta})^{-\alpha} \tag{5.5}$$

is bounded below by a positive number $F(r)$ for all $0 \le \theta < 2\pi$ (see Appendix). Note that these envelopes all exceed the bound (5.1) for Λ sufficiently close to μ (whatever we choose for B) but, since $\alpha > 1$, for any fixed φ they vanish as $\mu \to \Lambda$. This demonstrates that $V(\varphi,\Lambda)$ also vanishes as $\mu \to \Lambda$ (in fact, uniformly for bounded complex φ).

As Λ is lowered through the critical point $\Lambda = \mu$, the Tychonoff solution (5.2) takes the form of a divergent wave-packet that comes in from infinite φ (see Fig. 1). Lowering Λ still further, the solution becomes less oscillatory. At values of φ (much less than the wave-packet position), we still have $V \approx 0$, while for values φ, much larger than the wave-packet position, V rapidly tends to v.

[The fact that the series in (5.2) tends to 1 as $\varphi \to \infty$ seems far from obvious, but can be convincingly demonstrated numerically. Individual terms in (5.2) become very large but cancel each other to a high degree. For example, for the point $\varphi = 3.8\ \mu/a$ in the right hand plot in Fig. 1, individual terms grow to $10^{45}v$. To get accurate results required high digits accuracy and many terms, e.g., 90 digits and close to 300 terms for the right hand plot. Numerically, we established that $V \to v$ for large φ by working to even greater accuracy. For example, for $\Lambda = 0.6\ \mu$ we followed the solution out to $\varphi = 16$ and established that there, $V = 0.9999v$ to four decimal places. However, this required working to 244 digits accuracy and summing 1230 terms.]

VI. CONCLUSIONS

We summarize our main findings as follows. Nonsingular solutions $V(\varphi,\Lambda)$ to the linearized flow equation (1.2), that grow slower than (2.10),

$$\frac{1}{\sqrt{|\varphi|}} \exp\left(\frac{a^2\varphi^2}{2\Lambda^2}\right), \tag{6.1}$$

are square-integrable under the Sturm-Liouville measure. They can be expanded over polynomial eigenoperators, the Hermite polynomials (1.8), with the series converging in the square integrable sense. The flow towards the IR is unique and nonsingular. Generically, however, flows towards the ultraviolet fail at a singularity, after which the solution no longer exists, at least as a real solution. We proved this in Sec. II.

The uniqueness of the expansion over eigenoperators is an important property since it allows for the universality of the continuum limit, this being parameterized by the marginal/relevant couplings that can be uniquely identified in this expansion. However, if solutions grow at large φ in a way such as to exceed the above bound, they need no longer have a unique expansion over eigenoperators. In Sec. III, we demonstrated this by the solution (3.1). It has a convergent expansion over the HH eigenoperators (1.10) for $\Lambda > \mu$, such that all are relevant, but has a convergent expansion over the polynomial eigenoperators (1.8) when $\Lambda < \mu$, such that all but three of them are irrelevant.

If the solution grows faster than the square of the above bound, the flow towards the IR can end in a singularity and thus lead to flows that cannot be completed (i.e., such that there is an obstruction to integrating out all the modes). We saw this in Sec. IV, where we also saw that a convergent expansion over HH eigenoperators can lead to such singularities and thus incomplete flows. In Sec. V, we related this to the fact that solutions that grow faster than the square of the above no longer have a convergent Green's function representation.

Finally, in Sec. V we saw that if we allow growth faster than any exponential $e^{B\varphi^2}$ (with fixed constant B), then solutions are no longer uniquely determined by the initial "bare" potential $V(\varphi,\Lambda_0)$. New interactions can spontaneously appear at lower scales through "Tychonoff" wave-packets that travel in from $\varphi = \infty$.

It is tempting to try and find a physical rôle for such an effect, just as it was for the HH eigenoperators [1–20] (see our discussion in Sec. I), and indeed it is tempting to search for physical meaning in the other challenging effects we have just summarized. However, our objections [21–23] to HH interactions apply equally well to all these effects. In particular, if we use the flow equation for the Legendre effective action with IR cutoff [28–30]

$$\frac{\partial}{\partial\Lambda}\Gamma[\varphi] = \frac{1}{2}\mathrm{tr}\left[\mathcal{R} + \frac{\delta^2\Gamma}{\delta\varphi\delta\varphi}\right]^{-1}\frac{\partial\mathcal{R}}{\partial\Lambda}, \tag{6.2}$$

then for any interaction that exceeds (6.1), the right hand side is forced to vanish at large φ, no matter how small we make the interaction at any finite φ [23]. It follows that working with the linearized flow equation (1.2) is not justified at large φ. Instead, at sufficiently large φ, the flow equation collapses to $\partial_\Lambda\Gamma[\varphi] = 0$, i.e., mean-field evolution takes over such that, in fact, the action is frozen out and independent of Λ. For interactions that grow at large field like $e^{B\varphi^2}$, for some B, since they are frozen out, we find that at scales $\Lambda < a/\sqrt{2B}$ they will again be inside the bound (6.1) and thus have "fallen" back into the SL space where they can be expanded over the polynomial eigenoperators, as in (2.11) [21–23]. Interactions that grow faster than $e^{B\varphi^2}$ for any B[6], do not fall back into the SL space, but nevertheless

[6]For example $V \sim e^{C\varphi^4}$.

their exact evolution is very different from that described by the linearized flow equation (1.2).

We finish with more general remarks. Firstly, one has to bear in mind that a linearized perturbation of the fixed point action might be redundant [52,53]. Being reparameterizations, such perturbations about the Gaussian fixed point would have to contain $\Box\varphi$ as a factor. Since our perturbations involve only an effective potential, we know that none of them are redundant.

Secondly, beyond the linearized level, we need to worry about stability. Interactions that are unbounded from below are clearly dangerous (at the nonperturbative level). In this paper, we have for simplicity focused on even interactions. These are stable provided we choose the sign so that they are positive for large field.

Thirdly, we have concentrated exclusively on Wilsonian RG linearized around the Gaussian fixed point. This allows us to be completely rigorous. But it is natural to expect this at the qualitative level our conclusions hold more generally.

In our earlier work, we used the Local Potential Approximation [34,54] and its generalization to derivative expansion approximations [22,24,35,55] to analyze nonperturbative fixed points and their perturbations. Although such approximation schemes are uncontrolled, in practice they yield reasonably accurate results [35–44]. Around such nonperturbative fixed points Γ_*, the second derivative term in the linearized flow equation, that is the analogue of the right hand side in (1.2), now has a coefficient that depends on the fixed point action itself (through the $\delta^2_\varphi\Gamma_*$ terms above). Solving for the corresponding Sturm-Liouville measure, we found that SL perturbations (ones that are square-integrable under the SL measure) grow slower than

$$\tilde{\varphi}^q \exp(c\tilde{\varphi}^p) \tag{6.3}$$

at large field (where $c > 0$, $p > 0$, and q are nonuniversal) [21,22]. Eigenoperator solutions divide into two sets: one with quantized scaling dimensions whose large field dependence grows like a power of the field and which span this SL space, and one with nonquantized scaling dimensions whose dependence at large field grows like the square of (6.3). As recalled above, we already noted that the use of linearized equations for these latter perturbations is actually not justified.

From this, one can already begin to see how the results will generalize. The rôle of bound (6.1) is now played by (6.3). Within the derivative expansion, the large field behavior of the flow equation linearized around Γ_* should be amenable to mathematical analysis [49]. We expect that solutions that exceed the bound (6.3) no longer have a unique expansion over eigenoperators, that solutions that grow faster than the square of the bound can have flow to the IR that ends prematurely in a singularity, and finally, if solutions are allowed that grow faster than any exponential $\mathrm{e}^{B\varphi^p}$ (with fixed constant B), then Tychonoff-like wave-packets can spontaneously appear at lower scales.

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APPENDIX: CONVERGENCE AND ENVELOPES

Here we derive the envelope formula (5.4) and prove absolute convergence of the series (5.2). Our derivation closely follows the exposition of Tychonoff's proof as given in Chapt. 7 of [51].

First we note that the series (5.2) is absolutely convergent if the sum

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!}\left|\left(\frac{2a\varphi}{\mu}\right)^{2k} g^{(k)}\left(\frac{\mu^2-\Lambda^2}{\mu^2}\right)\right| \tag{A1}$$

converges, i.e., where all terms are taken positive. To prove that the above sum converges, we use Cauchy's representation of derivatives of analytic functions

$$g^{(k)}(t) = \frac{k!}{2\pi i}\oint dz \frac{\mathrm{e}^{-z^{-\alpha}}}{(z-t)^{k+1}}, \tag{A2}$$

where we take real $t > 0$, putting us in the regime $\Lambda < \mu$. Choosing the contour to be the circle $z = t(1 + r\mathrm{e}^{i\theta})$, where $0 \leq \theta < 2\pi$ and r is fixed in the range $0 < r < 1$, we have

$$\mathrm{Re}(-z^{-\alpha}) = -t^{-\alpha}\mathrm{Re}(1 + r\mathrm{e}^{i\theta})^{-\alpha}. \tag{A3}$$

Now if r is small enough, the last factor, which is (5.5), is bounded below by a positive constant, which we call $F(r)$. This can be determined by minimizing over θ. For example, if $\alpha = 2$, we find that we must have $r < \frac{1}{\sqrt{2}}$. Then we find that $F(r) = \frac{1}{2}(1-2r^2)/(1-r^2)^2$ for $r > \frac{1}{2}$, while for $0 < r < \frac{1}{2}$ we have $F(r) = 1/(1+r)^2$. Thus from (A2) we have that

$$|g^{(k)}(t)| \leq \frac{k!}{(rt)^k}\mathrm{e}^{-F(r)t^{-\alpha}}. \tag{A4}$$

Finally, since $k!/(2k)! < 1/k!$, we see that the kth term in (A1) is bounded above by

$$\frac{1}{k!}\left(\frac{4a^2\varphi^2}{(\mu^2-\Lambda^2)r}\right)^k \mathrm{e}^{-F(r)t^{-\alpha}}. \tag{A5}$$

Since the sum of these terms converges, it follows that the sum (A1) converges, and thus that the sum in (5.2) is absolutely convergent. In fact, the above is just a term-wise expansion of the exponential in (5.4) (up to the factor $|v|$). Since its sum is larger than (A1), which in turn is larger in magnitude than the sum in (5.2), we have also proven that $|V(\varphi,\Lambda)|$ is bounded by the envelopes (5.4).

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