Third post-Newtonian gravitational radiation from two-body scattering. II. Hereditary energy radiation

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(Received 21 March 2022; accepted 26 April 2022; published 17 May 2022)

We compute the hereditary part of the third post-Newtonian accurate gravitational energy radiation from hyperbolic scatterings (and parabolic scatterings) of nonspinning compact objects. We employ large angular momentum (j) expansion, and compute it to the relative $1/j^{11}$ order (so the first 12 terms). For the parabolic scattering case, the exact solution is computed. At the end, the completely collected expression of the energy radiation up to the third post-Newtonian and from $1/j^3$ to $1/j^{15}$ order, is presented including the instantaneous contribution.

DOI: 10.1103/PhysRevD.105.104035

I. INTRODUCTION

Accurate prediction of energy radiation from compact object binaries emitted via gravitational waves (GW) is one of the crucial inputs to predict dynamics of the binary system. The behavior of a gravitational field at future null infinity is not just dependent on *instantaneous* dynamics of the binaries at the same retarded time but also dependent on the past history of the system due to nonlinearity of Einstein's field equation [1]. The latter contribution is called *hereditary* [2–4] understood within the framework of post-Newtonian (PN) theory and multipolar post-Minkwoskian expansion [5,6] and effective field theory approach [7]. Because more attention has been paid on bound orbits in GW/multimessengers astronomy [8–12], the instantaneous and hereditary contributions of the energy and angular momentum fluxes from bound orbits, were already computed long time ago up to the third post-Newtonian (3PN) order (i.e. $1/c^6$ correction to the Newtonian term, where c is speed of light) in [13–15] long ago in both instantaneous/hereditary contributions. On the other hand, a highly accurate description on unbound orbits is getting more attention today in several complementary methodologies [16–21] because of its direct usage such as for detection in LIGO, LISA, and IPTA [22–25], and its astronomical applications to gravitational captures [26], and also the fact that analytic solutions describing dynamics of unbound orbits have comprehensive information on both bound and unbound orbits [27–29].

In this work, we compute the hereditary contribution of energy radiation ($\Delta\mathcal{E}$) during hyperbolic and parabolic scattering of nonspinning compact objects up to 3PN order. This is a sequel to Ref. [17], where 3PN instantaneous contribution of $\Delta\mathcal{E}$ and angular momentum radiation ($\Delta\mathcal{J}$) were computed. By Ref. [17] and this work, we complete 3PN energy radiation. Historically, the leading (Newtonian) order energy radiation was computed in [30], and the extension to 1PN order was made in [31]. And the post-Minkowskian (PM) result was computed at the leading G^3 order in [32] (G is Newton's constant). Recently, the hereditary contribution of both $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ was tackled in [33] partially up to $1/j^7$ order, at the leading order of each tail, tail-of-tail, and tail-squared pieces. In an independent way, we compute all pieces of the hereditary contribution up to 3PN order including next-to-leading order of tail piece, and provide up to $1/j^{15}$ or $1/j^{16}$ order in large-j (or, large eccentricity) expansion. Since taking large angular momentum approximation after PN approximation sequentially is equivalent to small energy approximation after PM expansion, this corresponds to compute up to G^{15} (for tail), and G^{16} (for tail-of-tail, tail-squared) orders. The reason that the aimed G order is different is that our purpose is not only providing a truncated solution in a certain G order but also providing a decent approximation with 12 terms for practical usage. Since tail piece starts at G^4 order, while tail-of-tail and tail-squared pieces start at G^5 order, they could end at the different orders. Also, for this reason, we work with two variables, angular momentum and eccentricity to characterize hyperbolic orbits. We will see that this choice exhibits remarkable convergence in Sec. VI.

This paper is organized as what follows. In Sec. II, we explain all inputs needed for entire computation. Next, we perform the main computation. The three different pieces are solved via several strategies adopted for each piece, leading order of tail in Sec. III, leading order of tail-of-tail and tail-squared in Sec. IV, and 1PN correction of tail in Sec. V. In Sec. VI, we provide the exact value of energy radiation from parabolic orbits, and then compare it to hyperbolic energy radiation. In Appendix A, some required

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integrals are computed exactly. In Appendix C, we collect all contributions up to 3PN and from G^3 to G^{15} including the instantaneous part. In the Supplemental Material [34], all results are provided in *Wolfram* language.

II. FORMALISM AND NOTATIONS

We are computing total energy radiation from scatterings of compact binaries of which each component masses are m_1, m_2, and their positions are x_1^i, x_2^i in an almost Minkowskian coordinate. Since the total energy radiation $\Delta\mathcal{E}$ is composed of several pieces schematically,

$$\Delta\mathcal{E} = \Delta\mathcal{E}_{\text{inst}} + \Delta\mathcal{E}_{\text{hered}}, \tag{1}$$

where $\Delta\mathcal{E}_{\text{hered}}$, of our interest, consists of subpieces,

$$\Delta\mathcal{E}_{\text{hered}} = \Delta\mathcal{E}_{\text{tail}} + \Delta\mathcal{E}_{(\text{tail})^2} + \Delta\mathcal{E}_{\text{tail(tail)}}. \tag{2}$$

Since $\Delta\mathcal{E}_{\text{tail}}$ starts at 1.5PN order, we need the leading term and its 1PN correction (2.5PN), while only leading contribution of $\Delta\mathcal{E}_{(\text{tail})^2} + \Delta\mathcal{E}_{\text{tail(tail)}}$ is required, because they start at 3PN order. To do this, we need 1PN quadrupole mass moment I_{ij}, Newtonian octupolar mass moment I_{ijk}, and Newtonian quadrupole current moment J_{ij} in terms of relative position $x^i = x_1^i - x_2^i$ of binaries in the center-of-mass frame, which can be found in [13], and we present them here for convenience,

$$I_{ij} = M\nu(x_i x_j)_{\text{STF}}\left[1 + v^2\left(\frac{29}{42} - \frac{29}{14}\nu\right) + \frac{GM}{R}\left(-\frac{5}{7} + \frac{8}{7}\nu\right)\right] + \frac{M\nu}{c^2}(v_i v_j)_{\text{STF}}R^2\left(\frac{11}{21} - \frac{11}{7}\nu\right) + \frac{M\nu}{c^2}(x_i v_j)_{\text{STF}}\frac{d(R^2)}{d\hat{t}}\left(-\frac{2}{7} + \frac{6}{7}\nu\right), \tag{3a}$$

$$I_{ijk} = -M\nu\sqrt{1-4\nu}(x_i x_j x_k)_{\text{STF}}, \tag{3b}$$

$$J_{ij} = M\nu\sqrt{1-4\nu}(x^i (x\times v)^j)_{\text{STF}}. \tag{3c}$$

Therein, the following are defined: $v^i = \frac{dx^i}{d\hat{t}}$ where $\hat{t}$ stands for the physical time, total mass $M = m_1 + m_2$, symmetric mass ratio $\nu = \frac{m_1 m_2}{M^2}$, $R = \sqrt{x^i x^j \delta_{ij}}$, and $()_{\text{STF}}$ means the symmetric and trace-free part. x^i in terms of physical time $\hat{t}$ up to 3PN can be found in Ref. [16] (we need only 1PN), and is written here with introducing reduced variables such as $r := \frac{R}{GM}$ and $t := \frac{t_{\text{phys}}}{GM}$. Note that r has a dimension of $\frac{\text{time}^2}{\text{length}^2}$ while t has $\frac{\text{time}^3}{\text{length}^3}$. We define the azimuthal angle as $\phi := \arctan(\frac{x^2}{x^1})$ assuming that $x^3 = 0$. The analytic expressions for these dynamical variables are given in time implicitly via an eccentric anomaly u $(-\infty \le u \le \infty)$ characterized by e_t (time eccentricity) in 1PN relation,

$$r = j^2\frac{e_t\cosh u - 1}{e_t^2 - 1} + \frac{1}{c^2}\frac{8 + 4e_t^2(-3+\nu) - e_t(e_t^2(-4+\nu) + 3\nu)\cosh u}{2(e_t^2-1)}, \tag{4a}$$

$$\phi = 2\arctan\left(\sqrt{\frac{e_t+1}{e_t-1}}\tanh\left(\frac{u}{2}\right)\right) + \frac{1}{c^2 j^2}\left(6\arctan\left(\sqrt{\frac{e_t+1}{e_t-1}}\tanh\left(\frac{u}{2}\right)\right) - \frac{e_t\sqrt{e_t^2-1}(-4+\nu)\sinh(u)}{(e_t\cosh(u)-1)}\right), \tag{4b}$$

$$t = \frac{1}{n}(e_t\sinh u - u), \tag{4c}$$

where

$$n = \frac{(e_t^2-1)^{3/2}}{j^3} + \frac{1}{c^2 j^5}\frac{(e_t^2-1)^{3/2}(3 + \nu + e_t^2(-9+5\nu))}{2}, \tag{5}$$

and e_t is dimensionless and given in reduced energy $E = \frac{\mathcal{E}}{M\nu}$, reduced angular momentum $j = \frac{\mathcal{J}}{GM^2\nu}$, as

$$e_t^2 = 1 + 2Ej^2 + \frac{E}{2c^2}(8 - 8\nu + (17-7\nu)(2Ej^2)). \tag{6}$$

Note that we will adopt e_t and j to characterize hyperbolic orbits, because e_t gives the simplest structure in t which makes the following computations easier, and also large-e_t expansion (which is equivalent to large-j expansion) with j exhibits a great convergence rate even at $e_t = 1$ as reported in [17].

III. LEADING ORDER TAIL CONTRIBUTION

In this section, we are going to get the analytic expressions of the Newtonian contribution of each tail contributions of energy radiation. The computation done in this section is influenced by [35–37]. In the tail contribution up to 3PN order, the explicit expressions [15] are given as

$$\Delta\mathcal{E}_{\rm tail} = \frac{1}{c^8}\frac{4G^2\mathcal{M}}{5}\frac{1}{G^6M^6}\int_{-\infty}^{\infty}dt\int_0^{\infty}d\tau I_{ij}^{(3)}(t)I_{ij}^{(5)}(t-\tau)\left[\log\left(\frac{c\tau GM}{2r_0}\right)+\frac{11}{12}\right]$$
$$+\frac{1}{c^{10}}\frac{4G^2\mathcal{M}}{189}\frac{1}{G^8M^8}\int_{-\infty}^{\infty}dt\int_0^{\infty}d\tau I_{ijk}^{(4)}(t)I_{ijk}^{(6)}(t-\tau)\left[\log\left(\frac{c\tau GM}{2r_0}\right)+\frac{97}{60}\right]$$
$$+\frac{1}{c^{10}}\frac{64G^2\mathcal{M}}{45}\frac{1}{G^6M^6}\int_{-\infty}^{\infty}dt\int_0^{\infty}d\tau J_{ij}^{(3)}(t)J_{ij}^{(5)}(t-\tau)\left[\log\left(\frac{c\tau GM}{2r_0}\right)+\frac{7}{6}\right], \tag{7}$$

where $\mathcal{M}$ is ADM mass

$$\mathcal{M} = M\left(1+\frac{\nu}{c^2}\frac{e_t^2-1}{2j^2}\right). \tag{8}$$

Note that the extra GMs in the above expressions, which cannot be found in [15], appear because of using t and τ as the reduced time variable. In favor of computation, we decompose $\Delta\mathcal{E}_{\rm tail}$ into several pieces in Fourier domain such as

$$\Delta\mathcal{E}_{(0)}+\Delta\mathcal{E}_{(1)} = \frac{2}{5}\frac{n^6\mathcal{M}}{c^8G^4M^6e_t^6}\int_0^{\infty}dp|\hat{I}_{ij}(p)|^2p^7, \tag{9a}$$

$$\Delta\mathcal{E}_{(2)} = \frac{2}{189}\frac{n^8\mathcal{M}}{c^{10}G^6M^8e_t^8}\int_0^{\infty}dp|\hat{I}_{ijk}(p)|^2p^9, \tag{9b}$$

$$\Delta\mathcal{E}_{(3)} = \frac{32}{45}\frac{n^6\mathcal{M}}{c^{10}G^4M^6e_t^6}\int_0^{\infty}dp|\hat{J}_{ij}(p)|^2p^7, \tag{9c}$$

where $\Delta\mathcal{E}_{(0)}$, $\Delta\mathcal{E}_{(2)}$, and $\Delta\mathcal{E}_{(3)}$ denote the leading (Newtonian) order contributions of the right-hand sides respectively, whereas $\Delta\mathcal{E}_{(1)}$ refers to 1PN correction of the right-hand side. Therein, we have defined the Fourier transform as

$$\hat{K}(p) := \int_{-\infty}^{+\infty}d\bar{l}e^{ip\bar{l}}K(\bar{l}), \tag{10a}$$

$$K(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}dpe^{-ip\bar{l}}\hat{K}(p), \tag{10b}$$

where the reduced mean anomaly is defined as $\bar{l} := \sinh u - \frac{u}{e_t} = \frac{n}{e_t}(t-t_0)$ in the preparation of large e_t expansion, and p and l are both dimensionless for the convenience of forthcoming computation. Also, we should mention that we use the following integration reported in [15] (with γ being Euler's constant):

$$\int_0^{+\infty}dte^{i\sigma t}\log(t) = -\frac{1}{\sigma}\left[\frac{\pi}{2}\mathrm{sign}(\sigma)+i(\log(|\sigma|)+\gamma)\right] \tag{11}$$

in the middle of obtaining the Fourier domain expressions in Eqs. (9). Among them, we compute three of them $\Delta\mathcal{E}_{(0)}$, $\Delta\mathcal{E}_{(2)}$, and $\Delta\mathcal{E}_{(3)}$ here, because they have analytic expressions in Fourier domain, whereas it is not the case for $\Delta\mathcal{E}_{(1)}$. We will come back to $\Delta\mathcal{E}_{(1)}$ in Sec. V.

Using Eqs. (4), we can write all components of multipole moments in terms of the eccentric anomaly u (particularly the form of e^{ku} with integers k). By considering $d\bar{l} = du(\cosh u - \frac{1}{e_t})$, every term encountered in the Fourier transformation could be written in form of the typical integral,

$$I_t := \int_{-\infty}^{+\infty}due^{ip\sinh u + ku - i\frac{p}{e_t}u}. \tag{12}$$

This is nothing but one of integral representations of the Hankel function of the first kind $H_a^{(1)}(x)$ (we will call it just the Hankel function henceforth),

$$H_a^{(1)}(x) = \frac{1}{i\pi}\int_{-\infty}^{+\infty}e^{x\sinh u - au}du. \tag{13}$$

Note that the integral on the right-hand side is only convergent when the order is in $-1 < \mathrm{Re}(a) < 1$, which is not our case (k could be bigger than 1 in I_t). It implies that the typical integral I_t is actually divergent in the usual integral sense. But the Hankel function is even analytic outside of $-1 < \mathrm{Re}(a) < 1$, hence, we will use the Hankel function as a regularized, hence finite value of the typical divergent integral by the argument of analytic continuation resulting in

$$I_t = i\pi H^{(1)}_{i\frac{p}{e_t}-k}(ip). \tag{14}$$

After successive uses of the relations

$$H_{a-1}^{(1)}(x)+H_{a+1}^{(1)}(x) = \frac{2a}{x}H_a^{(1)}(x), \tag{15}$$

to unify all the orders appearing in the integrands to either 0 or 1, we obtain the intermediate results of $|\hat{I}_{ij}(p)|^2$, $|\hat{I}_{ijk}(p)|^2$ and $|\hat{J}_{ij}(p)|^2$ at the leading order. As pointed out in [35], the large-e_t expansion drops the p dependence in the order of the Hankel function, which makes the computation much simpler. After taking the large-e_t expansion, every term can be dealt with by the following master integral (a similar expression is reported in Eq. (7.24) of [37]),

$$\int_{-\infty}^{+\infty} dp p^n H_a^{(1)}(ip) H_b^{(1)}(ip) = \frac{2^n e^{-i\frac{(1+a+b)\pi}{2}}}{i\pi\Gamma(1+n)} \Gamma\left(\frac{1-a-b+n}{2}\right)\Gamma\left(\frac{1+a-b+n}{2}\right) \times \Gamma\left(\frac{1-a+b+n}{2}\right)\Gamma\left(\frac{1+a+b+n}{2}\right), \quad (16)$$

Now that all the computation is reduced down to the straightforward but extensive machinery work, we only present the final results with the first 12 terms in the large e_t expansion $\Delta\mathcal{E}_{(0)}^{(l)}$ in Appendix B. [We use the superscript (l) to denote that this is not the exact solution but expressed in large-e_t (or, large-j) expansion.] We have found that the first three terms are coincident to Eq. (4.14) in [33]. By the exactly same procedure, we also get the first 12 terms of $\Delta\mathcal{E}_{(2)}$ and $\Delta\mathcal{E}_{(3)}$ presented in Appendix B.

IV. LEADING QUADRATIC TAIL CONTRIBUTION

In this section, we are going to obtain analytic expressions of (tail)^2 and tail (tail) contributions. In terms of mulitpole moments, the leading order terms of each piece [15] are

$$\Delta\mathcal{E}_{(\mathrm{tail})^2} = \frac{4G^3\mathcal{M}^2}{c^{11}5}\frac{1}{G^7M^7}\int_{-\infty}^{+\infty} dt\left(\int_0^{+\infty} d\tau I_{ij}^{(5)}(t-\tau)\left[\log\left(\frac{c\tau GM}{2r_0}\right)+\frac{11}{12}\right]\right)^2, \quad (17a)$$

$$\Delta\mathcal{E}_{\mathrm{tail(tail)}} = \frac{4G^3\mathcal{M}^2}{c^{11}5}\frac{1}{G^7M^7}\int_{-\infty}^{+\infty} dt I_{ij}^{(3)}(t)\int_0^{+\infty} d\tau I_{ij}^{(6)}(t-\tau)\left[\log^2\left(\frac{c\tau GM}{2r_0}\right)+\frac{57}{70}\log\left(\frac{c\tau GM}{2r_0}\right)+\frac{124627}{44100}\right]. \quad (17b)$$

Similarly, they are rearranged compactly in a Fourier domain into

$$\Delta\mathcal{E}_{(4)} = \frac{\mathcal{M}^2}{c^{11}G^4M^7}\frac{2}{5\pi}\frac{n^7}{e_t^7}\left[\frac{2\pi^2}{3}-\frac{214}{105}\left(\log\left(\frac{2nr_0}{ce_tGM}\right)+\gamma_E\right)-\frac{116761}{29400}\right]\left(\int_0^\infty dp|\hat{I}_{ij}|^2p^8\right), \quad (18a)$$

$$\Delta\mathcal{E}_{(5)} = -\frac{\mathcal{M}^2}{c^{11}G^4M^7}\frac{428}{525\pi}\frac{n^7}{e_t^7}\int_0^\infty dp|\hat{I}_{ij}|^2p^8\log p, \quad (18b)$$

where we use the relations [15], Eq. (11), and

$$\int_0^{+\infty} dt e^{i\sigma t}\log^2(t) = \frac{i}{\sigma}\left\{\frac{\pi^2}{6}-\left[\frac{\pi}{2}\mathrm{sign}(\sigma)+i(\log(|\sigma|)+\gamma)\right]^2\right\}. \quad (19)$$

Thanks to an even power of p in the integrand, $\Delta\mathcal{E}_{(4)}$ can be integrated in an exact way. Using the elementary integrals $E^i(n)$ defined in Appendix A,

$$\int_0^\infty dp|\hat{I}_{ij}|^2p^8 = 8\pi^2e_t^3j^8G^4M^6\nu^2\left[\frac{(3e_t^2-4)e_tE^y(5)}{(e_t^2-1)^3}+\frac{e_t^2E^z(4)}{(e_t^2-1)^3}+\frac{e_t^2E^z(6)}{(e_t^2-1)^2}-\frac{E^x(6)}{e_t^2-1}-\frac{(e_t^4-3e_t^2+3)E^x(4)}{3(e_t^2-1)^4}\right], \quad (20)$$

where the exact solutions of each elementary integrals are listed in Eqs. (A10). Gathering all pieces, we obtain the exact expression ($\mathcal{M}=M$ at this order) displayed in Eq. (B5). We have checked that the r_0 dependence in $\Delta\mathcal{E}_{(4)}$ is canceled with r_0 in Eq. (31.d) of [17], so the final result does not depend on r_0.

In the case of $\Delta\mathcal{E}_{(5)}$, since there is no closed form solution, we use the same strategy of computing the Newtonian tails explained in Sec. III. The picky $\log p$ can be dealt with by the derivative with respect to n in the master integral Eq. (16), and hence is reduced down to the machinery work as well. Likewise, we display the first 12 terms of the large-e_t expansion of $\Delta\mathcal{E}_{(5)}$ in Appendix B. The agreement is found in the first three terms with Eq. (4.14) in Ref. [33].[1]

V. 1PN TAIL CONTRIBUTION

Now, we need to tackle down the most tricky computation of $\Delta\mathcal{E}_{(1)}$ in this paper. Since we could not obtain Fourier transform of the quadrupole moment at the 1PN order in general, time domain integration is attempted.

[1]Only the second arXiv version of [33] for now.

Contrary to the way we have computed, we could not find the systematic and simple automatic way of integration i.e., master integral method, so the integrations should be treated in a rather heuristic way. For the first 4 terms, we could find exact values of the coefficients by symbolic integration, but we failed for the other 8 terms. Instead, numerical integration and Polynomial-time algorithm, involving the Sum-of-square and LQ decomposition schemes (PSLQ) [38] are adopted to find analytic form of the coefficients. Since we have encountered more than 10 thousands terms, we will explain the process of computation roughly.

By putting all dynamical variables Eq. (4) in the quadrupole moment Eq. (3a), we construct the integral over u and v which are given via the relations to the time variables

$$t = \frac{1}{n}(e_t \sinh u - u), \tag{21a}$$

$$t - \tau = \frac{1}{n}(e_t \sinh v - v). \tag{21b}$$

For the efficient computation, we subsequently choose two dynamical parameters (κ, q) in the integrand,

$$\kappa = \mathrm{e}^u, \tag{22a}$$

$$q = \frac{\mathrm{e}^v}{\kappa}. \tag{22b}$$

Schematically, the tail radiation in the time domain has the following form:

$$\Delta\mathcal{E} \sim \int_{-\infty}^{+\infty} dt \int_{-\infty}^{t} d(t-\tau)\mathcal{I}_1(t, t-\tau), \tag{23a}$$

$$\sim \int_{-\infty}^{+\infty} du \int_{-\infty}^{u} dv \mathcal{I}_2(u, v), \tag{23b}$$

$$\sim \int_{0}^{+\infty} d\kappa \int_{0}^{1} dq \mathcal{I}_3(\kappa, q), \tag{23c}$$

where $\mathcal{I}_{1,2,3}$ represent some integrands. One can easily see that the choice of q makes two integrals independent. All terms we encounter are classified into four categories,

$$I_1 = \frac{(q\kappa^2+1)^n \kappa^l}{(q-1)^a (q^2\kappa^2+1)^b}, \tag{24a}$$

$$I_2 = \frac{(q\kappa^2+1)^n \kappa^l \arctan(\frac{\kappa-1}{\kappa+1})}{(q-1)^a (q^2\kappa^2+1)^b}, \tag{24b}$$

$$I_3 = \frac{(q\kappa^2+1)^n \kappa^l \log^m(q)}{(q-1)^a (q^2\kappa^2+1)^c}, \tag{24c}$$

$$I_4 = \frac{(q\kappa^2+1)^n \kappa^l \arctan(\frac{\kappa-1}{\kappa+1}) \log^m(q)}{(q-1)^a (q^2\kappa^2+1)^b}, \tag{24d}$$

where m, l, a, b, c denote some positive integers whereas n is an integer. To solve them, we have found that

$$\begin{aligned} &\int_0^1 dq \frac{\log^n(q) q^m}{(q+a)(q+b)(q+c)(q+d)} \\ &= (-1)^{n+m+1} n! \Bigg[\frac{a^m \mathrm{Li}_{n+1}(\frac{-1}{a})}{(a-b)(a-c)(a-d)} \\ &\quad + \frac{b^m \mathrm{Li}_{n+1}(\frac{-1}{b})}{(b-a)(b-c)(b-d)} + \frac{c^m \mathrm{Li}_{n+1}(\frac{-1}{c})}{(c-a)(c-b)(c-d)} \\ &\quad + \frac{d^m \mathrm{Li}_{n+1}(\frac{-1}{d})}{(d-a)(d-b)(d-c)}\Bigg], \end{aligned} \tag{25}$$

where n, m are positive integers with $m \leq 3$, and Li_n is a polylogarithm. After taking successive derivatives with respect to a, b, c, d, we manage to integrate all types of integrand over q. In the cases of I_1, I_2, and I_3, further integration over κ is also possible analytically (hence, every coefficient of the ν^3 term is determined exactly). However, in the case of I_4, we encounter multilogarithm terms such as

$$\int dq I_4 \sim g(Q) \arctan(Q) \mathrm{Li}_n(f(Q)), \tag{26}$$

where $f(Q)$, $g(Q)$ are some rational functions of a polynomial of Q with $Q = \frac{\kappa-1}{\kappa+1}$. Given that arctan is also a logarithmic function, we need to do integration over two or more logarithms. When $n = 2$ (i.e., dilogarithm), we could get rid of Li_2 by integration by parts, but when $n \geq 3$, because the derivative of Li_n is still a polylogarithm, we could not find an analytic way of integration. Instead, for remaining terms, we integrate I_4 type integrals (q is already integrated) numerically with 600–1000 digits, and attempt to find their analytic forms using PSLQ algorithm. Since PSLQ algorithm is used for guessing a relation between integers, we need to make a list of candidates. For even orders e_t^n ($n = 8, 6, 4 \cdots$), the candidate list consists of combinations of even powers of π, and $\zeta(m)$ with odd integer m, such as

$$\{1, \pi^2, \pi^4, \ldots, \zeta(3), \pi^2\zeta(3), \cdots \zeta(5), \pi^2\zeta(5), \cdots\}, \tag{27}$$

and odd powers of π for odd powers of e_t,

$$\{\pi, \pi^3, \pi^5 \cdots\}. \tag{28}$$

The higher order of e_t we take, the longer candidate lists are required. We first guess analytic forms of numerical coefficients using first 400–800 digits, and check if the number of digits used in guessed integers is much smaller than 400–800. For example, if from this 40-digit number,

46.5094150204497302632144726006520928 0334, PSLQ guesses these two long integers (in each numerator and denominator) in total 55 digits,

$$\frac{4320335101191349704192837044}{291827644122831121457024899}\pi.$$

We regard it as a failure because it just stores 40 digits with 55 digits. But if PSLQ finds two digits such as $\frac{3}{2}\pi^3$, we regard it as trustworthy. After meeting this standard, we also check if they can predict the next 200 digits, and if so, we confirm them. Using the coefficients determined in this way, we compute the first 12 terms of $\Delta\mathcal{E}_{(1)}$ which can be found in Appendix B.

VI. RADIATION FROM PARABOLIC ORBITS

In this section, we are going to compute the hereditary part of energy radiation $\Delta\mathcal{E}^{(p)}_{\text{hered}}$ from parabolic orbits and this could be understood as the limit of $e_t \to 1$ (or, $E \to 0$) in the hyperbolic radiation. As pointed out in [17], the choice of the parametrization (e_t, h) [or (e, h)] of orbits allows us to quickly converge large eccentricity expansions even approximating well at $e_t = 1$. We check if this works even in the hereditary contribution as well as the validity of our hyperbolic computations.

The parabolic version of Keplerian parametrization can be derived by changing u to w via $u = (e_t^2 - 1)^{1/2} w$ in Eqs. (4). After taking the limit of $e_t \to 1$, one will get

$$r = \frac{j^2 w^2}{2} + \frac{j^2}{2} + \frac{1}{c^2}\left(\frac{\nu - 6}{2} + (1-\nu) w^2\right), \tag{29a}$$

$$t = \frac{j^3 w^3}{6} + \frac{j^3 w}{2} + \frac{j}{c^2}\left(-\frac{1}{2}(\nu - 1) w^3 - \frac{3}{2}(\nu - 1) w\right), \tag{29b}$$

$$\phi = 2\arctan(w) + \frac{2(3w^2\arctan(w) - \nu w + 4w + 3\arctan(w))}{c^2 j^2 (1 + w^2)}. \tag{29c}$$

By these, it is possible to construct every multipole moment in terms of w. In the computation of $\Delta\mathcal{E}^{(p)}_{(0)}$ and $\Delta\mathcal{E}^{(p)}_{(1)}$, most of the integrations can be done straightforwardly except

$$I_p \coloneqq \int_{-\infty}^{\infty} dw \frac{w(w^6 + 6w^4 - 18w^2 + 4)\arctan(w)}{(w^2+1)^7 (w^2+4)^{7/2}}, \tag{30}$$

which arises in 1PN order of the quadrupole tail part. After inserting the integral representation of

$$\arctan(w) = \int_0^1 dk \frac{w}{1 + w^2 k^2},$$

the successive integrations over w and k could yield the analytic result. By simplifying the result with the help of the second order of the Clausen function, we managed to obtain

$$I_p = \frac{385\,\mathrm{Cl}_2(\frac{\pi}{3})}{1024\sqrt{3}} - \frac{5221}{115200} - \frac{501151\pi}{2799360\sqrt{3}} + \frac{77\pi\log(3)}{1024\sqrt{3}}, \tag{31}$$

where

$$\mathrm{Cl}_2(z) \coloneqq -\int_0^z dx \log\left|2\sin\frac{x}{2}\right|. \tag{32}$$

Specially, $\mathrm{Cl}_2(\frac{\pi}{3})$ is the maximum value of $\mathrm{Cl}_2(z)$ when z is real.

As a result, we obtain the quadrupolar tail piece of energy radiation from parabolic orbits $\Delta\mathcal{E}^{(p)}$,

$$\Delta\mathcal{E}^{(p)}_{(0,1)} = \frac{M\nu^2\pi}{c^8 j^{10}}\left[\frac{169984}{45\sqrt{3}} + \frac{1}{c^2 j^2}\left(27720\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{7949344\pi}{1215} - \frac{41631152}{525\sqrt{3}} + 5544\pi\log(3) + \frac{2818048\nu}{63\sqrt{3}}\right)\right]. \tag{33}$$

In the case of contributions $\Delta\mathcal{E}^{(p)}_{(2)}$, $\Delta\mathcal{E}^{(p)}_{(3)}$, all computation is trivial. And for $\Delta\mathcal{E}^{(p)}_{(4)}$, it is easily obtained by putting 1 in e_t in Eq. (B5) because it is exact,

$$\Delta\mathcal{E}^{(p)}_{(2)} = \frac{M\nu^2(1-4\nu)\pi}{c^{10} j^{12}}\frac{1448960}{63\sqrt{3}}, \tag{34a}$$

$$\Delta\mathcal{E}^{(p)}_{(3)} = \frac{M\nu^2(1-4\nu)\pi}{c^{10} j^{12}}\frac{4096}{9\sqrt{3}}, \tag{34b}$$

$$\Delta\mathcal{E}^{(p)}_{(4)} = \frac{M\nu^2\pi}{c^{11} j^{13}}\left[-\frac{161249\gamma}{15} - \frac{161249}{15}\log\left(\frac{2r_0 n}{e_t GMc}\right) + \frac{10549\pi^2}{3} - \frac{175958827}{8400}\right], \tag{34c}$$

where we keep $\log(n)$, because it is divergent at $e_t = 1$, which will be canceled by $\Delta\mathcal{E}_{(5)}$.

The fifth piece $\Delta\mathcal{E}^{(p)}_{(5)}$ is left. After performing the inverse Fourier transform, $\Delta\mathcal{E}_{(5)}$ is reexpressed in a time domain as

$$\Delta\mathcal{E}_{(5)} = \frac{428}{525}\left[\left(\gamma + \log\left(\frac{n}{c^3 e_t}\right)\right)\int_{-\infty}^{+\infty} dt (I^{(4)}_{ij}(t))^2 + \int_{-\infty}^{+\infty} dt \int_t^{+\infty} d\tau I^{(5)}_{ij}(t+\tau) I^{(4)}_{ij}(\tau)\log(\tau c^3)\right]. \tag{35}$$

As previously computed, almost all terms can be dealt with by elementary integration methods, but the arctan term should be replaced by its integral representation again. One could get

$$\Delta\mathcal{E}^{(p)}_{(5)} = \frac{M\nu^2\pi}{c^{11}j^{13}}\left[\frac{161249\gamma}{15} + \frac{161249}{15}\log\left(\frac{n}{c^3 e_t}\right) - \frac{90210202}{1575} + \frac{161249}{5}\log(2cj) + \frac{161249}{30}\log(3)\right]. \tag{36}$$

Indeed, the $\log(n)$ term in $\Delta\mathcal{E}^{(p)}_{(4)}$ and $\Delta\mathcal{E}^{(p)}_{(5)}$ are canceled. Contrary to $\Delta\mathcal{E}^{(p)}_{(5)}$ having $\log(j)$ dependence, $\Delta\mathcal{E}^{(l)}_{(5)}$ does not have it. This missing $\log(j)$ comes in from $\log(n) = \log(\frac{(e_t^2-1)^{3/2}}{j^3})$ in $\Delta\mathcal{E}_{(4)}$ [Eq. (B5)], which also indicates that $\Delta\mathcal{E}_{(4)}$ and $\Delta\mathcal{E}_{(5)}$ should not be treated independently.

TABLE I. Seven digits comparisons. $G = M = c = j = r_0 = 1$ and $\nu = 1/8$ are chosen.

	$\Delta\mathcal{E}^{(p)}$	$\Delta\mathcal{E}^{(l)}\vert_{e_t=1}$
$\Delta\mathcal{E}_{(0)}$	107.0544	107.0545
$\Delta\mathcal{E}_{(1)}$	1240.453	1240.384
$\Delta\mathcal{E}_{(2)} + \Delta\mathcal{E}_{(3)}$	332.3571	332.3535
$\Delta\mathcal{E}_{(4)} + \Delta\mathcal{E}_{(5)}$	−1114.843	−1114.842

Finally, we compare the exact parabolic results $\Delta\mathcal{E}^{(p)}$ and the hyperbolic results in large-e_t expansion $\Delta\mathcal{E}^{(l)}$ at $e_t = 1$ in Table I. One could find that they have very close values, which shows the robustness of our computations as well as the efficiency of large-e_t expansion.

For completeness, we present the completed result of $\Delta\mathcal{E}^{(p)} = \Delta\mathcal{E}^{(p)}_{\text{inst}} + \Delta\mathcal{E}^{(p)}_{\text{hered}}$ from parabolic orbits up to 3PN combined with the instantaneous contribution (Eq. (43) of [17]),

$$\Delta\mathcal{E}^{(p)} = \frac{M\nu^2\pi}{c^5 j^7}\left\{\frac{170}{3} + \frac{1}{c^2 j^2}\left(\frac{13447}{20} - \frac{1127\nu}{3}\right) + \frac{1}{c^3 j^3}\frac{169984}{45\sqrt{3}} + \frac{1}{c^4 j^4}\left(\frac{5839651}{1008} - \frac{258051\nu}{40} + \frac{5481\nu^2}{4}\right) + \frac{1}{c^5 j^5}\left(27720\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{29317552}{525\sqrt{3}} + \frac{7949344\pi}{1215} + 5544\pi\log(3) - \frac{3092480\nu}{63\sqrt{3}}\right) + \frac{1}{c^6 j^6}\left[\frac{161249\log(2\sqrt{3}cj)}{15} + \frac{10549\pi^2}{3} + \frac{16880186749}{241920} + \left(-\frac{3972009943}{30240} + \frac{208813\pi^2}{160}\right)\nu + \frac{1157409\nu^2}{32} - \frac{29645\nu^3}{8}\right]\right\} + \mathcal{O}\left(\frac{1}{c^{12}}\right). \tag{37}$$

VII. DISCUSSION AND FUTURE WORK

We have computed the hereditary part of energy radiation from hyperbolic scatterings as well as parabolic scatterings. For hyperbolic computation, we rely on large-e_t (or large-j) expansion while the parabolic result is exact. As expected, the hyperbolic result in large-e_t is close to the value of the parabolic one numerically. It is also interesting to point out that the irrational values appearing in the parabolic case are π, log(2), log(3), $\sqrt{3}$, and $\mathrm{Cl}_2(\frac{\pi}{3})$ whereas π, log(2), γ, and $\zeta(2n+1)$ in the hyperbolic case. In order to complete 3PN radiation reaction to a scattering angle, impact parameter, we need to compute angular momentum radiation. This is going to be done in a subsequent paper using the same techniques elaborated in this paper except the memory effect which will probably require another treatment. We also expect that the practical knowledge on the integrations gained from this work might be helpful to extend our knowledge especially in nonlocal in time conservative dynamics, which has the similar structure with hereditary radiation.

ACKNOWLEDGMENTS

We thank Yannick Boetzel for the setup of the formalism at the beginning phase, Rafael A. Porto for discussions, and Hyung-Mok Lee for technical support. G. C. is supported by the ERC Consolidator Grant "Precision Gravity: From the LHC to LISA," provided by the European Research Council (ERC) under the European Union's H2020 research and innovation program, Grant No. 817791.

APPENDIX A: EXACT INTEGRALS

The aim of this section is to obtain the following element integrals:

$$E^x(n) \coloneqq \int_0^\infty dq q^n (H^{(1)}_{iq}(iqe_t))^2, \tag{A1a}$$

$$E^y(n) \coloneqq i\int_0^\infty dq q^n H_{iq}(iqe_t)\dot{H}^{(1)}_{iq}(iqe_t), \tag{A1b}$$

$$E^z(n) \coloneqq \int_0^\infty dq q^n (\dot{H}^{(1)}_{iq}(iqe_t))^2, \tag{A1c}$$

where $\dot{H}_a^{(1)}(x) = \frac{\partial}{\partial x} H_a^{(1)}(x)$. Inspired by appendix of Ref. [39], we start at the following relation between (l, u, e_t):

$$l = e_t \sinh u - u. \tag{A2}$$

This gives values of l from e_t and u explicitly. What we want to do is to obtain an explicit expression $u = U(l, e_t)$. First, we seek Fourier transform $K(q)$ of $e \sinh u$,

$$e_t \sinh u \coloneqq \int_{-\infty}^{+\infty} K(q) \mathrm{e}^{-iql} dq, \tag{A3}$$

or

$$\begin{aligned} K(q) &= \frac{e_t}{2\pi} \int \sinh u \mathrm{e}^{iql} dl, \\ &= -\frac{H_{iq}^{(1)}(iqe_t)}{2q}. \end{aligned} \tag{A4}$$

So,

$$U(l, e_t) = -l - \int_{-\infty}^{+\infty} \frac{H_{iq}^{(1)}(iqe_t)}{2q} \mathrm{e}^{-iql} dq. \tag{A5}$$

Now, we have two definitions for U: the implicit one Eq. (A2), and the explicit one Eq. (A5). If one takes an operator $\nabla_{e_t} \coloneqq \partial_{e_t}^2 + \frac{1}{e_t} \partial_{e_t}$ on the definitions, two seemingly different (but actually the same) expressions are obtained,

$$\nabla_{e_t} U = \frac{(e_t^2 - 1)\sinh(u)}{e_t(e_t \cosh(u) - 1)^3} \quad \text{(from the implicit side)}, \tag{A6}$$

$$= \frac{(e_t^2 - 1)}{2e_t^2} \int_{-\infty}^{+\infty} q H_{iq}^{(1)}(iqe_t) \mathrm{e}^{-iql} dq \quad \text{(from the explicit side)}, \tag{A7}$$

in which we apply

$$\dot{H}_a^{(1)}(x) = \frac{a H_a^{(1)}(x)}{x} - H_{a+1}^{(1)}(x).$$

If one repeats this several times, since

$$(\nabla_{e_t})^n U \sim \int_{-\infty}^{+\infty} q^{2n-1} H_{iq}^{(1)}(iqe_t) \mathrm{e}^{-iql} dq, \tag{A8}$$

from the explicit side, we multiply it with its complex conjugate $(*)$ to get the desirable form,

$$[(\nabla_{e_t})^n U][(\nabla_{e_t})^m U]^* \sim \iint dq dk q^{2n-1} k^{2m-1} H_{iq}^{(1)}(iqe_t) H_{ik}^{(1)*}(ike_t) \mathrm{e}^{-i(q-k)l}. \tag{A9}$$

On the other hand, one will get elementary expressions from the implicit side. After integrating over $-\infty < l < \infty$ on the both sides and considering $H_{iq}^{(1)}(iqe_t)$ is a purely imaginary number, one can get exact values of E^x type integrals. In order to get E^y and E^z types, taking derivatives with respect to e_t is sufficient. As a result, we list the required integrals below:

$$E^x(2) = \frac{13e_t^2 + 2}{12\pi(e_t^2 - 1)^3} + \frac{(e_t^2 + 4)e_t^2 \arccos(-\frac{1}{e_t})}{4\pi(e_t^2 - 1)^{7/2}}, \tag{A10a}$$

$$E^x(4) = -\frac{3691e_t^6 + 11082e_t^4 + 2568e_t^2 - 16}{960\pi(e_t^2 - 1)^6} - \frac{(27e_t^6 + 472e_t^4 + 592e_t^2 + 64)e_t^2 \arccos(-\frac{1}{e_t})}{64\pi(e_t^2 - 1)^{13/2}}, \tag{A10b}$$

$$\begin{aligned} E^x(6) = &-\frac{4954041e_t^{10} + 45033746e_t^8 + 65383216e_t^6 + 18070896e_t^4 + 565696e_t^2 + 1280}{161280\pi(e_t^2 - 1)^9} \\ &-\frac{(1125e_t^{10} + 45820e_t^8 + 189040e_t^6 + 161152e_t^4 + 27776e_t^2 + 512)e_t^2 \arccos(-\frac{1}{e_t})}{512\pi(e_t^2 - 1)^{19/2}}, \end{aligned} \tag{A10c}$$

$$E^y(3) = \frac{5e_t(11e_t^2 + 10)}{24\pi(e_t^2 - 1)^4} + \frac{e_t(3e_t^4 + 24e_t^2 + 8) \arccos(-\frac{1}{e_t})}{8\pi(e_t^2 - 1)^{9/2}}, \tag{A10d}$$

$$E^y(5) = \frac{e_t(7517e_t^6 + 39294e_t^4 + 26296e_t^2 + 1968)}{640\pi(e_t^2 - 1)^7} + \frac{e_t(135e_t^8 + 3520e_t^6 + 8160e_t^4 + 3072e_t^2 + 128) \arccos(-\frac{1}{e_t})}{128\pi(e_t^2 - 1)^{15/2}}, \tag{A10e}$$

$$E^z(4) = \frac{5469e_t^6 + 12598e_t^4 + 2392e_t^2 + 16}{960\pi e_t^2(e_t^2-1)^5} + \frac{(45e_t^6 + 632e_t^4 + 624e_t^2 + 64)\arccos(-\frac{1}{e_t})}{64\pi(e_t^2-1)^{11/2}}, \tag{A10f}$$

$$E^z(6) = \frac{6581763e_t^{10} + 53772430e_t^8 + 70576832e_t^6 + 18257424e_t^4 + 587456e_t^2 - 1280}{161280\pi e_t^2(e_t^2-1)^8} + \frac{(1575e_t^{10} + 58140e_t^8 + 217360e_t^6 + 169856e_t^4 + 28032e_t^2 + 512)\arccos(-\frac{1}{e_t})}{512\pi(e_t^2-1)^{17/2}}. \tag{A10g}$$

Note that for $E^x(n)$ and $E^z(n)$, n should be even, and odd for $E^y(n)$. Otherwise, this strategy does not work.

APPENDIX B: LIST OF PARTIAL RESULTS

We note the following partial results:

$$\begin{aligned}
\Delta\mathcal{E}^{(l)}_{(0)} = \frac{M\nu^2}{c^8 j^{10}}\Bigg[&\frac{3136e_t^6}{45} + \frac{297\pi^3 e_t^5}{20} + \left(\frac{88576\pi^2}{675} - \frac{64}{45}\right)e_t^4 \\
&+ \left(\frac{11741\pi^3}{24} - \frac{2755\pi^5}{64}\right)e_t^3 + \left(-\frac{27776}{225} + \frac{844288\pi^2}{675} - \frac{280576\pi^4}{2625}\right)e_t^2 \\
&+ \left(\frac{202289\pi^3}{96} - \frac{3419707\pi^5}{1152} + \frac{17885\pi^7}{64}\right)e_t + \left(\frac{55936}{1575} + \frac{421888\pi^2}{135} - \frac{509827072\pi^4}{212625} + \frac{104726528\pi^6}{496125}\right) \\
&+ \frac{1}{e_t}\left(\frac{3553649\pi^3}{960} - \frac{154014913\pi^5}{4608} + \frac{102671023\pi^7}{3200} - \frac{47703411\pi^9}{16384}\right) \\
&+ \frac{1}{e_t^2}\left(\frac{1856}{189} + \frac{4238336\pi^2}{945} - \frac{1355776\pi^4}{81} + \frac{7286226944\pi^6}{893025} - \frac{719847424\pi^8}{1091475}\right) \\
&+ \frac{1}{e_t^3}\left(\frac{36730841\pi^3}{7680} - \frac{7602883699\pi^5}{46080} + \frac{24445251593\pi^7}{38400} - \frac{406754528303\pi^9}{819200} + \frac{2879946531\pi^{11}}{65536}\right) \\
&+ \frac{1}{e_t^4}\left(\frac{5312}{1485} + \frac{5176832\pi^2}{945} - \frac{905998336\pi^4}{14175} + \frac{12913147904\pi^6}{127575} - \frac{209391976448\pi^8}{5457375} + \frac{2021654528\pi^{10}}{693693}\right) \\
&+ \frac{1}{e_t^5}\left(\frac{86668951\pi^3}{15360} - \frac{191346755029\pi^5}{368640} + \frac{426736282133\pi^7}{76800}\right. \\
&\left. - \frac{440356873312201\pi^9}{29491200} + \frac{2142139802884279\pi^{11}}{206438400} - \frac{4738568225391\pi^{13}}{5242880}\right) + \mathcal{O}(1/e_t^6)\Bigg],
\end{aligned} \tag{B1}$$

$$\begin{aligned}
\Delta\mathcal{E}^{(l)}_{(1)} = \frac{M\nu^2}{c^{10} j^{12}}\Bigg\{&e_t^8\left(-\frac{288256}{315} + \frac{185824\nu}{315}\right) + e_t^7\left(\frac{9216\pi}{35} - \frac{110367\pi^3}{560} + \frac{873\pi^3\nu}{7}\right) \\
&+ e_t^6\left[\frac{467596}{315} - \frac{6344704\pi^2}{4725} + \frac{2898\zeta(3)}{5} + \left(\frac{81856}{315} + \frac{1715456\pi^2}{1575}\right)\nu\right] \\
&+ e_t^5\left[\frac{2816\pi}{225} - \frac{216981733\pi^3}{36960} + \frac{514505\pi^5}{896} + \left(\frac{682223\pi^3}{168} - \frac{315785\pi^5}{896}\right)\nu\right] \\
&+ e_t^4\left[\frac{3920593}{3150} - \frac{22037087\pi^2}{3780} + \frac{177987584\pi^4}{202125} + \frac{54936\zeta(3)}{5} - \frac{8883\pi^2\zeta(3)}{8} - \frac{118017\zeta(5)}{8}\right. \\
&\left. + \left(-\frac{1670624}{1575} + \frac{49979392\pi^2}{4725} - \frac{5280768\pi^4}{6125}\right)\nu\right]
\end{aligned}$$

$$
\begin{aligned}
&+e_t^3\bigg[-\frac{79616\pi}{135}-\frac{202204565099\pi^3}{11088000}+\frac{2230868679727\pi^5}{57657600}-\frac{7622335\pi^7}{2048}\\
&+\left(\frac{6238315\pi^3}{336}-\frac{42527319\pi^5}{1792}+\frac{2272095\pi^7}{1024}\right)\nu\bigg]\\
&+e_t^2\bigg[-\frac{5903108}{2205}+\frac{2185941973\pi^2}{226800}+\frac{5354923998451\pi^4}{523908000}-\frac{11762012782592\pi^6}{8442559125}+\frac{76944\zeta(3)}{5}\\
&-\frac{443177\pi^2\zeta(3)}{8}+\frac{174195\pi^4\zeta(3)}{64}-\frac{5887923\zeta(5)}{8}+\frac{3857175\pi^2\zeta(5)}{64}+\frac{47405925\zeta(7)}{64}\\
&+\left(-\frac{1834496}{11025}+\frac{137875712\pi^2}{4725}-\frac{28483790848\pi^4}{1488375}+\frac{1145110528\pi^6}{694575}\right)\nu\bigg]\\
&+e_t\bigg[\frac{12832\pi}{45}-\frac{13673548523\pi^3}{1056000}+\frac{2331631085671213\pi^5}{5765760000}-\frac{181467504528103\pi^7}{429977600}+\frac{1265629617\pi^9}{32768}\\
&+\left(\frac{246285677\pi^3}{6720}-\frac{17308193471\pi^5}{64512}+\frac{12669225577\pi^7}{51200}-\frac{732535083\pi^9}{32768}\right)\nu\bigg]\\
&+\bigg[\frac{1423722529}{2116800}+\frac{735046449181\pi^2}{19051200}-\frac{14545675750451\pi^4}{41912640000}-\frac{1052053473232830449\pi^6}{38903312448000}\\
&+\frac{9956950540288\pi^8}{2916520425}+\frac{10752\zeta(3)}{5}-\frac{1682177\pi^2\zeta(3)}{4}+\frac{39521979\pi^4\zeta(3)}{160}-\frac{9183825\pi^6\zeta(3)}{1024}\\
&-\frac{22348923\zeta(5)}{4}+\frac{175025907\pi^2\zeta(5)}{32}-\frac{512457435\pi^4\zeta(5)}{2048}+\frac{2151124857\zeta(7)}{32}-\frac{10497111975\pi^2\zeta(7)}{2048}\\
&-\frac{126709233525\zeta(9)}{2048}+\left(\frac{804128}{3675}+\frac{1533979136\pi^2}{33075}-\frac{202340205568\pi^4}{1488375}+\frac{26006683648\pi^6}{416745}-\frac{12671254528\pi^8}{2546775}\right)\nu\bigg]\\
&+\frac{1}{e_t}\bigg[\frac{8\pi}{3}-\frac{13799463626119\pi^3}{14370048000}+\frac{108826027744604749\pi^5}{62270208000}-\frac{21125048445944211030463\pi^7}{2628134885376000}\\
&+\frac{5848151327145833\pi^9}{896860160}-\frac{152039294253\pi^{11}}{262144}+\bigg(\frac{341022193\pi^3}{6720}-\frac{878042967227\pi^5}{645120}\\
&+\frac{2979531597611\pi^7}{614400}-\frac{42327561853999\pi^9}{11468800}+\frac{21336464457\pi^{11}}{65536}\bigg)\nu\bigg]\\
&+\frac{1}{e_t^2}\bigg[\frac{334356791}{3326400}+\frac{199332473447\pi^2}{3386880}-\frac{373142823899\pi^4}{1587600}+\frac{3907498382463157\pi^6}{424569600000}\\
&+\frac{584827839010454989447\pi^8}{6570337213440000}-\frac{6152058407223296\pi^{10}}{541111756185}-\frac{27597339\pi^2\zeta(3)}{20}+\frac{1213492203\pi^4\zeta(3)}{320}\\
&-\frac{6498156357\pi^6\zeta(3)}{5120}+\frac{3215354373\pi^8\zeta(3)}{81920}-\frac{366650361\zeta(5)}{20}+\frac{5374036899\pi^2\zeta(5)}{64}\\
&-\frac{1812985623603\pi^4\zeta(5)}{51200}+\frac{10171018935\pi^6\zeta(5)}{8192}+\frac{66048647049\zeta(7)}{64}-\frac{7427392716051\pi^2\zeta(7)}{10240}\\
&+\frac{525021435477\pi^4\zeta(7)}{16384}-\frac{89655063257529\zeta(9)}{10240}+\frac{10562439115305\pi^2\zeta(9)}{16384}\\
&+\frac{126935300600955\zeta(11)}{16384}+\bigg(\frac{704192}{10395}+\frac{78030592\pi^2}{1323}-\frac{53212499968\pi^4}{99225}+\frac{137693298688\pi^6}{178605}\\
&-\frac{10766761459712\pi^8}{38201625}+\frac{308738523136\pi^{10}}{14567553}\bigg)\nu\bigg]
\end{aligned}
$$

$$+\frac{1}{e_t^3}\left[\frac{1054\pi}{135}+\frac{212624533804289\pi^3}{36951552000}+\frac{1157475570129868636 7\pi^5}{2490808320000}-\frac{25790664698470750454305469\pi^7}{394220232806400000}\right.$$
$$+\frac{20251212693594225849008305 37\pi^9}{1063869001600204 8000}-\frac{5826127395266536829411\pi^{11}}{42941664460800}+\frac{124362967035879\pi^{13}}{10485760}$$
$$+\left(\frac{6592502233\pi^3}{107520}-\frac{22857379501687\pi^5}{5160960}+\frac{52153564678267\pi^7}{1228800}-\frac{14962473720996481\pi^9}{137625600}\right.$$
$$\left.\left.+\frac{1787772436877977\pi^{11}}{24084480}-\frac{13523632850871\pi^{13}}{2097152}\right)\nu\right]+\mathcal{O}(1/e_t^4)\Bigg\}. \tag{B2}$$

$$\Delta\mathcal{E}_{(2)}^{(l)}=\frac{M\nu^2(1-4\nu)}{c^{10}j^{12}}\left[\frac{81856e_t^8}{945}+\frac{4053\pi^3e_t^7}{160}+\left(\frac{50560}{189}+\frac{275456\pi^2}{945}\right)e_t^6+\left(\frac{28654757\pi^3}{20160}-\frac{1292765\pi^5}{10752}\right)e_t^5\right.$$
$$+\left(-\frac{192064}{525}+\frac{4470784\pi^2}{945}-\frac{20166656\pi^4}{55125}\right)e_t^4+\left(\frac{163927481\pi^3}{16128}-\frac{795673307\pi^5}{64512}+\frac{2352595\pi^7}{2048}\right)e_t^3$$
$$+\left(-\frac{167936}{1225}+\frac{261379072\pi^2}{14175}-\frac{53227319296\pi^4}{4465125}+\frac{10682630144\pi^6}{10418625}\right)e_t^2$$
$$+\left(\frac{270990515\pi^3}{10752}-\frac{16781634175\pi^5}{86016}+\frac{3732414369\pi^7}{20480}-\frac{2161208385\pi^9}{131072}\right)e_t$$
$$+\left(\frac{3086144}{33075}+\frac{471584768\pi^2}{14175}-\frac{71593908224\pi^4}{637875}+\frac{1672914731008\pi^6}{31255875}-\frac{4699193344\pi^8}{1091475}\right)$$
$$+\frac{1}{e_t}\left(\frac{9587828293\pi^3}{258048}-\frac{3207169211179\pi^5}{2580480}+\frac{701408449909\pi^7}{147456}-\frac{509401541368309\pi^9}{137625600}+\frac{171695021883\pi^{11}}{524288}\right)$$
$$+\frac{1}{e_t^2}\left(\frac{255104}{10395}+\frac{288994304\pi^2}{6615}-\frac{52566433792\pi^4}{99225}+\frac{758425714688\pi^6}{893025}-\frac{12350325784576\pi^8}{38201625}+\frac{1074396135424\pi^{10}}{43702659}\right)$$
$$+\frac{1}{e_t^3}\left(\frac{118105384883\pi^3}{2580480}-\frac{96013915997417\pi^5}{20643840}+\frac{380628106184173\pi^7}{7372800}-\frac{232396062526713293\pi^9}{1651507200}\right.$$
$$\left.\left.+\frac{1136196879582428477\pi^{11}}{11560550400}-\frac{359312576835069\pi^{13}}{41943040}\right)+\mathcal{O}(1/e_t^4)\right], \tag{B3}$$

$$\Delta\mathcal{E}_{(3)}^{(l)}=\frac{M\nu^2(1-4\nu)}{c^{10}j^{12}}\left[\frac{512e_t^8}{135}+\frac{15\pi^3e_t^7}{16}+\left(\frac{1024}{135}+\frac{2048\pi^2}{225}\right)e_t^6+\left(\frac{10801\pi^3}{288}-\frac{1225\pi^5}{384}\right)e_t^5\right.$$
$$+\left(-\frac{1024}{75}+\frac{14336\pi^2}{135}-\frac{65536\pi^4}{7875}\right)e_t^4+\left(\frac{226961\pi^3}{1152}-\frac{186277\pi^5}{768}+\frac{46305\pi^7}{2048}\right)e_t^3$$
$$+\left(-\frac{1024}{525}+\frac{641024\pi^2}{2025}-\frac{130777088\pi^4}{637875}+\frac{1048576\pi^6}{59535}\right)e_t^2+\left(\frac{1521919\pi^3}{3840}-\frac{9146851\pi^5}{3072}+\frac{282281979\pi^7}{102400}\right.$$
$$\left.-\frac{8164233\pi^9}{32768}\right)e_t+\left(\frac{4096}{1575}+\frac{7026688\pi^2}{14175}-\frac{985563136\pi^4}{637875}+\frac{214433792\pi^6}{297675}-\frac{4194304\pi^8}{72765}\right)$$
$$+\frac{1}{e_t}\left(\frac{9909613\pi^3}{18432}-\frac{1446169151\pi^5}{92160}+\frac{42478149259\pi^7}{737280}-\frac{217594014011\pi^9}{4915200}+\frac{1025047023\pi^{11}}{262144}\right)$$
$$+\frac{1}{e_t^2}\left(\frac{1024}{1485}+\frac{83968\pi^2}{135}-\frac{2523136\pi^4}{405}+\frac{1194852352\pi^6}{127575}-\frac{2716860416\pi^8}{779625}+\frac{33554432\pi^{10}}{127413}\right)$$
$$+\frac{1}{e_t^3}\left(\frac{118881539\pi^3}{184320}-\frac{38059061669\pi^5}{737280}+\frac{3865364478731\pi^7}{7372800}\right.$$
$$\left.\left.-\frac{27120739832417\pi^9}{19660800}+\frac{10474810693061\pi^{11}}{11010048}-\frac{173590231815\pi^{13}}{2097152}\right)+\mathcal{O}(1/e_t^4)\right], \tag{B4}$$

$$\Delta\mathcal{E}_{(4)} = \frac{M\nu^2}{c^{11}j^{13}}\left[\left(\frac{163787e_t^6}{450}+\frac{180379e_t^4}{75}+\frac{166996e_t^2}{75}+\frac{62744}{225}\right)\sqrt{e_t^2-1}\right.$$
$$\left.+\left(\frac{297e_t^8}{10}+\frac{14008e_t^6}{15}+\frac{41368e_t^4}{15}+\frac{4352e_t^2}{3}+\frac{512}{5}\right)\arccos\left(-\frac{1}{e_t}\right)\right]$$
$$\times\left[\frac{2\pi^2}{3}-\frac{214}{105}\left(\log\left(\frac{2nr_0}{GMce_t}\right)+\gamma\right)-\frac{116761}{29400}\right], \tag{B5}$$

$$\Delta\mathcal{E}^{(l)}_{(5)} = \frac{M\nu^2}{c^{11}j^{13}}\left[e_t^8\left(\frac{10593\gamma\pi}{350}-107\pi+\frac{31779}{350}\pi\log(2)\right)+\frac{54784e_t^7}{354375}(5190\gamma-5047-5190\log(2))\right.$$
$$+\frac{107e_t^6}{151200}(-2603475\pi\zeta(3)+1344768\gamma\pi-2290582\pi+4034304\pi\log(2))$$
$$+\frac{54784e_t^5}{2480625}(-443880\zeta(3)+291795\gamma+45734-291795\log(2))$$
$$+\frac{107e_t^4}{302400}(-351258810\pi\zeta(3)+374243625\pi\zeta(5)+7942656\gamma\pi-6335000\pi+23827968\pi\log(2))$$
$$+\frac{6848e_t^3}{7441875}(-223001760\zeta(3)+220907520\zeta(5)+8625330\gamma+8763361-8625330\log(2))$$
$$+\frac{107e_t^2}{193536000}(-2243993740800\pi\zeta(3)+27140132920320\pi\zeta(5)-25257903843375\pi\zeta(7)$$
$$+2673868800\gamma\pi+186045664\pi+8021606400\pi\log(2))$$
$$+\frac{6848e_t}{573024375}(-101159732520\zeta(3)+631953315840\zeta(5)-531449856000\zeta(7)$$
$$+140502285\gamma+277115017-140502285\log(2))$$
$$+\left(-\frac{17869266109\pi\zeta(3)}{3600}+\frac{1338717347453\pi\zeta(5)}{4800}-\frac{483420808341489\pi\zeta(7)}{204800}+\frac{137148534994779\pi\zeta(9)}{65536}\right.$$
$$\left.+\frac{54784\gamma\pi}{525}+\frac{2277666521\pi}{16128000}+\frac{54784}{175}\pi\log(2)\right)$$
$$+\frac{428}{7449316875e_t}(-63394926154560\zeta(3)+1492742225694720\zeta(5)-6265138721587200\zeta(7)$$
$$+4835860807680000\zeta(9)+401531130\gamma+1718774371-401531130\log(2))$$
$$-\frac{15301}{3538944000e_t^2}(2814246420480\pi\zeta(3)-505641831137280\pi\zeta(5)+15738675691455360\pi\zeta(7)$$
$$-113566524198701760\pi\zeta(9)+98396833403743875\pi\zeta(11)+52736\pi)$$
$$+\frac{428}{4469590125e_t^3}(-80247248257176\zeta(3)+5268405431697408\zeta(5)-66851431345520640\zeta(7)$$
$$+230673046606184448\zeta(9)-169009790268211200\zeta(11)+2171169\gamma+1653152-2171169\log(2))$$
$$\left.+\mathcal{O}(1/e_t^4)\right], \tag{B6}$$

where $\zeta(z)$ is a Riemann-Zeta function.

APPENDIX C: TOTAL ENERGY RADIATION

In this Appendix, we collect all contributions of total energy radiation $\Delta\mathcal{E} = \Delta\mathcal{E}_{\rm inst} + \Delta\mathcal{E}_{\rm hered}$ up to 3PN and $1/j^{15}$ order including both instantaneous and hereditary contributions. Note that, to be consistent, we expand the contributions of which exact form are known (instantaneous part and $\Delta E_{(4)}$) and discard $1/j^{16}$ contributions of $\Delta E^{(l)}_{(2)}$ and $\Delta E^{(l)}_{(3)}$ already computed in Eq. (B3). Because of the size of the expression, we introduce shorthand notations,

$$p := \sqrt{\frac{2E}{c^2}} \tag{C1}$$

and J (not to be confused with the physical angular momentum $\mathcal{J}$) defined as

$$\frac{1}{J^n} := \frac{M\nu^2 p^{7-n}}{c^{n-2}} \frac{1}{j^n}, \tag{C2}$$

so that whatever value the positive integer n is, $\frac{1}{J^n}$ has a dimension of energy $\mathcal{E} \sim \text{mass}\,\frac{\text{length}^2}{\text{time}^2}$, and is formally as small as $\mathcal{O}(\frac{1}{c^5})$ (i.e., 2.5PN order, the leading order of energy radiation) and $\mathcal{O}(G^n)$, while p is dimensionless and as small as $\mathcal{O}(\frac{1}{c})$ without entailing G, hence which is going to serve as a PN parameter. Every coefficient of $1/J^n$ is dimensionless and polynomials of p, of which the highest power is 6 (that is, 3PN),

$$\begin{aligned}
\Delta\mathcal{E} = {} & \frac{\pi}{J^3}\left[\frac{37}{15} + p^2\left(\frac{2393}{840} - \frac{37\nu}{10}\right) + p^4\left(\frac{149}{36} - \frac{2393\nu}{672} + \frac{185\nu^2}{48}\right) + p^6\left(\frac{415147}{354816} - \frac{149\nu}{36} + \frac{45467\nu^2}{13440} - \frac{407\nu^3}{120}\right)\right] \\
& + \frac{1}{J^4}\left[\frac{1568}{45} + p^2\left(\frac{25468}{525} - \frac{1228\nu}{15}\right) + p^3\frac{3136}{45} + p^4\left(\frac{1894327}{44100} - \frac{92537\nu}{1050} + \frac{6809\nu^2}{60}\right)\right. \\
& \left. + p^5\left(\frac{308368}{315} - \frac{229168\nu}{315}\right) + p^6\left(\frac{40120357}{3880800} - \frac{13179269\nu}{211680} + \frac{1801097\nu^2}{16800} - \frac{59303\nu^3}{480}\right)\right] \\
& + \frac{\pi}{J^5}\left[\frac{122}{5} + p^2\left(\frac{15539}{280} - \frac{436\nu}{5}\right) + p^3\frac{297\pi^2}{20} + p^4\left(\frac{2995}{252} - \frac{109657\nu}{840} + \frac{3201\nu^2}{20}\right) + p^5\left(\frac{3681\pi^2}{20} - \frac{27207\pi^2\nu}{160}\right)\right. \\
& \left. + p^6\left(\frac{27548555963}{310464000} - \frac{10593\log(\frac{p}{2})}{350} + \frac{99\pi^2}{10} + \frac{106919\nu}{1344} - \frac{4059\pi^2\nu}{640} + \frac{865253\nu^2}{4480} - \frac{6893\nu^3}{32}\right)\right] \\
& + \frac{1}{J^6}\left[\frac{4672}{45} + p^2\left(\frac{29240}{63} - \frac{5176\nu}{9}\right) + p^3\left(\frac{9344}{45} + \frac{88576\pi^2}{675}\right) + p^4\left(-\frac{75949}{17010} - \frac{6952373\nu}{4725} + \frac{126623\nu^2}{90}\right)\right. \\
& + p^5\left(\frac{888256}{315} + \frac{1338112\pi^2}{945} - \frac{129216\nu}{35} - \frac{2617088\pi^2\nu}{1575}\right) \\
& \left. + p^6\left(\frac{567214004327}{187110000} - \frac{18955264\log(2p)}{23625} + \frac{177152\pi^2}{675} + \frac{9695063569\nu}{4762800} - \frac{212216\pi^2\nu}{1575} + \frac{28815049\nu^2}{10800} - \frac{563801\nu^3}{240}\right)\right] \\
& + \frac{\pi}{J^7}\left[\frac{85}{3} + p^2\left(\frac{2259}{8} - 265\nu\right) + p^3\left(\frac{1579\pi^2}{3} - \frac{2755\pi^4}{64}\right) + p^4\left(\frac{736055}{6048} - \frac{862691\nu}{672} + \frac{7075\nu^2}{8}\right)\right. \\
& + p^5\left(\frac{3346853\pi^2}{672} - \frac{712945\pi^4}{1792} - \frac{321683}{42}\pi^2\nu + \frac{2174405\pi^4\nu}{3584}\right) \\
& \left. + p^6\left(\frac{37762952557}{8467200} - \frac{337906\log(\frac{p}{2})}{315} + \frac{3158\pi^2}{9} - \frac{58957\zeta(3)}{32} + \frac{23867693\nu}{18144} - \frac{51947\pi^2\nu}{384} + \frac{3986149\nu^2}{1344} - \frac{88825\nu^3}{48}\right)\right] \\
& + \frac{1}{J^8}\left[\frac{3104}{75} + p^2\left(\frac{623092}{525} - \frac{63796\nu}{75}\right) + p^3\left(\frac{6208}{75} + \frac{68096\pi^2}{45} - \frac{280576\pi^4}{2625}\right) + p^4\left(\frac{5008183}{2268} - \frac{1036933\nu}{126} + \frac{81429\nu^2}{20}\right)\right. \\
& + p^5\left(\frac{3418768}{1575} + \frac{8569216\pi^2}{675} - \frac{15222272\pi^4}{18375} - \frac{1552496\nu}{315} - \frac{17636992}{675}\pi^2\nu + \frac{1237504}{735}\pi^4\nu\right) \\
& + p^6\left(\frac{1499994124247}{34927200} - \frac{2081792\log(2p)}{225} + \frac{136192\pi^2}{45} - \frac{60043264\zeta(3)}{6125} - \frac{613211773\nu}{151200}\right. \\
& \left.\left. - \frac{19516\pi^2\nu}{25} + \frac{60207133\nu^2}{2400} - \frac{8706467\nu^3}{800}\right)\right] \\
& + \frac{\pi}{J^9}\left[p^2\left(\frac{13447}{40} - \frac{1127\nu}{6}\right) + p^3\left(\frac{17213\pi^2}{6} - \frac{873523\pi^4}{288} + \frac{17885\pi^6}{64}\right) + p^4\left(\frac{11947909}{6480} - \frac{2838577\nu}{720} + \frac{5733\nu^2}{4}\right)\right. \\
& + p^5\left(\frac{523165\pi^2}{24} - \frac{89668187\pi^4}{4608} + \frac{451885\pi^6}{256} - \frac{969317}{16}\pi^2\nu + \frac{13906539}{256}\pi^4\nu - \frac{2524095}{512}\pi^6\nu\right)
\end{aligned}$$

$$+p^6\left(\frac{112825559083}{3628800}-\frac{263113\log(\frac{p}{2})}{45}+\frac{17213\pi^2}{9}-\frac{93466961\zeta(3)}{720}+\frac{8474935\zeta(5)}{64}\right.$$
$$\left.\left.-\frac{71395523\nu}{5184}-\frac{96145}{384}\pi^2\nu+\frac{23677969\nu^2}{1440}-\frac{20237\nu^3}{4}\right)\right]$$
$$+\frac{1}{J^{10}}\left[-\frac{15488}{1575}+p^2\left(\frac{1910576}{3675}-\frac{40464\nu}{175}\right)+p^3\left(-\frac{30976}{1575}+\frac{3042304\pi^2}{675}-\frac{76079104\pi^4}{30375}+\frac{104726528\pi^6}{496125}\right)\right.$$
$$+p^4\left(\frac{26004421}{2835}-\frac{149079818\nu}{11025}+\frac{649561\nu^2}{175}\right)$$
$$+p^5\left(\frac{1002112}{2205}+\frac{152894464\pi^2}{4725}-\frac{3895906304\pi^4}{297675}+\frac{3622043648\pi^6}{3472875}-\frac{14566784\nu}{11025}-\frac{568677376\pi^2\nu}{4725}\right.$$
$$\left.+\frac{76963606528\pi^4\nu}{1488375}-\frac{14488174592\pi^6\nu}{3472875}\right)$$
$$+p^6\left(\frac{113929481001857}{654885000}-\frac{651053056\log(2p)}{23625}+\frac{6084608\pi^2}{675}-\frac{16280928256\zeta(3)}{70875}+\frac{11205738496\zeta(5)}{55125}\right.$$
$$\left.\left.-\frac{8445373541\nu}{68040}-\frac{9184}{225}\pi^2\nu+\frac{7195113763\nu^2}{88200}-\frac{15402959\nu^3}{840}\right)\right]$$
$$+\frac{\pi}{J^{11}}\left[p^3\left(\frac{24717\pi^2}{5}-\frac{2235121\pi^4}{64}+\frac{25779537\pi^6}{800}-\frac{47703411\pi^8}{16384}\right)+p^4\left(\frac{5839651}{2016}-\frac{258051\nu}{80}+\frac{5481\nu^2}{8}\right)\right.$$
$$+p^5\left(\frac{1742341\pi^2}{48}-\frac{168183211\pi^4}{1152}+\frac{607042663\pi^6}{5120}-\frac{691453665\pi^8}{65536}-\frac{20913841\pi^2\nu}{120}\right.$$
$$\left.+\frac{3876083701\pi^4\nu}{4608}-\frac{1150599893\pi^6\nu}{1600}+\frac{8441537391\pi^8\nu}{131072}\right)$$
$$+p^6\left(\frac{8986587257}{115200}-\frac{251878\log(\frac{p}{2})}{25}+\frac{16478\pi^2}{5}-\frac{239157947\zeta(3)}{160}+\frac{12215817747\zeta(5)}{800}-\frac{114395585661\zeta(7)}{8192}\right.$$
$$\left.\left.-\frac{465343901\nu}{6720}+\frac{255717}{640}\pi^2\nu+\frac{10035909\nu^2}{320}-\frac{42399\nu^3}{8}\right)\right]$$
$$+\frac{1}{J^{12}}\left[\frac{928}{189}+p^2\left(-\frac{931324}{6615}+\frac{292\nu}{7}\right)+p^3\left(\frac{1856}{189}+\frac{4238336\pi^2}{945}-\frac{1355776\pi^4}{81}+\frac{7286226944\pi^6}{893025}-\frac{719847424\pi^8}{1091475}\right)\right.$$
$$+p^4\left(\frac{231340289}{47628}-\frac{56967727\nu}{13230}+\frac{545935\nu^2}{756}\right)$$
$$+p^5\left(\frac{1214483489}{2116800}+\frac{1470364547101\pi^2}{19051200}-\frac{2325888630656051\pi^4}{41912640000}-\frac{2502327660950486689\pi^6}{38903312448000}+\frac{5407030771712\pi^8}{2916520425}\right.$$
$$+\frac{10752\zeta(3)}{5}-\frac{1682177\pi^2\zeta(3)}{4}+\frac{39521979\pi^4\zeta(3)}{160}-\frac{9183825\pi^6\zeta(3)}{1024}-\frac{22348923\zeta(5)}{4}+\frac{175025907\pi^2\zeta(5)}{32}$$
$$-\frac{512457435\pi^4\zeta(5)}{2048}+\frac{2151124857\zeta(7)}{32}-\frac{10497111975\pi^2\zeta(7)}{2048}-\frac{126709233525\zeta(9)}{2048}+\frac{5120464\nu}{11025}$$
$$\left.-\frac{5834608384\pi^2\nu}{33075}+\frac{505889411072\pi^4\nu}{1488375}-\frac{11148967936\pi^6\nu}{77175}+\frac{9611902976\pi^8\nu}{848925}\right)$$
$$+p^6\left(\frac{979767288724297}{3667356000}-\frac{907003904\log(2p)}{33075}+\frac{8476672\pi^2}{945}-\frac{290136064\zeta(3)}{189}+\frac{779626283008\zeta(5)}{99225}\right.$$
$$\left.\left.-\frac{154047348736\zeta(7)}{24255}-\frac{493693095713\nu}{1905120}+\frac{696344}{315}\pi^2\nu+\frac{18870027479\nu^2}{211680}-\frac{70214105\nu^3}{6048}\right)\right]$$

$$
\begin{aligned}
&+\frac{\pi}{J^{13}}\bigg[p^3\bigg(\frac{10549\pi^2}{4}-\frac{35497847\pi^4}{240}+\frac{297848551\pi^6}{480}-\frac{101390485757\pi^8}{204800}+\frac{2879946531\pi^{10}}{65536}\bigg)\\
&+p^5\bigg(\frac{1323421\pi^2}{40}-\frac{2953014799\pi^4}{7680}+\frac{17649804463\pi^6}{19200}-\frac{13648067236109\pi^8}{22937600}+\frac{839620089\pi^{10}}{16384}\\
&-\frac{33199903}{160}\pi^2\nu+\frac{965028169}{192}\pi^4\nu-\frac{344986768681\pi^6\nu}{19200}+\frac{31455504778007\pi^8\nu}{2293760}-\frac{634501586565\pi^{10}\nu}{524288}\bigg)\\
&+p^6\bigg(\frac{72363445187}{1075200}-\frac{161249\log(\frac{p}{2})}{30}+\frac{10549\pi^2}{6}-\frac{3798269629\zeta(3)}{600}+\frac{141137663381\zeta(5)}{480}\\
&-\frac{243140348991507\zeta(7)}{102400}+\frac{137148534994779\zeta(9)}{65536}-\frac{3972009943\nu}{60480}+\frac{208813\pi^2\nu}{320}+\frac{1157409\nu^2}{64}-\frac{29645\nu^3}{16}\bigg)\bigg]\\
&+\frac{1}{J^{14}}\bigg[-\frac{10816}{3465}+p^2\bigg(\frac{1386104}{17325}-\frac{57256\nu}{3465}\bigg)\\
&+p^3\bigg(-\frac{21632}{3465}+\frac{312832\pi^2}{315}-\frac{222912512\pi^4}{4725}+\frac{27701936128\pi^6}{297675}-\frac{6236143616\pi^8}{165375}+\frac{2021654528\pi^{10}}{693693}\bigg)\\
&+p^4\bigg(-\frac{3251704661}{2182950}+\frac{332628437\nu}{363825}-\frac{230933\nu^2}{2310}\bigg)\\
&+p^5\bigg(\frac{87104}{1485}+\frac{35586304\pi^2}{1323}-\frac{2014613504\pi^4}{19845}+\frac{261816320\pi^6}{5103}-\frac{17126785024\pi^8}{3472875}+\frac{1103101952\pi^{10}}{14567553}\\
&-\frac{153152\nu}{1485}-\frac{1090105088\pi^2\nu}{6615}+\frac{193032687616\pi^4\nu}{99225}-\frac{56361549824\pi^6\nu}{18225}+\frac{44841496346624\pi^8\nu}{38201625}-\frac{1299282132992\pi^{10}\nu}{14567553}\bigg)\\
&+p^6\bigg(\frac{8870217669331141}{87405318000}-\frac{66946048\log(2p)}{11025}+\frac{625664\pi^2}{315}-\frac{47703277568\zeta(3)}{11025}+\frac{2964107165696\zeta(5)}{33075}\\
&-\frac{1334534733824\zeta(7)}{3675}+\frac{1946853310464\zeta(9)}{7007}-\frac{35726902969\nu}{388080}+984\pi^2\nu+\frac{40648137491\nu^2}{1940400}-\frac{8654887\nu^3}{5040}\bigg)\bigg]\\
&+\frac{\pi}{J^{15}}\bigg[p^3\bigg(-\frac{272920219\pi^4}{960}+\frac{11072684623\pi^6}{2400}-\frac{52303041073283\pi^8}{3686400}+\frac{266066506940663\pi^{10}}{25804800}-\frac{4738568225391\pi^{12}}{5242880}\bigg)\\
&+p^5\bigg(\frac{504009\pi^2}{32}-\frac{3136589467\pi^4}{5376}-\frac{2787258331\pi^6}{2400}+\frac{4428020055371\pi^8}{368640}-\frac{61360374917367889\pi^{10}}{5780275200}\\
&+\frac{19996958843433\pi^{12}}{20971520}-\frac{693891}{8}\pi^2\nu+\frac{259903968599\pi^4\nu}{17920}-\frac{1131734007261\pi^6\nu}{6400}+\frac{4888954393057889\pi^8\nu}{9830400}\\
&-\frac{84543857131670761\pi^{10}\nu}{240844800}+\frac{1285287637996791\pi^{12}\nu}{41943040}\bigg)\\
&+p^6\bigg(\frac{13990148029}{6912000}-\frac{29202463433\zeta(3)}{2400}+\frac{5246870699213\zeta(5)}{2400}-\frac{41808589714402511\zeta(7)}{614400}\\
&+\frac{201120530875501809\zeta(9)}{409600}-\frac{446094799380943713\zeta(11)}{1048576}\bigg)\bigg]+\mathcal{O}(1/c^{12},G^{16}).
\end{aligned}
\tag{C3}
$$

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