Repulsive to attractive fluctuation-induced forces in disordered Landau-Ginzburg model

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Critical fluctuations of some order parameters describing a fluid generate long-range forces between boundaries. Here, we discuss fluctuation-induced forces associated with a disordered Landau-Ginzburg model defined in a d-dimensional slab geometry $\mathbb{R}^{d-1} \times [0, L]$. In the model the strength of the disordered field is defined by a nonthermal control parameter. We study a nearly critical scenario, using the distributional zeta-function method, where the quenched free energy is written as a series of moments of the partition function. In the Gaussian approximation, we show that for each moment of the partition function, and for some specific strength of the disorder, the nonthermal fluctuations, associated with an order parameterlike quantity, become long ranged. We demonstrate that the sign of the fluctuation-induced force between boundaries depends in a nontrivial way on the strength of the aforementioned nonthermal control parameter.

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I. INTRODUCTION

Fluctuation-induced forces are a strikingly universal phenomena since macroscopic boundaries that change the spectrum of a fluctuating medium may present such type of associated forces. Examples of this phenomena are the Casimir forces generated by quantum fluctuations [1–6]. Situations such as a bounded medium-experiencing thermal fluctuations near to a critical regime with long-range correlations, or Goldstone modes of a broken continuous symmetry, may lead to the appearance of fluctuation-induced forces [7–13].

Critical regimes are also achieved in fluids and magnetic systems with quenched disordered fields [14,15]. Considering this scenario, and inspired by the critical Casimir effect, in this work we study the associated induced force that appears in a system described by the disordered Landau-Ginzburg model defined in a d-dimensional slab geometry, which is driven to the criticality by nonthermal fluctuations. In a confined system approaching a second-order phase transition, when the length scale of the fluctuations are very large, an influence from the boundaries may appear. The fluctuation spectrum associated with the order parameterlike quantities becomes highly sensitive to the geometry of the boundaries. The terminology for order parameterlike quantities will be discussed later.

In order to deal with critical regimes driven by quenched disorder fields, we employ the distributional zeta-function method. This methods leads to a representation of the quenched free energy where the main contribution is given by a series. Each term of this series is a moment of the partition function with its own ground state. Therefore the multivalley free energy landscape of some disordered systems can be easily obtained [16–23].

Our purpose is to discuss the sign of the force between the boundaries, for the case of Dirichlet boundary conditions in the nearly critical scenario. To proceed, in each moment of the partition function we compute the saddle-point contribution and discuss Gaussian fluctuations around such saddle points. Next, we deal with the series of the eigenvalues of Laplace operators. Using generalized zeta-functions, and an analytic regularization procedure, we develop a global approach following Ref. [24]. This procedure shows that there are specific moments of the partition function which are contributing to the force between the boundaries, induced by geometric restrictions, i.e., the constraints in the fluctuation spectra of each specific moment. Although this global approach does not show the connection between the structure of the divergences and the geometry of the boundaries, its simplicity reveals the relation between the intensity of the effect and the correlation lengths of the fluctuations associated with the order parameterlike quantities in some moments of the partition function. Also, it shows the link among the dimension of the space, the boundary conditions, and the structure of the divergences in the associated spectral zeta-functions.

Repulsive and attractive critical Casimir forces depending on the boundary conditions were discussed in Ref. [25]. In Ref. [26], for systems described by an $O(N)\varphi^4$ model in a d-dimensional film geomery, it was proved that there is a

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crossover from attractive to repulsive induced forces, as a function of the distance between the boundaries. For quantum fields, a similar result can be found in Ref. [27], which discussed the sign of the Casimir force between two plates, a perfectly conducting one and an infinite permeable plate. In Ref. [28] it was proved that a repulsive Casimir force appears when the boundaries are dielectric materials with nontrivial magnetic susceptibility. Finally, Refs. [29,30] discussed the dependence of the sign of the Casimir energy on the dimensionality of space, type of boundary condition, and other variables. Our main result is a connection between the sign of the fluctuation-induced force and the strength of the nonthermal control parameter. This result, which shows the behavior of the sign change of the fluctuation-induced forces (i.e., the variation between their attractive and repulsive nature) and its explicit dependence on the strength of the disorder, as far as we know, is new in the literature.

Note that although we are in the statistical field theory framework, we are not using an ultraviolet cutoff in the model. Using the argument of universality in the critical behavior, where the results of macroscopic measurements must be independent of the cutoff parameter, we can remove a natural physical cutoff and use an analytic regularization procedure to obtain finite results. Although these two methods, the cutoff method and analytic regularization procedure, are quite different, it is possible to compare them and prove the analytic equivalence between them in some specific situations [31–34]. One comment is in order. To implement the renormalization program in systems where translational invariance is broken, it is a requirement to introduce counterterms which are surface interactions [35–39]. Since in this work we adopt a global approach, we are not introducing these boundary contributions in the model.

This paper is organized as follows. In section II we discuss the Landau-Ginzburg model defined in the continuum, in the presence of a quenched disorder, and the distributional zeta-function method. In section III, in this scenario of confined fluctuations near the critical regime, the spectral zeta-function method and an analytic regularization procedure are discussed. Conclusions are given in section IV. Henceforth we work with units such that $\hbar = c = k_B = 1$.

II. LANDAU-GINZBURG MODEL WITH DISORDERED FIELDS

We discuss a confined random field fluid system assuming a Landau-Ginzburg model with Z_2 symmetry in a d-dimensional slab geometry $\mathbb{R}^{d-1} \times [0, L]$. The quenched disorder field is linearly coupled with a scalar field. The cases of the Dirichlet, Neumann Laplacian, and periodic boundary conditions are discussed. The case of periodic boundary conditions for Bose fields is closely related to a finite temperature field theory [40]. In the statistical field theory scenario, the action functional $S(\varphi)$ for the one component scalar field is given by

$$S = \int d^d\mathbf{x} \left[\frac{1}{2}\varphi(\mathbf{x})\left(-\Delta + m_0^2\right)\varphi(\mathbf{x}) + \frac{\lambda_0}{4!}\varphi^4(\mathbf{x})\right]. \quad (1)$$

The symbol Δ denotes the Laplacian in $\mathbb{R}^d$, and λ_0 and m_0^2 are respectively the coupling constant and a parameter that give the distance of the model from the critical point. We call it the square mass of the model. Note that we are using the action $S(\varphi) = \beta H(\varphi)$, where $H(\varphi)$ is the Hamiltonian of the model. The action is the energy measured in units of temperature. The generating functional of correlation functions for one disorder realization in the presence of an external source $j(\mathbf{x})$ is defined as

$$Z(j, h) = \int [d\varphi] \exp\left(-S(\varphi, h) + \int d^d x j(\mathbf{x})\varphi(\mathbf{x})\right), \quad (2)$$

where $[d\varphi]$ is a formal Lebesgue measure, given by $[d\varphi] = \prod_{\mathbf{x}} d\varphi(\mathbf{x})$ and the action functional in the presence of the disorder is

$$S(\varphi, h) = S(\varphi) + \int d^d x h(\mathbf{x})\varphi(\mathbf{x}) \quad (3)$$

for $h(\mathbf{x}) \in L^2(\mathbb{R}^n)$. In the above equation, $S(\varphi)$ is the pure Landau-Ginzburg action functional and $h(\mathbf{x})$ is a quenched random field. This is the simplest scalar model with a disorder field linearly coupled with the scalar field of the theory. We would like to point out that one can discuss also the quenched random mass model given by

$$S(\varphi, \eta) = S(\varphi) + \frac{\rho}{4}\int d^d x \eta(\mathbf{x})\varphi^2(\mathbf{x}). \quad (4)$$

This model is known as the random-temperature disorder, where a small density of impurities lead to randomness in the local transition temperature. In this work we will discuss only the quenched random field model. Measured in units of temperature, the disordered free energy for one disorder realization is $W(j, h) = -\ln Z(j, h)$. Performing the average over the ensemble of all realizations of the disorder we have

$$\mathbb{E}[W(j, h)] = \int [dh] P(h) \ln Z(j, h), \quad (5)$$

where $[dh] = \prod_x dh(x)$ is a functional measure. The probability distribution of the disorder is written as $[dh]P(h)$, being

$$P(h) = p_0 \exp\left(-\frac{1}{2\sigma^2}\int d^d x (h(\mathbf{x}))^2\right). \quad (6)$$

The quantity σ is a positive parameter associated with the disorder and p_0 is a normalization constant. This defines a delta correlated process as

$$\mathbb{E}[h(\mathbf{x})h(\mathbf{y})] = \sigma^2\delta^d(\mathbf{x}-\mathbf{y}). \tag{7}$$

For a given probability distribution of the disorder, one is mainly interested in obtaining the average free energy. For a general disorder probability distribution, using the disordered functional integral $Z(j,h)$ given by Eq. (2), the distributional zeta-function, $\Phi(s,j)$, is defined as

$$\Phi(s,j) = \int [dh]P(h)\frac{1}{Z(j,h)^s}, \tag{8}$$

for $s \in \mathbb{C}$, with this function being defined in the region where the above integral converges. As was proved in [16,17], $\Phi(s)$ function is defined for $\mathrm{Re}(s) \geq 0$. Therefore the integral is defined in the half-complex plane, and an analytic continuation is not necessary. The average generating functional can be written as

$$\mathbb{E}[W(j,h)] = -(d/ds)\Phi(s,j)|_{s=0^+}, \qquad \Re(s) \geq 0, \tag{9}$$

where one defines the complex exponential $n^{-s} = \exp(-s\log n)$ with $\log n \in \mathbb{R}$. Using analytic tools, again in units of temperature, the quenched free energy of a system in the presence of an external field is $F_q(j) = -\mathbb{E}[W(j,h)]$, where

$$\mathbb{E}[W(j,h)] = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}a^k}{kk!}\mathbb{E}[(Z(j,h))^k] - \ln(a) - \gamma + R(a,j). \tag{10}$$

The quantity a is a dimensionless arbitrary constant, γ is the Euler-Mascheroni constant, and $R(a)$ is given by

$$R(a,j) = -\int [dh]P(h)\int_a^{\infty}\frac{dt}{t}\exp(-Z(j,h)t). \tag{11}$$

Integrating over the disorder, each moment of the partition function can be written as

$$\mathbb{E}[(Z(j,h))^k] = \int\prod_{i=1}^{k}[d\varphi_i]\exp(-S_{\mathrm{eff}}(\varphi_i,j_i)), \tag{12}$$

where the effective action $S_{\mathrm{eff}}(\varphi_i)$ describes a k-field component field theory.

$$S_{\mathrm{eff}}(\varphi_i^{(k)},j_i^{(k)}) = \int d^dx\Bigg[\sum_{i=1}^{k}\left(\frac{1}{2}\varphi_i^{(k)}(\mathbf{x})(-\Delta+m_0^2)\varphi_i^{(k)}(\mathbf{x}) + \frac{\lambda_0}{4!}(\varphi_i^{(k)}(\mathbf{x}))^4\right) - \frac{\sigma^2}{2}\sum_{i,j=1}^{k}\varphi_i^{(k)}(\mathbf{x})\varphi_j^{(k)}(\mathbf{x}) - \sum_{i=1}^{k}\varphi_i^{(k)}(\mathbf{x})j_i^{(k)}(\mathbf{x})\Bigg]. \tag{13}$$

In order to avoid unnecessary complications, and for practical purposes, we assume the following configuration of the scalar fields $\varphi_i^{(k)}(x)=\varphi_j^{(k)}(\mathbf{x})=\varphi^{(k)}(\mathbf{x})$ in the function space and also $j_i^{(k)}(\mathbf{x}) = j_l^{(k)}(\mathbf{x}) = j(\mathbf{x})$. Although it can be shown that under the assumption of different fields, the theory is invariant under the permutation group of (k) elements, and the generic scale invariance is obtained (for free theories and their generalizations). The aforementioned configuration has proved its validity when the multivalley free energy landscape in systems with multiplicative quenched disorder is obtained (see i.e., Ref [23]). Therefore, all the terms of the series have the same structure. The next step will be to assume the Gaussian approximation, expand each functional integral around the minimum up to the lowest-order quadratic term, and integrate out the fluctuations. Assuming that in each moment of the partition function the fields are equal, we have that the kth moment of the partition function is written as

$$\mathbb{E}[(Z(j,h))^k] = \left[\int [d\varphi]\exp\left(-S^{(k)}(\varphi^{(k)},j)\right)\right]^k. \tag{14}$$

In this case, the new effective action is written as

$$S^{(k)}(\varphi^{(k)},j) = \int d^dx\left(\frac{1}{2}\varphi^{(k)}(\mathbf{x})\left(-\Delta+m_0^2-k\sigma^2\right)\varphi^{(k)}(\mathbf{x}) + \frac{\lambda_0}{4!}(\varphi^{(k)}(\mathbf{x}))^4 - \varphi^{(k)}(x)j(\mathbf{x})\right). \tag{15}$$

In what follows we define each contribution $W^{(k)}(j)$ as

$$W^{(k)}(j) \equiv c_k\mathbb{E}[(Z(j,h))^k] \tag{16}$$

being $c_k(a) = (-1)^{k+1}a^k/kk!$. For simplicity we shall adopt the convention $c_k(a) = c_k$.

In this situation, we have three different contributions of the terms of the series depending on the sign of $m_0^2 - k\sigma^2$ for $m_0^2 > \sigma^2$: **(i)** the case where $m_0^2 - k\sigma^2 > 0$, **(ii)** the case where $m_0^2 \cong k\sigma^2$ is a situation similar to a second-order phase transition and defines k_c, and **(iii)** the case where this quantity is negative and one has to shift the field to a new minimum, i.e., $\phi^{(k)}(\mathbf{x}) = \varphi^{(k)}(\mathbf{x}) \pm \vartheta^{(k)}$, with $\vartheta^{(k)} = (6(k\sigma^2 - m_0^2)/\lambda_0)^{1/2}$. This case is similar to the spontaneous symmetry breaking in statistical field theory and the behavior of each term of the series is described by the $(+)$ or $(-)$ cases. From now on, we choose the minus sign above. In this case, we find a positive squared mass with self-interaction terms $(\phi^{(k)}(\mathbf{x}))^3$ and $(\phi^{(k)}(\mathbf{x}))^4$.

Here, we consider the three level approximation. In each moment of the partition function order parameterlike quantities are defined, i.e., $\varphi_0^{(k)}(\mathbf{x})$. In this way, for any moment of the partition function, this three level contribution is

$$-\Delta\varphi_0^{(k)}(\mathbf{x}) + (m_0^2 - k\sigma^2)\varphi_0^{(k)}(\mathbf{x}) + \frac{\lambda_0}{3!}(\varphi_0^{(k)}(\mathbf{x}))^3 = j(\mathbf{x}). \tag{17}$$

The Fourier transform of the susceptibility-like quantity is

$$\chi^{(k)}(\mathbf{q}) = \frac{1}{\mathbf{q}^2 + m_0^2 - k\sigma^2 + \frac{1}{2}\lambda_0(\varphi_0^{(k)})^2}. \tag{18}$$

When we have the terms in the series where $k < k_c$, we obtain that $\varphi_0^{(k)}(\mathbf{x}) = 0$ and

$$\chi^{(k)}(\mathbf{q}) = \frac{1}{\mathbf{q}^2 + m_0^2 - k\sigma^2}. \tag{19}$$

The correlation length for $k < k_c$ is therefore

$$\xi_<^{(k)}(\sigma, m_0) = (m_0^2 - k\sigma^2)^{-\frac{1}{2}}. \tag{20}$$

The Fourier transform of the susceptibility-like quantity is

$$\chi^{(k)}(\mathbf{q}) = \frac{1}{\mathbf{q}^2 + 2(k\sigma^2 - m_0^2)}. \tag{21}$$

Then the correlation length, when $k > k_c$, reads as

$$\xi_>^{(k)} = (2(k\sigma^2 - m_0^2))^{-\frac{1}{2}}. \tag{22}$$

Note that we are computing the saddle-point contribution and we will take into account Gaussian fluctuations around said saddle point. Although the critical exponents using this approximation are not correct for dimensions below the critical dimension, here we are interested in computing the fluctuation-induced force between the boundaries. Radiative corrections are negligible in this scenario.

III. FLUCTUATION-INDUCED FORCE IN SYSTEMS WITH DISORDER

In this section we will discuss the nearly critical scenario of the system, in order to present the fluctuation-induced force between the boundaries. The next step is to assume the Gaussian approximation. For $k\sigma^2 > m_0^2$ we expand each functional integral around the minimum up to the lowest-order quadratic term and integrate out the fluctuations.

Starting from the elliptic operator $-\Delta + 2(k\sigma^2 - m_0^2)$ we define,

$$D(\mathbf{x}, \mathbf{y}, k) \equiv (-\Delta + 2(k\sigma^2 - m_0^2))\delta^d(\mathbf{x} - \mathbf{y}). \tag{23}$$

Within (23) we define the inverse kernel $K(\mathbf{x}, \mathbf{z}; k)$ as

$$\int d^d z K(\mathbf{x}, \mathbf{z}; k) D(\mathbf{z}, \mathbf{y}; k) = \delta^d(\mathbf{x} - \mathbf{y}). \tag{24}$$

Therefore up to the Gaussian approximation we can write

$$\mathbb{E}[W(j,h)] = \sum_{k=1}^{\infty} \frac{c_k}{(\det D(k))^{k/2}} \times \left[\exp\left(-\int d^d x \int d^d y j(\mathbf{x}) K(\mathbf{x}, \mathbf{y}; k) j(\mathbf{y})\right)\right]^k. \tag{25}$$

With the theory in finite-size geometry in one dimension, we have the spatial coordinate $x_d = z$ compactified, and a slab defined as

$$\Omega = [\mathbf{x} \equiv (x_1, x_2, \ldots, x_{d-1}, z) : 0 \leq z \leq L] \subset \mathbb{R}^d.$$

To implement the renormalization procedure in Euclidean field theory, where the field depends on $d-1$ unbounded coordinates and one bounded coordinate in the interval $[0, L]$, one has to work in a mixed representation for the Schwinger functions. One can show that the amputated one-loop two-point Schwinger function can be decomposed in a translational invariant contribution and another one that breaks the translational invariance. Since we are interested in discussing a global approach using the spectral zeta-function of the Laplacian, the formalism used for periodic or antiperiodic boundary conditions is the same used for Dirichlet boundary conditions.

The Fourier transform of the susceptibility-like quantity $\chi^{(k)}((\mathbf{x} - \mathbf{y})_{||}, z, z')$ reads

$$\chi^{(k)}(\mathbf{q}_{||}, n) = \frac{1}{(\mathbf{q}_{||})^2 + (\frac{2n\pi}{L})^2 + 2(k\sigma^2 - m_0^2)}. \tag{26}$$

Each of moment of the partition function contributes to the quenched free energy by means of a functional determinant. To evaluate each of these functional determinants, the formalism of spectral zeta-function is a standard procedure [41–44]. Suppose an infinite sequence of non-zero real or complex numbers λ_n. If the sequence of numbers is zeta regularizable, we define the regularized product $\prod_n \lambda_n$. The zeta regularized product of these numbers is defined as $\exp(-\zeta'(0))$, where this generalized zeta-function is given by

$$\zeta(s) = \sum_n \lambda_n^{-s}, \qquad \Re(s) > s_0 \tag{27}$$

for $s \in \mathbb{C}$. This function is being defined in the region of the complex plane where the sum converges and

$\zeta'(0) = \frac{d}{ds}\zeta(s)|_{s\to 0^+}$, by analytic extension. In this framework, one can write

$$[\det D(k)]^{-\frac{k}{2}} = \exp\left[\frac{k}{2}(\zeta'(0,k))\right]. \tag{28}$$

Due to the fact that we are assuming flat boundaries, here we will discuss each contribution for the free energy using an analytic regularization procedure, calculating $\zeta(-\frac{1}{2},k)$ instead of $\zeta'(0,k)$ [45]. One comment is in order. There is a relationship between the Casimir energy and the one-loop effective action. These two quantities differ by a contribution proportional to the second fundamental form, which is zero for a d-dimensional slab geometry.

Let us assume a thermodynamic limit with respect to the surface area, i.e., $L_1, L_2, \ldots, L_{d-1} \gg L_d$, and $2(k\sigma^2 - m_0^2) > 0$. To proceed, one defines the spectral zeta-function $\zeta_d(s,k)$ as

$$\zeta_d(s,k) = \frac{1}{(2\pi)^{d-1}}\left(\prod_{i=1}^{d-1} L_i\right)\int \prod_{i=1}^{d-1} dq_i \times \sum_{n\in\mathbb{Z}}\left(q_1^2 + \cdots + q_{d-1}^2 + \left(\frac{2\pi n}{L_d}\right)^2 + 2(k\sigma^2 - m_0^2)\right)^{-s} \tag{29}$$

for $s \in \mathbb{C}$. We like to point out that in order to be rigorous we should have included the term μ^{2s+1}, where μ has dimension of mass, in the above expression, to keep dimensionality consistence. But in order to avoid unnecessary nomenclature and given that we are only looking at the situation where $s = 1/2$, we can omit this term and stick to our notation.

With the finite length $L_d = L$ and by performing the angular part of the integral over the continuous mode spectrum of the $(d-1)$ non-compact dimensions, we get that

$$\int d\Omega_{d-1} = \frac{2(\pi)^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{30}$$

Note that we are assuming $d \geq 2$. We would like to stress that one can show that the second-order phase transition in $d = 2$ is suppressed, since the two-point correlation functions belong to the space of locally integrable functions, in the sense of generalized functions, and therefore must have integrable singularities only at coinciding points. Since $G_0(\mathbf{x}-\mathbf{y}, m_0^2) = -\frac{1}{2\pi}\ln(m_0|\mathbf{x}-\mathbf{y}|)$ in $d = 2$ dimensions, the theory violates the regularity condition, that is one condition to define a field theory. Therefore the nearly critical scenario is reached when the correlation lengths of the fluctuations of the order parameterlike quantities satisfies $\xi_>^{(k)} > L$.

Let us define $\lfloor\kappa\rfloor$ as the largest integer $\leq \kappa$ for any $\kappa \in \mathbb{R}$. In other words, $\lfloor\kappa\rfloor$ is the integer r for which $r \leq \kappa < r+1$. Within this notation, we notice that we have a set of moments such that

$$\left\lfloor\frac{m_0^2}{\sigma^2}\right\rfloor \leq k \leq \left\lfloor\frac{1}{\sigma^2}\left(\frac{1}{2L^2} + m_0^2\right)\right\rfloor. \tag{31}$$

We are interested in discussing the contribution of the moments of the partition function where $k \geq \lfloor\frac{m_0^2}{\sigma^2}\rfloor$, i.e., where each of the order parameterlike quantities does not vanish.

At this point we introduce the spectral zeta-function per unit area

$$Z_d(s,k) = \frac{\zeta_d(s,k)}{A(d)(\prod_{i=1}^{d-1} L_i)} \tag{32}$$

where the factor $A(d)$ is defined as

$$A(d) = \frac{1}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}. \tag{33}$$

The expression for $Z_d(s,k)$ is written as

$$Z_d(s,k) = \left(\frac{L}{\sqrt{4\pi}}\right)^{2s}\int_0^\infty dp\, p^{d-2} \times \sum_{n\in\mathbb{Z}}\left(\pi n^2 + \frac{L^2}{4\pi}\left(p^2 + 2(k\sigma^2 - m_0^2)\right)\right)^{-s}. \tag{34}$$

To proceed, let us rewrite $Z_d(s,k)$ in a way that is suitable for our analysis. After a Mellin transform, and renaming some quantities, we can rewrite the spectral zeta-function per unit area as

$$Z_d(s,k) = \frac{B(s,d)}{2\Gamma(\frac{d-1}{2})}\frac{1}{L^{d-2s-1}}\int_0^\infty dr r^{d-2} \times \int_0^\infty dt t^{s-1}\exp\left(-(m^2(k) + r^2)t\right)\Theta(t), \tag{35}$$

with the dimensionless quantities $m^2(k) = \frac{L^2}{2\pi}(k\sigma^2 - m_0^2)$ and $r^2 = \frac{L^2}{4\pi}p^2$. Also $B(s,d)$ is defined as

$$B(s,d) = 2(\sqrt{4\pi})^{d-2s-1}\frac{\Gamma(\frac{d-1}{2})}{\Gamma(s)} \tag{36}$$

and the theta function $\Theta(v)$ defined as

$$\Theta(v) = \sum_{n\in\mathbb{Z}}\exp(-\pi n^2 v), \tag{37}$$

an example of a modular form, appears. The quantity

$$m^2(k) = \frac{L^2}{4\pi(\xi_>^{(k)})^2} \tag{38}$$

defines the finite-size scaling, i.e., close to the critical point. Finite-size effects are controlled by the ratio $L/\xi_>^{(k)}$. Splitting the integral in the t variable from Eq. (35) into two contributions and performing the integral in the r variable, we can recast the spectral zeta-function per unit area as $Z_d(s,k) = Z_d^{(1)}(s,k) + Z_d^{(2)}(s,k)$, where

$$Z_d^{(1)}(s,k) = C(s,d)\int_0^1 dt t^{s-\frac{d}{2}-\frac{1}{2}} \exp(-m^2(k)t)\Theta(t), \tag{39}$$

and

$$Z_d^{(2)}(s,k) = C(s,d)\int_1^\infty dt t^{s-\frac{d}{2}-\frac{1}{2}} \exp(-m^2(k)t)\Theta(t) \tag{40}$$

and where $C(s,d) = (1/L^{d-2s-1})B(s,d)$. Changing variables in the integral $Z_d^{(1)}(s,k)$ and using the symmetry of the theta function we have

$$Z_d^{(1)}(s,k) = C(s,d)\int_1^\infty dt t^{-s+\frac{d}{2}-1} \exp\left(-\frac{m^2(k)}{t}\right)\Theta(t). \tag{41}$$

From the definition of the psi-function $\psi(v) = \sum_{n=1}^\infty \exp(-\pi n^2 v)$, such that $\psi(v) = \frac{1}{2}(\Theta(v) - 1)$, we can rewrite $Z_d(s,k)$ as having four contributions, $I_d^{(i)}(s,k), i = 1,\ldots,4$. Therefore

$$Z_d(s,k) = C(s,d)(2I_d^{(1)}(s,k) + 2I_d^{(2)}(s,k) + I_d^{(3)}(s,k) + I_d^{(4)}(s,k)). \tag{42}$$

In order to be more explicit, let us evidence these integrals:

$$I_d^{(1)}(s,k) = \int_1^\infty dt t^{s-\frac{d}{2}-\frac{1}{2}} \exp(-m^2(k)t)\psi(t), \tag{43}$$

$$I_d^{(2)}(s,k) = \int_1^\infty dt t^{-s+\frac{d}{2}-1} \exp\left(-\frac{m^2(k)}{t}\right)\psi(t), \tag{44}$$

$$I_d^{(3)}(s,k) = \int_1^\infty dt t^{s-\frac{d}{2}-\frac{1}{2}} \exp(-m^2(k)t), \tag{45}$$

and finally

$$I_d^{(4)}(s,k) = \int_1^\infty dt t^{-s+\frac{d}{2}-1} \exp\left(-\frac{m^2(k)}{t}\right). \tag{46}$$

For the case of the Dirichlet Laplacian where only the integrals $I_d^{(1)}(s,k)$ and $I_d^{(2)}(s,k)$ appear, and using the fact that $\psi(t) = O(e^{-\pi t})$ as $t \to \infty$, the integrals $I_d^{(1)}(s,k)$ and $I_d^{(2)}(s,k)$ represent an everywhere regular function of s for $m^2(k) \in \mathbb{R}_+$. The upper bound ensures uniform convergence of the integrals on every bounded domain in $\mathbb{C}$. As in the standard quantum field theory scenario, the contribution to the average free energy from each moment of the partition function of the system can be evaluated for $s = -\frac{1}{2}$.

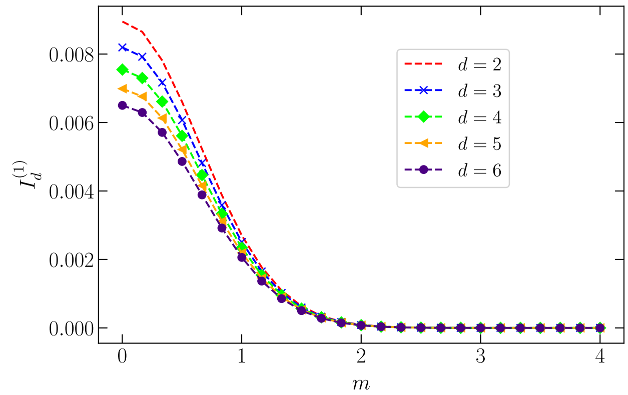

FIG. 1. Behavior of $I_d^{(1)}(s,k)$ for $s = -1/2$, for arbitrary dimensionality of the space and also dimensionless quantity $m(k) = m$.

In Fig. (1) we depict the behavior of the integral given by $I_d^{(1)}(s,k)$ for an arbitrary dimensionality of space and dimensionless quantity $m(k)$. For completeness we discuss the $d = 2$ case. We can see that the integral vanishes when the value of the dimensionless quantity $m(k)$ satisfies $m(k) > 2$. On the other hand the contribution of the integral $I_d^{(2)}(s,k)$ is depicted in Fig. (2). The contribution of the integral $I_d^{(2)}(s,k)$ for $s = -1/2$ vanishes for $m(k) > 2.5$.

For the case of the Neumann Laplacian and also the periodic boundary conditions, not only the integrals $I_d^{(1)}(s,k)$ and $I_d^{(2)}(s,k)$ but also the integrals $I_d^{(3)}(s,k)$

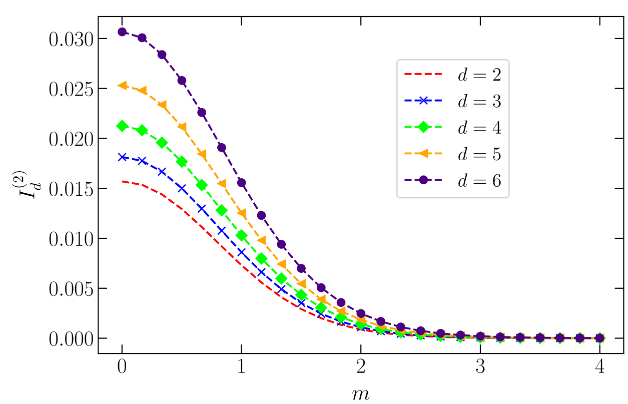

FIG. 2. Behavior of $I_d^{(2)}(s,k)$ for $s = -1/2$, for arbitrary dimensionality of the space and dimensionless quantity $m(k) = m$.

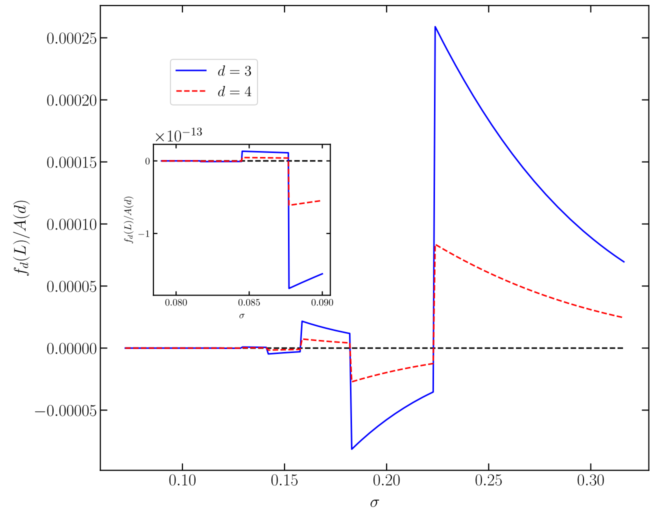

FIG. 3. Ratio $f_d(L)/A(d)$ in function of disorder strength σ for $L = 10$.

and $I_d^{(4)}(s,k)$ appear. In the absence of the exponential decay of the $\psi(v)$ function and for $m^2(k) \in \mathbb{R}^+$ we have to discuss the polar structure of the integrals $I_d^{(3)}(s,k)$ and $I_d^{(4)}(s,k)$. Note that we are assuming that $m^2(k)$ is small, but different from zero. One can show that the contribution of $I_d^{(3)}(s,k)$ is finite for odd dimensional space. Also the contribution of $I_d^{(4)}(s,k)$ is finite only for even dimensional space. Thus, it is not possible to define the Casimir-like energy per unit area associated with the Neumann Laplacian using an analytic regularization procedure in the Gaussian approximation [46]. This obstruction is related to the presence of the zero mode [47].

With this in mind, we present the main result of this paper. For Dirichlet boundary conditions we can write $F_d(L)$ as

$$F_d(L) = -\sum_{k=1}^{\infty} c_k(a) \exp\left[\frac{k}{2}\zeta_d\left(-\frac{1}{2}, k\right)\right]. \quad (47)$$

Examining the leading contribution of the series representation for the quenched free energy, where the correlation length of the fluctuations attains its maximum value, and with a suitable choice $a = \exp(|\zeta_d(-\frac{1}{2}, k_1)|)$, we can write that the force per unit area is given by

$$f_d(L) = \frac{(-1)^{k_1+1}}{2k_1!} \frac{1}{\left(\prod_{i=1}^{d-1} L_i\right)} \frac{\partial}{\partial L}\zeta_d\left(-\frac{1}{2}, k_1\right), \quad (48)$$

where $k_1 = \lfloor \frac{m_0^2}{\sigma^2} \rfloor$. Using the definition of $Z_d(s,k)$ we can rewrite (48) as follows:

$$f_d(L) = \frac{(-1)^{k_1+1}}{2k_1!} A(d) \frac{\partial}{\partial L} Z_d\left(-\frac{1}{2}, k_1\right). \quad (49)$$

In order to explore the behavior of the force (49) obtained by the distributional zeta-function method, in Fig. 3 we depict the ratio $f_d(L)/A(d)$ in function of disorder strength σ for a fixed L. A clear dependence of the fluctuation of induced forces, from attractive to repulsive, is observed with respect to the parameter σ which characterizes the strength of disorder. We can observe how the fluctuations increase their amplitude, as the system becomes more disordered. The inset of Fig. 3 is showing that the change from attractive to repulsive forces is present even with a small amount of disorder. A similar behavior is expected for other dimensions.

For completeness, in Fig. 4 we depict these sign-fluctuations of induced forces for different values of the parameter L. We can observe, as an expected behavior, the magnitude decreasing the amplitude of fluctuations, as the geometric-slab parameter is getting larger. However, we are obtaining an interesting result for how the form of the change between the repulsive to attractive nature of the induced forces is being affected by the parameter L. We are going from a type of rectangular-shape fluctuation to another smoother type.

We finally remark that this result of the fluctuation-induced forces, attractive or repulsive, depends on the strength of the disorder and is, as far as we know, new in the literature. This sign changing of the fluctuation-induced force should be testable in experiments [48–50].

IV. CONCLUSIONS

Using the distributional zeta-function method, we discussed fluctuation-induced forces associated with a disordered Landau-Ginzburg model defined in a d-dimensional slab geometry. Assuming the Gaussian approximation in each moment of the partition function, we obtain a nearly critical scenario. For some specific strength of the disorder, the fluctuations associated with an order parameterlike quantity in a specific moment of the partition function become long ranged. The induced-force per unit area in the case of the Dirichlet boundary condition depends on the contribution coming from the leading term, with the largest correlation length of the fluctuations. The sign of the induced force depends on $\lfloor \frac{m_0^2}{\sigma^2} \rfloor$ being odd or even. A similar situation is obtained for a case with a dielectric surface and a permeable one, with large dielectric constant ϵ and large permeability μ respectively. The transition between the attractive or repulsive behavior depends on the ratio $\sqrt{\frac{\mu}{\epsilon}}$. Our result that the fluctuation-induced forces, attractive or repulsive, depend on the strength of the disorder is new in literature.

To conclude we would like to point out that the quenched disorder generates fluctuations, which differ significantly from the thermal fluctuations. For pure, translational

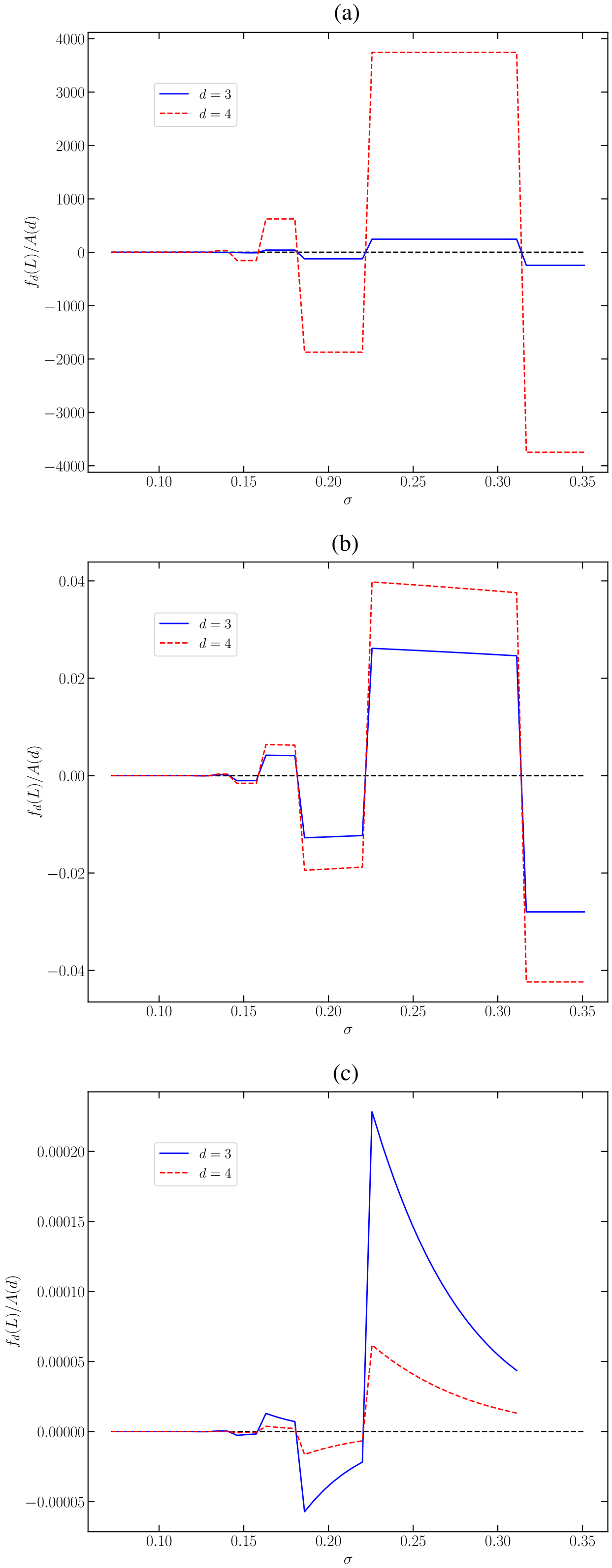

FIG. 4. Ratio $f_d(L)/A(d)$ in function of disorder strength σ for: a) $L = 0.3$, b) $L = 3$, and c) $L = 15$.

invariant systems, with dimension of the order parameter being one, driven by thermal fluctuations, there is a unique temperature where the system becomes critical. For systems with quenched disorder, the correlation function associated with the order parameter remains long ranged for an enumerable set of values of the disorder.

This led to the question of the analytic structure of this disordered Landau-Ginzburg model. From the series representation of the average generating functional of connected correlation functions, one can obtain a series representation for the average generating functional of vertex functions. For a fixed disorder, i.e., σ fixed, there is always a term in the series with $m_0^2 - k\sigma^2 = 0$. The argument is as follows: we define a sequence of critical σ points, which has an accumulation point at $\sigma = 0$. Notice that this occurs for any m_0^2. Therefore, we have infinitely many terms in the series that contribute a divergent susceptibility, an infinite correlation length with the power law decay of the correlation functions. The average generating functional of vertex functions has an infinite number of singularities. In the complex σ plane, this accumulation of singularities defines a natural boundary of analyticity, where there is no possibility of analytic extension [51,52]. Actually, the limit $\sigma \to 0$ can not be achieved. The appearance of a natural boundary of a similar nature is studied in the prime number spectra in quantum field theory [53].

A natural continuation of this work is to discuss fluctuation-induced forces in finite-size critical systems, defined on a slab $\mathbb{R}^{d-1} \times [0, L]$, with quenched disorder at the surfaces [54,55]. Finally, let us note that the case where quantum and disorder-induced fluctuations are present in a system with randomness must be investigated. Therefore, we have plans to discuss in a succeeding work the effects of disorder, in a system at low temperatures described by quantum field theory, prepared in the spontaneously broken phase. Being more specific, the idea is to investigate the low-temperature behavior of a system in a spontaneously broken symmetry phase described by a Euclidean quantum $\lambda\varphi^4_{d+1}$ model with quenched disorder in one-loop approximation. To study the low-temperature behavior of the system, since the disorder is strongly correlated in imaginary time [56–59], one can use the equivalence between a disorder Euclidean quantum $\lambda\varphi^4_{d+1}$ model with a classical model defined on a space $\mathbb{R}^d \times S^1$, with anisotropic quenched disorder. Such a model with spatially non-uniform disorder has some similarities to the McCoy-Wu random Ising model, an anisotropic two-dimensional classical Ising model with random exchange along one direction but uniform along the other [60,61]. These subjects are under investigation by the authors.

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