Slowly rotating black holes in nonlinear electrodynamics

David Kubizňák,[1,2,*] Tayebeh Tahamtan,[1,†] and Otakar Svítek[1,‡]
[1]Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Prague, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
[2]Perimeter Institute, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada

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We show how (at least, in principle) one can construct electrically and magnetically charged slowly rotating black hole solutions coupled to nonlinear electrodynamics (NLE). Our generalized Lense–Thirring ansatz is, apart from the static metric function f and the electrostatic potential ϕ inherited from the corresponding spherical solution, characterized by two new functions h (in the metric) and ω (in the vector potential) encoding the effect of rotation. In the linear Maxwell case, the rotating solutions are completely characterized by a static solution, featuring $h = (f - 1)/r^2$ and $\omega = 1$. We show that when the first is imposed, the ansatz is inconsistent with any restricted (see below) NLE but the Maxwell electrodynamics. In particular, this implies that the (standard) Newman–Janis algorithm cannot be used to generate rotating solutions for any restricted nontrivial NLE. We present a few explicit examples of slowly rotating solutions in particular models of NLE, as well as briefly discuss the NLE charged Taub-NUT spacetimes.

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I. INTRODUCTION

Theories of nonlinear electrodynamics (NLE) are classical field theories that naturally generalize the linear Maxwell theory. Dating back to the beginning of the twentieth century, first such theories emerged as an attempt to tame the divergencies associated with pointlike charges and cure the problem of infinite self-energy in Maxwell's theory. While this original problem has later been resolved by invention of renormalization, NLE remains at the theoretical forefront to these days. Perhaps the best known example of NLE is the Born–Infeld theory [1], which has many unique and remarkable properties [2], and naturally arises in the context of string theory [3,4] and early universe cosmology [5]. Other models of NLE were proposed to resolve the spacetime singularity [6]—providing a physical source for the regular black holes [7], to capture the basic features of the QED at the classical level [8], to describe dual strings in flat spacetime [9], or most recently as a maximally symmetric alternative to the Maxwell theory [10,11] and its deformation [12].

Of course, all the above models can be straightforwardly extended to a curved spacetime. At present, there exists a plethora of static *spherically symmetric* solutions of the Einstein-NLE system—such solutions are known for the Born–Infeld theory [13–15], for logarithmic Lagrangians [16], for a square root Lagrangian [17,18], for regular black hole models [6,19,20] (see also [21,22] for nonminimal coupling models), and other theories [23–26]. Recently, also dynamical solutions lacking any symmetry were constructed [27], and their further generalizations including electromagnetic radiation were studied in [28,29]. All these solutions are, however, twist free, and it would be extremely valuable to obtain rotating generalizations of the static spherically symmetric cases thus providing an NLE version of the Kerr–Newman solution.

So far there have been a number of attempts at constructing *rotating* black holes coupled to NLE. An obvious candidate to this end is to try to generate such solutions from the corresponding static ones by employing the (possibly upgraded) *Newman–Janis trick* [30,31]. While this trick successfully leads to a Kerr–Newman solution, it does not preserve the Einstein field equations for arbitrary source [32] nor works for the vacuum solutions in the presence of modified gravity theories [33,34]. In particular, charged spacetimes generated in this way, e.g., [35–39], do not satisfy the corresponding Einstein-NLE equations [35,40,41]. At the same time, a task of solving the corresponding equations of motion directly seems, due to their innate nonlinearity, quite formidable. That is why here we approach the rotating generalization more modestly—by considering a *slow rotation approximation*.[1]

*dkubiznak@perimeterinstitute.ca
†tahamtan@utf.mff.cuni.cz
‡ota@matfyz.cz

[1]While some of the studies claim to have considered slowly rotating solutions in NLE, e.g., [42–45], as we see below, this is not really the case because of the incorrect ansatz for such solutions.

As we show below, Maxwell's theory is surprisingly unique in providing (even slowly) rotating solutions in a "natural way". Specifically, we prove a "No Go theorem" that, in particular, rules out the standard Newman–Janis trick as a way of deriving rotating NLE solutions within a large family of NLE models. Based on this theorem, one can immediately dismiss a number of attempts at constructing rotating NLE black holes published previously in the literature. We also present a complete set of ordinary differential equations (ODEs) governing slowly rotating Einstein-NLE solutions in a broader setting and use them to derive two new explicit solutions. These solutions may, at least, in principle, be used in the future as "test grounds" for finding a possible generalization of the Newmann–Janis trick for the NLE theories (if it exists).

In order to construct slowly rotating solutions in NLE, we employ the *generalized Lense–Thirring* ansatz for the metric [34,46,47]

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + 2ar^2 h \sin^2\theta dt d\varphi + r^2 d\Omega^2, \tag{1}$$

where a is the rotation parameter, $N = N(r)$ and $f = f(r)$ are two independent metric functions, and $d\Omega^2 = \sin^2\theta d\varphi^2 + d\theta^2$ is the volume element on the sphere. As we see, for any NLE, one can set

$$N = 1, \tag{2}$$

generalizing the result of [48] for spherical Maxwell and Born–Infeld black holes. Moreover, for the charged slowly rotating solutions in the Einstein–Maxwell theory we have

$$h = \frac{f-1}{r^2}, \tag{3}$$

where f is the corresponding static metric function, $f = 1 - \frac{2M}{r} + \frac{e^2+p^2}{r^2}$, where e, p are the electric, magnetic charges, and M stands for the mass. Interestingly, as we see in Sec. III, the metrics generated by the Newman–Janis algorithm are, in the slow rotation approximation, of the form (37) with (2) and (3). However such a form, namely, that h is given by the corresponding static metric function f via (3), is consistent *only* in the Maxwell theory among all NLEs of the restricted form (13) below. This, in particular, means that not only is the Newman–Janis algorithm unable to construct full rotating solutions in NLE, it actually fails already at the lowest linear in a order.

Our paper is organized as follows. In the next section, we summarize the basics of NLE theories and list their equations. In Sec. III, we review the corresponding static solutions, as well as review the rotating metrics generated by the Newman–Janis formalism and their slowly rotating approximation. We then show how (at least, in principle) one can construct electrically (Sec. IV) and magnetically (Sec. V) charged solutions in any NLE, as well as establish the uniqueness of the Maxwell theory as the only NLE whose slowly rotating solutions can be written in the above form. We conclude in Sec. VI. In Appendix A, we construct slowly rotating magnetized solutions in the "Square Root" model of NLE, and Appendix B contains the discussion of NLE charged Taub-NUT solutions.

II. THEORIES OF NLE

Let us first review the basics of nonlinear electrodynamics. Any such theory is formulated in terms of the two invariants of the electromagnetic field,

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \qquad \mathcal{P} = \frac{1}{2} F_{\mu\nu} (*F)^{\mu\nu}, \tag{4}$$

where the field strength $F_{\mu\nu}$ is given in terms of the vector potential A_μ by the familiar expression, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Whereas $\mathcal{S}$ is a true scalar, the invariant $\mathcal{P}$ is only a pseudoscalar. To restore parity invariance, we thus consider theories that depend on $\mathcal{P}$ via its "square"; that is, we assume that the NLE theory is characterized by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2). \tag{5}$$

As we see, the latter assumption significantly simplifies the subsequent discussion of the slowly rotating solutions. In addition, one might require that the theory of NLE should approach that of Maxwell in the weak field approximation, imposing

$$\lim_{F_{\mu\nu}\to 0} \mathcal{L} = \frac{1}{2}\mathcal{S} + O(\mathcal{S}^2, \mathcal{P}^2), \tag{6}$$

which is known as the *principle of correspondence*. (This requirement is violated, for example, by the Square Root Lagrangian discussed in the Appendix.)

Introducing the following notation:

$$\mathcal{L}_\mathcal{S} = \frac{\partial \mathcal{L}}{\partial \mathcal{S}}, \qquad \mathcal{L}_\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{P}} = 2\mathcal{P}\mathcal{L}_{\mathcal{P}^2} = 2\mathcal{P}\frac{\partial \mathcal{L}}{\partial \mathcal{P}^2}, \tag{7}$$

the *generalized Maxwell* equations read

$$d * E = 0, \qquad dF = 0, \tag{8}$$

where

$$E_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = 2(\mathcal{L}_S F_{\mu\nu} + \mathcal{L}_P * F_{\mu\nu}). \tag{9}$$

Moreover, upon minimally coupling to the Einstein–Hilbert term,

$$I = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}(R - 4\mathcal{L}), \tag{10}$$

we obtain the following *Einstein equations*:

$$H_{\mu\nu} = G_{\mu\nu} - 8\pi T_{\mu\nu} = 0, \tag{11}$$

where the generalized EM energy-momentum tensor reads

$$T^{\mu\nu} = \frac{1}{4\pi}(2F^{\mu\sigma}F^{\nu}{}_{\sigma}\mathcal{L}_S + \mathcal{P}\mathcal{L}_P g^{\mu\nu} - \mathcal{L}g^{\mu\nu}). \tag{12}$$

We refer to Eqs. (8) and (11) as the Einstein-NLE equations.

In what follows, we also consider a simpler class of theories, obtained by considering *restricted Lagrangians* that are independent of the invariant $\mathcal{P}$, that is,

$$\mathcal{L} = \mathcal{L}(\mathcal{S}). \tag{13}$$

The corresponding equations of motion straightforwardly follow from the above.

III. NEWMAN–JANIS ALGORITHM: GENERATING ROTATING SOLUTIONS FROM THE STATIC ONES?

In the NLE literature, many rotating black hole "solutions" have been generated from the static ones by applying the (standard—"coined for Maxwell's theory") Newman–Janis algorithm. In this section, we review this approach and its limitations. We start by considering the static spherically symmetric solutions.

A. Static solutions

Consider a general static spherically symmetric metric element,

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \tag{14}$$

where $N = N(r)$ and $f = f(r)$ are two independent metric functions, and $d\Omega^2 = \sin^2\theta d\varphi^2 + d\theta^2$ is the volume element on the sphere. It was shown in [48] for the Born–Infeld theory, and is similarly valid for any NLE, that

$$N = 1. \tag{15}$$

The argument goes as follows. Let l denote a radial null vector of the metric element (14). Then, we have $R_{\mu\nu}l^\mu l^\nu \propto N'$. Thus, Eq. (15) can be imposed provided

$$T_{\mu\nu}l^\mu l^\nu = 0. \tag{16}$$

The NLE electromagnetic stress tensor $T_{\mu\nu}$, (12), consists of two terms, first proportional to $g_{\mu\nu}$ and the second proportional to $F_{\mu\alpha}F_\nu{}^\alpha$. When multiplied by $l^\mu l^\nu$, the first term trivially vanishes, while the latter term is proportional to $w_\alpha w^\alpha$, where $w_\alpha = F_{\alpha\mu}l^\mu$. However, for radial magnetic fields, we have $w_\alpha = 0$, whereas for radial electric fields $w^2 = 0$, implying that (16) is satisfied for any NLE and (15) can be imposed [48].

To find the spherical solution for given NLE, we thus consider the spherical element (14) with $N = 1$, supplemented by the corresponding vector potential. Considering both electric e and magnetic p charges, the ansatz for the vector potential reads

$$A = e\phi dt + p\cos\theta d\varphi, \tag{17}$$

where $\phi = \phi(r)$ is the function characterizing the electrostatic potential. We then find that the invariants (4) are given by

$$\mathcal{S} = -e^2\phi'^2 + \frac{p^2}{r^4}, \qquad \mathcal{P} = -\frac{2ep\phi'}{r^2}. \tag{18}$$

The t component of the Maxwell Eq. (8), $(\nabla\cdot E)_t = 0$, then yields

$$\phi'' + \phi'\frac{d}{dr}\lg\left(\frac{4p^2\mathcal{L}_{\mathcal{P}^2}}{r^2} - r^2\mathcal{L}_\mathcal{S}\right) = 0. \tag{19}$$

In fact, without specifying NLE, one can integrate this equation once, to obtain

$$\mathcal{L}_\mathcal{S} = \frac{4p^2\mathcal{L}_{\mathcal{P}^2}}{r^4} + \frac{\beta}{r^2\phi'}, \tag{20}$$

where β is a dimensionless integration constant. Since $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}_{\mathcal{P}^2}$ depend on ϕ (or more precisely its first derivatives) but not on f, this equation can be (at least, in principle) integrated to obtain ϕ. Once ϕ is known, the metric function f can be obtained from the Einstein equation, say $H_{rr} = 0$,

$$f' + \frac{f}{r} + B(r) = 0, \tag{21}$$

where

$$B(r) = 4re^2\phi'^2\mathcal{L}_\mathcal{S} - 2r(2\mathcal{P}^2\mathcal{L}_{\mathcal{P}^2} - \mathcal{L}) - \frac{1}{r}, \tag{22}$$

which yields a solution

$$f = -\frac{\int B(r)rdr}{r} - \frac{2M}{r}, \tag{23}$$

where M is an integration constant. The remaining equations are then automatically satisfied. We refer to [49] for the discussion of thermodynamics of these solutions.

B. Newman–Janis algorithm

Starting from a static solution (14), there is a hope (fulfilled in the Maxwell/vacuum case) that one could obtain the corresponding rotating solution by the Newman–Janis algorithm, e.g., [30,31,39,50] (see also [51]). The "recipe" goes as follows: (i) Start from a general spherical spacetime characterized by two metric functions $f = f(r)$ and $g = g(r)$,

$$ds^2 = -f dt^2 + \frac{dr}{f} + g d\Omega^2, \tag{24}$$

and proceed to the Eddington–Finkelstein coordinates, (u, r, θ, φ),

$$du = dt - \frac{dr}{f}. \tag{25}$$

The corresponding inverse metric can then be written as

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu, \tag{26}$$

where the (complex) null frame reads

$$l = \partial_r, \qquad n = \partial_u - \frac{f}{2}\partial_r, \qquad m = \frac{1}{\sqrt{2g}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\right). \tag{27}$$

ii) Perform a complex coordinate transformation,

$$u \to u - ia\cos\theta, \qquad r \to r + ia\cos\theta, \tag{28}$$

where a is the rotation parameter, and replace $f \to F(r, a, \theta)$ and $g \to \Sigma(r, a, \theta)$. Of course, the transformation (28) affects the null frame (27), as $\partial_\theta \to \partial_\theta + ia\sin\theta(\partial_u - \partial_r)$; that is,

$$\begin{aligned} &l \to \partial_r, \qquad n \to \partial_u - \frac{F}{2}\partial_r, \\ &m \to \frac{1}{\sqrt{2\Sigma}}\left(\partial_\theta + ia\sin\theta(\partial_u - \partial_r) + \frac{i}{\sin\theta}\partial_\varphi\right). \end{aligned} \tag{29}$$

Expression (26) then defines the new metric by inversion. (iii) Return back to the Boyer–Lindquist coordinates,

$$du = dt + \lambda(r)dr, \qquad d\varphi = d\varphi + \chi(r)dr, \tag{30}$$

requiring that the only nondiagonal component of the metric is that of $g_{t\varphi}$. Together with imposing $g = r^2$, this fixes the above functions F and Σ. The resulting metric then takes the following Carter's form [52]:

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ \frac{\sin^2\theta}{\Sigma}[(r^2 + a^2)d\varphi - adt]^2, \end{aligned} \tag{31}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 f + a^2. \tag{32}$$

Note that such a "solution" is completely characterized by a single metric function f of the corresponding static solution.

Of course, in the NLE case, the generated metric also has to be supplemented by the corresponding "rotating" vector potential A. For example, the following proposal,

$$A = \frac{p\cos\theta}{\Sigma}[(r^2 + a^2)d\varphi - adt], \tag{33}$$

has been used in [39] to construct the rotating magnetically charged solutions.

However, the above Newman–Janis generated rotating spacetime does not solve the corresponding NLE equations, e.g., [40]. In fact, as we see, this is true even at the linear $O(a)$ level. To show this, consider a slowly rotating limit of the above metric and potential, obtaining thus,

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + 2ar^2\sin^2\theta h dt d\varphi + r^2 d\Omega^2, \tag{34}$$

$$A = p\cos\theta\left(d\varphi - \frac{a\omega}{r^2}dt\right), \tag{35}$$

where, in the above, we have $\omega = 1$, and

$$h = \frac{f-1}{r^2}. \tag{36}$$

As we see in the next section, the slowly rotating magnetically charged solutions of NLE can be obtained in the form (34) and (35). However, the restriction (36), following from the standard Newman–Janis algorithm, is too strong and consistent only with the Maxwell theory [among all restricted theories (13)]. The same conclusion about (36) remains valid also in the electrically charged case. In other words, the standard Newman–Janis algorithm fails to produce rotating solutions for any nontrivial NLE. This is similar to the recent observation [34] that the Einstein gravity is the only theory (up to quartic corrections in curvature) that admits the slowly rotating spacetimes of the form (34) with h given by (36).

IV. SLOWLY ROTATING ELECTRIC SOLUTIONS

In this and the next sections, we show how (at least, in principle) one can construct slowly rotating spacetimes coupled to any NLE. To this purpose, we consider the electrically and magnetically charged cases separately (and leave the more complicated dyonic case to the future studies). In both cases, we impose the generalized Lense–Thirring ansatz for the metric,

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + 2ar^2 \sin^2\theta h dt d\varphi + r^2 d\Omega^2, \quad (37)$$

taking into account that at least at the $O(a)$ order, the condition $N = 1$, (15), has to remain valid. In what follows, we consistently work to the linear order in the rotation parameter a.

A. Finding electric solutions

For the electric solutions, we choose the following ansatz for the vector potential:

$$A = e\phi(dt - a\omega \sin^2\theta d\varphi), \quad (38)$$

where ϕ corresponds to the static solution, and $\omega = \omega(r)$ captures the effect of rotation. In this case, we find

$$\mathcal{S} = -e^2\phi'^2 + O(a^2), \qquad \mathcal{P} = -\frac{4e^2\phi\phi'\omega\cos\theta}{r^2} a + O(a^3). \quad (39)$$

Again, since $\mathcal{P}$ is linear in a, and $\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2)$, then at a given $O(a)$ order $T_{\mu\nu}$ takes the following simplified form:

$$4\pi T_{\mu\nu} = f^0 F_{\mu\sigma} F_\nu{}^\sigma + g^0 g_{\mu\nu}, \quad (40)$$

where

$$f^0 = 2\mathcal{L}_\mathcal{S}|_{\mathcal{P}=0}, \qquad g^0 = -\mathcal{L}|_{\mathcal{P}=0}, \quad (41)$$

both being functions of ϕ' (independent of ω and f).

Considering Einstein equations $H_{rr} = 0 = H_{\varphi\varphi}$, and solving them algebraically for f^0 and g^0, we obtain

$$f^0 = \frac{\zeta}{4e^2 r^2 \phi'^2}, \qquad g^0 = \frac{rf'' + 2f'}{4r}, \quad (42)$$

where

$$\zeta = r^2 f'' - 2f + 2. \quad (43)$$

At the same time, $(\nabla \cdot E)_t = 0$ can algebraically be solved for $\mathcal{L}'_\mathcal{S}$ and yields

$$\mathcal{L}'_\mathcal{S} = -\frac{\mathcal{L}_\mathcal{S}(r\phi'' + 2\phi')}{r\phi'}. \quad (44)$$

The remaining nontrivial equations are then $H_{t\varphi} = 0$ and $(\nabla \cdot E)_\varphi = 0$ (both being of the order a). They explicitly give

$$r^4\phi' h'' + 4r^3\phi' h' - \phi\zeta\omega' - \phi'\zeta\omega = 0, \quad (45)$$

$$A\omega'' + B\omega' + C\omega - r^4\phi'^2\mathcal{L}_\mathcal{S} h' = 0, \quad (46)$$

where

$$\begin{aligned} A &= r^2\phi\phi' f\mathcal{L}_\mathcal{S}, \\ B &= -r\mathcal{L}_\mathcal{S}(rf\phi\phi'' - r\phi\phi' f' - 2rf\phi'^2 + 2f\phi\phi'), \\ C &= \phi'(-8e^2\phi\phi'^2\mathcal{L}_{\mathcal{P}^2} + r\phi'\mathcal{L}_\mathcal{S}(rf' - 2f) - 2\phi\mathcal{L}_\mathcal{S}), \end{aligned} \quad (47)$$

and $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}_{\mathcal{P}^2}$ are expressed at $\mathcal{P} = 0$; that is, they are functions of ϕ' but not ω.

Thus, we have a simple procedure for determining the slowly rotating electrically charged solutions. The functions f and ϕ are those of the corresponding static solution, given by (23) and (20), after setting $p = 0$. Equations (45) and (46) then represent coupled ordinary differential equations for "rotating" functions ω and h. Obviously, one can express h' from the second equation, and by plugging this back to the first one, obtain a third-order ODE for ω. As discussed in conclusions, such an equation is guaranteed to have a "nice" solution for ω, which then yields h by integrating (46).

B. Maxwell uniqueness

Let us now impose the conditions (3), $h = (f-1)/r^2$, upon which Eq. (45) gives

$$[\phi(\omega - 1)]'\zeta = 0. \quad (48)$$

In other words, we find that for any nontrivial NLE, we have to have

$$\omega = 1 + \frac{c}{\phi}, \quad (49)$$

for some (dimensionful) constant c. Plugging this into Eq. (46) then yields

$$r\mathcal{L}_\mathcal{S}\phi' + \mathcal{L}_\mathcal{S}(\phi + c) + 4e^2\phi'^2\mathcal{L}_{\mathcal{P}^2}(\phi + c) = 0. \quad (50)$$

For restricted class of theories, (13), we have $\mathcal{L}_{\mathcal{P}^2} = 0$,[2] and the latter equation can be integrated to give

$$\phi = \frac{1}{r} - c. \quad (51)$$

[2]More generally, the theory $\mathcal{L}(\mathcal{S}, \mathcal{P}^2)$ is admissible provided the solution of (50) is consistent with the solution of (20) (with $p = 0$). This requirement seems rather restrictive. In particular, we have checked that it is not satisfied for the ModMax theory [10,11]—this theory thus does not admit slowly rotating electric solutions with $h = (f-1)/r^2$. On the other hand, a theory given by $\mathcal{L} = (\mathcal{S}^4 + \mathcal{P}^4)^{1/4}$ admits trivially Maxwell-like solutions.

It is now obvious that the constant c is unphysical and only corresponds to the gauge for the vector potential—it can be gauged away by $A \to A + d\lambda$ where $\lambda = ect$. Thus, without loss of generality, we have

$$\omega = 1, \qquad \phi = \frac{1}{r}. \tag{52}$$

Equation (20) (with $p = 0$) then yields the Maxwell theory. Thus, we have proved the following:

Theorem.—For restricted class of theories, (13), the only NLE consistent with $h = (f-1)/r^2$ for the ansatz (37) and (38) is the Maxwell theory.

In particular, this implies:

Collorary.—Electrically charged spacetimes generated by the standard Newman–Janis algorithm do not solve the corresponding NLE equations following from (13), not even at the linear $O(a)$ level.

C. Special NLE with $\omega = 1$

In the above, we have established that imposing $h = (f-1)/r^2$ leads to the Maxwell theory and, in particular, implies that one has to have $\omega = 1$. Let us now ask the "opposite": imposing

$$\omega = 1, \tag{53}$$

can we have any nontrivial NLE? The partial motivation to study this question stems from the NLE literature, e.g., [42,43], where the assumption (53) is automatically assumed. As we now show, apart from the Maxwell theory, there is yet another special NLE consistent with (53), given by (61) below. This theory is, however, distinct from the NLE theories studied in [42,43], invalidating thus some of the results in these papers. This theory also provides an example of NLE where slowly rotating electric solutions can be explicitly constructed.

To construct our special NLE, let us return to Eq. (45) and impose (53). In this case, this equation can be integrated for h, and gives

$$h = \frac{f-1}{r^2} - \frac{2M_0}{r^3} + h_0, \tag{54}$$

where M_0 and h_0 are the integration constants. Here, h_0 can be reabsorbed by redefining φ, namely, $d\varphi \to d\varphi - ah_0 dt$. In other words, h_0 is not physical, and we can set $h_0 = 0$. On the other hand, M_0 seems physical as it "redefines" the asymptotic angular momentum. Of course, one possibility is to consider $M_0 = 0$, in which case, we are back to the Maxwell case. On the other hand, considering M_0 nontrivial, Eq. (46) then yields

$$4e^2 \mathcal{L}_{\mathcal{P}^2} \phi \phi'^2 + \mathcal{L}_{\mathcal{S}} (r + 3M_0)\phi' + \phi \mathcal{L}_{\mathcal{S}} = 0. \tag{55}$$

Focusing on the restricted theories (13), the latter can be integrated and gives

$$\phi = \frac{1}{r + 3M_0}. \tag{56}$$

The corresponding metric function f is then obtained by integrating (20) where the lhs is given by one half of the first expression in (42). This then yields

$$f = 1 + \frac{4\beta e^2}{9M_0^2} - \frac{2M + 8\beta e^2/(9M_0)}{r} - \frac{8r\beta e^2}{27M_0^3} + r^2\left(\Lambda - \frac{8\beta e^2}{81M_0^4}\lg\left(\frac{r}{r+3M_0}\right)\right), \tag{57}$$

where M and Λ are the integration constants. Interestingly, for large r, this has the following expansion:

$$f \approx 1 - \frac{2M}{r} + \Lambda r^2 - \frac{2\beta e^2}{r^2} + \frac{24\beta e^2 M_0}{5r^3} + O\left(\frac{1}{r^4}\right), \tag{58}$$

which upon setting $\Lambda = 0$ and $\beta = -1/2$ has the required Reissner–Nordstrom asymptotic behavior. For small enough positive M_0, we have (up to) two horizons, shielding singularity at $r = 0$. Note also that the electromagnetic field is regular on the horizon, by the token of (39).

The corresponding theory can easily be constructed from (20). Namely, we have

$$\mathcal{L}_{\mathcal{S}} = -\frac{1}{2r^2\phi'} = \frac{(r+3M_0)^2}{2r^2} = \frac{1}{2}(1-s)^{-2}, \tag{59}$$

where

$$s = \left(-\frac{\mathcal{S}}{\mathcal{S}_0}\right)^{\frac{1}{4}}, \qquad \mathcal{S}_0 = \frac{e^2}{(3M_0)^4}. \tag{60}$$

Equation (59) can be integrated to yield

$$\mathcal{L} = 2\mathcal{S}_0\left(\frac{s^3 + 3s^2 - 4s - 2}{2(1-s)} - 3\lg(1-s) + 1\right), \tag{61}$$

which obeys (6). Of course, in here, $\mathcal{S}_0$ is the fundamental coupling constant that gives rise to the modification related to M_0 above.

To conclude, among restricted NLE theories (13), there are two theories that yield the Lense–Thirring solutions with $\omega = 1$: Maxwell theory and the theory defined by the Lagrangian (61). Surprisingly, this Lagrangian is identical to the one obtained in [28] as the only NLE model admitting electromagnetic radiation in the Robinson–Trautman class of spacetimes.

V. SLOWLY ROTATING MAGNETIC SOLUTIONS

A. Finding magnetic solutions

To find the magnetic solutions, we supplement the metric (37) with the following ansatz for the vector potential:

$$A = p\cos\theta\left(d\varphi - \frac{a\omega}{r^2}dt\right), \tag{62}$$

where $\omega = \omega(r)$ is a new vector potential function. The field invariants (4) now read

$$\mathcal{S} = \frac{p^2}{r^4} + O(a^2), \qquad \mathcal{P} = \frac{2p^2\cos\theta(r\omega' - 2\omega)}{r^5}a + O(a^3). \tag{63}$$

Note that since $\mathcal{P}$ is linear in a, and $\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2)$, then at a given $O(a)$ order $T_{\mu\nu}$ takes a simplified form,

$$4\pi T_{\mu\nu} = f^0 F_{\mu\sigma}F_\nu{}^\sigma + g^0 g_{\mu\nu}, \tag{64}$$

where

$$f^0 = 2\mathcal{L}_\mathcal{S}|_{\mathcal{P}=0}, \qquad g^0 = -\mathcal{L}|_{\mathcal{P}=0}, \tag{65}$$

both being explicit functions of r (independent of ω and f). Equation (23) with $e = 0$ immediately gives

$$f = 1 - \frac{2M}{r} - \frac{2\int r^2\mathcal{L}(r)dr}{r}. \tag{66}$$

Solving algebraically, $H_{rr} = 0 = H_{\varphi\varphi}$ for f^0, g^0 yields

$$f^0 = \frac{r^2\zeta}{4p^2}, \qquad g^0 = \frac{1}{2r^2}(rf' + f - 1), \tag{67}$$

where ζ is given by (43), $\zeta = r^2 f'' - 2f + 2$. Eliminating further $\frac{d}{dr}\mathcal{L}_\mathcal{S}$ from $(\nabla\cdot E)_t = 0$, and plugging these to $H_{t\varphi} = 0$ and $(\nabla\cdot E)_t = 0$, gives the following two $O(a)$ equations:

$$r^4 f h'' + 4r^3 f h' - r^2\zeta h - \zeta\omega = 0, \tag{68}$$

$$A\omega'' + B\omega' + C\omega - 2r^6\mathcal{L}_\mathcal{S} h = 0, \tag{69}$$

where

$$\begin{aligned} A &= (r^6\mathcal{L}_\mathcal{S} - 4p^2r^2\mathcal{L}_{\mathcal{P}^2})f, \\ B &= rf(r^5\mathcal{L}'_\mathcal{S} - 2r^4\mathcal{L}_\mathcal{S} - 4p^2r\mathcal{L}'_{\mathcal{P}^2} + 24p^2\mathcal{L}_{\mathcal{P}^2}), \\ C &= f(8p^2r\mathcal{L}'_{\mathcal{P}^2} - 2r^5\mathcal{L}'_\mathcal{S} + 2r^4\mathcal{L}_\mathcal{S} - 40p^2\mathcal{L}_{\mathcal{P}^2}) - 2r^4\mathcal{L}_\mathcal{S}. \end{aligned} \tag{70}$$

Equations (68) and (69) represent two coupled ordinary differential equations for ω and h. While these equations can be easily decoupled, they result in higher(4th)-order linear ODEs with variable coefficients. (We briefly comment on finding the corresponding solutions in conclusions.) This procedure is illustrated in Appendix A where we construct slowly rotating magnetized black holes in the "Square Root" NLE.

B. Maxwell uniqueness

Let us now impose (3), $h = (f-1)/r^2$. Then, Eq. (68) immediately yields

$$(\omega - 1)\zeta = 0, \tag{71}$$

and for any nontrivial NLE, we must have

$$\omega = 1. \tag{72}$$

Returning back to $(\nabla\cdot E)_t = 0$ then yields that

$$r^5\mathcal{L}'_\mathcal{S} - 4p^2 r\frac{d}{dr}\mathcal{L}_{\mathcal{P}^2} + 20p^2\mathcal{L}_{\mathcal{P}^2} = 0. \tag{73}$$

Obviously, for the restricted class of theories (13), we have just proved that one has to have $\mathcal{L}_\mathcal{S} = \text{const.}$, which is only consistent with the Maxwell theory.[3] We have thus proved the following:

Theorem.—Among all restricted NLE theories (13), Maxwell theory is the only one that admits the magnetically charged slowly rotating solutions of the form (37), (62) with the restriction $h = (f-1)/r^2$. In particular, this means that the standard Newman–Janis algorithm fails to produce solutions already at the linear $O(a)$ level.

This theorem, in particular, invalidates "solutions" constructed in [37,39].

VI. CONCLUSION

In this paper, we have analysed slowly rotating generalizations of static spacetimes sourced by NLE. To this end, we have presented a generalized Lense–Thirring ansatz for the metric (37) together with a simple ansatz for the vector potential in electric (38) and magnetic (62) cases and shown that with these one can solve (at least, in principle) the corresponding Einstein-NLE equations to the linear order in the rotation parameter. The dyonic case seems more complicated, and we leave it for the future studies. To illustrate this procedure, we have found two explicit examples where the corresponding solutions can be found in a closed form. (The detailed analyses of these solutions are left for future studies.)

We have also proved the "No Go Theorem" which establishes that the Maxwell theory is the only NLE among all theories (13) that admits function h given by the "natural" expression (3). This, in particular, shows that the standard Newman–Janis algorithm [which leads to the form (3) in the slow rotation approximation] fails to produce rotating solutions in NLE even at the lowest (linear) level in rotation parameter a, as well as shows that a number of (slowly rotating) metrics constructed in previous studies cannot satisfy the corresponding equations

[3]Again, Eq. (73) is not satisfied for the ModMax theory,and trivially works for $\mathcal{L} = (\mathcal{S}^4 + \mathcal{P}^4)^{1/4}$.

of motion. We have also pointed out that (i) some of the previous attempts to construct rotating solutions in NLE used too simplistic ($\omega = 1$) ansatz for the vector potential, while (ii) some other previously constructed solutions actually do not present slowly rotating black holes but rather correspond to weakly NUT-charged solutions (see Appendix B where these solutions are constructed for the general case). This effectively leaves no solutions at all for (slowly) rotating black holes in NLE (see, however, the recent progress in [53,54]).

Our work opens several new directions for future studies. First, the above No Go Theorem strictly speaking only regards the restricted NLEs (13). It is possible that once a more general setting of (5) is considered, there are other theories which allow for (3), see Eq. (50) and the corresponding discussion in the footnote. It would be interesting to construct examples of such nontrivial theories.

Second, while we have shown that the standard Newman–Janis algorithm does not give rise to the corresponding rotating solutions, we cannot dismiss a possibility that an appropriately *modified Newman–Janis* algorithm cannot be formulated for NLE theories. The explicit slowly rotating solutions found in this paper may provide a test ground for finding such an algorithm.

Third, during our investigation, we have stumbled upon a *special* example of NLE, whose slowly rotating electric solutions are distinguished by the "Maxwell-like" ($\omega = 1$) form of the gauge potential. Interestingly, this is the same theory discovered recently in [28] as the only theory of NLE admitting radiation in the Robinson–Trautman class of spacetimes. This theory certainly deserves further attention in the future.

Fourth, as we have seen above, the general solution for the slowly rotating NLE hinges on solving a higher-order linear ODE with variable coefficients. Assuming the coefficients of the equation are continuous functions, the general theory of ODE guarantees the existence and uniqueness of a solution provided some initial conditions are prescribed. Furthermore, the solution of such nth-order ODE would be at least C^{n-1}. In this way, the procedures detailed above provide a solution to our stated problem. However, one would like to have more then just existence result. To this end, one can transform the higher-order ODE into a first-order system and employ some of the approximate solution methods. One such method is the Magnus expansion [55] which gives the solution as an exponential of a series containing integrals of nested commutators of the coefficient matrix. The convergence is controlled by a suitable norm of the coefficient matrix, and even truncated solutions often capture the main features of the complete solution.

Historically, it took almost 50 years to upgrade the slowly rotating charged solutions of Lense–Thirring to the full nonlinear Kerr–Newmann geometry. It will be interesting to see if some of the hereby presented slowly rotating solutions can be promoted to full (possibly analytic) charged and rotating solutions in some nontrivial NLE.

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APPENDIX A: MAGNETIC SOLUTIONS IN SQUARE ROOT ELECTRODYNAMICS

In this Appendix, we construct a slowly rotating magnetized solution in the so called *Square Root* model of NLE, characterized by

$$\mathcal{L} = -\beta\sqrt{S}, \tag{A1}$$

where β is a dimensionful coupling constant, with dimensions $1/L$. This Lagrangian represents a strong field regime of many models of NLE, Born–Infeld theory for example. It was originally proposed by Nielsen and Olesen [9] to treat the so-called dual string in flat spacetime. It also gives rise [56] to the confinement potential [57], see also [58] for recent developments on nonlinear gauge theories containing "Square Root" Lagrangians. Of course, the model can also be generalized to curved spacetime and was recently discussed in [59–61]. Note that when considering only the magnetic field all energy conditions are satisfied unlike the case of pure radial electric field.

To construct the magnetized solution, we adopt the ansatz (37) together with the potential (62) and follow the procedure outlined in Sec. V. Namely, the static metric function f, (66), is given by

$$f = 1 - \frac{2M}{r} + 2\beta p. \tag{A2}$$

Note that this modification of the Schwarzschild solution is related to the solid angle deficit/excess (depending on the sign of βp). Such a solution also represents the geometry outside the core of the so-called global monopole, a spacetime defect created by a gravitating triplet of scalar fields whose original $O(3)$ symmetry is spontaneously broken to $U(1)$. A global monopole was extensively discussed in the literature, see, e.g., [62–64] for original works and some more recent work by two of the authors [65,66].

For $M = 0$, Eqs. (68) and (69) yield the following solutions for h and ω:

$$\omega = \frac{\omega_1}{r} + \omega_2 r^2 + \omega_3 r^{(\frac{1}{2}+q)} + \omega_4 r^{(\frac{1}{2}-q)},$$
$$h = -\frac{\omega_1}{r^3} - \omega_2 - 2\omega_3 \beta p r^{(q-\frac{3}{2})} - 2\omega_4 \beta p r^{(-q-\frac{3}{2})}, \quad \text{(A3)}$$

where ω_i's are the integration constants, and

$$q = \frac{\sqrt{4p^2\beta^2 + 36p\beta + 17}}{4p\beta + 2}. \quad \text{(A4)}$$

Obviously, we can eliminate ω_2 by redefining φ, and so we set $\omega_2 = 0$. Writing $\omega_1 = 2M_0$ and setting $\omega_3 = 0 = \omega_4$ for simplicity (though it would be interesting to study the physical meaning of these terms), we thus recover the following simple solution:

$$h = -\frac{2M_0}{r^3}, \qquad \omega = \frac{2M_0}{r}. \quad \text{(A5)}$$

When $M \neq 0$, the terms with "strange powers" of r in (A3) are replaced by hypergeometric functions. However, setting again $\omega_3 = 0 = \omega_4$, the solution (A5) remains valid also in this case.

APPENDIX B: TAUB-NUT SOLUTIONS IN NLE

Lorentzian Taub-NUT spacetimes [67,68] represent an interesting class of axisymmetric (electro-vacuum) solutions of Einstein equations. Such solutions are characterized by the appearance of the so called Misner strings [69]—the singular rotating sources of angular momentum [70]. As these strings extend all the way to infinity, the Taub-NUT solutions are not asymptotically flat. They also feature various pathologies, such as the existence of closed timelike curves in the vicinity of Misner strings. As we see below, some of these solutions were in the NLE literature confused with the slowly rotating black hole solutions. To demonstrate this, we study the charged Taub-NUT solutions coupled to a general NLE (5).

Namely, we seek the charged Taub-NUT solution in the following form:

$$ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + \frac{dr^2}{f} + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$A = \phi(dt + 2n\cos\theta d\varphi), \quad \text{(B1)}$$

where we have denoted the NUT parameter by n, choosing a symmetric distribution for Misner strings (which are located on both the north-pole and south-pole axes). The solution is characterized by a single metric function $f = f(r)$, and single gauge potential function $\phi = \phi(r)$.

Using this ansatz, we find the following expressions for the invariant $\mathcal{S}$ and $\mathcal{P}$:

$$\mathcal{S} = -\phi'^2 + \frac{4n^2\phi^2}{(n^2 + r^2)^2}, \qquad \mathcal{P} = -\frac{4n\phi\phi'}{n^2 + r^2}. \quad \text{(B2)}$$

The t component of the generalized Maxwell equation (8) then yields the following ODE:

$$\phi'' + \phi' \frac{d}{dr}\lg(-(n^2 + r^2)\mathcal{L}_\mathcal{S}) + \frac{2n\phi}{(n^2 + r^2)^2}\left(\frac{(n^2 + r^2)\mathcal{L}'_\mathcal{P}}{\mathcal{L}_\mathcal{S}} + 2n\right) = 0. \quad \text{(B3)}$$

Since $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}_\mathcal{P}$ depend only on ϕ but not f, this equation can (at least, in principle) be integrated to yield solution for ϕ. The $H_{rr} = 0$ Einstein equation then yields a first-order equation for f,

$$f' + \frac{r^2 - n^2}{r(n^2 + r^2)} f - \frac{1}{r}(1 - 8n\mathcal{L}_\mathcal{P}\phi\phi' - 2(n^2 + r^2)(\mathcal{L} + 2\mathcal{L}_\mathcal{S}\phi'^2)) = 0, \quad \text{(B4)}$$

and the remaining equations are automatically satisfied. The explicit examples were constructed, e.g., for the Born–Infeld theory [71] (see also [72]) or the recently constructed ModMax theory [73].

In particular, let us consider a small n expansion, expanding the metric and the gauge potential to the linear order in n. In this case, the *weakly NUT-charged* solution is fully characterized by the static metric function f and static electric potential of the corresponding NLE, determined from

$$\phi'' + \phi' \frac{d}{dr}\lg(-r^2\mathcal{L}_\mathcal{S}) = 0,$$
$$f' + \frac{f}{r} - \frac{1}{r}(1 - 2r^2(\mathcal{L} + 2\mathcal{L}_\mathcal{S}\phi'^2)) = 0, \quad \text{(B5)}$$

c.f. Eqs. (20) and (23). [As always, here we assumed that $\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2)$ and so $\mathcal{L}_\mathcal{P} \sim O(n)$.] In particular, for a specific NLE, the corresponding solutions were constructed in [44,45] and confused with the slowly rotating black holes.

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