

## Gauge anomalies of finite groups

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We show how the theory of characters can be used to analyze an anomaly corresponding to chiral fermions carrying an arbitrary representation of a gauge group that is finite, but otherwise arbitrary. By way of example, we do this for some groups of relevance for the study of quark and lepton masses and mixings.

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Finite symmetry groups are ubiquitous in physics, e.g., for stabilizing particles such as the proton or dark matter or for explaining the patterns of masses and mixings of quarks and leptons. The lore of quantum gravity says that they should be gauged and the lore of gauge symmetry says that they should be free of anomalies that can arise when they act on chiral fermions, at least if they are to be linearly realized *in vacuo*.

The study of such anomalies was pioneered by Graham Ross in collaboration with Luis Ibañez [1], who studied cyclic groups by embedding them as subgroups of a spontaneously broken  $U(1)$  (see also [2]). The extension to an arbitrary finite Abelian group follows immediately, since such a group is isomorphic to a product of cyclic groups, but the general case of groups that are not necessarily Abelian is still poorly understood. This is unfortunate, since groups with irreducible representations whose dimensions exceed one, which is a *sine qua non* for applications to flavor physics, are necessarily non-Abelian.

Here we show that one can perform a complete analysis of the anomalies of an arbitrary finite group using the theory of characters. To give a flavor of the power of this approach, consider the following situation: given a group and a representation of it, there must exist a unique largest subgroup that is anomaly free with respect to that representation. By considering all representations, one generates a list of possible anomaly-free subgroups. As the examples considered at the end of this paper show, this list can be found using the theory of characters with a line or two of menial algebra. (To match representations with subgroups in the list takes another line or two.)

Let us begin by recalling the current state of the art but phrasing things in a way which is both mathematically *kosher* and as general as possible. Suppose the spacetime dimension is 4 and that the fermions (all with the same chirality) carry the complex representation  $\rho$  of  $G$ . An argument [3] along the lines of Fujikawa's [4], shows that the transformation corresponding to  $g \in G$  is free of a mixed anomaly between  $G$  and gravity if and only if  $\det \rho(g) = \pm 1$ . The curious factor of  $\pm 1$  arises ultimately from an index theorem, but we can see that it must be present by following Ross and Ibañez and considering  $G$  to be a cyclic group arising as the linearly realized subgroup of a spontaneously broken  $U(1)$  that is itself anomaly free. The freedom to include a minus sign then arises from the possibility that a single charged chiral fermion can acquire a Majorana mass and thus decouple at low energy.

No further anomalies are detected by Fujikawa's argument (we will discuss further possible anomalies at the end) unless  $G$  has a nontrivial Lie algebra, but then  $G$  is no longer finite. In that case, mixed anomalies can arise between the normal subgroup  $G_0$  consisting of the component path connected to the identity element in  $G$  and the group of connected components  $G/G_0$ , which is finite assuming  $G$  is compact. A useful stepping stone in carrying over the analysis to this more general case is to study the slightly more general condition  $(\det \rho(g))^m = 1$  with  $m \in \mathbb{N}$  and we shall do so in what follows. So for the gravitational anomaly discussed above,  $m = 2$ , while for the case where  $G = G_0 \times G/G_0$  is a product, the corresponding values of  $m$  for the mixed anomaly are given in Ref. [5] in terms of the Dynkin indices of the representations of  $G_0$ . For the  $N$ -dimensional fundamental representation of  $SU(N)$ , for example, we get  $m = 1$ .

A recent preprint [6] (see also [5,7–9]) goes on to establish the following: (i) the set of  $g \in G$  such that  $(\det \rho(g))^m = 1$  forms a subgroup of  $G$ , which we call the *anomaly-free subgroup*<sup>1</sup> and denote  $G_\rho^m$ ; (ii)  $G_\rho^m$  is normal;

<sup>1</sup>Italics are used to indicate that a definition is being given.

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and (iii)  $G_\rho^m$  contains the *derived subgroup*  $G'$  (being the normal subgroup generated by *commutators*, i.e., elements in  $\{ghg^{-1}h^{-1} | g, h \in G\}$ ). Moreover, the first of these three facts implies that we may form the set of cosets  $G/G_\rho^m$ , the second implies that  $G/G_\rho^m$  forms a group (which we call the *anomalous quotient group*), and the third implies that it is Abelian. It is then further shown in [6] that the anomalous quotient group is cyclic.

Our point of departure is to observe that all of these facts follow almost immediately once one notices that the map  $g \mapsto (\det \rho(g))^m$  defines a homomorphism  $\pi_\rho^m$  from  $G$  to the Abelian group  $S^1 \in \mathbb{C}$  of complex numbers with unit modulus, with group law given by complex multiplication. Indeed,  $G_\rho^m$  is then the kernel of  $\pi_\rho^m$ , so is normal, and  $G/G_\rho^m$  is isomorphic to the image of  $\pi_\rho^m$ , so is isomorphic to a subgroup of  $S^1$ . But all subgroups of the Abelian group  $S^1$  are of necessity Abelian and all finite subgroups of  $S^1$  are moreover cyclic,<sup>2</sup> so  $G/G_\rho^m$  is cyclic.

Regarding the observation in [6] that  $G_\rho^m \supset G'$ , this is implied by our observation that  $G/G_\rho^m$  is Abelian, but it will be useful for what follows to spell out the connection in more detail. To wit, we have that  $G'$  is a normal subgroup, so we can form the quotient group  $G/G'$ , which turns out to be Abelian and which we call the *Abelianization* of  $G$ . Its importance lies in the fact that  $G/G'$ , equipped with the natural projection map  $\pi': G \rightarrow G/G'$ , is universal among Abelian groups  $A$  equipped with a map  $\sigma: G \rightarrow A$ . In other words, given any such  $A$  and  $\sigma$ , there exists a unique map  $\sigma'$  such that<sup>3,4</sup>

$$\sigma = \sigma' \circ \pi'. \tag{1}$$

To go further, we notice that the map  $g \mapsto (\det \rho(g))^m$  defines not just a homomorphism, but also a character, meaning that the heavy machinery of the theory of characters can be brought to bear. Recall that, given a representation  $\rho$ , the *character*  $\chi$  afforded by  $\rho$  is the map  $G \rightarrow \mathbb{C}: g \mapsto \text{tr} \rho(g)$  defined by taking the trace of the linear operator  $\rho(g)$ . Its *degree* is given by its value on the identity

<sup>2</sup>Proof. Parametrize the elements of  $S^1$  by an angle in  $[0, 2\pi]$  and let  $\theta$  be the smallest element in a finite subgroup. If the subgroup is not cyclic, then it is not generated by  $\theta$ , so there must exist an element  $\phi$  such that  $n\theta < \phi < (n+1)\theta \Rightarrow 0 < \phi - n\theta < \theta$ . So  $\theta$  is not the smallest element, which is a contradiction.

<sup>3</sup>Proof.  $G/G'$  is Abelian since  $ghG' = gh h^{-1} g^{-1} h g G' = hg G'$ . Given  $\sigma$ , define  $\sigma'(gG') := \sigma(g)$ . This is well-defined since if  $hG' = gG'$ , then there exists a commutator  $jkj^{-1}k^{-1}$  (or a product of commutators) such that  $h = gjkj^{-1}k^{-1}$ . But then  $\sigma'(hG') = \sigma(h) = \sigma(gjkj^{-1}k^{-1}) = \sigma(g) = \sigma'(gG')$  (a similar argument works for a product of generators). It is unique because if  $\sigma''$  is another such function, we must have  $\sigma''(gG') = \sigma''\pi(g) = \sigma(g) = \sigma'\pi(g) = \sigma'(gG')$ .

<sup>4</sup>In high-falutin' terms, we might say that Abelianization is a functor from the category of groups to the category of Abelian groups that is left adjoint to the functor that forgets that a group is Abelian.

element in  $G$  or equivalently by the dimension of the vector space carrying the representation  $\rho$ . Characters of degree one, also called *linear characters*, are special because we then have, colloquially, that “ $\text{tr} \rho = \rho$ ”; as a result they land in  $S^1 \subset \mathbb{C}$  and define homomorphisms  $G \rightarrow S^1$ .

More precisely, there is a one-to-one correspondence between linear characters and homomorphisms  $G \rightarrow S^1$  and this allows us to study anomalies in complete generality. Indeed, the total number of linear characters of  $G$  is finite (being of degree one, they are necessarily irreducible characters, so their number is bounded above by the number of irreducible characters, which is equal to the number of conjugacy classes, which in turn is bounded above by the number  $|G|$  of elements of  $G$ , which is finite). Moreover, as we shall soon see, the corresponding maps  $G \rightarrow S^1$  are easily explicitly determined, as are their kernels and images. To study the anomalous properties of any particular representation  $\rho$ , we merely need to explicitly match up  $\pi_\rho^m$  with a linear character on our list. This too is easily done. In fact, because every representation of a finite group is completely reducible, it is often possible to make statements about the anomalies of all representations at once, as we shall see when we discuss some examples.

Before we describe the gory details, it is perhaps useful to spell out exactly what it achieves in terms of allowing us to analyze the anomalies of a specified representation  $\rho$  of a specified group  $G$ . To do that, we need to ask what it might actually mean to “specify”  $\rho$  and  $G$ .

At the most explicit level, we might suppose that we are given (perhaps as a result of inspecting a Lagrangian of physical interest) a set of generators and relations for  $G$  along with a set of matrices, one for each generator, forming a representation of  $G$ . Here one might think that our methods do not bring much, since one can directly compute the  $m$ th powers of the determinants of matrices and compare with unity. But even here, our methods offer a slight advantage, in that we need only carry out such a computation on a set of generators of the Abelianization  $G/G'$ .<sup>5</sup>

Less explicitly, we might be given the character table for  $G$  and the character  $\chi$  afforded by  $\rho$ . Here, *a priori*, we would need to first reconstruct  $\rho$  and then compute its determinants (on conjugacy classes) as above. With our methods, we can extract the anomaly from  $\chi$  directly. Of course, since the character table gives us every irreducible character, we could also figure out the anomaly of any representation (or its character) by reducing it. This is made particularly straightforward by the fact that the determinant of a reducible character is given by the product of the determinants of its summands.

In fact, it is not even necessary to know the character table, since we can (and will shortly show) reconstruct the

<sup>5</sup>A set of generators of minimal size has one generator for each factor of  $G/G'$  expressed as  $\Pi_i \mathbb{Z}_{k_i}$ , where each  $k_i$  divides  $k_{i+1}$ .

necessary part of it (namely the linear characters) directly. So let us suppose we are in the worst-case scenario where we know  $G$  only in the form of some generators and relations and are interested in an arbitrary representation. Our first goal is to find the linear characters. By the universal property in (1), these can be obtained by precomposing the linear characters of the Abelianization  $G/G'$  with the map  $\pi'$ . So we use the given generators and relations for  $G$  to find  $G'$  and then  $G/G'$ , which we express as a product of cyclic groups, i.e.,  $G/G' \cong \prod_i \mathbb{Z}_{k_i}$ , where each  $k_i$  divides  $k_{i+1}$ , and choose a generator for each factor. The  $|G/G'|$  distinct linear characters of  $G/G'$  are then obtained by assigning a  $k_i$ th root of unity to the  $i$ th factor. Precomposing these with  $\pi'$  then determines the linear characters on  $G$ .<sup>6</sup>

From here, it is easy to answer the question of which anomaly free subgroups arise when we consider the set of all possible representations. This amounts to considering which linear characters arise in the image of  $\pi^m$  and is easily settled since  $\pi^1$  obviously surjects onto the linear characters. So the map  $\pi^m$  hits precisely the linear characters that are  $m$ -fold products of linear characters (with multiplication of characters defined pointwise in the target). The kernels of these yield the possible anomaly-free subgroups.

For a second question that is easy to answer, suppose we are given just one character  $\chi$  and we wish to compute the anomaly free subgroup of a representation afforded by it.

Given  $\chi$  we may define a linear character as its *determinant*  $\det \chi: G \rightarrow S^1: g \mapsto \det \rho(g)$ , where  $\rho$  is any representation affording  $\chi$ . This may be computed explicitly from  $\chi$  as follows. Supposing  $\chi$  has degree  $n$ , then we have the following beastly formula relating the determinant of the character of an element to the characters of the element's powers (obtained straightforwardly from the formula for an arbitrary square matrix given in [10]):

$$\det \chi(g) = \sum_{\{k_1, \dots, k_l \in \mathbb{N} \mid \sum l k_l = n\}} \prod_{l=1}^n \frac{(-1)^{k_l+1}}{l^{k_l} k_l!} (\chi(g^l))^{k_l}. \quad (2)$$

For  $n = 2$ , for example, we have

$$\det \chi(g) = \frac{\chi^2(g) - \chi(g^2)}{2}, \quad (3)$$

while for  $n = 3$  we have

$$\det \chi(g) = \frac{\chi^3(g) - 3\chi(g^2)\chi(g) + 2\chi(g^3)}{6}. \quad (4)$$

<sup>6</sup>We note that the map (of sets) from the set of conjugacy classes of  $G$  to its Abelianization defined by  $[g] \rightarrow gG'$  is surjective, but not injective unless  $G'$  is the trivial group.

With the determinant in hand, we can simply raise it to the desired power  $m$  and read off the kernel to extract the anomaly-free subgroup.

With power tools in hand, let us now apply them to find all the anomaly free subgroups of irreducible characters for various groups that have appeared in the literature on flavor physics.

We start with the group  $S^3$  of permutations of three objects. It has three conjugacy classes, labeled by the cycle lengths  $(\cdot)$ ,  $(\cdot, \cdot)$ , and  $(\cdot, \cdot, \cdot)$ . The derived subgroup is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ , consisting of the union of the conjugacy classes  $(\cdot)$  and  $(\cdot, \cdot)$  and the map  $\pi'$  sends those classes to the trivial element  $1 \in \mathbb{Z}/2\mathbb{Z} \cong G/G'$  and sends the class  $(\cdot, \cdot, \cdot)$  to the nontrivial element  $-1$ . The two linear characters of  $\mathbb{Z}/2\mathbb{Z}$  are defined by  $\chi_0(1) = \chi_0(-1) = 1$  and  $\chi_1(1) = 1, \chi_1(-1) = -1$ . After precomposing with  $\pi'$ , we find the linear characters of  $S_3$  given by  $\chi_0((\cdot)) = \chi_0((\cdot, \cdot)) = \chi_0((\cdot, \cdot, \cdot)) = 1$  and  $\chi_0((\cdot)) = \chi_0((\cdot, \cdot, \cdot)) = 1, \chi_0((\cdot, \cdot)) = -1$ . There is one further irreducible character, which can be found using orthogonality to be  $\chi_2((\cdot)) = 2, \chi_2((\cdot, \cdot)) = -1, \chi_2((\cdot, \cdot, \cdot)) = 0$ . Its determinant can only differ from 1 on the conjugacy class  $(\cdot, \cdot)$ , for which (3) yields  $\det \chi_2((\cdot, \cdot)) = \frac{\chi_2^2((\cdot, \cdot)) - \chi_2((\cdot, \cdot)^2)}{2} = \frac{0-2}{2} = -1$ . Thus we identify  $\ker \det \chi_2 = \chi_1$ . So for odd  $m$ , the irreducible representations affording  $\chi_1$  and  $\chi_2$  are anomalous with the anomaly-free subgroup  $S'_3 \cong \mathbb{Z}/3\mathbb{Z}$ , while for even  $m$  all representations are anomaly free.

This example is somewhat boring since the list of possible anomaly-free subgroups has just two entries, viz.  $S_3$  and  $S'_3$ . A more interesting example from flavor physics [11] is the quaternion group, of order 8. Henceforth we take the liberty of starting from the character table, which can be found these days at the touch of a button using GAP [12]. The table for  $Q_8$  is shown at the top left in Fig. 1. A glance at the linear characters shows that the Abelianization is isomorphic to the Klein group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  of order 4, so the derived subgroup is of order 2 and is the subset  $\{1, -1\} \subset Q_8$ .<sup>7</sup> But the three possible anomaly-free subgroups for  $m$  odd are larger (because the anomalous quotient must be cyclic), being all isomorphic to  $\mathbb{Z}/4$ . One is  $\{\pm 1, \pm i\}$  and the other two are obtained by the outer automorphisms of  $Q_8$ , which permute  $i, j$ , and  $k$ . For  $m$  even, all representations are anomaly free.

For the group  $A_4$ , of order 12, of alternating permutations of four objects, used in e.g., [14,15], the character table appears at the top right in Fig. 1. The three linear characters indicate that  $A_4/A'_4 \cong \mathbb{Z}/3\mathbb{Z}$ . Thus  $|A'_4| = 4$  and inspection of  $\chi_1$  shows us that  $A'_4 = [(\cdot)] \cup [(12)(34)] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . For the remaining character (of degree 3), it suffices to compute the determinant on the permutation

<sup>7</sup>We shall not need it for our trifles, but a useful fact in this context [13] is that  $g \in G$  is a commutator (ergo a generator of  $G'$ ) if and only if  $\sum_i \chi_i(g)/\chi_i(1) \neq 0$ , where  $i$  indexes the irreducible characters.

$Q_8$	1	-1	i	j	k	det	$A_4$	(.)	(12)(34)	(123)	(132)	det
$\chi_0$	1	1	1	1	1	$\chi_0$	$\chi_0$	1	1	1	1	$\chi_0$
$\chi_1$	1	1	1	-1	-1	$\chi_1$	$\chi_1$	1	1	$\omega$	$\omega^2$	$\chi_1$
$\chi_2$	1	1	-1	1	-1	$\chi_2$	$\chi_2$	1	1	$\omega^2$	$\omega$	$\chi_2$
$\chi_3$	1	1	-1	-1	1	$\chi_3$	$\chi_3$	3	-1	0	0	$\chi_0$
$\chi_4$	2	-2	0	0	0	$\chi_0$						

  

$SL(2, F_3)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$	det
$\chi_0$	1	1	1	1	1	1	1	$\chi_0$
$\chi_1$	1	1	$\omega^2$	$\omega$	1	$\omega^2$	$\omega$	$\chi_1$
$\chi_2$	1	1	$\omega$	$\omega^2$	1	$\omega$	$\omega^2$	$\chi_2$
$\chi_3$	2	-2	-1	-1	0	1	1	$\chi_0$
$\chi_4$	2	-2	$-\omega$	$-\omega^2$	0	$\omega$	$\omega^2$	$\chi_1$
$\chi_5$	2	-2	$-\omega^2$	$-\omega$	0	$\omega^2$	$\omega$	$\chi_2$
$\chi_6$	3	3	0	0	-1	0	0	$\chi_0$

FIG. 1. Character tables for  $Q_8$  (top left),  $A_4$  (top right), and  $SL(2, F_3)$  (bottom) with conjugacy classes in the top row labeled by representative group elements. The characters are labeled in the left-hand column and their respective determinants appear in the right-hand column.

(123). Since  $\chi_3((123)) = 0$ , our formula collapses to  $\det \chi_3((123)) = \frac{2\chi_3((123)^3)}{6} = \frac{\chi_3((.))}{3} = 1$ , so  $\det \chi_3 = \chi_0$ . For  $m = 1, 2 \pmod 3$ , the irreducible representations affording  $\chi_1$  and  $\chi_2$  are anomalous with anomaly-free subgroup  $A'_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

For the group  $SL(2, F_3)$ , of order 24, consisting of  $2 \times 2$  matrices with elements in the field  $F_3$ , used in [16,17], the character table appears at the bottom of Fig. 1. We immediately read off that the Abelianization is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ . The derived subgroup thus has order 8 and must consist of the union of the conjugacy classes containing  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ , and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , which is isomorphic to  $Q_8$ . To compute the determinants, it is sufficient to consider the conjugacy class containing  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . A simple calculation then yields the right hand column in the table. For  $m$  a multiple of 3, all representations are anomaly free; otherwise, only the irreducible representations with characters  $\chi_{0,3,6}$  are anomaly free, while the others have an anomaly-free subgroup  $SL(2, F_3)' \cong Q_8$ . Those who care for such things may now continue *ad nauseam*.

Finally, we must return to the thorny question of whether there might exist further anomalies, undetected by Fujikawa’s argument. Such anomalies, which necessarily can arise only if we consider spacetimes with nontrivial topology, are signaled by a nontrivial value of the exponentiated Atiyah-Patodi-Singer eta invariant. This is a

bordism invariant for finite  $G$ , but it is not known how to compute it in general (for cyclic groups, see [18,19]).

Even if it is nontrivial, it may be possible to cancel the resulting anomalies without changing the degrees of freedom by coupling to a topological quantum field theory. Again, it is not known what form such a theory may take, in general. One well-understood class of examples amounts to replacing  $G$  by an extension  $\tilde{G}$  (and  $\rho$  by its “restriction” along  $\tilde{G} \rightarrow G$ ), where the role of the topological theory is to obstruct the lifting of an arbitrary principal  $G$  bundle to a principal  $\tilde{G}$  bundle [18,19]. Again, we may return to Ibañez and Ross [1] for a pertinent example, where there exists an  $l \in \mathbb{N}$  such that the extension  $\mathbb{Z}/nl\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  can be used to cancel the pure gauge anomaly for  $\mathbb{Z}/n\mathbb{Z}$  [2]. Our arguments show that this cannot happen for the anomaly that we have considered: any such anomaly corresponds to a linear character  $G \rightarrow S^1$ , which cannot be sent to the trivial character by pullback along any extension  $\tilde{G} \rightarrow G$ . A similar argument from the bordism point of view appears in [19].

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