One-loop tunneling-induced energetics

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Tunneling between degenerate vacua is allowed in finite-volume quantum field theory, and it features remarkable energetic properties, which result from the competition of different dominant configurations in the partition function. We derive the one-loop effective potential based on two homogeneous vacua of the bare theory, and we discuss the resulting null energy condition violation in $O(4)$-symmetric Euclidean spacetime, as a result of a nonextensive effective action.

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I. INTRODUCTION

Tunneling is an important aspect of quantum mechanics, which is suppressed by a large number of degrees of freedom in quantum field theory (QFT), where spontaneous symmetry breaking (SSB) plays a fundamental role instead. Strictly speaking though, SSB requires an infinite volume to occur, and tunneling happens in any finite volume, although it can take a huge amount of time to settle in a macroscopic system, in which case SSB is a much better approximation.

In path integral quantization, the equilibrium state arising from tunneling can be described when taking into account several saddle points, which leads to remarkable energetic features, as a consequence of symmetry restoration and its relation to convexity [1]. One consequence of the competition between different saddle points is a nonextensive effective action [2], which leads to a violation of the null energy condition (NEC) [3]. NEC violation is known in QFT [4], as in the Casimir effect for example, which has been used in a cosmological context to describe the possibility of dynamically generating a spacetime expansion [5]. The nonextensive nature of the effective action has been used in [6] to provide a dynamical mechanism for a cosmological bounce [7], without the need for exotic matter or modified gravity. Assuming a contraction of the Universe, the NEC violation mechanism described in [6] switches on when the causal volume has shrunk enough for tunneling to become significant, consequently leading to a spacetime expansion. Tunneling is then suppressed when the causal volume becomes large enough, and SSB takes over for the following steps of the cosmological evolution.

In the present article we extend the flat-spacetime study [2] to one loop, and we show that quantum corrections do not change the qualitative predictions of the "tree-level semiclassical approximation" (ignoring fluctuation factors), regarding the dynamical generation of a nonextensive one-particle-irreducible (1PI) effective action. Since this study is done at finite volume, it does not involve any thermodynamical limit, and the famous Maxwell cut which features a flat effective potential cannot be obtained. It is interesting to note that the Maxwell cut appears in the Wilsonian effective potential [8] independently of the volume though; however, the Wilsonian effective potential is equivalent to the 1PI effective potential for large volume only, and we focus here on the latter potential.

We comment here on a possible ambiguity between real time and Euclidean time dependence. We are looking at the equilibrium effective action, obtained, in principle, for a large real time. On the other hand, a finite Euclidean time represents the inverse temperature of the equilibrium finite-temperature system, and it is independent of the typical spacial length involved in the system. Tunneling at finite temperature is studied in [9], where the Euclidean time can be large, therefore allowing (Euclidean) time-dependent saddle points to develop, while the space volume is kept finite in order to keep a significant tunneling rate. The present work assumes an $O(4)$-symmetric Euclidean spacetime instead, where the length in the "time" direction corresponds to the typical time needed for quantum fluctuations to travel through the three-dimensional box in which the scalar field lives, and for which the equilibrium is assumed to be reached.

Related to the finite-temperature analogy, we stress that the symmetry restoration mechanism we study here is not related to the Kibble-Zurek mechanism [10], which describes high-temperature symmetry restoration in a large three-dimensional volume. Tunneling-induced symmetry restoration is valid at any temperature (including zero temperature), therefore in situations where SSB would happen if the volume was infinite.

In Sec. II we discuss generic features of finite-volume effects on QFT. Although we consider degenerate vacua, the introduction of a source j to define the 1PI effective theory lifts this degeneracy, which, in principle, leads to the formation of bubbles of different vacua, as described originally in [11]. The radius of such a bubble goes to infinity for a vanishing source though, which is not consistent with a finite volume if we focus on the vicinity of the true vacuum $\phi = 0$, which is mapped to $j = 0$ through the Legendre transform. Hence bubbles cannot form and, given the $O(4)$ symmetry between Euclidean spacetime components, only homogeneous saddle points play a role in the partition function.

We calculate the one-loop 1PI effective potential in Sec. III, starting from the semiclassical approximation for the partition function, based on the two homogeneous saddle points. An important technical point consists in renormalizing the connected generating functionals for each individual saddle point first, before performing the Legendre transform. As expected, the effective potential obtained from the interplay of the two saddle points is convex. Another important feature is a nontrivial spacetime volume dependence, such that the effective action is not extensive.

Section IV describes the consequences of the above features, in particular, the NEC violation. This violation is a dynamical process, arising from quantum fluctuations, and it is a consequence of finite spacetime volume.

II. FINITE VOLUME EFFECTS

We consider a single real scalar field in a double-well potential, described by the action

$$S[\phi] = \int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{24}(\phi^2 - v^2)^2 + j\phi, \quad (1)$$

featuring two vacua which are degenerate when the source j vanishes.

A. Saddle points

Tunneling between the two vacua should be taken into account in the situation where fluctuations above these overlap [6], which, assuming an $O(4)$-symmetric Euclidean spacetime, is controlled by the dimensionless parameter

$$A \equiv \frac{\lambda L^4 v^4}{24\hbar}. \quad (2)$$

Tunneling happens for any finite volume and thus any finite parameter A, but with a probability decreasing exponentially with A. For this reason, the larger A, the longer one should wait for the equilibrium to be reached.

Besides homogeneous solutions of the equation of motion, one should, in principle, consider other saddle points which depend on the four-dimensional Euclidean radial coordinate $r = \sqrt{x^\mu x^\mu}$. We impose periodic boundary conditions in the finite four-dimensional volume where the scalar field exists, so that a shot saddle point [12] cannot be taken into account. For a nonvanishing source $j \neq 0$, one could, in principle, have bubbles of different vacua, but we explain here why we can disregard these bubbles.

The radius of the four-dimensional bubble is obtained by minimizing the bubble action, which corresponds to a compromise between the energy inside the bubble and the surface tension at the bubble wall [11]. This leads to

$$R \simeq \frac{3v^2}{2j}\sqrt{\frac{3}{\lambda}} \quad (3)$$

and should be smaller than L, which leads to the minimum value for the source

$$j \geq \frac{3v^2}{L}\sqrt{\frac{3}{\lambda}}. \quad (4)$$

Because of symmetry restoration via tunneling, we focus on the true vacuum $\phi_c = 0$, which maps to $j = 0$ through the Legendre transform used in the derivation of the 1PI effective action. Thus we restrict the present study to sources smaller than the lower bound (4), for which bubble saddle points cannot form. In what follows we therefore consider homogeneous saddle points only, which satisfy periodic boundary conditions.

The equation of motion for these saddle points is

$$\phi^3 - v^2\phi + \frac{6j}{\lambda} = 0, \quad (5)$$

with a number of real solutions depending on the source. We introduce the critical source $j_c \equiv \lambda v^3/(9\sqrt{3})$, as well as the dimensionless source $k \equiv j/j_c$:

(i) If $|k| > 1$, there is one (real) solution, which is

$$\varphi_0 = -\mathrm{sign}(k)\frac{2}{\sqrt{3}}\cosh\left(\frac{1}{3}\cosh^{-1}(|k|)\right). \quad (6)$$

This corresponds to the regime where the partition function is defined above one saddle point, and the 1PI effective potential has the known expression (39) given later in this article.

(ii) If $|k| < 1$ there are two solutions, which are

$$\phi_1(k) = \frac{2v}{\sqrt{3}}\cos(\pi/3 - (1/3)\arccos(k))$$
$$\phi_2(k) = \frac{2v}{\sqrt{3}}\cos(\pi - (1/3)\arccos(k)) = -\phi_1(-k), \tag{7}$$

This is the regime we focus on in this article. We note that, in the limit $k \to 1^-$, the semiclassical approximation for the partition function might not be accurate since fluctuations above the "false vacuum" (the one with the highest energy) are large. In this article we restrict our studies to values $k \ll 1$ though, where the semiclassical approximation is reliable.

B. Discrete versus continuous momentum components

Quantum corrections about a uniform saddle point ϕ_i are, in principle, quantized in a finite box, and with periodic boundary conditions the corresponding fluctuation determinant is

$$\frac{1}{\sqrt{\det(\delta^2 S[\phi_i])}} = \exp\left(-\frac{1}{2}\sum_{n_\mu}\ln\left(\frac{(2\pi/L)^2 n_\mu n_\mu + U''(\phi_i)}{(2\pi/L)^2 n_\mu n_\mu + \lambda v^2/3}\right)\right), \tag{8}$$

where n_μ is a set of integers, each going from $-\infty$ to ∞. In what follows we will approximate the sum over discrete wave vector components $2\pi n_\mu/L$ by an integral over continuous components. To justify this, we focus, for simplicity, on the summation over the index n_0, and we introduce the functions

$$F(\Phi) = \sum_{n_0=-\infty}^{\infty}\ln(n_0^2 + \Phi^2)$$
$$G(\Phi) = \int_{-\infty}^{\infty} dx\ \ln(x^2 + \Phi^2), \tag{9}$$

where $\Phi^2 = n_1^2 + n_2^2 + n_3^2 + (L/2\pi)^2 U''(\phi_i)$. The derivatives of these functions are easy to calculate:

$$F'(\Phi) = 2\Phi\sum_{n_0=-\infty}^{\infty}\frac{1}{n^2+\Phi^2} = 2\pi\ \coth(\pi\Phi)$$
$$G'(\Phi) = 2\Phi\int_{-\infty}^{\infty}\frac{dx}{x^2+\Phi^2} = 2\pi, \tag{10}$$

and one can see that they are approximately identical for $\Phi \gg 1/\pi$, which is indeed the case if the integers n_1, n_2, n_3 are not small. Quantum corrections are dominated by large integers n_1, n_2, n_3 though, and are therefore well approximated by continuous summation over wave vector components.

III. INTERPLAY OF SADDLE POINTS

In this section we derive the one-loop effective action based on two homogeneous saddle points, extending the work of [2] in the situation of one scalar flavor.

A. Semiclassical approximation

The semiclassical approximation for the partition function evaluated from the two uniform saddle points assumes that different saddle points are "far enough" in field configuration space for the quadratic fluctuations above these saddle points to be independent,

$$Z[k] \simeq \sum_{i=1,2}\frac{\exp(-S[\phi_i]/\hbar)}{\sqrt{\det(\delta^2 S[\phi_i])}} = \sum_{i=1,2}\exp(-\Sigma[\phi_i]/\hbar), \tag{11}$$

where the individual connected-graph generating functionals are

$$\Sigma[\phi_i] \equiv S[\phi_i] + \frac{\hbar}{2}\mathrm{Tr}\{\ln(\delta^2 S[\phi_i])\}$$
$$= S[\phi_i] + \hbar\frac{V}{2}\int\frac{d^4p}{(2\pi)^4}\ln\left(\frac{p^2 + U''(\phi_i(k))}{p^2 + U''(\phi_i(0))}\right)$$
$$= Vj\phi_i + \frac{\lambda V}{24}(\phi_i^2 - v^2)^2 + \frac{\hbar\lambda^2 v^4}{288\pi^2}V\int_0^X xdx\ \ln\left(\frac{x+a_i}{x+1}\right). \tag{12}$$

In the latter expression,

$$X = \frac{3\Lambda^2}{\lambda v^2} \quad \text{and} \quad a_i = \frac{3}{2}\frac{\phi_i^2}{v^2} - \frac{1}{2}, \tag{13}$$

and Λ is an ultraviolet cutoff.

B. Renormalized connected-graph generating functional

Instead of renormalizing the 1PI effective action Γ, we first renormalize the individual connected-graph generating functionals $\Sigma[\phi_i]$ and then perform the Legendre transform to find the renormalized effective action Γ. This procedure avoids a potential confusion arising from mixing the different loop orders from both $\Sigma[\phi_i]$. Keeping the dominant terms in Λ in the above loop integral, we obtain

$$\frac{1}{V}\Sigma[\phi_i] = j\phi_i - \frac{\lambda v^2}{12}\left(1 - \frac{\hbar}{16\pi^2}\left(3\frac{\Lambda^2}{v^2} + \frac{1}{2}\lambda\ln\left(\frac{\Lambda^2}{\lambda v^2}\right)\right)\right)\phi_i^2 + \frac{\lambda}{24}\left(1 - \frac{3\hbar\lambda}{32\pi^2}\ln\left(\frac{\Lambda^2}{\lambda v^2}\right)\right)\phi_i^4 + \frac{\hbar\lambda^2 v^4}{2304\pi^2}\left(3\frac{\phi_i^2}{v^2} - 1\right)^2\ln\left(\frac{3}{2}\frac{\phi_i^2}{v^2} - \frac{1}{2}\right), \tag{14}$$

where source-independent terms are omitted, as well as terms which vanish in the limit $\Lambda \to \infty$. We then define the renormalized parameters

$$\lambda_R \equiv \lambda - \frac{3\hbar\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\lambda v^2}\right)$$
$$v_R^2 = v^2 - \frac{\hbar v^2}{16\pi^2}\left(3\frac{\Lambda^2}{v^2} - \lambda\ln\left(\frac{\Lambda^2}{\lambda v^2}\right)\right), \tag{15}$$

and we obtain

$$\Sigma[\phi_i] = \Sigma_R[\phi_i] + \mathcal{O}(\hbar^2), \tag{16}$$

where

$$\frac{1}{V}\Sigma_R[\phi_i] = j\phi_i - \frac{\lambda_R v_R^2}{12}\phi_i^2 + \frac{\lambda_R}{24}\phi_i^4 + \frac{\hbar\lambda_R^2 v_R^4}{2304\pi^2}\left(3\frac{\phi_i^2}{v_R^2} - 1\right)^2\ln\left(\frac{3}{2}\frac{\phi_i^2}{v_R^2} - \frac{1}{2}\right). \tag{17}$$

In the latter expression, the saddle points ϕ_i are still expressed in terms of the bare parameters λ and v. But one can write, in terms of the renormalized dimensionless source $k_R = 9\sqrt{3}j/(\lambda_R v_R^3)$ and the renormalized saddle point $\phi_{iR}(k_R) = \phi_i(k) + \mathcal{O}(\hbar)$,

$$\Sigma_R[\phi_i(k)] = \Sigma_R[\phi_{iR}(k_R)] + (\phi_i(k) - \phi_{iR}(k_R))\frac{\partial\Sigma_R}{\partial\phi_i}\bigg|_{\phi_{iR}(k_R)} + \mathcal{O}(\hbar^2)$$
$$= \Sigma_R[\phi_{iR}(k_R)] + (\phi_i(k) - \phi_{iR}(k_R))\frac{\partial\Sigma}{\partial\phi_i}\bigg|_{\phi_i(k)} + \mathcal{O}(\hbar^2). \tag{18}$$

Since the equations of motion satisfied by the saddle points are $\partial\Sigma/\partial\phi_i = \mathcal{O}(\hbar)$, we finally obtain

$$\Sigma_R[\phi_i(k)] = \Sigma_R[\phi_{iR}(k_R)] + \mathcal{O}(\hbar)^2, \tag{19}$$

and the saddle points in the expression (17) can be read as functions of λ_R and v_R. We introduce the dimensionless quantities

$$\varphi_i \equiv \frac{\phi_{iR}(k_R)}{v_R} \quad \text{and} \quad A_R \equiv \frac{\lambda_R V v_R^4}{24\hbar}, \tag{20}$$

in terms of which we write the final expression for the renormalized individual connected-graph generating functionals as

$$\Sigma_R[\varphi_i] = \hbar A_R\left(\frac{8k_R}{3\sqrt{3}}\varphi_i - 2\varphi_i^2 + \varphi_i^4 + \frac{\hbar\lambda_R}{96\pi^2}(3\varphi_i^2 - 1)^2\ln\left(\frac{3}{2}\varphi_i^2 - \frac{1}{2}\right)\right), \tag{21}$$

where

$$\varphi_1(k_R) = \frac{2}{\sqrt{3}}\cos(\pi/3 - (1/3)\arccos(k_R))$$
$$\varphi_2(k_R) = \frac{2}{\sqrt{3}}\cos(\pi - (1/3)\arccos(k_R)) = -\varphi_1(-k_R). \tag{22}$$

Below we derive the effective potential obtained from the two functionals (21) in the semiclassical approximation (11).

C. One-loop 1PI effective action

Here we follow the usual steps leading to the 1PI effective action. Starting from the partition function (11), the classical field is given by

$$\frac{\phi_c}{v_R} = -\frac{3\sqrt{3}}{8A_R}\frac{\partial\ln(Z)}{\partial k_R}, \tag{23}$$

and an expansion in the source k_R gives

$$\frac{\phi_c}{v_R} = f_1 k_R + f_3 k_R^3 + \mathcal{O}(k_R^5), \tag{24}$$

where, at one loop,

$$f_1 = -\frac{1 + 8A_R}{3\sqrt{3}} + \frac{\hbar\lambda_R}{192\sqrt{3}\pi^2}(7 + 16A_R)$$
$$f_3 = \frac{4}{243\sqrt{3}}(128A_R^3 + 12A_R - 3) - \frac{\hbar\lambda_R}{15552\sqrt{3}\pi^2}(2048A_R^3 + 384A_R - 93). \tag{25}$$

In order to perform the Legendre transform, we invert the latter relation as

$$k_R = -\frac{3\sqrt{3}}{8}g_2\left(\frac{\phi_c}{v_R}\right) - \frac{\sqrt{3}}{16}g_4\left(\frac{\phi_c}{v_R}\right)^3 + \mathcal{O}(\phi_c/v_R)^5, \tag{26}$$

where the one-loop coefficients are

$$g_2 = \frac{8}{1+8A_R} + \frac{\hbar\lambda_R}{8\pi^2}\frac{16A_R+7}{(1+8A_R)^2}$$
$$g_4 = \frac{64(128A_R^3+12A_R-3)}{(1+8A_R)^4} + \frac{\hbar\lambda_R}{4\pi^2}\frac{16384A_R^4+12288A_R^3+936A_R-243}{(1+8A_R)^5}. \quad (27)$$

The renormalized effective action for a constant classical field $\Gamma = VU_{\mathrm{eff}}$ is obtained by integrating the equation

$$\frac{dU_{\mathrm{eff}}}{d\phi_c} = -j \to \frac{d\Gamma}{d\phi_c} = -\frac{8\hbar A_R}{3\sqrt{3}}\frac{k_R}{v_R}, \quad (28)$$

such that, finally,

$$\Gamma[\phi_c] = \hbar A_R\left(g_0 + \frac{g_2}{2}\left(\frac{\phi_c}{v_R}\right)^2 + \frac{g_4}{24}\left(\frac{\phi_c}{v_R}\right)^4 + \mathcal{O}(\phi_c/v_R)^6\right), \quad (29)$$

where g_0 is a constant. We note that the limit $\hbar \to 0$ leads to the result derived in [2], where the effective action is derived in the tree-level semiclassical approximation, i.e., with fluctuation factors which are ignored. The one-loop effective action satisfies the same fundamental features as the one obtained in [2]:

(i) Consistently with general arguments, the action (29) is convex since $g_2 > 0$ and $g_4 > 0$ for large A_R.

(ii) The action (29) is not extensive, as a result of the nontrivial volume dependence of the renormalized constants g_2, g_4, through the parameter A_R.

D. Resummation for infinitesimal source

It is interesting to note that, in the limit of an infinitesimal source, one can obtain an analytical expression for the one-loop effective action. If we consider terms linear in k_R only in the renormalized connected-graph generating functional, we obtain

$$\Sigma_R[\varphi_1(k_R)] = \frac{8\hbar A_R}{3\sqrt{3}\eta}k_R = -\Sigma_R[\varphi_2(k_R)], \quad (30)$$

where source-independent terms are dropped, and

$$\frac{1}{\eta} \equiv 1 - \frac{\hbar\lambda_R}{64\pi^2}. \quad (31)$$

The steps described above can be applied to Eq. (30) without the need for an expansion in k_R, and they lead to the classical field

$$\eta\frac{\phi_c}{v_R} = -\tanh\left(\frac{8A_R}{3\sqrt{3}\eta}k_R\right). \quad (32)$$

This relation is inverted as

$$\frac{8A_R}{3\sqrt{3}}k_R = \frac{\eta}{2}\ln\left(\frac{1-\eta\phi_c/v_R}{1+\eta\phi_c/v_R}\right), \quad (33)$$

and it leads to the one-loop effective action

$$\tilde{\Gamma}[\phi_c] = \hbar A_R g_0 + \frac{\hbar}{2}\left(1-\eta\frac{\phi_c}{v_R}\right)\ln\left(1-\eta\frac{\phi_c}{v_R}\right) + \frac{\hbar}{2}\left(1+\eta\frac{\phi_c}{v_R}\right)\ln\left(1+\eta\frac{\phi_c}{v_R}\right), \quad (34)$$

where the integration constant is chosen to match the effective action (29). This result was derived in [13] in the tree-level semiclassical approximation, and the one-loop result (34) simply consists in the replacement $v \to v_R/\eta$, which leaves the functional form of the effective action unchanged. The expression (34) apparently makes sense for $|\phi_c| \le v_R/\eta$ (although it is not differentiable at $|\phi_c| = v_R/\eta$), but because of the one-to-one mapping between ϕ_c and $k_R \ll 1$, the effective action (34) is actually valid for $|\phi_c| \ll v_R/\eta$ only.

As a side comment, we show here that the effective action (34) corresponds to a resummation of all the orders in ϕ_c, in the limit of large (but finite) volume $A_R \gg 1$. To see this, let us express the functional Σ_R in terms of the original variables

$$\Sigma_R = V\left(\epsilon j v_R + a\frac{j^2}{v_R^2} + \cdots\right), \quad (35)$$

where $\epsilon = \pm 1$ (depending on which vacuum one focuses on), a is a dimensionless constant independent of V, and dots represent higher orders in j, which are also independent of V. Since we are interested in the vicinity of the vacuum $\phi_c = 0$, it is enough to choose a source which satisfies $|Vjv_R| \ll 1$. In this situation, the quadratic term in the action Σ_R is of the order

$$\frac{j^2}{v_R^2}V = |Vjv_R|\frac{|j|}{v_R^3} \ll |Vjv_R|\frac{1}{Vv_R^4}, \quad (36)$$

which is therefore negligible compared to the linear term Vjv_R for large volume V. As a consequence, the limit $A_R \gg 1$ of the effective action (29) should be identical to the Taylor expansion of the resummation (34). One can check that this is indeed the case since

$$\Gamma[\phi_c] = \hbar A_R g_0 + \frac{\hbar}{2}\left(1+\frac{\hbar\lambda_R}{32\pi^2}+\mathcal{O}(A_R^{-1})\right)\left(\frac{\phi_c}{v_R}\right)^2 + \frac{\hbar}{24}\left(2+\frac{\hbar\lambda_R}{8\pi^2}+\mathcal{O}(A_R^{-1})\right)\left(\frac{\phi_c}{v_R}\right)^4 + \mathcal{O}(\phi_c/v_R)^6$$
$$= \hbar A_R g_0 + \frac{\hbar}{2}(1+\mathcal{O}(A_R^{-1}))\left(\eta\frac{\phi_c}{v_R}\right)^2 + \frac{\hbar}{12}(1+\mathcal{O}(A_R^{-1}))\left(\eta\frac{\phi_c}{v_R}\right)^4 + \mathcal{O}(\phi_c/v_R)^6, \quad (37)$$

where higher orders in $\hbar$ are neglected. On the other hand, we have

$$\tilde{\Gamma}[\phi_c] = \hbar A_R g_0 + \frac{\hbar}{2}\left(\eta\frac{\phi_c}{v_R}\right)^2 + \frac{\hbar}{12}\left(\eta\frac{\phi_c}{v_R}\right)^4 + \mathcal{O}(\phi_c/v_R)^6, \quad (38)$$

such that $\Gamma[\phi_c]$ and $\tilde{\Gamma}[\phi_c]$ are identical up to terms of order A_R^{-1}, and the convergence as A_R increases can be seen on Fig. 1. We also note that this identification is valid to all orders in ϕ_c, although we show here the property up to the order 4 only. We stress here that deriving the effective potential for higher powers of the classical field requires higher orders in the source and thus should take into account bubble saddle points, which is not done here. So the expression (34) is actually valid for $|\phi_c| \ll v_R$, although a similar resummation might exist in the presence of bubbles, which is a topic to be explored in a future work.

IV. ENERGETICS

In this section we discuss the energetic properties arising from the nonextensive feature of the effective action (29).

A. Matching with the single-saddle-point regime

The effective potential for $|k_R| > 1$ is based on a single saddle point and has the usual expression in terms of the renormalized parameters λ_R, v_R,

$$U_{\text{eff}}^{(|k_R|>1)}(\phi_c) = \frac{\lambda_R}{24}(\phi_c^2 - v_R^2)^2 + \frac{\hbar\lambda_R^2}{2304\pi^2}(3\phi_c^2 - v_R^2)^2 \ln\left(\frac{3}{2}\frac{\phi_c^2}{v_R^2} - \frac{1}{2}\right), \quad (39)$$

where the origin of energies is chosen such that $U_{\text{eff}}^{(|k_R|>1)}(v_R) = 0$.

The constant g_0 in Eq. (29) is obtained by imposing continuity of the effective potential at $|k_R| = 1$, which corresponds to the boundary between the regime with one saddle point and the regime with two homogeneous saddle points. Taking the limit of Eq. (23) for $|k_R| \to 1$ yields the corresponding classical field

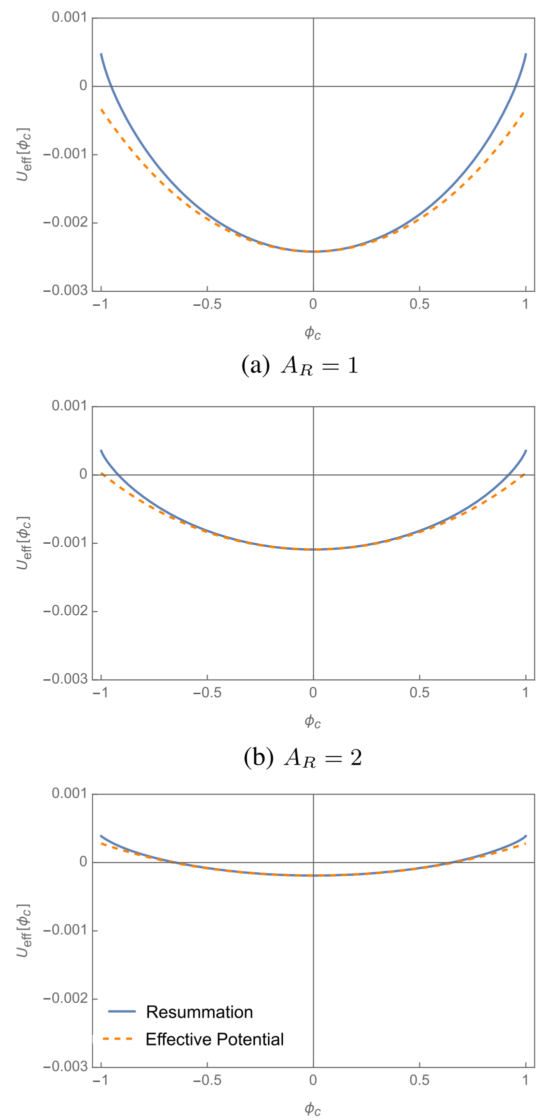
$$\phi_c(|k_R| \to 1) = \pm\frac{v_R}{\sqrt{3}}\frac{1-2e^{3A_R}}{1+e^{3A_R}}, \quad (40)$$

which, for large A_R, takes the value $\phi_c(|k_R| \to 1) \simeq \pm 2v_R/\sqrt{3}$. The requirement

$$U_{\text{eff}}^{(|k_R|>1)}(2v_R/\sqrt{3}) = U_{\text{eff}}^{(|k_R|<1)}(2v_R/\sqrt{3}) \quad (41)$$

then leads to

$$g_0 \simeq \frac{1}{9} - \frac{2}{3}g_2 - \frac{2}{27}g_4 + \frac{3\hbar\lambda_R}{32\pi^2}\ln\left(\frac{3}{2}\right), \quad (42)$$



(a) $A_R = 1$

(b) $A_R = 2$

(c) $A_R = 5$

FIG. 1. Plots of the effective potential and the resummation for $\lambda = 0.1$, $\hbar = v_R = 1$, and a range of values for A_R.

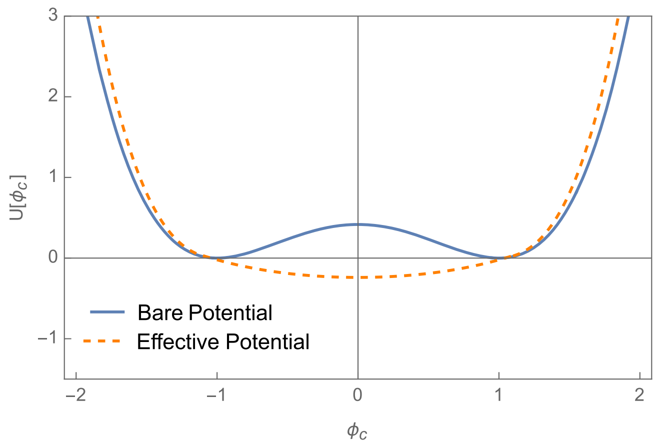


FIG. 2. Plots of the bare and effective potentials for $\hbar = A_R = v_R = 1$ and $\lambda_R = 10$. This choice of λ_R corresponds to a strong coupling regime, but it is chosen for demonstrative purposes.

and this expression is used below to compare the effective potential with the bare potential (see Fig. 2), and with the resummation (34) for different values of A_R (see Fig. 1).

B. Null energy condition

The fact that all known matter satisfies the NEC is an important conjecture since it appears as one of the assumptions for the derivation of singularity theorems [14]. For a homogeneous fluid with density ρ and pressure p, the NEC reads $\rho + p \geq 0$, and the fluid we consider here is the ground state $\phi_c = 0$. Although a full study of the mechanism presented here should be extended to curved spacetime, we can already see by the result (29) how tunneling can lead to a dynamical NEC violation, as a consequence of the nonextensive nature of the effective theory [6]. Following the thermodynamical approach, we have

$$\begin{aligned} \rho + p &= \frac{\Gamma[0]}{V} - \frac{\partial \Gamma[0]}{\partial V} \\ &= -\frac{\lambda_R v_R^4}{24} V \frac{\partial g_0}{\partial V} \\ &= -\frac{A_R \lambda_R v_R^4}{81(1+8A_R)^5}\left(P_1(A_R) + \frac{\hbar\lambda_R}{\pi^2}\frac{P_2(A_R)}{1+8A_R}\right), \end{aligned} \tag{43}$$

with

$$\begin{aligned} P_1(x) &= 16(5632x^3 + 1344x^2 + 504x - 99) \\ P_2(x) &= 26624x^4 + 28928x^3 + 3744x^2 + 2556x - 639, \end{aligned} \tag{44}$$

and one can see that $\rho + p < 0$. Also, this NEC violation is a finite volume effect since for $A_R \gg 1$ we have

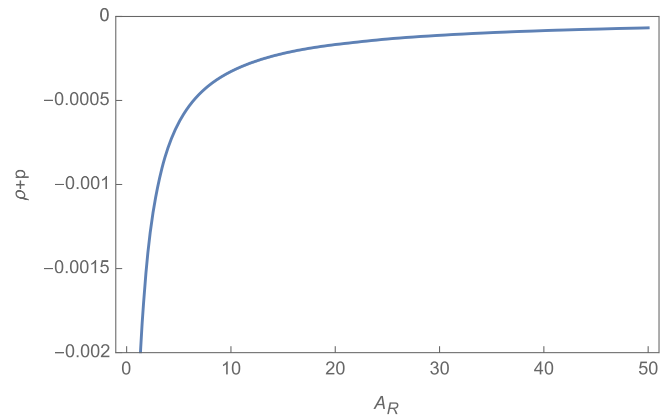


FIG. 3. Plot of $\rho + p$ as a function of A_R for $\lambda = 0.1$ and $\hbar = v_R = 1$. The NEC is violated for a finite spacetime volume V, and it is recovered asymptotically when $V \to \infty$.

$$\begin{aligned} \rho + p &\simeq -\frac{\lambda_R v_R^4}{324 A_R}\left(11 + \frac{13\hbar\lambda_R}{32\pi^2}\right) \\ &= -\frac{22\hbar}{27V} + \mathcal{O}(\hbar^2). \end{aligned} \tag{45}$$

The result (43) is sketched as a function of A_R in Fig. 3, and we finish this section with two important comments:

(i) From the expressions (43) and (45), one can see that the NEC is violated for all finite values of A_R, although we find numerically that the constant g_0 becomes positive for $A_R \gtrsim 7$ when $\lambda = 0.1$ and $\hbar = v_R = 1$. NEC violation is therefore independent of the origin of energies, which is expected in flat spacetime.

(ii) Although the NEC is violated, the averaged NEC (ANEC) is not. The latter is a weaker condition than the former, and requires instead

$$\int d\lambda(\rho + p) \geq 0, \tag{46}$$

where the integral runs along a null geodesic. As explained in [15] with specific examples related to the Casimir effect or a more generic confining potential, the environmental energy needed to maintain the scalar field confined compensates the negative value of $\rho + p$ inside the box, leading to the ANEC being satisfied.

V. CONCLUSION

We generalized the results of [2,6] by taking into account one-loop corrections in the semiclassical approximation for the partition function. A consistent renormalization of the model requires that we redefine parameters before implementing the Legendre transform, which is specific to the presence of several saddle points. Our results show that the present NEC violation mechanism is stable under quantum fluctuations, and it is a fundamental feature due to the structure of the partition function, independently of the accuracy of the latter.

The next steps we plan to make include the following:

(i) Large volume L^4: Although tunneling is suppressed, if one waits a long enough (real) time for the equilibrium to be reached, the present mechanism should hold. In this case one should take into account additional saddle points for a nonvanishing source though, in the form of bubbles with different vacua [11]. Among the next studies is to evaluate the contribution of the NEC violation effect described here to dark energy.

(ii) Curved spacetime: The $O(4)$ symmetry between spacetime coordinates is not valid if one focuses on a Friedman-Lemaitre-Robertson-Walker metric, for example, and the finite-temperature study [9] needs to be extended beyond flat spacetime. An appropriate causal quantization volume should be defined, as well as a comparison between the tunneling rate and the spacetime expansion rate. One should also include the nonextensive feature of the effective action in the energy-momentum tensor of the fluid to be coupled to (classical) gravity.

(iii) Real-time tunneling: Ideally, these studies should be done in a Minkowski (or Lorentzian) metric. Tunneling in real time is more involved though (see [16] for a review), but it would allow us to go beyond equilibrium field theory, which could be game-changing in the study of the early Universe.

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