Quantum flavor vacuum in the expanding universe: A possible candidate for cosmological dark matter?

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We analyze fermion mixing in the framework of field quantization in curved spacetime. We compute the expectation value of the energy-momentum tensor of mixed fermions on the flavor vacuum. We consider spatially flat Friedmann-Lemaître-Robertson-Walker metrics, and we show that the energy-momentum tensor of the flavor vacuum is diagonal and conserved. Therefore, it can be interpreted as the effective energy-momentum tensor of a perfect fluid. In particular, assuming a fixed de Sitter background, the equation of state of the fluid is consistent with that of dust and cold dark matter. Our results establish a new link between quantum effects and classical fluids, and indicate that the flavor vacuum of mixed fermions may represent a new component of dark matter.

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I. INTRODUCTION

Among the open issues of modern cosmology is the understanding of the dark components of the universe. Most of the total energy density is shared between "dark energy" that drives the accelerated expansion of the universe [1–7] and "dark matter," the nonbaryonic matter that is responsible for holding galaxies and clusters together [8–25]. A conclusive explanation for this "dark universe" is still lacking. Several proposals, often very different in nature and perspective, have been put forward to account for dark matter. New particles such as axions, axionlike particles [26–36], and supersymmetric partners [37], arising from extensions of the standard model, might explain the missing matter. Another possible dark matter component is represented by massive compact objects such as primordial black holes [38]. Other proposals rely on pure quantum field theoretical effects and the nontrivial structure of the vacuum of flavor fermion mixing [39–43]. The relevance of fermion fields in astrophysical and cosmological contexts is well represented by the role of neutrinos. These particles are a valuable source of information as they are expected to play an important role in investigating astrophysical processes. They can be used as a test of the standard cosmological model [44,45] even at early time [44,46] and have been recognized as a possible component of dark energy [47]. Moreover, the corresponding vacuum has been identified as a possible dark matter component in flat space [39,40]. Neutrinos are known to oscillate among their three flavors, and their general oscillation formulas in curved spacetime have been derived in Refs. [48–52] together with a number of other aspects of this phenomenon. We remark that flavor mixing and massive neutrino oscillations are phenomena beyond the standard model.

In the following, starting from the analysis of the possible contribution of the flavor vacuum to the dark matter, previously studied in Minkowski spacetime [39,40], we study the behavior of the flavor vacuum in the case of a curved background. We compute, in the context of an homogeneous and isotropic spacetime, represented by a spatially flat Friedmann-Lemaître-Robertson-Walker metric, the expectation value of the energy-momentum tensor of neutrinos on the flavor (mixed) vacuum. We show that this expectation value is a diagonal tensor and satisfies the Bianchi identities. Consequently, it behaves as an effective stress energy tensor akin to that of a perfect fluid and can enter the Einstein equations as a regular source term. In particular, assuming a fixed de Sitter background, the equation of state of such a fluid is that of dust or cold dark matter ($w = 0$).

These results are not a mere generalization of the studies conducted in flat space [39–43], since the underlying quantum field theory (QFT) of fermion mixing is much more involved, with respect to flat space, and the properties of the flavor vacuum depend critically on the curved background considered. The computation performed on the de Sitter background already gives an indication that the flavor vacuum may play a role within

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the dark sector of the universe. Indeed, the flavor vacuum brings along an additional energy density, and, being pressureless, may be associated with a dark matter component. According to our results, a possible constituent of dark matter is then represented by the energy of the vacuum of mixed fields. In future works we will analyze the energy-momentum tensor associated with the fermion flavor vacuum in spacetimes that describe large scale structures, such as spiral galaxies. In that context the fluid related to the flavor vacuum can be more properly identified with a dark matter component.

The paper is organized as follows. Section II contains a summary of the properties of the quantum Dirac equation in curved spacetime. Section III contains the field quantization of the flavor fields and the introduction of the oscillation components. In Sec. IV the component of the expectation value of the quantum stress energy tensor is derived. Section V contains the application to a specific background metric, i.e., the de Sitter spacetime and the derivation of the corresponding exact solution for the component of the stress energy tensor, together with some consideration on its regularization. Finally Sec. VI is dedicated to the conclusions. In the following, we will use the $+---$ signature, and we will assume that greek indices run from $0,\ldots,3$; lowercase Latin indices from $1,\ldots,3$; and uppercase Latin indices from $1,\ldots,4$. These last indices will represent tetrad indices. The symbol $[,]$ denotes the matrix commutator so that, e.g., $[\gamma_A,\gamma_B]=\gamma_A\gamma_B-\gamma_B\gamma_A$.

II. THE DIRAC EQUATION AND ITS SOLUTIONS IN FLAT FLRW SPACETIME

For the reader's convenience we start this section by setting the notation and introducing the metric of interest. We then elaborate on the Dirac equation and discuss the general form of the solution. We will focus on the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime (we use the $+---$ signature) described by the metric $g_{\mu\nu}=\mathrm{diag}(1,-C^2(t),-C^2(t),-C^2(t))$. where $C(t)$ is the scale factor. For our purposes it will be useful to express this metric in terms of the conformal time τ defined as $d\tau=\frac{dt}{C(t)}$ with range $\tau\in(-\infty,\infty)$ corresponding to $t\in(-\infty,\infty)$. Using τ, the line element reads

$$ds^2=C^2(\tau)[d\tau^2-dx^2-dy^2-dz^2] \tag{1}$$

when expressed in Cartesian spatial coordinates and in conformal time τ.

In order to write the Dirac equation, we need to choose a tetrad field e^A_μ with the property $g_{\mu\nu}=e^A_\mu e^B_\nu\eta_{AB}$. Given the metric of Eq. (1), a convenient choice of tetrads is

$$e^A_\mu=C(\tau)\delta^A_\mu, \tag{2}$$

where the Kronecker symbol δ^A_μ signals that the only nonzero component of e^A is the one for which the Lorentz index A and the spacetime index μ coincide.

Using the tetrads above one can define the generalization of gamma matrices γ^A to the case of curved spacetime $\tilde{\gamma}^\mu=e^\mu_A\gamma^A$.

Finally, in order to write the Dirac equation, we also need the spin connections $\omega^{AB}_\mu=e^A_\nu\Gamma^\nu_{\sigma\mu}e^{\sigma B}+e^A_\nu\partial_\mu e^{\nu B}$. The spinorial covariant derivative is defined with the aid of the spin connections as $D_\mu\psi=\partial_\mu\psi+\Gamma_\mu\psi$ and $\Gamma_\mu=\frac{1}{8}\omega^{AB}_\mu[\gamma_A,\gamma_B]$. We can now write down the Dirac equation

$$i\tilde{\gamma}^\mu(x)D_\mu\psi-m\psi=0. \tag{3}$$

The above equation can be generated by the Lagrangian

$$\mathcal{L}=\sqrt{-g}\left\{\frac{i}{2}[\bar{\psi}\tilde{\gamma}^\mu(x)D_\mu\psi-D_\mu\bar{\psi}\tilde{\gamma}^\mu(x)\psi]-m\bar{\psi}\psi\right\}, \tag{4}$$

and we can define the energy-momentum tensor of the spinor field as the variation of the above Lagrangian with respect to ψ and $\bar{\psi}$

$$T_{\mu\nu}=\frac{i}{2}\{\bar{\psi}\tilde{\gamma}_\mu(x)D_\nu\psi+\bar{\psi}\tilde{\gamma}_\nu(x)D_\mu\psi - D_\mu\bar{\psi}\tilde{\gamma}_\nu(x)\psi-D_\nu\bar{\psi}\tilde{\gamma}_\mu(x)\psi\}. \tag{5}$$

In the metric (1) the Dirac equation reads

$$\left(i\gamma^0\partial_\tau+\frac{3i}{2}\frac{\partial_\tau C}{C}\gamma^0+i\gamma^j\partial_j-mC\right)\psi=0. \tag{6}$$

We remark that the Dirac equation (6) holds for the quantum field theoretic free Dirac field of mass m. At odds with the *classical* Dirac field, we cannot drop the spatial derivatives by arguing that space-dependent quantities cannot enter the right-hand side of the Einstein equation if the metric depends only on time. Indeed, *all* the solutions to Eq. (6) must be considered for the field expansion, including those which depend explicitly on the spatial coordinates. Consistency with the (time-dependent only) metric is then achieved for the expectation value of the physical observables, including the energy-momentum tensor, which turn out to depend only upon time.

The spatial dependence of this equation suggests that we look for plane wave solutions $\psi\propto e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$, where the mode label $\boldsymbol{p}$ is a 3-vector that can be thought of as the would-be plane wave momentum when $C(\tau)$ reduces to a constant, and "$\boldsymbol{a}\cdot\boldsymbol{b}$" is a shorthand notation for $\sum_{j=1,2,3}a_jb_j$. We remark that the actual momentum that is instantaneously carried by the mode with label $\boldsymbol{p}$ is the comoving momentum[1] $\frac{\boldsymbol{p}}{C(\tau)}$.

Using the helicity eigenbispinors ξ_λ defined as

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}\xi_\lambda = \lambda\xi_\lambda, \tag{7}$$

the solution of (6) can be written as the combination of the positive and negative energy solutions in the form

$$\begin{aligned} u_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{x}) &= e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\begin{pmatrix} f_p(\tau)\xi_\lambda(\hat{p}) \\ g_p(\tau)\lambda\xi_\lambda(\hat{p})\end{pmatrix}, \\ v_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{x}) &= e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\begin{pmatrix} g_p^*(\tau)\xi_\lambda(\hat{p}) \\ -f_p^*(\tau)\lambda\xi_\lambda(\hat{p})\end{pmatrix}. \end{aligned} \tag{8}$$

Here $\hat{p} = \frac{\boldsymbol{p}}{p}$ denotes the unit vector in the direction of $\boldsymbol{p}$, $\boldsymbol{\sigma}$ is the vector of Pauli matrices, and $\lambda = \pm 1$. Notice that for each $\lambda = \pm 1$, the quantity $\xi_\lambda^\dagger(\hat{p})\sigma_i\xi_\lambda(\hat{p})$ is an odd function of p_i, for $i = 1, 2, 3$. In particular, $\xi_\lambda^\dagger(\hat{p})\sigma_i\xi_\lambda(\hat{p})$ changes sign when the momentum is reversed $\boldsymbol{p}\to-\boldsymbol{p}$. The statement can be proven by direct calculation, as shown in Appendix B. Inserting Eq. (8) in Eq. (6), and using the defining property of the helicity bispinors, we can write (6) as the system

$$\begin{aligned} i\partial_\tau f_{\boldsymbol{p}} &= \left(mC - \frac{3i}{2}\frac{\partial_\tau C}{C}\right)f_{\boldsymbol{p}} + pg_{\boldsymbol{p}}, \\ i\partial_\tau g_{\boldsymbol{p}} &= \left(-mC - \frac{3i}{2}\frac{\partial_\tau C}{C}\right)g_{\boldsymbol{p}} + pf_{\boldsymbol{p}}, \\ i\partial_\tau f_p^* &= \left(-mC - \frac{3i}{2}\frac{\partial_\tau C}{C}\right)f_p^* - pg_p^*, \\ i\partial_\tau g_p^* &= \left(mC - \frac{3i}{2}\frac{\partial_\tau C}{C}\right)g_p^* - pf_p^*. \end{aligned} \tag{9}$$

The above equations show that the functions f, g depend only on the modulus p of the 3-momentum. Thus, from now on, we will drop the vector index $\boldsymbol{p}$ in favor of the scalar index p.

The scalar product between two solutions of the Dirac equation A, B is defined as

$$(A,B)_\tau = \int_{\Sigma_\tau} d^3x\sqrt{-g}\bar{A}\tilde{\gamma}^\tau(x)B, \tag{10}$$

where the integration is to be carried over a hypersurface Σ_τ of constant conformal time τ. If A and B are solutions of the *same* Dirac equation, the scalar product $(A,B)_\tau$ is independent of τ. This is no longer true if A and B are solutions of distinct Dirac equations. Taking into account that the determinant is $g = -C^8$ and adopting the normalization

$$|f_p|^2 + |g_p|^2 = \frac{1}{(2\pi C)^3}, \tag{11}$$

we obtain the following orthonormality and completeness relations, respectively (for details, see Appendix A):

$$\begin{aligned} (u_{\boldsymbol{p},\lambda}, u_{\boldsymbol{q},\lambda'})_\tau &= \delta^3(\boldsymbol{p}-\boldsymbol{q})\delta_{\lambda,\lambda'}, \\ (u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{q},\lambda'})_\tau &= (v_{\boldsymbol{p},\lambda}, u_{\boldsymbol{q},\lambda'})_\tau = 0, \\ (v_{\boldsymbol{p},\lambda}, v_{\boldsymbol{q},\lambda'})_\tau &= \delta^3(\boldsymbol{p}-\boldsymbol{q})\delta_{\lambda,\lambda'}, \end{aligned} \tag{12}$$

$$\sum_\lambda (u_{\boldsymbol{p},\lambda}u^\dagger_{\boldsymbol{p},\lambda} + v_{\boldsymbol{p},\lambda}v^\dagger_{\boldsymbol{p},\lambda}) = \frac{1}{(2\pi C)^3}\begin{pmatrix}\mathbb{I} & 0 \\ 0 & \mathbb{I}\end{pmatrix}. \tag{13}$$

Equation (13) shows that the set of solutions (8) provides a basis not only for the solution space of the Dirac equation but also for the space of 4-spinors.

The system of equations (9) can be further simplified by introducing the functions

$$\phi_p = C^{\frac{3}{2}}f_p, \qquad \gamma_p = C^{\frac{3}{2}}g_p \tag{14}$$

since then the system becomes

$$\begin{aligned} \partial_\tau\phi_p &= -imC\phi_p - ip\gamma_p, \\ \partial_\tau\gamma_p &= imC\gamma_p - ip\phi_p, \\ \partial_\tau\phi_p^* &= imC\phi_p^* + ip\gamma_p^*, \\ \partial_\tau\gamma_p^* &= -imC\gamma_p^* + ip\phi_p^*, \end{aligned}$$

and the normalization condition (11) becomes

$$|\phi_p|^2 + |\gamma_p|^2 = \frac{1}{(2\pi)^3}. \tag{15}$$

The first two of Eqs. (15) can be combined to give two second-order equations for ϕ_p,

$$\partial_\tau^2\phi_p + (im\partial_\tau C + p^2 + m^2C^2)\phi_p = 0. \tag{16}$$

In the same way, we can obtain a second-order equation for ϕ_p^*.

We conclude the section by introducing a bilinear form of the solutions of the Dirac equation, which is convenient

[1]This is most conveniently seen by inserting the plane wave ansatz in Eq. (6) and reverting to the coordinate time t:

$$\left[i\gamma^0\left(\partial_t + \frac{3i}{2}\frac{\partial_t C}{C}\right) - \frac{\gamma^j p_j}{C} - m\right]\psi_{\boldsymbol{p}} = 0.$$

In this form the equation resembles the Dirac equation in flat space, with a time-dependent potential $\frac{3i}{2}\frac{\partial_t C}{C}$ and with the quantity $\frac{p_j}{C}$ playing the role of an instantaneous momentum.

for the study of the energy-momentum tensor. Given two solutions A, B of the Dirac equation, we define the auxiliary tensor as

$$L_{\mu\nu}(A,B) = \bar{A}\tilde{\gamma}_\mu(x)D_\nu B + \bar{A}\tilde{\gamma}_\nu(x)D_\mu B - D_\mu\bar{A}\tilde{\gamma}_\nu(x)B - D_\nu\bar{A}\tilde{\gamma}_\mu(x)B. \tag{17}$$

The properties of the auxiliary tensor are analyzed in Appendix C.

III. QUANTIZATION OF FLAVOR FIELDS

In the following, we start by quantizing a single Dirac field of definite mass, and then we report the analysis of the main properties of two flavor mixed fields. It is important to stress that here we adopt the flavor Fock space quantization of the flavor fields in a curved background [48]. In such a setting, the representations in terms of annihilators with definite mass (*mass representation*) and the representation in terms of annihilators with definite flavor (*flavor representation*) are unitarily inequivalent, regardless of the background metric considered (Minkowski, FLRW, etc.). The *flavor vacuum*, annihilated by all the destruction operators with definite flavor, has the structure of a condensate of particle-antiparticle pairs with definite masses. Such a structure characterizes the mixing phenomenon. The analysis of the flavor mixing in QFT and the unitary inequivalence between flavor and mass representations in the Minkowski metric are reported in [53] and in references therein. The current paper specifically pertains to the flavor vacuum on a curved background. In addition to the particle creation phenomena related to the expansion of the universe, we also need to consider the condensate structure of the flavor vacuum that we remark is to be taken into account regardless of the background metric.

Our approach to the quantization of the flavor fields is different from the quantization employed in previous works, as [54]. There the authors work within a unique representation (the mass representation) and the creation operators for fields with definite masses and fields with definite flavor, at a fixed time, act upon the same Fock space and define a unique vacuum state (the mass vacuum). Inequivalent representations only come into play for distinct times, due to the expansion-related particle creation.

In our case, two inequivalent representations (the mass and the flavor ones) are present even at the same time. This point can be understood by considering the flat spacetime limit of the energy density. Indeed, in Refs. [39–43], it has been shown that the expectation value of the Hamiltonian density computed on the flavor vacuum is nonvanishing even in the flat spacetime limit (where there is no particle creation due to expansion). This result is due to the condensate structure of the flavor vacuum. On the contrary, on the mass vacuum that the authors of Ref. [54] use, the expectation value of the (normal-ordered) Hamiltonian density vanishes in the flat spacetime limit. Such a circumstance is an obvious consequence of the absence of particle creation. Here, we generalize the results of [39] to the case of a curved background.

Finally, in order to avoid any confusion, we remark the Bogoliubov coefficients discussed in Sec. IV are the *mixing coefficients*, provided by the inner products of modes with different masses at the same, given time [48]. These are not to be confused with the Bogoliubov coefficients, as those appearing in [54], that relate the field expansions at *distinct* times.

A. Dirac field

Since the set of solutions (8) $\{u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{p},\lambda}\}_{\boldsymbol{p},\lambda}$ is complete, any solution of the (linear) Dirac equation can be written as a linear combination of these modes. In particular, this is true for the Dirac field: $\psi(x) = \sum_\lambda \int d^3p(A_{\boldsymbol{p},\lambda}u_{\boldsymbol{p},\lambda} + B^*_{-\boldsymbol{p},\lambda}v_{\boldsymbol{p},\lambda})$. In this equation, the notation $B^*_{-\boldsymbol{p},\lambda}$ is chosen to remark that $v_{\boldsymbol{p},\lambda}$ describes an antiparticle with momentum $-\boldsymbol{p}$. Notice that all the spacetime dependence is in the modes, while the coefficients are independent of both space and time coordinates.

Quantization is achieved, as usual, by promoting the field, and thus the expansion coefficients to operators

$$\psi(x) = \sum_\lambda \int d^3p(A_{\boldsymbol{p},\lambda}u_{\boldsymbol{p},\lambda} + B^\dagger_{-\boldsymbol{p},\lambda}v_{\boldsymbol{p},\lambda}). \tag{18}$$

and imposing the canonical anticommutation relations. The momentum conjugate to $\psi(x)$, according to the Lagrangian (4), is $\pi_\psi(x) = iC^3\psi^\dagger(x)$, so that the canonical anticommutation relations to be imposed are

$$\{\psi_A(\tau,\boldsymbol{x}), \pi_{\psi B}(\tau,\boldsymbol{x}')\} = iC^3\{\psi_A(\tau,\boldsymbol{x}), \psi^\dagger_B(\tau,\boldsymbol{x}')\} = i\delta_{AB}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{19}$$

Here the indices A and B are referred to as the spinor components $A, B = 1, 2, 3, 4$. It is easy to show that the relations (19) are satisfied if one imposes the following anticommutation relations on the coefficients:

$$\{A_{\boldsymbol{p},\lambda}, A^\dagger_{\boldsymbol{q},\lambda'}\} = \{B_{\boldsymbol{p},\lambda}, B^\dagger_{\boldsymbol{q},\lambda'}\} = \delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}) \tag{20}$$

with all the other anticommutators vanishing (see Appendix D …). The field expansion (18) defines the vacuum state $|0\rangle$ as the state annihilated by all the annihilation operators $A_{\boldsymbol{p},\lambda}|0\rangle = B_{\boldsymbol{p},\lambda}|0\rangle = 0, \ \forall\ \boldsymbol{p},\lambda$. It is important to stress that the definition of the vacuum state depends critically on the choice of the field modes. Specifically, its particle interpretation is tied to the boundary conditions specified on the solutions (8). Another kind of field expansion is possible if one assumes a specific time evolution of the modes, as is done within the adiabatic approximation (see, e.g., Refs. [55–58]). Contrary to our field expansion (18), the annihilation operators (and thus the vacuum) are thereby endowed with a specific time dependence. The expansion (18), instead, does not assume any particular time dependence and is, therefore, more general. The two expansions can be made to coincide at a given time. The quantized energy-momentum tensor is obtained by inserting the field expansion in the definition (5):

$$T_{\mu\nu} = \frac{i}{2}\sum_{\lambda,\lambda'}\int d^3p\int d^3q\{A^\dagger_{\boldsymbol{p},\lambda}A_{\boldsymbol{q},\lambda'}L_{\mu\nu}(u_{\boldsymbol{p},\lambda},u_{\boldsymbol{q},\lambda'}) + A^\dagger_{\boldsymbol{p},\lambda}B^\dagger_{-\boldsymbol{q},\lambda'}L_{\mu\nu}(u_{\boldsymbol{p},\lambda},v_{\boldsymbol{q},\lambda'}) \\ + B_{-\boldsymbol{p},\lambda}A_{\boldsymbol{q},\lambda'}L_{\mu\nu}(v_{\boldsymbol{p},\lambda},u_{\boldsymbol{q},\lambda'}) + B_{-\boldsymbol{p},\lambda}B^\dagger_{-\boldsymbol{q},\lambda'}L_{\mu\nu}(v_{\boldsymbol{p},\lambda},v_{\boldsymbol{q},\lambda'})\}, \tag{21}$$

where we have used the definition (17) of $L_{\mu\nu}$. The Hamiltonian density, defined with respect to ∂_τ corresponds to the component T^τ_τ of the above equation. It follows immediately that the vacuum is not an eigenstate of the Hamiltonian unless $L^\tau_\tau(v_{\boldsymbol{p},\lambda},u_{\boldsymbol{q},\lambda'}) = 0$, due to the $A^\dagger_{\boldsymbol{p},\lambda}B^\dagger_{-\boldsymbol{q},\lambda'}$ term. In particular, the vacuum is unstable under the creation of particle-antiparticle pairs with opposite momenta, a result known from the previous analyses. This is a reflection of the noninvariance under time translations. On the other hand, from the relation $L_{\tau i}(u_{\boldsymbol{p},\lambda},v_{\boldsymbol{p},\lambda}) = 0$, derived in Appendix C [cf. Eq. (C14)], it is clear that the vacuum respects the residual translational symmetry in the spatial coordinates, i.e., $\boldsymbol{P}|0\rangle = 0$.

B. The flavor fields

Up to now, our considerations have been restricted to a single Dirac field of definite mass. To introduce the flavor fields, we follow [48] and start by introducing two Dirac fields with distinct masses m_1, m_2. We consider two flavors for simplicity but the analysis can easily be generalized to three flavors. The theory of two free massive Dirac fields is just the product of two copies of the theory for a single Dirac field. All the relations discussed above remain valid, provided we assign a new index $j = 1, 2$ to all the quantities involved, $\psi_j, m_j, A_{\boldsymbol{p},\lambda;j}, u_{\boldsymbol{p},\lambda;j}$, and so on. All the previous relations are index-wise valid for $j = 1,\ 2$. The mass vacuum that we now denote explicitly as $|0_M\rangle$ is annihilated by all the annihilators for each index j. The total energy-momentum tensor is simply the sum of two copies of Eq. (21), one for each j. We also require that each mode of field 1 is related to the corresponding mode of field 2, with the same labels $\boldsymbol{p}$, λ by the substitution $m_1 \to m_2$, and vice versa.

The flavor fields are then introduced via the rotation

$$\begin{aligned}\psi_e(x) &= \cos(\theta)\psi_1(x) + \sin(\theta)\psi_2(x),\\ \psi_\mu(x) &= \cos(\theta)\psi_2(x) - \sin(\theta)\psi_1(x),\end{aligned} \tag{22}$$

where θ is the two-flavor mixing angle. At the quantum level, the rotation is employed by the mixing generator

$$\mathcal{I}_\theta(\tau) = \exp\{\theta[(\psi_1,\psi_2)_\tau - (\psi_2,\psi_1)_\tau]\}. \tag{23}$$

Here $(\psi_2,\psi_1)_\tau$ stands for the scalar product at the τ hypersurface (recall that for fields of distinct masses the product *does* depend on τ). The flavor fields are then

$$\begin{aligned}\psi_e(x) &= \mathcal{I}_\theta^{-1}\psi_1(x)\mathcal{I}_\theta,\\ \psi_\mu(x) &= \mathcal{I}_\theta^{-1}\psi_2(x)\mathcal{I}_\theta.\end{aligned} \tag{24}$$

The action of the generator also defines the flavor annihilators ($A_{\boldsymbol{p},\lambda;e} = \mathcal{I}_\theta^{-1}A_{\boldsymbol{p},\lambda;1}\mathcal{I}_\theta$ and similar) and the flavor vacuum as

$$|0_F\rangle = \mathcal{I}_\theta^{-1}|0_M\rangle. \tag{25}$$

Notice that, contrary to the mass vacuum, the flavor vacuum has an explicit τ dependence. The terminology “flavor vacuum” is justified in that this state is annihilated by all the flavor annihilators.

IV. VACUUM EXPECTATION VALUE OF THE ENERGY-MOMENTUM TENSOR ON THE FLAVOR VACUUM

We are specifically interested in the contributions that flavor mixing induces on the energy-momentum tensor. More precisely, we ask what is the expectation value of the energy-momentum tensor on the state corresponding to the absence of flavor neutrinos at a given time $|O_F(\tau_0)\rangle$. We stress that in this calculation, we assume a fixed but arbitrary expansion of the mass fields, and therefore a fixed but arbitrary choice of the mass vacuum. The effect of a change in the mass representation on the flavor fields is known [48], and once the result is computed for a given representation, one can implement the adequate transformations to get the result in other mass representations. Likewise, we keep the time τ_0 arbitrary and distinct from the time argument τ of the energy-momentum tensor. The quantity we wish to compute is then

$$\mathbb{T}_{\mu\nu} = \langle 0_F(\tau_0)|T_{\mu\nu}|0_F(\tau_0)\rangle, \tag{26}$$

where $T_{\mu\nu}$ is given by Eq. (21). We remark that the only sensible definition for the energy-momentum tensor is the one (21) in terms of the fields with definite mass. Let us analyze the typical term in Eq. (26). It has the form

$$\langle 0_F(\tau_0)|A^{\dagger}_{\boldsymbol{p},\lambda;1}A_{\boldsymbol{q},\lambda';1}|0_F(\tau_0)\rangle. \tag{27}$$

Using the definition of the flavor vacuum, this equals

$$\begin{aligned}\langle 0_M|\mathcal{I}_\theta(\tau_0)A^{\dagger}_{\boldsymbol{p},\lambda;1}A_{\boldsymbol{q},\lambda';1}\mathcal{I}^{-1}_\theta(\tau_0)|0_M\rangle &= \langle 0_M|\mathcal{I}_\theta(\tau_0)A^{\dagger}_{\boldsymbol{p},\lambda;1}\mathcal{I}^{-1}_\theta(\tau_0)\mathcal{I}_\theta(\tau_0)A_{\boldsymbol{q},\lambda';1}\mathcal{I}^{-1}_\theta(\tau_0)|0_M\rangle\\ &= \langle 0_M|\mathcal{I}^{-1}_{-\theta}(\tau_0)A^{\dagger}_{\boldsymbol{p},\lambda;1}\mathcal{I}_{-\theta}(\tau_0)\mathcal{I}^{-1}_{-\theta}(\tau_0)A_{\boldsymbol{q},\lambda';1}\mathcal{I}_{-\theta}(\tau_0)|0_M\rangle,\end{aligned} \tag{28}$$

where we have used that [Eq. (23)] $\mathcal{I}^{-1}_\theta = \mathcal{I}_{-\theta}$. Now the operator $\mathcal{I}^{-1}_{-\theta}(\tau_0)A_{\boldsymbol{p},\lambda;1}\mathcal{I}_{-\theta}(\tau_0)$ is just the mass annihilator transformed according to a mixing transformation with angle $-\theta$. Knowing the transformation rule in terms of θ [48], we can easily write down the transformed operators:

$$\begin{aligned}\mathcal{I}^{-1}_{-\theta}(\tau_0)A_{\boldsymbol{p},\lambda;1}\mathcal{I}_{-\theta}(\tau_0) &= \cos(\theta)A_{\boldsymbol{p},\lambda;1} - \sin(\theta)(\Lambda^*_{\boldsymbol{p}}(\tau_0)A_{\boldsymbol{p},\lambda;2} + \Xi_{\boldsymbol{p}}(\tau_0)B^{\dagger}_{-\boldsymbol{p},\lambda;2}),\\ \mathcal{I}^{-1}_{-\theta}(\tau_0)A_{\boldsymbol{p},\lambda;2}\mathcal{I}_{-\theta}(\tau_0) &= \cos(\theta)A_{\boldsymbol{p},\lambda;2} + \sin(\theta)(\Lambda_{\boldsymbol{p}}(\tau_0)A_{\boldsymbol{p},\lambda;1} - \Xi_{\boldsymbol{p}}(\tau_0)B^{\dagger}_{-\boldsymbol{p},\lambda;1}),\\ \mathcal{I}^{-1}_{-\theta}(\tau_0)B_{-\boldsymbol{p},\lambda;1}\mathcal{I}_{-\theta}(\tau_0) &= \cos(\theta)B_{-\boldsymbol{p},\lambda;1} - \sin(\theta)(\Lambda^*_{\boldsymbol{p}}(\tau_0)B_{-\boldsymbol{p},\lambda;2} - \Xi_{\boldsymbol{p}}(\tau_0)A^{\dagger}_{\boldsymbol{p},\lambda;2}),\\ \mathcal{I}^{-1}_{-\theta}(\tau_0)B_{-\boldsymbol{p},\lambda;2}\mathcal{I}_{-\theta}(\tau_0) &= \cos(\theta)B_{-\boldsymbol{p},\lambda;2} + \sin(\theta)(\Lambda_{\boldsymbol{p}}(\tau_0)B_{-\boldsymbol{p},\lambda;1} + \Xi_{\boldsymbol{p}}(\tau_0)A^{\dagger}_{\boldsymbol{p},\lambda;1}),\end{aligned} \tag{29}$$

while the transformation rule for the adjoint operators can be obtained by considering the adjoint equations. The Bogoliubov coefficients are defined as the inner products

$$\begin{aligned}\delta^3(\mathbf{0})\Lambda_{\boldsymbol{p}}(\tau) &= (u_{\boldsymbol{p},\lambda;2}, u_{\boldsymbol{p},\lambda;1})_\tau = (v_{\boldsymbol{p},\lambda;1}, v_{\boldsymbol{p},\lambda;2})_\tau,\\ \delta^3(\mathbf{0})\Xi_{\boldsymbol{p}}(\tau) &= (u_{\boldsymbol{p},\lambda;1}, v_{\boldsymbol{p},\lambda;2})_\tau = -(u_{\boldsymbol{p},\lambda;2}, v_{\boldsymbol{p},\lambda;1})_\tau,\end{aligned} \tag{30}$$

where the notation anticipates that they do not depend on λ (we will see that they actually depend only on the modulus of the momentum p). The $\delta^3(\mathbf{0})$ factor is a reminiscence of the more general expressions involving distinct momenta $\Lambda_{\boldsymbol{p},\boldsymbol{q}} \propto \delta^3(\boldsymbol{p}-\boldsymbol{q})$ [48]. The delta factor is absorbed by a corresponding momentum integration in Eqs. (29), leaving only the finite coefficients defined via Eq. (30). These coefficients satisfy $|\Lambda_{\boldsymbol{p}}|^2 + |\Xi_{\boldsymbol{p}}|^2 = 1$ for all $\boldsymbol{p}$, τ. With the aid of Eqs. (29), we can evaluate all the expectation values appearing in Eq. (26):

$$\begin{aligned}\langle 0_F(\tau_0)|A^{\dagger}_{\boldsymbol{p},\lambda;j}A_{\boldsymbol{q};\lambda';j}|0_F(\tau_0)\rangle &= \sin^2\theta|\Xi_{\boldsymbol{p}}(\tau_0)|^2\delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}), \quad \forall\ j,\\ \langle 0_F(\tau_0)|B^{\dagger}_{-\boldsymbol{p},\lambda;j}B_{-\boldsymbol{q};\lambda';j}|0_F(\tau_0)\rangle &= \sin^2\theta|\Xi_{\boldsymbol{p}}(\tau_0)|^2\delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}), \quad \forall\ j,\\ \langle 0_F(\tau_0)|A^{\dagger}_{\boldsymbol{p},\lambda;1}B^{\dagger}_{-\boldsymbol{q};\lambda';1}|0_F(\tau_0)\rangle &= \sin^2\theta\Xi^*_{\boldsymbol{p}}(\tau_0)\Lambda_{\boldsymbol{p}}(\tau_0)\delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}),\\ \langle 0_F(\tau_0)|A^{\dagger}_{\boldsymbol{p},\lambda;2}B^{\dagger}_{-\boldsymbol{q};\lambda';2}|0_F(\tau_0)\rangle &= -\sin^2\theta\Xi^*_{\boldsymbol{p}}(\tau_0)\Lambda^*_{\boldsymbol{p}}(\tau_0)\delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}),\\ \langle 0_F(\tau_0)|B_{-\boldsymbol{p},\lambda;1}A_{\boldsymbol{q};\lambda';1}|0_F(\tau_0)\rangle &= \sin^2\theta\Xi_{\boldsymbol{p}}(\tau_0)\Lambda^*_{\boldsymbol{p}}(\tau_0)\delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}),\\ \langle 0_F(\tau_0)|B_{-\boldsymbol{p},\lambda;2}A_{\boldsymbol{q};\lambda';2}|0_F(\tau_0)\rangle &= -\sin^2\theta\Xi_{\boldsymbol{p}}(\tau_0)\Lambda_{\boldsymbol{p}}(\tau_0)\delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q}).\end{aligned} \tag{31}$$

It is convenient to give the result by splitting out the pure mixing component:

$$\mathbb{T}_{\mu\nu} = \mathbb{T}^{(MIX)}_{\mu\nu} + \mathbb{T}^{(N)}_{\mu\nu}, \tag{32}$$

$$\begin{aligned}\mathbb{T}^{(MIX)}_{\mu\nu} = \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\{&|\Xi_{\boldsymbol{p}}(\tau_0)|^2\sum_{j=1,2}(L_{\mu\nu}(u_{\boldsymbol{p},\lambda;j}, u_{\boldsymbol{p},\lambda;j}) - L_{\mu\nu}(v_{\boldsymbol{p},\lambda;j}, v_{\boldsymbol{p},\lambda;j}))\\ &+ \Xi^*_{\boldsymbol{p}}(\tau_0)\Lambda_{\boldsymbol{p}}(\tau_0)L_{\mu\nu}(u_{\boldsymbol{p},\lambda;1}, v_{\boldsymbol{p},\lambda;1}) + \Xi_{\boldsymbol{p}}(\tau_0)\Lambda^*_{\boldsymbol{p}}(\tau_0)L_{\mu\nu}(v_{\boldsymbol{p},\lambda;1}, u_{\boldsymbol{p},\lambda;1})\\ &- \Xi^*_{\boldsymbol{p}}(\tau_0)\Lambda^*_{\boldsymbol{p}}(\tau_0)L_{\mu\nu}(u_{\boldsymbol{p},\lambda;2}, v_{\boldsymbol{p},\lambda;2}) - \Xi_{\boldsymbol{p}}(\tau_0)\Lambda_{\boldsymbol{p}}(\tau_0)L_{\mu\nu}(v_{\boldsymbol{p},\lambda;2}, u_{\boldsymbol{p},\lambda;2})\},\end{aligned} \tag{33}$$

$$\mathbb{T}^{(N)}_{\mu\nu} = \frac{i}{2}\sum_{\lambda}\sum_{j=1,2}\int d^3p L_{\mu\nu}(v_{\boldsymbol{p},\lambda;j}, v_{\boldsymbol{p},\lambda;j}). \tag{34}$$

The last term comes from applying the anticommutator relation $\{B^{\dagger}_{-\boldsymbol{p},\lambda;j}, B_{-\boldsymbol{q},\lambda';j}\} = \delta_{\lambda\lambda'}\delta^3(\boldsymbol{p}-\boldsymbol{q})$ to the $BB^{\dagger}$ term. While the remaining terms, regrouped under the symbol $\mathbb{T}^{(MIX)}_{\mu\nu}$, show an explicit dependence on $\sin^2\theta$, and are therefore zero in the absence of mixing, the last term is present independently of mixing. Indeed, it is easy to check that $\mathbb{T}^{(N)}_{\mu\nu}$ represents the expectation value of the energy-momentum tensor on the *mass* vacuum:

$$\mathbb{T}^{(N)}_{\mu\nu} = \langle 0_M|T_{\mu\nu}|0_M\rangle. \tag{35}$$

The origin of this term is an ordering ambiguity in the energy-momentum tensor of the quantized fields. To understand its significance, let us consider the flat space limit, where $v_{\boldsymbol{p},\lambda;j} \propto e^{i\omega_{p;j}t}$ with $\omega_{p;j} = \sqrt{p^2+m_j^2}$. The auxiliary tensor becomes

$$L_{\mu\nu}(A,B) = \bar{A}\gamma_\mu\partial_\nu B + \bar{A}\gamma_\nu\partial_\mu B - \partial_\mu\bar{A}\gamma_\nu B - \partial_\nu\bar{A}\gamma_\mu B, \tag{36}$$

where both the Dirac matrices and the derivatives are the ordinary flat space ones. In particular, one has

$$\begin{aligned} L_{00}(v_{\boldsymbol{p},\lambda;j}, v_{\boldsymbol{p},\lambda;j}) &= 2(v^{\dagger}_{\boldsymbol{p},\lambda;j}\partial_t v_{\boldsymbol{p},\lambda;j} - \partial_t v^{\dagger}_{\boldsymbol{p},\lambda;j} v_{\boldsymbol{p},\lambda;j}) \\ &= 4i\omega_{p;j} v^{\dagger}_{\boldsymbol{p},\lambda;j} v_{\boldsymbol{p},\lambda;j} = 4i\omega_{p;j}, \end{aligned} \tag{37}$$

where we have made use of the normalization $v^{\dagger}_{\boldsymbol{p},\lambda;j}v_{\boldsymbol{p},\lambda;j} = 1$. Then

$$\mathbb{T}^{(N)}_{00} = -2\sum_{\lambda}\sum_{j=1,2}\int d^3p\,\omega_{p;j} \qquad (<0). \tag{38}$$

We can see that $\mathbb{T}^{(N)}_{00}$ corresponds to the diverging negative energy that is removed by the normal ordering prescription in flat space. In order that the energy-momentum tensor $T_{\mu\nu}$ approaches the normal-ordered flat-space expression in the limit, one must subtract $\mathbb{T}^{(N)}_{\mu\nu}$. Then we define a renormalized energy-momentum tensor by

$$T^r_{\mu\nu} = T_{\mu\nu} - \mathbb{T}^{(N)}_{\mu\nu}, \tag{39}$$

whose expectation value is

$$\langle 0_F(\tau_0)|T^r_{\mu\nu}|0_F(\tau_0)\rangle = \mathbb{T}^{(MIX)}_{\mu\nu}. \tag{40}$$

We now derive the general properties of the Bogoliubov coefficients, we show that $\mathbb{T}^{(MIX)}_{\mu\nu}$ behaves as a perfect fluid, and we determine the functional expression of $\mathbb{T}_{\tau\tau}$ and $\mathbb{T}^{\mu}_{\mu}$.

A. General properties of the Bogoliubov coefficients

Much can still be said without specifying the precise metric, i.e., without providing an explicit form for the scale factor C. Plugging the solutions (8) into the definition (30), we have

$$\Lambda_p(\tau) = (2\pi C)^3(f^*_{p;2}f_{p;1} + g^*_{p;2}g_{p;1}), \tag{41}$$

where we have switched to the notation Λ_p to highlight that it depends only upon the modulus p. In a similar way

$$\Xi_p(\tau) = (2\pi C)^3(f^*_{p;1}g^*_{p;2} - g^*_{p;1}f^*_{p;2}). \tag{42}$$

From Eqs. (41) and (42) we can verify that

$$|\Lambda_p(\tau)|^2 + |\Xi_p(\tau)|^2 = 1, \tag{43}$$

where in the last step we have made use of the normalization condition (A3). We conclude this subsection by expressing the Bogoliubov coefficients in terms of the reduced functions ϕ_p, γ_p:

$$\begin{aligned} \Lambda_p(\tau) &= (2\pi)^3(\phi^*_{p;2}\phi_{p;1} + \gamma^*_{p;2}\gamma_{p;1}), \\ \Xi_p(\tau) &= (2\pi)^3(\phi^*_{p;1}\gamma^*_{p;2} - \gamma^*_{p;1}\phi^*_{p;2}). \end{aligned} \tag{44}$$

B. Diagonality of the energy-momentum tensor

Using the result [Eq. (44)], we can prove that $\mathbb{T}_{\mu\nu}$ is nonzero only when $\mu=\nu$. This result has the important consequence that $\mathbb{T}^{\mu}_{\nu}$ can be interpreted as the energy-momentum tensor of a perfect fluid.

We start by proving that $\mathbb{T}_{\tau i} = 0$ for $i = 1, 2, 3$. Since we can always write $L_{\tau i}(a,b) = p_i h_{a,b}(p)$ (see Appendix C for details), each of the terms in the integrand of Eq. (32) is of the form $p_i\mathcal{F}(p)$, with $\mathcal{F}(p)$ a function of p only. We therefore have an odd function of p_i integrated over an even domain $p_i \in (-\infty,+\infty)$ and the integral vanishes:

$$\mathbb{T}_{\tau i} = 0. \tag{45}$$

The situation is similar for $\mathbb{T}_{ij}$ with $i \neq j$ and $i, j = 1, 2, 3$. In this case [see Eq. (C15) of Appendix C], namely $L_{ij}(a,b) = p_i p_j l_{a,b}(p)$ implies that each term under the integral sign is of the form $p_i p_j \mathcal{G}(p)$ with $\mathcal{G}(p)$ a function of p alone. It is clear that also in this case, the integral over the even domain $(p_i, p_j) \in (-\infty,+\infty)\times(-\infty,+\infty)$ vanishes, yielding $\mathbb{T}_{ij} = 0$ for $i \neq j$. Notice that the same conclusion does not apply to $\mathbb{T}_{ii}$, since in this case the integrand is an even function $p_i^2\mathcal{A}(p)$ of p_i. It is easy to verify that $\mathbb{T}_{ii}$ is the same for each $i = 1, 2, 3$, consistently with the isotropy of the underlying metric. Because of the manifest diagonality and isotropy of the energy-momentum tensor, there are really only two independent components of $\mathbb{T}_{\mu\nu}$, i.e., $\mathbb{T}_{\tau\tau}$ and

$\mathbb{T}_{ii}$ for a given i. Alternatively one can consider the $\mathbb{T}_{\tau\tau}$ component and the trace $\mathbb{T}^{\mu}_{\mu}$, as by definition

$$\mathbb{T}^{\mu}_{\mu} = g^{\mu\nu}\mathbb{T}_{\mu\nu} = C^{-2}\mathbb{T}_{\tau\tau} - 3C^{-2}\mathbb{T}_{ii},$$

and then

$$\mathbb{T}_{ii} = \frac{\mathbb{T}_{\tau\tau} - C^2\mathbb{T}^{\mu}_{\mu}}{3}, \tag{46}$$

where no sum is intended over the index i and we have used the isotropy of $\mathbb{T}_{\mu\nu}$.

It is worth stressing that, as evident from the definition and the expression of the auxiliary tensor, $\mathbb{T}_{\mu\nu}$ depends only on the time coordinate τ and parametrically on the arbitrary fixed time τ_0.

We conclude the section by giving the explicit functional form of $\mathbb{T}_{\tau\tau}$ and $\mathbb{T}^{\mu}_{\mu}$. From Eqs. (32) and (44) we have

$$\begin{aligned}\mathbb{T}_{\tau\tau}[\phi_{p;j},\gamma_{p;j}] = 2iC^{-2}\sin^2\theta\sum_{\lambda}\int d^3p\{&|\Xi_p(\tau_0)|^2\sum_{j=1,2}(\phi^*_{p;j}\partial_\tau\phi_{p;j} + \gamma^*_{p;j}\partial_\tau\gamma_{p;j} - \partial_\tau\phi^*_{p;j}\phi_{p;j} - \partial_\tau\gamma^*_{p;j}\gamma_{p;j})\\ &+ 2i\Im[\Xi^*_p(\tau_0)\Lambda_p(\tau_0)(\phi^*_{p;1}\partial_\tau\gamma^*_{p;1} - \gamma^*_{p;1}\partial_\tau\phi^*_{p;1}) - \Xi^*_p(\tau_0)\Lambda^*_p(\tau_0)(\phi^*_{p;2}\partial_\tau\gamma^*_{p;2} - \gamma^*_{p;2}\partial_\tau\phi^*_{p;2})]\}\\ &- iC^{-2}\sum_{\lambda}\sum_{j=1,2}\int d^3p[\phi^*_{p;j}\partial_\tau\phi_{p;j} + \gamma^*_{p;j}\partial_\tau\gamma_{p;j} - \partial_\tau\phi^*_{p;j}\phi_{p;j} - \partial_\tau\gamma^*_{p;j}\gamma_{p;j}]\end{aligned} \tag{47}$$

and

$$\begin{aligned}\mathbb{T}^{\mu}_{\mu}[\phi_{p;j},\gamma_{p;j}] = 4iC^{-3}\sin^2\theta\sum_{\lambda}\int d^3p\Big\{&-i|\Xi_p(\tau_0)|^2\sum_{j=1,2}m_j(|\phi_{p;j}|^2 - |\gamma_{p;j}|^2)\\ &+ 2i\Im[im_2\Xi^*_p(\tau_0)\Lambda^*_p(\tau_0)\phi^*_{p;2}\gamma^*_{p;2} - im_1\Xi^*_p(\tau_0)\Lambda_p(\tau_0)\phi^*_{p;1}\gamma^*_{p;1}]\Big\}\\ &- 2C^{-3}\sum_{\lambda}\sum_{j=1,2}m_j\int d^3p(|\phi_{p;j}|^2 - |\gamma_{p;j}|^2).\end{aligned} \tag{48}$$

Note that both the $\tau\tau$ component (47) and the trace (48) can also be seen as functionals of $\{\phi_p, \partial_\tau\phi_p\}$. In fact, from Eqs. (15) we have

$$\begin{aligned}\gamma_p(\eta) &= \frac{i}{p}(\partial_\eta\phi_p(\eta) + imC(\eta)\phi_p(\eta)),\\ \partial_\eta\gamma_p(\eta) &= \frac{i}{p}(\partial^2_\eta\phi_p(\eta) + im\partial_\eta C(\eta)\phi_p(\eta)\\ &\quad + imC(\eta)\partial_\eta\phi_p(\eta)),\end{aligned} \tag{49}$$

which can be substituted in the expressions above for $\mathbb{T}_{\mu\nu}$. From the diagonality of the expectation value it is also straightforward to prove its covariant conservation $\nabla_\mu\mathbb{T}^{\mu\nu} = 0$ (see Appendix E for the details).

V. APPLICATIONS: EXPONENTIAL EXPANSION

We have shown that independent of the specific scale factor C, the energy-momentum tensor associated with the flavor vacuum $\mathbb{T}^{(MIX)}_{\mu\nu}$ satisfies a number of important properties: (i) it is diagonal, (ii) it is covariantly conserved, and (iii) it depends only on time τ. Then $\mathbb{T}^{(MIX)}_{\mu\nu}$ for the metric (1) corresponds to the energy-momentum tensor of a perfect fluid with time-dependent energy density and pressure. In this section, in order to better understand the properties of $\mathbb{T}^{(MIX)}_{\mu\nu}$, we assume a specific evolution of the scale factor C and compute the corresponding expectation value of the energy-momentum tensor on the flavor vacuum. In doing so, we are neglecting the backreaction due to the flavor fields, i.e., the modifications induced on the metric by the energy-momentum tensor of Eq. (26). The computation based on a fixed background metric is undoubtedly an approximation, but it is useful to get an insight into the kind of contribution that emerges from the flavor vacuum. The self-consistent way to deal with $\mathbb{T}_{\mu\nu}$ is to insert Eq. (26), together with all the relevant matter terms, on the right-hand side of the Einstein equations and then solve simultaneously for the scale factor C and the Dirac modes. This kind of calculation will be performed elsewhere.

A. Positive energy solutions

Here, in particular, we study the energy-momentum tensor corresponding to an exponential evolution of the scale factor $C(t) = e^{H_0 t}$, with H_0 a constant with dimensions of mass. The great advantage of the so-picked scale factor is that the corresponding Dirac equation can be solved analytically without resorting to any approximation, allowing for an in-depth analysis of $\mathbb{T}_{\mu\nu}^{(MIX)}$. Transforming to conformal time we have

$$\tau = -\frac{1}{H_0}e^{-H_0 t}, \qquad C = -\frac{1}{H_0\tau}. \tag{50}$$

Notice that the conformal time τ is always negative. Inserting Eq. (50) into Eq. (16) we obtain

$$\tau^2\partial_\tau^2\phi_p + \left(p^2\tau^2 + \frac{im}{H_0} + \frac{m^2}{H_0^2}\right)\phi_p = 0, \tag{51}$$

or, introducing the positive variable $s = -p\tau$,

$$s^2\partial_s^2\phi_p + \left(s^2 + \frac{im}{H_0} + \frac{m^2}{H_0^2}\right)\phi_p = 0. \tag{52}$$

This is a Bessel-like equation, whose general solution can be written as

$$\phi_p(s) = s^{\frac{1}{2}}(C_1 J_\nu(s) + C_2 J_{-\nu}(s)), \quad \nu = \frac{1}{2}\left(1 - \frac{2im}{H_0}\right) \tag{53}$$

with $J_\nu(s)$ denoting the Bessel function of order ν and C_1, C_2 arbitrary complex constants. In order to specify the solution we need to impose some kind of boundary conditions, which in turn determine the positive energy solutions. We require that the modes of Eq. (53) be positive energy with respect to ∂_τ at early times, i.e., for $\tau \to -\infty$ (where $C \to 0$). With this choice the mass vacuum corresponds to the absence of massive neutrinos at early times. As $\tau \to -\infty$ the mass terms can be neglected, and Eq. (51) becomes

$$\partial_\tau^2\phi_p + p^2\phi_p = 0. \tag{54}$$

The positive energy solution with respect to ∂_τ is evidently $\phi_p^+ \propto e^{-ip\tau}$ or, in terms of the s variable, $\phi_p^+(s) \propto e^{is}$. We then impose the requirement $\lim_{s\to+\infty}\phi_p(s) \propto e^{is}$. Recalling that s is a positive real variable, we can employ the large argument expansion of the Bessel functions [59]

$$J_\nu(s) \simeq \sqrt{\frac{2}{\pi s}}\cos\left(s - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } s \to +\infty. \tag{55}$$

In this way, we show that the combination satisfying the requirement is given by

$$\begin{aligned}\phi_p(s) &= N_p s^{\frac{1}{2}}\left(J_\nu(s) - ie^{\frac{\pi m}{H_0}}J_{-\nu}(s)\right)\\ &\simeq N_p\sqrt{\frac{2}{\pi}}\left(-ie^{\frac{\pi m}{2H_0}}\right)\cosh\left(\frac{\pi m}{H_0}\right)e^{is},\end{aligned} \tag{56}$$

where N_p is a normalization constant and the last equivalence holds in the limit $s \to +\infty$. Inserting Eq. (56) in the first of Eqs. (49), we deduce

$$\gamma_p(s) = N_p s^{\frac{1}{2}}\left(-iJ_{\nu-1}(s) + e^{\frac{\pi m}{H_0}}J_{1-\nu}(s)\right), \tag{57}$$

where we have made use of the differential relations satisfied by the Bessel functions [59]. In order to fix N_p we impose the normalization condition (15) in the $s \to +\infty$ limit. Using the large argument expansion once again, we obtain $|N_p|^2 = \frac{1}{32\pi^2\cosh^2(\frac{\pi m}{H_0})}e^{-\frac{\pi m}{H_0}}$. Finally, choosing N_p real, we can write the positive energy solutions as

$$\begin{aligned}\phi_p(s) &= \frac{1}{4\pi}\frac{e^{-\frac{\pi m}{2H_0}}}{\cosh(\frac{\pi m}{H_0})}\sqrt{\frac{s}{2}}\left(J_\nu(s) - ie^{\frac{\pi m}{H_0}}J_{-\nu}(s)\right),\\ \gamma_p(s) &= \frac{1}{4\pi}\frac{e^{-\frac{\pi m}{2H_0}}}{\cosh(\frac{\pi m}{H_0})}\sqrt{\frac{s}{2}}\left(-iJ_{\nu-1}(s) + e^{\frac{\pi m}{H_0}}J_{1-\nu}(s)\right)\end{aligned} \tag{58}$$

with $\nu = \frac{1}{2}(1 - \frac{2im}{H_0})$ and $s = -p\tau$.

B. Bogoliubov coefficients

We have two sets of solutions $\phi_{p,j}, \gamma_{p,j}$, one for each value of the mass m_j, with $j = 1, 2$ from Eqs. (58). The compatibility requirement then implies that each of the $\phi_{p,j}, \gamma_{p,j}$ has the same form for $j = 1, 2$, except that one has m_j wherever the mass appears, including the function index $\nu_j = \frac{1}{2}(1 - \frac{2im_j}{H_0})$. We can compute the Bogoliubov coefficients $[\Lambda_p(\tau), \Xi_p(\tau)]$ straight away from Eqs. (44):

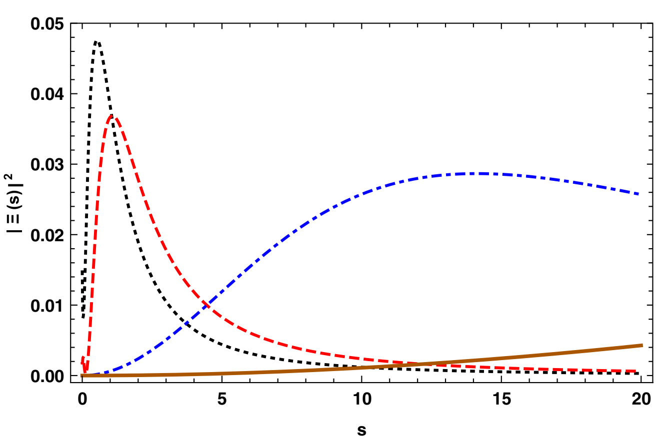

FIG. 1. Squared modulus of the Bogoliubov coefficient $|\Xi_p|^2$ as a function of s for sample values of the masses (all in units of H_0): (black dotted line) $m_1 = 0.7$, $m_2 = 1.4$; (red dashed line) $m_1 = 1$, $m_2 = 2$; (blue dot-dashed line) $m_1 = 10$, $m_2 = 20$; and (dark orange solid line), $m_1 = 100$, $m_2 = 300$. The momentum dependence is implicit in $s = -p\tau$.

$$\Lambda_p(\tau) = \frac{\pi s}{4}\frac{e^{-\frac{\pi}{2H_0}(m_1+m_2)}}{\cosh(\frac{\pi m_1}{H_0})\cosh(\frac{\pi m_2}{H_0})}\Big\{J^*_{\nu_2}(s)J_{\nu_1}(s) + J^*_{\nu_2-1}(s)J_{\nu_1-1}(s) + ie^{\frac{\pi m_1}{H_0}}[J^*_{\nu_2-1}(s)J_{1-\nu_1}(s) - J^*_{\nu_2}(s)J_{-\nu_1}(s)]$$
$$+ ie^{\frac{\pi m_2}{H_0}}[J^*_{-\nu_2}(s)J_{\nu_1}(s) - J^*_{1-\nu_2}(s)J_{\nu_1-1}(s)] + e^{\frac{\pi}{H_0}(m_1+m_2)}[J^*_{-\nu_2}(s)J_{-\nu_1}(s) + J^*_{1-\nu_2}(s)J_{1-\nu_1}(s)]\Big\},$$
$$\Xi_p(\tau) = \frac{\pi s}{4}\frac{e^{-\frac{\pi}{2H_0}(m_1+m_2)}}{\cosh(\frac{\pi m_1}{H_0})\cosh(\frac{\pi m_2}{H_0})}\Big\{i[J^*_{\nu_1}(s)J^*_{\nu_2-1}(s) - J^*_{\nu_1-1}(s)J^*_{\nu_2}(s)] + e^{\frac{\pi m_2}{H_0}}[J^*_{\nu_1}(s)J^*_{1-\nu_2}(s) + J^*_{\nu_1-1}(s)J^*_{-\nu_2}(s)]$$
$$- e^{\frac{\pi m_1}{H_0}}[J^*_{-\nu_1}(s)J^*_{\nu_2-1}(s) + J^*_{1-\nu_1}(s)J^*_{\nu_2}(s)] + ie^{\frac{\pi}{H_0}(m_1+m_2)}[J^*_{-\nu_1}(s)J^*_{1-\nu_2}(s) - J^*_{1-\nu_1}(s)J^*_{-\nu_2}(s)]\Big\}. \tag{59}$$

To give a flavor of the behavior of the Bogoliubov coefficients we have plotted $|\Xi_p(s)|^2$ as a function of s for sample values of m_1, m_2 in Fig. 1.

We shall be particularly interested in the late time expression of the Bogoliubov coefficients, namely for $s \to 0^+$ (the corresponding limit is $\tau \to 0^-$, or $t \to +\infty$). For its determination we make use of the small argument expansion of the Bessel functions $J_\nu(s) \simeq (\frac{s}{2})^\nu \frac{1}{\Gamma(1+\nu)}$ with Γ denoting the Euler gamma function. From Eqs. (59), it is easy to find that at the leading order for $s \to 0^+$ one has

$$\Lambda_p(s) \simeq \frac{\pi}{2}\frac{e^{-\frac{\pi}{2H_0}(m_1+m_2)}}{\cosh(\frac{\pi m_1}{H_0})\cosh(\frac{\pi m_2}{H_0})}\left[\frac{e^{i(\frac{m_2-m_1}{H_0})\log(\frac{s}{2})}}{\Gamma^*(\nu_2)\Gamma(\nu_1)} + e^{\frac{\pi}{H_0}(m_1+m_2)}\frac{e^{-i(\frac{m_2-m_1}{H_0})\log(\frac{s}{2})}}{\Gamma^*(1-\nu_2)\Gamma(1-\nu_1)}\right],$$
$$\Xi_p(s) \simeq \frac{\pi}{2}\frac{e^{-\frac{\pi}{2H_0}(m_1+m_2)}}{\cosh(\frac{\pi m_1}{H_0})\cosh(\frac{\pi m_2}{H_0})}\left[\frac{e^{\frac{\pi m_2}{H_0}}e^{-i(\frac{m_2-m_1}{H_0})\log(\frac{s}{2})}}{\Gamma^*(\nu_1)\Gamma^*(1-\nu_2)} - \frac{e^{\frac{\pi m_1}{H_0}}e^{i(\frac{m_2-m_1}{H_0})\log(\frac{s}{2})}}{\Gamma^*(1-\nu_1)\Gamma^*(\nu_2)}\right]. \tag{60}$$

C. Explicit form of the energy-momentum tensor

For the explicit calculation it is convenient to refer to the splitting of Eq. (32) and compute $\mathbb{T}^{(MIX)}_{\mu\nu}$ and $\mathbb{T}^{(N)}_{\mu\nu}$ separately. Moreover, keeping in mind the results of the previous sections, it is sufficient to compute the $\tau\tau$ component and the trace in order to fully determine $\mathbb{T}_{\mu\nu}$. Inserting the solutions (58) in Eq. (47) we find, after a lengthy but straightforward calculation,

$$\mathbb{T}^{(MIX)}_{\tau\tau}(\tau) = i\sin^2\theta\sum_\lambda\int d^3p|\Xi_p(\tau_0)|^2\left(\frac{H_0^2p^2\tau^3}{16\pi^2}\right)\sum_{j=1,2}\frac{e^{-\frac{\pi m_j}{H_0}}}{\cosh^2(\frac{\pi m_j}{H_0})}\Big\{\Big[2(J^*_{\nu_j}J_{\nu_j-1} - J^*_{\nu_j-1}J_\nu) + \frac{\nu_j^*-\nu_j}{s}|J_{\nu_j}|^2$$
$$+\frac{\nu_j-\nu_j^*}{s}|J_{\nu_j-1}|^2\Big] + ie^{\frac{\pi m_j}{H_0}}\Big[2(J^*_{\nu_j}J_{1-\nu_j} + J^*_{-\nu_j}J_{\nu_j-1} + J^*_{\nu_j-1}J_{-\nu_j} + J^*_{1-\nu_j}J_{\nu_j}) + \frac{\nu_j-\nu_j^*}{s}J^*_{\nu_j}J_{-\nu_j}$$
$$+\frac{\nu_j^*-\nu_j}{s}J^*_{-\nu_j}J_{\nu_j} + \frac{\nu_j-\nu_j^*}{s}J^*_{\nu_j-1}J_{1-\nu_j} + \frac{\nu_j^*-\nu_j}{s}J^*_{1-\nu_j}J_{\nu_j-1}\Big] + e^{\frac{2\pi m_j}{H_0}}\Big[2(J^*_{1-\nu_j}J_{-\nu_j} - J^*_{-\nu_j}J_{1-\nu_j})$$
$$+\frac{\nu_j^*-\nu_j}{s}|J_{-\nu_j}|^2 + \frac{\nu_j-\nu_j^*}{s}|J_{1-\nu_j}|^2\Big]\Big\}$$
$$+\frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\Big\{\Xi^*_p(\tau_0)\Lambda_p(\tau_0)\left(\frac{H_0^2p^2\tau^3e^{-\frac{\pi m_1}{H_0}}}{8\pi^2\cosh^2(\frac{\pi m_1}{H_0})}\right)\Big[\Big(-i(J^*_{\nu_1})^2 - i(J^*_{\nu_1-1})^2 + i\frac{2\nu_1^*-1}{s}J^*_{\nu_1}J^*_{\nu_1-1}\Big)$$
$$+e^{\frac{\pi m_1}{H_0}}\Big(2(J^*_{\nu_1}J^*_{-\nu_1} - J^*_{\nu_1-1}J^*_{1-\nu_1}) + \frac{2\nu_1^*-1}{s}J^*_{\nu_1}J^*_{1-\nu_1} + \frac{1-2\nu_1^*}{s}J^*_{-\nu_1}J^*_{\nu_1-1}\Big)$$
$$+ie^{\frac{2\pi m_1}{H_0}}\Big((J^*_{-\nu_1})^2 + (J^*_{1-\nu_1})^2 + \frac{2\nu_1^*-1}{s}J^*_{-\nu_1}J^*_{1-\nu_1}\Big)\Big] - \text{c.c.}\Big\}$$
$$-\frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\{\Xi^*_p(\tau_0)\Lambda^*_p(\tau_0)\left(\frac{H_0^2p^2\tau^3e^{-\frac{\pi m_2}{H_0}}}{8\pi^2\cosh^2(\frac{\pi m_2}{H_0})}\right)\Big[\Big(-i(J^*_{\nu_2})^2 - i(J^*_{\nu_2-1})^2 + i\frac{2\nu_2^*-1}{s}J^*_{\nu_2}J^*_{\nu_2-1}\Big)$$
$$+e^{\frac{\pi m_2}{H_0}}\Big(2(J^*_{\nu_2}J^*_{-\nu_2} - J^*_{\nu_2-1}J^*_{1-\nu_2}) + \frac{2\nu_2^*-1}{s}J^*_{\nu_2}J^*_{1-\nu_2} + \frac{1-2\nu_2^*}{s}J^*_{-\nu_2}J^*_{\nu_2-1}\Big)$$
$$+ie^{\frac{2\pi m_2}{H_0}}\Big((J^*_{-\nu_2})^2 + (J^*_{1-\nu_2})^2 + \frac{2\nu_2^*-1}{s}J^*_{-\nu_2}J^*_{1-\nu_2}\Big)\Big] - \text{c.c.}\Big\}. \tag{61}$$

In the above equation we have left the Bogoliubov coefficients implicit and the argument of the Bessel functions $s = -p\tau$ has been omitted for a better visualization. We recall that the helicity sum is over $\lambda = \pm 1$ and that in general $\tau \neq \tau_0$. An analogous calculation can be performed for the trace:

$$\begin{aligned}\mathbb{T}^{\mu(MIX)}_{\mu} &= i\sin^2\theta\sum_\lambda \int d^3p|\Xi_p(\tau_0)|^2\left(\frac{iH_0^3\tau^3 s}{8\pi^2}\right)\sum_{j=1,2}\left(\frac{m_j e^{\frac{-\pi m_j}{H_0}}}{\cosh^2(\frac{\pi m_j}{H_0})}\right)\Big\{|J_{\nu_j}|^2 - |J_{\nu_j-1}|^2 \\ &\quad + ie^{\frac{\pi m_j}{H_0}}(J^*_{-\nu_j}J_{\nu_j} - J^*_{\nu_j}J_{-\nu_j} + J^*_{1-\nu_j}J_{\nu_j-1} - J^*_{\nu_j-1}J_{1-\nu_j}) + e^{\frac{2\pi m_j}{H_0}}(|J_{-\nu_j}|^2 - |J_{1-\nu_j}|^2)\Big\} \\ &\quad + \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\Big\{\Xi^*_p(\tau_0)\Lambda_p(\tau_0)\left(\frac{im_1 sH_0^3\tau^3 e^{-\frac{\pi m_1}{H_0}}}{4\pi^2\cosh^2(\frac{\pi m_1}{H_0})}\right)\Big[iJ^*_{\nu_1}J^*_{\nu_1-1} + e^{\frac{\pi m_1}{H_0}}J^*_{\nu_1}J^*_{1-\nu_1} \\ &\quad - e^{\frac{\pi m_1}{H_0}}J^*_{-\nu_1}J^*_{\nu_1-1} + ie^{\frac{2\pi m_1}{H_0}}J^*_{-\nu_1}J^*_{1-\nu_1}\Big] - \text{c.c.}\Big\} \\ &\quad - \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\Big\{\Xi^*_p(\tau_0)\Lambda^*_p(\tau_0)\left(\frac{im_2 sH_0^3\tau^3 e^{-\frac{\pi m_2}{H_0}}}{4\pi^2\cosh^2(\frac{\pi m_2}{H_0})}\right)\Big[iJ^*_{\nu_2}J^*_{\nu_2-1} + e^{\frac{\pi m_2}{H_0}}J^*_{\nu_2}J^*_{1-\nu_2} \\ &\quad - e^{\frac{\pi m_2}{H_0}}J^*_{-\nu_2}J^*_{\nu_2-1} + ie^{\frac{2\pi m_2}{H_0}}J^*_{-\nu_2}J^*_{1-\nu_2}\Big] - \text{c.c.}\Big\}. \end{aligned} \tag{62}$$

We are particularly interested in the late time ($\tau \to 0^-$) expression of the above equations. According to the definition (26), these represent the contribution of the flavor vacuum state, defined at an earlier time $\tau_0 < \tau$, to the energy-momentum tensor at late times. We then perform the small argument expansion $J_\nu(-p\tau) \simeq (\frac{-p\tau}{2})^\nu \frac{1}{\Gamma(1+\nu)}$ for all the Bessel functions appearing in Eqs. (61) and (62), and keep only the terms of lowest order in the variable τ. At order τ, the $\tau\tau$ component is found to be

$$\begin{aligned}\mathbb{T}^{(MIX)(1)}_{\tau\tau}(\tau) &\simeq i\sin^2\theta\sum_\lambda\int d^3p|\Xi_p(\tau_0)|^2\left(i\frac{H_0\tau}{2\pi^3}\right)\sum_{j=1,2}m_j\tanh\left(\frac{\pi m_j}{H_0}\right) \\ &\quad + \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\left[\Xi^*_p(\tau_0)\Lambda_p(\tau_0)\left(\frac{-im_1H_0\tau}{2\pi^3\cosh(\frac{\pi m_1}{H_0})}\right) - \text{c.c.}\right] \\ &\quad - \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\left[\Xi^*_p(\tau_0)\Lambda^*_p(\tau_0)\left(\frac{-im_2H_0\tau}{2\pi^3\cosh(\frac{\pi m_2}{H_0})}\right) - \text{c.c.}\right].\end{aligned} \tag{63}$$

The corresponding lowest order in the trace is $\propto \tau^3$. To see why this is the case, recall that by definition

$$\mathbb{T}^\mu_\mu = C^{-2}\mathbb{T}_{\tau\tau} - C^{-2}\sum_{l=1}^3\mathbb{T}_{ll} = H_0^2\tau^2\mathbb{T}_{\tau\tau} - H_0^2\tau^2\sum_{l=1}^3\mathbb{T}_{ll}, \tag{64}$$

so that in correspondence with $\mathbb{T}_{\tau\tau} \propto \tau$ one has $\mathbb{T}^\mu_\mu \propto \tau^3$. To this order the trace is

$$\begin{aligned}\mathbb{T}^{\mu(MIX)(1)}_{\mu} &\simeq i\sin^2\theta\sum_\lambda\int d^3p|\Xi_p(\tau_0)|^2\left(\frac{iH_0^3\tau^3}{2\pi^3}\right)\sum_{j=1,2}m_j\tanh\left(\frac{\pi m_j}{H_0}\right) \\ &\quad + \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\left[\Xi^*_p(\tau_0)\Lambda_p(\tau_0)\left(\frac{-im_1H_0^3\tau^3}{2\pi^3\cosh(\frac{\pi m_1}{H_0})}\right) - \text{c.c.}\right] \\ &\quad - \frac{i}{2}\sin^2\theta\sum_\lambda\int d^3p\left[\Xi^*_p(\tau_0)\Lambda^*_p(\tau_0)\left(\frac{-im_2H_0^3\tau^3}{2\pi^3\cosh(\frac{\pi m_2}{H_0})}\right) - \text{c.c.}\right].\end{aligned} \tag{65}$$

Inserting Eqs. (63) and (65) in Eq. (46), we can deduce the important result $\mathbb{T}^{(MIX)(1)}_{ii} = 0, \ \forall\ i$. In other words, at first order in τ, the spatial components of the energy-momentum tensor vanish. Then the equation of state reads, at lowest order in τ,

$$w^{(1)} = \frac{\mathbb{T}_i^{i(MIX)(1)}}{\mathbb{T}_\tau^{\tau(MIX)(1)}} = 0, \tag{66}$$

i.e., *the energy-momentum tensor associated with the flavor vacuum satisfies, at late times ($\tau \to 0^-$), the equation of state of a pressureless perfect fluid*. It is important to stress that this result does not depend on the value of the momentum integrals, and therefore is independent of any regularization. Moreover, as it is evident from (66), it holds for any choice of the reference time τ_0.

D. Regularization

All the momentum integrals in the expression for the energy-momentum tensor, both for the general case and the for the late time approximation, are to be performed over the whole of $\mathbb{R}^3$ and are formally divergent. To extract a finite result, we need some form of regularization. The most immediate way to regularize the momentum integrations is to introduce an ultraviolet momentum cutoff $\mathcal{K}$. Usually, the cutoff is chosen in correspondence with a "new physics" energy scale, beyond which the "low energy" quantum field theory description breaks down. In our case, $\mathcal{K}$ ought to be related to the scale at which the semiclassical approximation breaks down, i.e., at the quantum gravity scale $\mathcal{K} \simeq M_{pl}$ which is of the order of the Planck mass M_{pl}. However, we can expect the cutoff to be at a much lower scale for what concerns the proper mixing term. Indeed, at very high energies, the mass difference between the neutrino states becomes negligible, and the oscillation frequency $\propto \frac{1}{p}$ approaches zero, implying that there is no oscillation. The same result also holds in quantum field theory, where the mixing Bogoliubov coefficient Ξ_p generally approaches zero at high energies (therefore yielding no contribution to the energy-momentum tensor). The heuristic argument above provides a physical reason to adopt the cutoff regularization and also justifies the adoption of a cutoff scale $\mathcal{K} \ll M_{pl}$, at least for what concerns the mixing.

Before proceeding we need to clarify that the cutoff must be imposed upon the comoving momentum $\boldsymbol{p}_{\text{PHYS}} = \frac{\boldsymbol{p}}{C}$ rather than the mode label p.

The imposition of a cutoff $\mathcal{K}_0$ on p_{PHYS} then translates into a sort of "comoving" cutoff for the mode label p

$$p_{\text{CUTOFF}} = \mathcal{K}_0 C = \frac{-\mathcal{K}_0}{H_0 \tau} = \mathcal{K}(\tau), \tag{67}$$

which is strictly positive (recall that $\tau < 0$). For the late time energy-momentum tensor, in the approximation in which $\tau_0 < \tau$ is also at late times $\tau_0 \to 0^-$, we can give a simple analytical form of the regularized integrals. Performing the integrals in Eqs. (63) and (65), with the comoving cutoff of Eq. (67), we obtain

$$\begin{aligned}
\mathbb{T}_{\tau\tau}^{(MIX)(1)} = {} & \frac{-\sin^2\theta H_0 \tau \mathcal{K}^3(\tau)}{3\pi^2} \left[\frac{e^{\frac{\pi(m_2-m_1)}{H_0}} + e^{-\frac{\pi(m_2-m_1)}{H_0}}}{\cosh(\frac{\pi m_1}{H_0})\cosh(\frac{\pi m_2}{H_0})} \left(m_1 \tanh\left(\frac{\pi m_1}{H_0}\right) + m_2 \tanh\left(\frac{\pi m_2}{H_0}\right) \right) \right. \\
& \left. - 2\frac{m_1 \tanh(\frac{\pi m_2}{H_0})}{\cosh^2(\frac{\pi m_1}{H_0})} - 2\frac{m_2 \tanh(\frac{\pi m_1}{H_0})}{\cosh^2(\frac{\pi m_2}{H_0})} \right] + \sin^2\theta H_0 \tau \mathcal{K}^3(\tau) \left[\left(\frac{m_1 \tanh(\frac{\pi m_1}{H_0}) + m_2 \tanh(\frac{\pi m_2}{H_0})}{\cosh^2(\frac{\pi m_1}{H_0})\cosh^2(\frac{\pi m_2}{H_0})} \right) \right. \\
& \left. \times \left(\frac{1}{\Gamma(\nu_1)\Gamma^*(\nu_2)} \right)^2 \left(\frac{1}{3 + 2i\frac{m_2-m_1}{H_0}} \right) \left(\frac{-\mathcal{K}(\tau)\tau_0}{2} \right)^{\frac{2i(m_2-m_1)}{H_0}} + \text{c.c.} \right] \\
& + i\sin^2\theta \mathcal{K}^3(\tau) \left\{ \left[\left(\frac{-i m_1 H_0 \tau e^{\frac{-\pi m_1}{H_0}}}{2\cosh^3(\frac{\pi m_1}{H_0})\cosh^2(\frac{\pi m_2}{H_0})} \right) \left(\frac{1}{\Gamma(\nu_1)\Gamma^*(\nu_2)} \right)^2 \left(\frac{1}{3 + 2i\frac{m_2-m_1}{H_0}} \right) \left(\frac{-\mathcal{K}(\tau)\tau_0}{2} \right)^{\frac{2i(m_2-m_1)}{H_0}} \right. \right. \\
& \left. \left. - \left(\frac{-i m_1 H_0 \tau e^{\frac{-\pi m_1}{H_0}}}{2\cosh^3(\frac{\pi m_1}{H_0})\cosh^2(\frac{\pi m_2}{H_0})} \right) \left(\frac{1}{\Gamma^*(\nu_1)\Gamma(\nu_2)} \right)^2 \left(\frac{1}{3 - 2i\frac{m_2-m_1}{H_0}} \right) \left(\frac{-\mathcal{K}(\tau)\tau_0}{2} \right)^{\frac{-2i(m_2-m_1)}{H_0}} \right] - \text{c.c.} \right\} \\
& - i\sin^2\theta \mathcal{K}^3(\tau) \left\{ \left[\left(\frac{-i m_2 H_0 \tau e^{\frac{-\pi m_2}{H_0}}}{2\cosh^3(\frac{\pi m_2}{H_0})\cosh^2(\frac{\pi m_1}{H_0})} \right) \left(\frac{1}{\Gamma(\nu_1)\Gamma^*(\nu_2)} \right)^2 \left(\frac{1}{3 + 2i\frac{m_2-m_1}{H_0}} \right) \left(\frac{-\mathcal{K}(\tau)\tau_0}{2} \right)^{\frac{2i(m_2-m_1)}{H_0}} \right. \right. \\
& \left. \left. - \left(\frac{-i m_2 H_0 \tau e^{\frac{-\pi m_2}{H_0}}}{2\cosh^3(\frac{\pi m_2}{H_0})\cosh^2(\frac{\pi m_1}{H_0})} \right) \left(\frac{1}{\Gamma^*(\nu_1)\Gamma(\nu_2)} \right)^2 \left(\frac{1}{3 - 2i\frac{m_2-m_1}{H_0}} \right) \left(\frac{-\mathcal{K}(\tau)\tau_0}{2} \right)^{\frac{-2i(m_2-m_1)}{H_0}} \right] - \text{c.c.} \right\}.
\end{aligned} \tag{68}$$

The trace can be deduced immediately from Eq. (68) by simply multiplying by the factor C^{-2}, since the pressure is zero for the late time energy-momentum tensor [see Eq. (66)].

A visual indication of the physical content of Eq. (68) is provided in Fig. 2, where the mixing energy density $\mathbb{T}_\tau^{\tau(MIX)}$ from Eq. (68) is plotted against conformal time τ for sample values of the parameters. We notice that ρ_{MIX}, for the parameters and

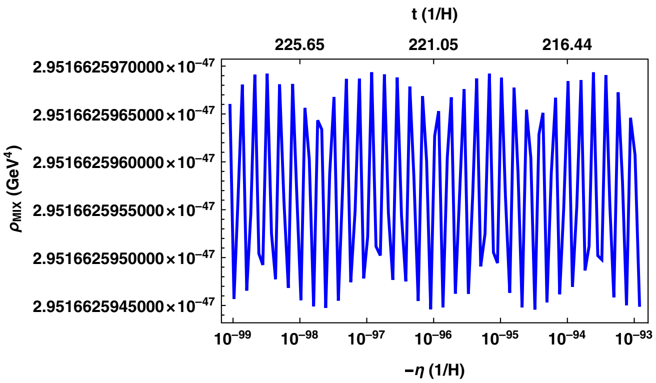

FIG. 2. Logarithmic scale plot of the energy density $\rho_{MIX} = \mathbb{T}_{\tau}^{\tau(MIX)}$ from Eq. (68) as a function of conformal time τ for sample values of the parameters. The corresponding coordinate time t is reported above. We have used a cutoff $\mathcal{K}_0 = 246$ GeV of the order of the electroweak scale, neutrino masses $m_1 = 15.25 H_0$, $m_2 = 22.25 H_0$, and the expansion rate $H_0 = 10^{-3}$ eV.

the range considered in Fig. 2, is nearly constant, except for tiny oscillations of relative magnitude $\delta\rho_{MIX}/\rho_{MIX} \simeq 10^{-9}$. The oscillations come from the imaginary exponentials in Eq. (68). Such a simple evolution pattern can be expected to disappear when other time ranges are considered, and the full expression from Eq. (61) is employed, entailing a much more intricate time evolution. The results obtained in this work indicate that the vacuum energy associates with neutrino mixing in curved space and might represent a dark matter component. Notice that for a momentum cutoff of the order of the electroweak scale $\mathcal{K}_0 = 246$ GeV, for neutrino masses m_1, m_2 such that $\Delta m_{12}^2 \simeq 10^{-5}$ eV^2 and for a value of $H_0 \simeq 10^{-3}$ eV, we obtain an energy density $\mathbb{T}_{\tau}^{\tau(MIX)}$ which is compatible with the upper bound on the dark matter content of the universe. A value of $H_0 \simeq 10^{-3}$ eV might have been reached during the very early phases of the universe, i.e., during the first second after the big bang.

Two important points need to be noted. First, the results obtained do provide only an indication of the possible role of the flavor vacuum as a dark matter component. By no means do they suffice to *identify* a dark matter component with the flavor vacuum. Such a claim obviously requires that the analysis be conducted on other metrics. Second, the contribution to the energy density of Eq. (68) is due to the *flavor vacuum* of neutrinos. Such a vacuum has the structure of a condensate of particle-antiparticle pairs of neutrinos with definite masses. Therefore this state has the same number of leptons and antileptons, and no lepton asymmetry is involved in this state.

The contribution due to the flavor vacuum is clearly distinct from the one that neutrinos themselves bring along, which certainly corresponds to states with a finite number of particles, and not to the vacuum.

We remark, in addition, that we have computed the expectation value of the energy-momentum tensor on the flavor vacuum state. Such a contribution, being associated with the vacuum state, is independent of the annihilation of fermion-antifermion pairs with definite flavor ($\nu_e - \bar{\nu}_e, \nu_\mu - \bar{\nu}_\mu$), since the flavor vacuum is in itself a condensate of particle-antiparticle pairs with definite masses ($\nu_1, \bar{\nu}_1, \nu_2, \bar{\nu}_2$). Note also that the rate of the fermion-antifermion annihilation $f + \bar{f} \to 2\gamma$ is extremely suppressed for flavor neutrinos (see, e.g., [60]), which are to date the only elementary fermions known to oscillate.

Moreover, as the energy density (68) is associated with the vacuum state, it represents an additional contribution to the energy density of any neutrino state, for any distribution of neutrinos and antineutrinos with definite flavor. Therefore this additional contribution is present regardless of the thermodynamics of neutrinos, and in particular, is independent of the chemical potential μ.

VI. CONCLUSIONS

In this paper we have considered the quantum field theory of fermion mixing on curved spacetime, and we have computed the expectation value $\mathbb{T}_{\mu\nu}^{(MIX)}$ of the energy-momentum tensor of fermion fields on the flavor vacuum in a flat FLRW background. $\mathbb{T}_{\mu\nu}^{(MIX)}$ behaves as an effective energy-momentum tensor that satisfies the Bianchi identities (i.e., its divergence with respect to one of the two indices vanishes) and therefore can be employed as a regular source for the Einstein equations. In particular, it turns out that $\mathbb{T}_{\mu\nu}^{(MIX)}$ can be interpreted as the energy-momentum tensor of a barotropic fluid.

In this picture, therefore, quantum effects of the fermion fields can be associated, at classical level, with additional barotropic fluids whose thermodynamical properties depend on the geometry of the spacetime other than the specific features of the fermion field itself. In light of the matter field interpretation of the dark components of the universe this result seems quite encouraging, as it links directly quantum effects to effective classical fluids. Thus our results show explicitly that there can be a bridge between quantum properties of matter and dark matter/energy. However, the presence of a connection between classical fluids and quantum effect is not automatically sufficient alone to conclude that quantum effects are the prime cause of dark matter/energy. In any case, our findings provide an indication that the vacuum energy associated with neutrino mixing can represent a component of dark matter.

Here we are specifically interested in determining the form that the energy-momentum tensor, associated with flavor vacuum, assumes in cosmological FLRW metrics. In order to grasp the behavior of the energy-momentum tensor of the flavor vacuum, we have then assumed a specific form of the metric. In doing so, we have clearly considered that the term due to the flavor vacuum is not the dominant source of the Einstein equations, which are instead determined by some other source that forces the specific form of the metric.

In particular, considering a de Sitter underlying metric, we have derived the components of $\mathbb{T}_{\mu\nu}^{(MIX)}$ exactly using Bessel functions. It turns out that at first order in the time parameter only $\mathbb{T}_{\tau\tau}^{(MIX)}$ is different from zero. Hence, in this case the mixing of fermion fields can be associated with a zero pressure (dust or cold dark matter) fluid. We remark that the choice of the de Sitter metric is dictated, primarily, by the fact that exact analytical solutions of the Dirac equation can be found in this context. The actual identification of such a contribution as a dark matter component requires that the study be conducted on metrics that are adequate to the description of galaxies. Such an analysis will be performed in a forthcoming paper, where we will also consider the contribution due to the flavor vacuum as the dominant source in the Einstein equations. Nevertheless, the results presented here clearly hint at a dark-matter-like behavior of the flavor vacuum in a curved background.

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APPENDIX A: ORTHONORMALITY AND COMPLETENESS OF THE SOLUTION OF DIRAC EQUATION

Knowing that the determinant is $g = -C^8$, we can easily compute the scalar product between the solutions (8):

$$\begin{aligned}(u_{\boldsymbol{p},\lambda}, u_{\boldsymbol{q},\lambda'})_\tau &= \int_{\Sigma_\tau} d^3x C^3 u^\dagger_{\boldsymbol{p},\lambda} u_{\boldsymbol{q},\lambda'} \\ &= \int_{\Sigma_\tau} d^3x C^3 e^{-i(\boldsymbol{p}-\boldsymbol{q})\cdot \boldsymbol{x}} \xi^\dagger_\lambda \xi_{\lambda'} (f^*_p f_q + \lambda\lambda' g^*_p g_q) \\ &= (2\pi)^3 C^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) \delta_{\lambda,\lambda'} (|f_p|^2 + |g_p|^2). \end{aligned} \tag{A1}$$

In the last step we have used the definition of the Dirac delta function and the orthonormality of the helicity bispinors. In a similar way it is easy to show that $(u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{q},\lambda'})_\tau = 0$, $(v_{\boldsymbol{p},\lambda}, u_{\boldsymbol{q},\lambda'})_\tau = 0$, and

$$(v_{\boldsymbol{p},\lambda}, v_{\boldsymbol{q},\lambda'})_\tau = (2\pi)^3 C^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) \delta_{\lambda,\lambda'} (|f_p|^2 + |g_p|^2). \tag{A2}$$

Equations (A1) and (A2) suggest that we adopt the following normalization:

$$|f_p|^2 + |g_p|^2 = \frac{1}{(2\pi C)^3} \tag{A3}$$

in order that the orthonormality of the solutions is satisfied. The set of solutions (8) is complete:

$$\begin{aligned}\sum_\lambda (u_{\boldsymbol{p},\lambda} u^\dagger_{\boldsymbol{p},\lambda} + v_{\boldsymbol{p},\lambda} v^\dagger_{\boldsymbol{p},\lambda}) &= \sum_\lambda \left[\begin{pmatrix} f_p \xi_\lambda \\ g_p \lambda \xi_\lambda \end{pmatrix} \begin{pmatrix} f^*_p \xi^\dagger_\lambda & g^*_p \lambda \xi^\dagger_\lambda \end{pmatrix} + \begin{pmatrix} g^*_p \xi_\lambda \\ -f^*_p \lambda \xi_\lambda \end{pmatrix} \begin{pmatrix} g_p \xi^\dagger_\lambda & -f_p \lambda \xi^\dagger_\lambda \end{pmatrix} \right] \\ &= \sum_\lambda \xi_\lambda \xi^\dagger_\lambda \begin{pmatrix} |f_p|^2 + |g_p|^2 & 0 \\ 0 & |f_p|^2 + |g_p|^2 \end{pmatrix} \\ &= \frac{1}{(2\pi C)^3} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}. \end{aligned} \tag{A4}$$

In the last step we have used the completeness of the helicity basis $\sum_\lambda \xi_\lambda \xi^\dagger_\lambda = \mathbb{I}$, with $\mathbb{I}$ the 2×2 identity matrix, and the normalization condition (A3).

APPENDIX B: PROPERTIES OF HELICITY EIGENBISPINORS

We prove that for each $\lambda = \pm 1$ the quantity $\xi^\dagger_\lambda(\hat{p})\sigma_i \xi_\lambda(\hat{p})$ is an odd function of p_i, for $i = 1, 2, 3$. In particular, $\xi^\dagger_\lambda(\hat{p})\sigma_i \xi_\lambda(\hat{p})$ changes sign when the momentum is reversed $\boldsymbol{p} \to -\boldsymbol{p}$.

Proof: The statement can be proven by direct calculation. For the $\lambda = +1$ case, we have

$$\begin{aligned}\xi^\dagger_+ \sigma_1 \xi_+ &= \begin{pmatrix} e^{i\frac{\phi_p}{2}} \cos\left(\frac{\theta_p}{2}\right) & e^{-i\frac{\phi_p}{2}} \sin\left(\frac{\theta_p}{2}\right) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\frac{\phi_p}{2}} & \cos\left(\frac{\theta_p}{2}\right) \\ e^{i\frac{\phi_p}{2}} & \sin\left(\frac{\theta_p}{2}\right) \end{pmatrix} \\ &= \cos\left(\frac{\theta_p}{2}\right) \sin\left(\frac{\theta_p}{2}\right) (e^{i\phi_p} + e^{-i\phi_p}) = \sin(\theta_p)\cos(\phi_p) = \frac{p_1}{p}, \end{aligned} \tag{B1}$$

$$\xi_+^\dagger\sigma_2\xi_+ = \begin{pmatrix} e^{i\frac{\phi_p}{2}}\cos\left(\frac{\theta_p}{2}\right) & e^{-i\frac{\phi_p}{2}}\sin\left(\frac{\theta_p}{2}\right)\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi_p}{2}} & \cos\left(\frac{\theta_p}{2}\right) \\ e^{i\frac{\phi_p}{2}} & \sin\left(\frac{\theta_p}{2}\right)\end{pmatrix}$$

$$= \cos\left(\frac{\theta_p}{2}\right)\sin\left(\frac{\theta_p}{2}\right)(ie^{-i\phi_p} - ie^{i\phi_p}) = \sin(\theta_p)\sin(\phi_p) = \frac{p_2}{p}, \tag{B2}$$

$$\xi_+^\dagger\sigma_3\xi_+ = \begin{pmatrix} e^{i\frac{\phi_p}{2}}\cos\left(\frac{\theta_p}{2}\right) & e^{-i\frac{\phi_p}{2}}\sin\left(\frac{\theta_p}{2}\right)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi_p}{2}} & \cos\left(\frac{\theta_p}{2}\right) \\ e^{i\frac{\phi_p}{2}} & \sin\left(\frac{\theta_p}{2}\right)\end{pmatrix}$$

$$= \cos^2\left(\frac{\theta_p}{2}\right) - \sin^2\left(\frac{\theta_p}{2}\right) = \cos(\theta_p) = \frac{p_3}{p}, \tag{B3}$$

and similarly, for the $\lambda = -1$ case we have

$$\xi_-^\dagger\sigma_1\xi_- = \begin{pmatrix} e^{i\frac{\phi_p}{2}}\sin\left(\frac{\theta_p}{2}\right) & -e^{-i\frac{\phi_p}{2}}\cos\left(\frac{\theta_p}{2}\right)\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi_p}{2}} & \sin\left(\frac{\theta_p}{2}\right) \\ -e^{i\frac{\phi_p}{2}} & \cos\left(\frac{\theta_p}{2}\right)\end{pmatrix}$$

$$= -\cos\left(\frac{\theta_p}{2}\right)\sin\left(\frac{\theta_p}{2}\right)(e^{i\phi_p} + e^{-i\phi_p}) = -\sin(\theta_p)\cos(\phi_p) = \frac{-p_1}{p}, \tag{B4}$$

$$\xi_-^\dagger\sigma_2\xi_- = \begin{pmatrix} e^{i\frac{\phi_p}{2}}\sin\left(\frac{\theta_p}{2}\right) & -e^{-i\frac{\phi_p}{2}}\cos\left(\frac{\theta_p}{2}\right)\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi_p}{2}} & \sin\left(\frac{\theta_p}{2}\right) \\ -e^{i\frac{\phi_p}{2}} & \cos\left(\frac{\theta_p}{2}\right)\end{pmatrix}$$

$$= -\cos\left(\frac{\theta_p}{2}\right)\sin\left(\frac{\theta_p}{2}\right)(ie^{-i\phi_p} - ie^{i\phi_p}) = -\sin(\theta_p)\sin(\phi_p) = \frac{-p_2}{p}, \tag{B5}$$

$$\xi_-^\dagger\sigma_3\xi_- = \begin{pmatrix} e^{i\frac{\phi_p}{2}}\sin\left(\frac{\theta_p}{2}\right) & -e^{-i\frac{\phi_p}{2}}\cos\left(\frac{\theta_p}{2}\right)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi_p}{2}} & \sin\left(\frac{\theta_p}{2}\right) \\ -e^{i\frac{\phi_p}{2}} & \cos\left(\frac{\theta_p}{2}\right)\end{pmatrix}$$

$$= -\cos^2\left(\frac{\theta_p}{2}\right) + \sin^2\left(\frac{\theta_p}{2}\right) = -\cos(\theta_p) = \frac{-p_3}{p}. \tag{B6}$$

This result can be summarized in the equation

$$\xi_\lambda^\dagger\boldsymbol{\sigma}\xi_\lambda = \frac{\lambda\boldsymbol{p}}{p}. \tag{B7}$$

APPENDIX C: THE AUXILIARY TENSOR

This appendix is devoted to exploring the properties of the auxiliary tensor

$$L_{\mu\nu}(A,B) = \bar{A}\tilde{\gamma}_\mu(x)D_\nu B + \bar{A}\tilde{\gamma}_\nu(x)D_\mu B - D_\mu\bar{A}\tilde{\gamma}_\nu(x)B - D_\nu\bar{A}\tilde{\gamma}_\mu(x)B. \tag{C1}$$

The first property is obvious from the definition $L_{\mu\nu}(A,B) = L_{\nu\mu}(A,B)$ for any A, B. The second property can be seen at once by comparing the definition (C1) with the form of the energy-momentum tensor (5). Since $T_{\mu\nu}$ is real, we immediately deduce that $L_{\mu\nu}(A,A)$ is pure imaginary for each solution A. The third property can easily be deduced by tracing the definition

$$L_\mu^\mu(A,B) = g^{\mu\nu}L_{\mu\nu}(A,B) = 2(\bar{A}\tilde{\gamma}^\mu(x)D_\mu B - D_\mu\bar{A}\tilde{\gamma}_\mu(x)B) = -4im\bar{A}B, \tag{C2}$$

where we have used the Dirac equation and its adjoint in the last step. Next we give a more explicit form for $L_{\tau\tau}(A,B)$. Given the expression of the $\tilde{\gamma}_\mu$ matrices, we have

$$L_{\tau\tau}(A,B) = 2(\bar{A}\tilde{\gamma}_\tau(x)D_\tau B - D_\tau\bar{A}\tilde{\gamma}_\tau(x)B) = 2C(\bar{A}\gamma^0\partial_\tau B - \partial_\tau\bar{A}\gamma^0 B) = 2C(A^\dagger\partial_\tau B - \partial_\tau A^\dagger B). \tag{C3}$$

In particular, we are interested in the auxiliary tensors $L_{\tau\tau}(u(v),u(v))$ and the traces $L^\mu_\mu(u(v),u(v))$ computed on the solutions (8). For the traces we have

$$\begin{aligned} L^\mu_\mu(u_{\boldsymbol{p},\lambda},u_{\boldsymbol{p},\lambda}) &= -4im\bar{u}_{\boldsymbol{p},\lambda}u_{\boldsymbol{p},\lambda} = -4imu^\dagger_{\boldsymbol{p},\lambda}\gamma^0 u_{\boldsymbol{p},\lambda} \\ &= -4im\begin{pmatrix} f^*_p\xi^\dagger_\lambda & g^*_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}\begin{pmatrix} f_p\xi_\lambda \\ g_p\lambda\xi_\lambda\end{pmatrix} \\ &= -4im(|f_p|^2-|g_p|^2) = -4imC^{-3}(|\phi_p|^2-|\gamma_p|^2), \end{aligned} \tag{C4}$$

$$\begin{aligned} L^\mu_\mu(u_{\boldsymbol{p},\lambda},v_{\boldsymbol{p},\lambda}) &= -4im\bar{u}_{\boldsymbol{p},\lambda}v_{\boldsymbol{p},\lambda} = -4imu^\dagger_{\boldsymbol{p},\lambda}\gamma^0 v_{\boldsymbol{p},\lambda} \\ &= -4im\begin{pmatrix} f^*_p\xi^\dagger_\lambda & g^*_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}\begin{pmatrix} g^*_p\xi_\lambda \\ -f^*_p\lambda\xi_\lambda\end{pmatrix} \\ &= -8imf^*_pg^*_p = -8imC^{-3}\phi^*_p\gamma^*_p, \end{aligned} \tag{C5}$$

$$\begin{aligned} L^\mu_\mu(v_{\boldsymbol{p},\lambda},u_{\boldsymbol{p},\lambda}) &= -4im\bar{v}_{\boldsymbol{p},\lambda}u_{\boldsymbol{p},\lambda} = -4imv^\dagger_{\boldsymbol{p},\lambda}\gamma^0 u_{\boldsymbol{p},\lambda} \\ &= -4im\begin{pmatrix} g_p\xi^\dagger_\lambda & -f_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}\begin{pmatrix} f_p\xi_\lambda \\ g_p\lambda\xi_\lambda\end{pmatrix} \\ &= -8imf_pg_p = -8imC^{-3}\phi_p\gamma_p = -(L^\mu_\mu(u_{\boldsymbol{p},\lambda},v_{\boldsymbol{p},\lambda}))^*, \end{aligned} \tag{C6}$$

$$\begin{aligned} L^\mu_\mu(v_{\boldsymbol{p},\lambda},v_{\boldsymbol{p},\lambda}) &= -4im\bar{v}_{\boldsymbol{p},\lambda}v_{\boldsymbol{p},\lambda} = -4imv^\dagger_{\boldsymbol{p},\lambda}\gamma^0 v_{\boldsymbol{p},\lambda} \\ &= -4im\begin{pmatrix} g_p\xi^\dagger_\lambda & -f_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}\begin{pmatrix} g^*_p\xi_\lambda \\ -f^*_p\lambda\xi_\lambda\end{pmatrix} \\ &= -4im(|g_p|^2-|f_p|^2) = -4imC^{-3}(|\gamma_p|^2-|\phi_p|^2) \\ &= -L^\mu_\mu(u_{\boldsymbol{p},\lambda},u_{\boldsymbol{p},\lambda}). \end{aligned} \tag{C7}$$

We then compute the $\tau\tau$ components

$$\begin{aligned} L_{\tau\tau}(u_{\boldsymbol{p},\lambda},u_{\boldsymbol{p},\lambda}) &= 2C[u^\dagger_{\boldsymbol{p},\lambda}\partial_\tau u_{\boldsymbol{p},\lambda} - \partial_\tau u^\dagger_{\boldsymbol{p},\lambda}u_{\boldsymbol{p},\lambda}] \\ &= 2C\left[\begin{pmatrix} f^*_p\xi^\dagger_\lambda & g^*_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix}\partial_\tau f_p\xi_\lambda \\ \partial_\tau g_p\lambda\xi_\lambda\end{pmatrix} - \begin{pmatrix}\partial_\tau f^*_p\xi^\dagger_\lambda & \partial_\tau g^*_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix} f_p\xi_\lambda \\ g_p\lambda\xi_\lambda\end{pmatrix}\right] \\ &= 2C[f^*_p\partial_\tau f_p - \partial_\tau f^*_pf_p + g^*_p\partial_\tau g_p - \partial_\tau g^*_pg_p] \\ &= 2C^{-2}[\phi^*_p\partial_\tau\phi_p - \partial_\tau\phi^*_p\phi_p + \gamma^*_p\partial_\tau\gamma_p - \partial_\tau\gamma^*_p\gamma_p], \end{aligned} \tag{C8}$$

$$\begin{aligned} L_{\tau\tau}(u_{\boldsymbol{p},\lambda},v_{\boldsymbol{p},\lambda}) &= 2C[u^\dagger_{\boldsymbol{p},\lambda}\partial_\tau v_{\boldsymbol{p},\lambda} - \partial_\tau u^\dagger_{\boldsymbol{p},\lambda}v_{\boldsymbol{p},\lambda}] \\ &= 2C\left[\begin{pmatrix} f^*_p\xi^\dagger_\lambda & g^*_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix}\partial_\tau g^*_p\xi_\lambda \\ -\partial_\tau f^*_p\lambda\xi_\lambda\end{pmatrix} - \begin{pmatrix}\partial_\tau f^*_p\xi^\dagger_\lambda & \partial_\tau g^*_p\lambda\xi^\dagger_\lambda\end{pmatrix}\begin{pmatrix} g^*_p\xi_\lambda \\ -f^*_p\lambda\xi_\lambda\end{pmatrix}\right] \\ &= 2C[f^*_p\partial_\tau g^*_p - \partial_\tau f^*_pg^*_p - g^*_p\partial_\tau f^*_p + \partial_\tau g^*_pf^*_p] \\ &= 4C^{-2}[\phi^*_p\partial_\tau\gamma^*_p - \gamma^*_p\partial_\tau\phi^*_p], \end{aligned} \tag{C9}$$

$$L_{\tau\tau}(v_{\boldsymbol{p},\lambda},u_{\boldsymbol{p},\lambda}) = -(L_{\tau\tau}(u_{\boldsymbol{p},\lambda},v_{\boldsymbol{p},\lambda}))^*, \tag{C10}$$

$$L_{\tau\tau}(v_{\boldsymbol{p},\lambda}, v_{\boldsymbol{p},\lambda}) = -L_{\tau\tau}(u_{\boldsymbol{p},\lambda}, u_{\boldsymbol{p},\lambda}). \tag{C11}$$

Notice that as a consequence $L_{\tau\tau}(u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{p},\lambda}) = 0$ if and only if $\gamma_p \propto \phi_p$. This is the case, for instance, in Minkowski spacetime, where $\phi_p \propto \gamma_p \propto e^{-i\omega_p t}$. For a general C, the proportionality does not hold, and this has dramatic consequences on the vacuum of the theory (eventually leading to particle creation).

In addition, one can show that $L_{\tau i}(u(v), u(v))$ is an odd function of the momentum $\boldsymbol{p}$ and that $L_{ij}(u(v), u(v))$ is an odd function with respect to both p_i and p_j. The proof requires a lengthy but straightforward calculation:

$$\begin{aligned}
L_{\tau i}(u_{\boldsymbol{p},\lambda},u_{\boldsymbol{p},\lambda}) &= \bar{u}_{\boldsymbol{p},\lambda}\tilde{\gamma}_\tau D_i u_{\boldsymbol{p},\lambda} + \bar{u}_{\boldsymbol{p},\lambda}\tilde{\gamma}_i D_\tau u_{\boldsymbol{p},\lambda} - D_i\bar{u}_{\boldsymbol{p},\lambda}\tilde{\gamma}_\tau u_{\boldsymbol{p},\lambda} - D_\tau\bar{u}_{\boldsymbol{p},\lambda}\tilde{\gamma}_i u_{\boldsymbol{p},\lambda} \\
&= C u^\dagger_{\boldsymbol{p},\lambda}\left(\partial_i + \frac{1}{8}\omega_i^{A,B}[\gamma_A,\gamma_B]\right)u_{\boldsymbol{p},\lambda} - C u^\dagger_{\boldsymbol{p},\lambda}\gamma^0\gamma^i\partial_\tau u_{\boldsymbol{p},\lambda} - C\left(\partial_i u^\dagger_{\boldsymbol{p},\lambda}\gamma^0 - u^\dagger_{\boldsymbol{p},\lambda}\frac{\gamma^0}{8}\omega_i^{A,B}[\gamma_A,\gamma_B]\right)\gamma^0 u_{\boldsymbol{p},\lambda} + C\partial_\tau u^\dagger_{\boldsymbol{p},\lambda}\gamma^0\gamma^i u_{\boldsymbol{p},\lambda} \\
&= C u^\dagger_{\boldsymbol{p},\lambda}\left(i p_i + \frac{\partial_\tau C}{2C}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}\right)u_{\boldsymbol{p},\lambda} - C u^\dagger_{\boldsymbol{p},\lambda}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}\partial_\tau u_{\boldsymbol{p},\lambda} \\
&\quad - C u^\dagger_{\boldsymbol{p},\lambda}\left(-i p_i + \frac{\partial_\tau C}{2C}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}\right)u_{\boldsymbol{p},\lambda} + C\partial_\tau u^\dagger_{\boldsymbol{p},\lambda}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}u_{\boldsymbol{p},\lambda} \\
&= 2 i p_i C u^\dagger_{\boldsymbol{p},\lambda}u_{\boldsymbol{p},\lambda} + C\left[\partial_\tau u^\dagger_{\boldsymbol{p},\lambda}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}u_{\boldsymbol{p},\lambda} - u^\dagger_{\boldsymbol{p},\lambda}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}\partial_\tau u_{\boldsymbol{p},\lambda}\right] \\
&= 2 i p_i C[|f_p|^2 + |g_p|^2] + C\lambda[\partial_\tau f_p^* g_p + \partial_\tau g_p^* f_p - f_p^*\partial_\tau g_p - g_p^*\partial_\tau f_p]\xi_\lambda^\dagger\sigma_i\xi_\lambda \\
&= p_i\left\{\frac{i}{\pi^3 C^2} + \frac{\lambda}{p}[C(\partial_\tau f_p^* g_p + \partial_\tau g_p^* f_p - \text{c.c.})]\right\}.
\end{aligned} \tag{C12}$$

In the last step, we have made use of the normalization condition (A3) and the property (B7). As it is evident from Eq. (C12), each $\tilde{\gamma}_i$ factor and each spatial derivative ∂_i brings along a factor p_i. Then for each $a, b = u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{p},\lambda}$ one has

$$L_{\tau i}(a,b) = p_i h_{a,b}(p), \tag{C13}$$

with $h_{a,b}(p)$ a function of the modulus p alone. In particular,

$$L_{\tau i}(u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{p},\lambda}) = 0. \tag{C14}$$

Similarly, for each $a, b = u_{\boldsymbol{p},\lambda}, v_{\boldsymbol{p},\lambda}$

$$L_{ij}(a,b) = p_i p_j l_{a,b}(p), \tag{C15}$$

with $l_{a,b}(p)$ a function of the modulus p alone.

APPENDIX D: CANONICAL ANTICOMMUTATION RELATIONS

Demonstration of the canonical anticommutation relations (19):

$$\begin{aligned}
\{\psi_A(\tau,\boldsymbol{x}),\pi_{\psi B}(\tau,\boldsymbol{x}')\} &= iC^3\sum_{\lambda,\lambda'}\int d^3p\int d^3q[\{A_{\boldsymbol{p},\lambda},A^\dagger_{\boldsymbol{q},\lambda'}\}(u_{\boldsymbol{p},\lambda}u^\dagger_{\boldsymbol{q},\lambda'})_{AB} + \{B^\dagger_{-\boldsymbol{p},\lambda},B_{-\boldsymbol{q},\lambda'}\}(v_{\boldsymbol{p},\lambda}v^\dagger_{\boldsymbol{q},\lambda'})_{AB}] \\
&= iC^3\sum_\lambda\int d^3p[(u_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{x})u^\dagger_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{x}'))_{AB} + (v_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{x})v^\dagger_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{x}'))_{AB}] \\
&= iC^3\sum_\lambda\int d^3p e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}[(u_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{0})u^\dagger_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{0}))_{AB} + (v_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{0})v^\dagger_{\boldsymbol{p},\lambda}(\tau,\boldsymbol{0}))_{AB}] \\
&= iC^3\frac{1}{(2\pi)^3C^3}\delta_{AB}\int d^3p e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\
&= i\delta_{AB}\delta^3(\boldsymbol{x}-\boldsymbol{x}').
\end{aligned} \tag{D1}$$

In the fourth step we have made use of the completeness relation [Eq. (A4)], writing the 4×4 identity matrix explicitly as δ_{AB}.

APPENDIX E: CONSERVATION OF THE ENERGY-MOMENTUM TENSOR

In this appendix we prove explicitly the covariant conservation of the energy-momentum tensor associated with the flavor vacuum. We show that

$$\nabla_\mu \mathbb{T}^{\mu\nu} = 0 \tag{E1}$$

with ∇_μ denoting the covariant derivative. There is no need here to distinguish between $\mathbb{T}^{(MIX)}_{\mu\nu}$ and $\mathbb{T}^{(N)}_{\mu\nu}$, since both satisfy Eq. (E1), and so does the full energy-momentum tensor. Preliminarily we derive the connection coefficients for the metric of Eq. (1). It is easy to see from the definition that the only nonvanishing coefficients are

$$\Gamma^\tau_{\tau\tau} = \Gamma^\tau_{ii} = \Gamma^i_{\tau i} = \Gamma^i_{i\tau} = \frac{\dot{C}}{C}. \tag{E2}$$

Here the dot denotes the derivative with respect to conformal time τ, and no sum is intended over repeated indices. Notice that the coefficients depend only on τ. In terms of the connection coefficients, the covariant divergence reads

$$\nabla_\mu \mathbb{T}^{\mu\nu} = \partial_\mu \mathbb{T}^{\mu\nu} + \Gamma^\mu_{\mu\sigma}\mathbb{T}^{\sigma\nu} + \Gamma^\nu_{\mu\sigma}\mathbb{T}^{\mu\sigma}. \tag{E3}$$

(i) ($\nu = i$) For $\nu = i$, with $i = 1, 2, 3$, Eq. (E3) becomes

$$\nabla_\mu \mathbb{T}^{\mu i} = \partial_\mu \mathbb{T}^{\mu i} + \Gamma^\mu_{\mu\sigma}\mathbb{T}^{\sigma i} + \Gamma^i_{\mu\sigma}\mathbb{T}^{\mu\sigma}. \tag{E4}$$

From the diagonality of $\mathbb{T}^{\mu\nu}$ proved above, we can write

$$\nabla_\mu \mathbb{T}^{\mu i} = \partial_i \mathbb{T}^{ii} + \sum_\mu \Gamma^\mu_{\mu i}\mathbb{T}^{ii} + \sum_\mu \Gamma^i_{\mu\mu}\mathbb{T}^{\mu\mu}, \tag{E5}$$

where no sum is intended over repeated indices and the summations are written out explicitly to avoid confusion. The first term on the right-hand side of Eq. (E5) is zero, since $\mathbb{T}^{\mu\nu}$ depends only on τ. Similarly, from Eq. (E2) we know that $\Gamma^\mu_{\mu i} = 0 = \Gamma^i_{\mu\mu}$ for each $\mu = 0, 1, 2, 3$, and each $i = 1, 2, 3$, so that also the second and the third terms on the right-hand side of Eq. (E5) vanish. Then

$$\nabla_\mu \mathbb{T}^{\mu i} = 0 \quad \forall\ i. \tag{E6}$$

(ii) ($\nu = \tau$) Only a slightly longer calculation is needed to prove the statement for $\nu = \tau$. Starting from Eq. (E3) we have

$$\begin{aligned}\nabla_\mu \mathbb{T}^{\mu\tau} &= \partial_\mu \mathbb{T}^{\mu\tau} + \Gamma^\mu_{\mu\sigma}\Gamma^{\sigma\tau} + \Gamma^\tau_{\mu\sigma}\mathbb{T}^{\mu\sigma} \\ &= \partial_\tau \mathbb{T}^{\tau\tau} + \Big(\Gamma^\tau_{\tau\tau} + \sum_i \Gamma^i_{i\tau}\Big)\mathbb{T}^{\tau\tau} + \Gamma^\tau_{\tau\tau}\mathbb{T}^{\tau\tau} \\ &\quad + \sum_i \Gamma^\tau_{ii}\mathbb{T}^{ii} \\ &= \partial_\tau \mathbb{T}^{\tau\tau} + 5\Gamma^\tau_{\tau\tau}\mathbb{T}^{\tau\tau} + 3\Gamma^\tau_{\tau\tau}\mathbb{T}^{ii},\end{aligned} \tag{E7}$$

where we have used the diagonality of $\mathbb{T}^{\mu\nu}$ and Eqs. (E2). For our purposes it is convenient to rewrite Eq. (E7) in terms of $\mathbb{T}_{\tau\tau}$ and the trace $\mathbb{T}^\mu_\mu$. To this end we employ Eqs. (46) and (E2) and lower the indices through the metric of Eq. (1), obtaining

$$\nabla_\mu \mathbb{T}^{\mu\tau} = \partial_\tau (C^{-4}\mathbb{T}_{\tau\tau}) + 6C^{-5}\dot{C}\mathbb{T}_{\tau\tau} - C^{-3}\dot{C}\mathbb{T}^\mu_\mu. \tag{E8}$$

From Eq. (32), we know that each of the terms above is the integral of the auxiliary tensor components $L_{\tau\tau}, L^\mu_\mu$ weighted by τ-independent coefficients (recall that the Bogoliubov coefficients are evaluated at a fixed time τ_0). It is therefore sufficient to prove that

$$\begin{aligned}&\partial_\tau (C^{-4} L_{\tau\tau}(a,b)) + 6C^{-5}\dot{C} L_{\tau\tau}(a,b) \\ &\quad - C^{-3}\dot{C} L^\mu_\mu(a,b) = 0\end{aligned} \tag{E9}$$

for each $a, b = u_{\boldsymbol{p},\lambda;j}, v_{\boldsymbol{p},\lambda;j}$, to show that the divergence (E8) vanishes. To this end we need the second-order equations

$$\begin{aligned}\partial^2_\tau \phi_{p;j} &= -(im_j\dot{C} + p^2 + m_j^2 C^2)\phi_{p;j}, \\ \partial^2_\tau \gamma_{p;j} &= -(-im_j\dot{C} + p^2 + m_j^2 C^2)\gamma_{p;j}.\end{aligned} \tag{E10}$$

The first of these equations is simply Eq. (16), and the second can likewise be deduced from the system (15). For $L_{\mu\nu}(u_{\boldsymbol{p},\lambda;j}, u_{\boldsymbol{p},\lambda;j})$ we have, using the properties of the auxiliary tensor,

$$
\begin{aligned}
&\partial_\tau(C^{-4}L_{\tau\tau}(u_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j}))+6C^{-5}\dot{C}L_{\tau\tau}(u_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})-C^{-3}\dot{C}L^\mu_\mu(u_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})\\
&=2\partial_\tau[C^{-6}(\phi^*_{p;j}\overleftrightarrow{\partial}_\tau\phi_{p;j}+\gamma^*_{p;j}\overleftrightarrow{\partial}_\tau\gamma_{p;j})]+12C^{-7}\dot{C}(\phi^*_{p;j}\overleftrightarrow{\partial}_\tau\phi_{p;j}+\gamma^*_{p;j}\overleftrightarrow{\partial}_\tau\gamma_{p;j})+4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)\\
&=2C^{-6}\partial_\tau(\phi^*_{p;j}\overleftrightarrow{\partial}_\tau\phi_{p;j}+\gamma^*_{p;j}\overleftrightarrow{\partial}_\tau\gamma_{p;j})+4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)\\
&=2C^{-6}(\phi^*_{p;j}\partial^2_\tau\phi_{p;j}-\phi_{p;j}\partial^2_\tau\phi^*_{p;j}+\gamma^*_{p;j}\partial^2_\tau\gamma_{p;j}-\gamma_{p;j}\partial^2_\tau\gamma^*_{p;j})+4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)\\
&=2C^{-6}\{\phi^*_{p;j}[-(im_j\dot{C}+p^2+m^2C^2)]\phi_{p;j}-\phi^*_{p;j}[-(-im_j\dot{C}+p^2+m^2C^2)]\phi_{p;j}\\
&\quad+\gamma^*_{p;j}[-(-im_j\dot{C}+p^2+m^2C^2)]\gamma_{p;j}-\phi^*_{p;j}[-(im_j\dot{C}+p^2+m^2C^2)]\gamma_{p;j}\}+4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)\\
&=2C^{-6}\{|\phi_{p;j}|^2(-2im_j\dot{C})+|\gamma_{p;j}|^2(2im_j\dot{C})\}4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)\\
&=-4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)+4im_jC^{-6}\dot{C}(|\phi_{p;j}|^2-|\gamma_{p;j}|^2)=0.
\end{aligned}
$$

In the fourth step we have used Eqs. (E10) and their complex conjugates, and the τ argument of the functions has been suppressed for notational simplicity. Note that from the properties $L_{\tau\tau}(v_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})=-L_{\tau\tau}(u_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})$ and $L^\mu_\mu(v_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})=-L^\mu_\mu(u_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})$ the same relation is satisfied by the components of $L_{\mu\nu}(v_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})$. Similarly one has

$$
\begin{aligned}
&\partial_\tau(C^{-4}L_{\tau\tau}(u_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j}))+6C^{-5}\dot{C}L_{\tau\tau}(u_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})-C^{-3}\dot{C}L^\mu_\mu(u_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})\\
&=\partial_\tau(4C^{-6}\phi^*_{p;j}\overleftrightarrow{\partial}_\tau\gamma^*_{p;j})+24C^{-7}\dot{C}(4C^{-6}\phi^*_{p;j}\overleftrightarrow{\partial}_\tau\gamma^*_{p;j})+8im_jC^{-6}\dot{C}\phi^*_{p;j}\gamma^*_{p;j}\\
&=4C^{-6}\partial_\tau(C^{-6}\phi^*_{p;j}\overleftrightarrow{\partial}_\tau\gamma^*_{p;j})+8im_jC^{-6}\dot{C}\phi^*_{p;j}\gamma^*_{p;j}\\
&=4[\phi^*_{p;j}\partial^2_\tau\gamma^*_{p;j}-\gamma^*_{p;j}\partial^2_\tau\phi^*_{p;j}]+8im_jC^{-6}\dot{C}\phi^*_{p;j}\gamma^*_{p;j}\\
&=4C^{-6}\{\phi^*_{p;j}[-(im_j\dot{C}+p^2+m^2C^2)]\gamma^*_{p;j}-\phi^*_{p;j}[-(-im_j\dot{C}+p^2+m^2C^2)]\gamma^*_{p;j}\}+8im_jC^{-6}\dot{C}\phi^*_{p;j}\gamma^*_{p;j}\\
&=-8im_jC^{-6}\dot{C}\phi^*_{p;j}\gamma^*_{p;j}++8im_jC^{-6}\dot{C}\phi^*_{p;j}\gamma^*_{p;j}=0.
\end{aligned}
$$

In the fourth step we have used the complex conjugates of Eqs. (E10). Finally, because of the properties $L_{\tau\tau}(v_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})=-L^*_{\tau\tau}(u_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})$ and $L^\mu_\mu(v_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})=-L^{\mu*}_\mu(u_{\boldsymbol{p},\lambda;j},v_{\boldsymbol{p},\lambda;j})$, the same relation is satisfied by the components of $L_{\mu\nu}(v_{\boldsymbol{p},\lambda;j},u_{\boldsymbol{p},\lambda;j})$. This is sufficient to prove the statement for $\nu=\tau$.

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