Cuscuta-Galileon cosmology: Dynamics, gravitational constants, and the Hubble constant

Kei-ichi Maeda[*] and Sirachak Panpanich[†]

Department of Pure and Applied Physics, Graduate School of Advanced Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan

(Received 22 February 2022; accepted 21 April 2022; published 12 May 2022)

We discuss cosmology based on a Cuscuta-Galileon gravity theory, which preserves just two degrees of freedom. Although there exist no additional degrees of freedom, introduction of a potential of a scalar field changes the dynamics. The scalar field is completely determined by matter fields. Giving an exponential potential as an example, we discuss the cosmological dynamics. The gravitational "constant" $G_{\rm F}$ appeared in the effective Friedmann equation becomes time dependent. We also present how to construct a potential when we know the evolution of the Hubble parameter. When we assume the ΛCDM cosmology for the background evolution, we find the potential form. We then analyze the density perturbations, which equation is characterized only by a change of the gravitational "constant" $G_{\rm eff}$, which also becomes time dependent. From the observational constraints such as the constraint from the big-bang nucleosynthesis and the constraint on time-variation of gravitational constant, we restrict the parameters in our models. The time dependence of the gravitational constant in the effective Friedmann equation, we may have a chance to explain the Hubble tension problem.

DOI: 10.1103/PhysRevD.105.104022

I. INTRODUCTION

In order to explain the accelerated expansion of the Universe [1,2] we require a mysterious energy, so-called dark energy. The dark energy candidates are a cosmological constant [3], a scalar field [4,5], a vector field [6,7], a massive tensor field [8–10], or even modification on the general relativity [11]. However, until now, these candidates or deviations from general relativity (GR) have not been detected in the solar system scale [12].

To address the above problem, many modified gravity theories with various types of screening mechanisms, such as chameleon screening [13,14], symmetron screening [15], and Vainshtein screening [16,17], have been proposed. These mechanisms use an effective potential to vary a mass of a scalar field [13,14] or coupling between a matter field and a scalar field [15,18], also nonlinear form of equation of motion of the scalar field leads to suppression on a fifth force [16,17,19–23].

Another solution to the previous problem is constructing new gravitational theories which propagate only two degrees of freedom as GR. Recently, two types of theories have been developed: One is called minimally modified gravity [24–29] whose gravitational Hamiltonian is constrained to provide only two degrees of freedom. The other one is called cuscuton gravity theory [30,31] or its extended version [32,33]. The extended cuscuton theory is generalization of the original cuscuton theory in the context of the beyond Horndeski theories [34], in which the second-order time derivatives of a scalar field in the equation of motion disappears, thus the scalar field is a nondynamical field. Both theories have some relation as shown in the Ref. [27].

In this work we consider the modified gravity with two degrees of freedom in the extended cuscuton framework. To find cosmological solutions we have to define explicit form of theory, one example has been given in the Ref. [33]. We are interested in the explicit form inspired from the Cuscuta-Galileon gravity [35] which is a Galileon generalization of the original cuscuton gravity. Its cosmological dynamics of the model has been studied in Ref. [36] where the Cuscuta-Galileon provides the sequence of the thermal history of the Universe successfully; however, the model actually has three degrees of freedom. Therefore, it is interesting to investigate cosmological solutions of the Cuscuta-Galileon gravity which has only two degrees of freedom whether the model still provides the thermal history of Universe correctly or not.

The paper is organized as follows. In Sec. II, we will give our Cuscuta-Galileon gravity theory and show that it has two dynamical degrees of freedom. In Sec. III, we apply it to cosmological model and present the effective Friedmann equation assuming the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric. In order to study the cosmological dynamics, in Sec. IV, we analyze the cosmological

[*]maeda-at-waseda.jp
[†]sirachakp-at-aoni.waseda.jp

evolution assuming the exponential potential. We then discuss the evolution of the Hubble expansion parameter and the effective gravitational constant in the Friedmann equation. The evolution of the Hubble parameter shows the tendency to fill the gap appearing in the Hubble tension problem. In Sec. V, we also present how to construct the potential when we know the evolution of the Hubble parameter and apply it to obtain the ΛCDM model.

In Sec. VI, we analyze the density perturbations. We find the gravitational constant in the evolution equation of the density contrast is modified and becomes time-dependent. We then give the constraints on the parameters in the theories from observation. The discussion and remarks follow in Sec. VII.

We also present the rescaling property in this model in Appendix A, the overview of the original cuscuton gravity theory with the construction of a potential when we know the evolution of the Hubble parameter in Appendix B, the detailed analysis of the cosmological dynamics for the exponential potential in Appendix C, the analysis for the case with a vacuum energy in Appendix D, and some peculiarity in the vacuum case in Appendix E.

II. CUSCUTA-GALILEON THEORY

We discuss the Cuscuta-Galileon gravity, in which the minimum contribution of a Galileon-type scalar field is included in the cuscuton gravity theory. The action is given by

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm PL}^2R+\alpha_2M_{\rm PL}^2\sqrt{-X}+\alpha_3M_{\rm PL}\ln\left(-\frac{X}{\Lambda^4}\right)\Box\phi-V(\phi)+3\alpha_3^2X\right]+S_M(g_{\mu\nu},\psi_M), \tag{2.1}$$

where X is defined as

$$X\equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

and α_2 and α_3 are dimensionless coupling constants, respectively, while Λ is a cutoff-scale constant with mass dimension. This model is one of a special case of the extended cuscuton gravity theory [32]; however, it is not the same as their application to dark energy [33]. The original cuscuton model is obtained by setting $\alpha_2=\mu^2/M_{\rm PL}^2$ and $\alpha_3=0$. This action has also found by covariantization of the minimally modified gravity [27].

Taking the variation of the above action with respect to the scalar field ϕ and the metric $g_{\mu\nu}$, we find the following basic equations:

$$\alpha_2M_{\rm PL}^2\frac{1}{\sqrt{-X}}\left[\Box\phi-\frac{1}{2X}\nabla X\cdot\nabla\phi\right]+\alpha_3M_{\rm PL}\left[-2\nabla\left(\frac{\Box\phi}{X}\right)\cdot\nabla\phi-2\frac{(\Box\phi)^2}{X}+\Box(\ln(-X))\right]-6\alpha_3^2\Box\phi-V_{,\phi}=0,$$

$$M_{\rm PL}^2G_{\mu\nu}=T_{\mu\nu}-g_{\mu\nu}V+\alpha_2M_{\rm PL}^2\frac{1}{\sqrt{-X}}[-g_{\mu\nu}X+\partial_\mu\phi\partial_\nu\phi]+\alpha_3M_{\rm PL}\left[-2\partial_\mu\phi\partial_\nu\phi\frac{\Box\phi}{X}+\frac{2}{X}\partial_{(\mu}X\partial_{\nu)}\phi-g_{\mu\nu}\frac{1}{X}(\nabla X\cdot\nabla\phi)\right]+3\alpha_3^2(g_{\mu\nu}X-2\partial_\mu\partial_\nu\phi).$$

Assuming the conservation of energy-momentum of matter field, i.e., $\nabla^\nu T_{\mu\nu}=0$, and using the Bianchi identity $\nabla^\nu G_{\mu\nu}\equiv 0$, we recover the first equation for ϕ from the second Einstein equations. Hence only the Einstein equations are independent in the present model. We do not have additional degrees of freedom in addition to the Einstein equations, The scalar field ϕ does not carry new degree of freedom just as the original cuscuton. We will prove it below.

Note that this model is completely different from the original one ($\alpha_3=0$). Because as we show in Appendix A, we can always set $\alpha_3=1$ without loss of generality, which means that the perturbation approach for the original theory does not provide an appropriate approximation even for the case of $|\alpha_3|\ll 1$. However we shall keep α_3 in the text in order to see the coupling dependence. Since the results for $\alpha_3<0$ can be obtained by the change of the sign of ϕ, we assume $\alpha_3\geq 0$ in this paper.

A. Degrees of freedom

According to the method in Refs. [37–39] we use the $3+1$ decomposition metric and choose the unitary gauge:

$$ds^2=-N^2dt^2+h_{ij}(dx^i+N^idt)(dx^j+N^jdt),\quad \phi=\phi(t),$$

the action (2.1) can be written in the Arnowitt-Deser-Misner (ADM) form as

$$S=\int dtd^3xN\sqrt{h}\left[\frac{1}{2}M_{\rm PL}^2({}^3R+K^{ij}K_{ij}-K^2)+\left(\frac{\alpha_2M_{\rm PL}^2|\dot\phi|}{N}-V(\phi)-\frac{3\alpha_3^2\dot\phi^2}{N^2}\right)+\left(-\frac{2\alpha_3M_{\rm PL}|\dot\phi|}{N}+C_1\right)K\right].$$

Note that in this section we will not consider contribution from the matter Lagrangian. ${}^3R$ is the three-dimensional Ricci scalar, K_{ij} is the extrinsic curvature, K is the trace of K_{ij}, and C_1 is an integration constant.

Following calculations in Ref. [40] since the scalar field is a function of time, the fundamental variables are only N, N^i, and h^{ij} which are the lapse function, the shift vector, and the three-dimensional metric, respectively. Their conjugate momenta are

$$\pi_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0, \qquad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{N}^i} = 0,$$
$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}^{ij}} = \frac{1}{2}\sqrt{h}\left(-\frac{2\alpha_3 M_{\rm PL}|\dot{\phi}|}{N} + C_1\right)h^{ij} - \frac{1}{2}M_{\rm PL}^2\sqrt{h}(Kh^{ij} - K^{ij}).$$

Thus the primary constraints are π_N and π_i. Using the Legendre transformation, the Hamiltonian is given by

$$H = \int d^3x(\mathcal{H} + N^i\mathcal{H}_i + \lambda_N\pi_N + \lambda^i\pi_i), \tag{2.2}$$

where

$$\mathcal{H} = N\sqrt{h}\left[\frac{2}{M_{\rm PL}^2}\left(\frac{\pi^{ij}\pi_{ij}}{h} - \frac{\pi^2}{2h}\right) - \frac{1}{2}M_{\rm PL}^2\,{}^3R - \left(\frac{\alpha_2 M_{\rm PL}^2|\dot{\phi}|}{N} - V(\phi)\right) + \frac{\pi}{M_{\rm PL}^2\sqrt{h}}\left(-\frac{2\alpha_3 M_{\rm PL}|\dot{\phi}|}{N} + C_1\right) - \frac{3}{4M_{\rm PL}^2}\left(C_1^2 - \frac{4\alpha_3 M_{\rm PL}C_1|\dot{\phi}|}{N}\right)\right]$$
$$\mathcal{H}_i = -2h_{ik}D_j\pi^{kj},$$

and λ_N and λ^i are Lagrange multipliers.

The secondary constraints are given by

$$0 = \dot{\pi}_N = -\frac{\partial H}{\partial N} \approx -\frac{\partial \mathcal{H}}{\partial N} \equiv \mathcal{C},$$
$$0 = \dot{\pi}_i = -\frac{\partial H}{\partial N^i} \approx \mathcal{H}_i.$$

The $\approx$ means equality when the constraints are imposed. However the momentum constraint is not a first-class constraint because one of the Poisson brackets with other constraints does not vanish. Therefore we introduce

$$\bar{\mathcal{H}}_i = \mathcal{H}_i + \pi_N\partial_i N.$$

On the constraint surface we find $\bar{\mathcal{H}}_i = \mathcal{H}_i(\approx 0)$ because of $\pi_N = 0$. Then we can consider $\bar{\mathcal{H}}_i$ as the momentum constraint. The Poisson brackets of constraints are (see definition of the Poisson bracket in Refs. [36,40])

$$\{\pi_i(x), \pi_N(x')\} = 0,$$
$$\{\pi_i(x), \bar{\mathcal{H}}_j(x')\} = 0,$$
$$\{\pi_i(x), \mathcal{C}(x')\} = 0,$$
$$\{\bar{\mathcal{H}}_i[f^i], \bar{\pi}_N[\varphi]\} = \int d^3y\pi_N f^i\partial_i\varphi \approx 0,$$
$$\{\bar{\mathcal{H}}_i[f^i], \mathcal{C}[\varphi]\} = \int d^3y\mathcal{C}f^i\partial_i\varphi \approx 0,$$
$$\{\pi_N(x), \mathcal{C}(x')\} = \frac{\partial^2\mathcal{H}}{\partial N^2}\delta(x - x'),$$

where we have used the smeared constraint forms which are defined as

$$\bar{\mathcal{H}}_i[f^i] \equiv \int d^3x f^i(x)\bar{\mathcal{H}}_i(x)$$
$$\bar{\pi}_N[\varphi] \equiv \int d^3x\varphi(x)\pi_N(x)$$
$$\mathcal{C}[\varphi] \equiv \int d^3x\varphi(x)\mathcal{C}(x).$$

Since $\mathcal{H}$ in the Hamiltonian (2.2) is a linear function of the lapse function, i.e., $\partial^2\mathcal{H}/\partial N^2 = 0$, the last Poisson bracket is equal to zero. Consequently, all of constraints are the first-class constraints.

We have 10 variables which is equal to 20 dimensions in phase space with 8 first-class of constraints. Thus degrees of freedom of the theory can be calculated by

$$\text{d.o.f.} = \frac{1}{2}(\text{variables} \times 2 - \text{1st class} \times 2 - \text{2nd class}) = \frac{1}{2}(10 \times 2 - 8 \times 2 - 0) = 2. \tag{2.3}$$

As a result, the theory has 2 degrees of freedom.

III. DYNAMICS OF FLRW SPACETIME

We consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric and choose the unitary gauge as

$$ds^2 = -N(t)^2dt^2 + a(t)^2d\boldsymbol{x}^2, \qquad \phi = \phi(t).$$

Substituting the metric into the above action, and then varying with respect to ϕ, N, and a, after setting $N = 1$ we find

$$6\alpha_3 M_{\rm PL}(3H^2 + \dot{H}) - V_{,\phi} - 3\alpha_2 M_{\rm PL}^2 H\epsilon_{\dot{\phi}} + 6\alpha_3^2(3H\dot{\phi} + \ddot{\phi}) = 0, \tag{3.1}$$

$$3M_{\rm PL}^2H^2-\rho-V(\phi)+6\alpha_3M_{\rm PL}H\dot{\phi}+3\alpha_3^2\dot{\phi}^2=0, \tag{3.2}$$

$$3M_{\rm PL}^2H^2+2M_{\rm PL}^2\dot{H}+P-V(\phi)+\alpha_2M_{\rm PL}^2|\dot{\phi}| +\alpha_3(2M_{\rm PL}\ddot{\phi}-3\alpha_3\dot{\phi}^2)=0, \tag{3.3}$$

where $\epsilon_{\dot{\phi}}\equiv{\rm sgn}(\dot{\phi})$, and ρ and P are total matter density and pressure, respectively. Since $\epsilon_{\dot{\phi}}$ changes the value discretely, we should assume that $\dot{\phi}\neq 0$ and it does not change the sign during the evolution of the universe. We may have two branch solutions. The solution with $\phi=$ constant is incompatible with the timelike ansatz. However we can discuss the limiting case as $\dot{\phi}\to 0$, which will give two different solutions unless H vanishes.

We may assume that matter components consist of perfect fluids such as matter and radiation, that is,

$$\rho=\sum_i\rho_i, \qquad P=\sum_i P_i, \quad \text{with} \quad P_i=w_i\rho_i, \tag{3.4}$$

where w_i describes the equation of state of ith matter component. The matter (ρ_m, P_m) and radiation (ρ_r, P_r) are given by $w_m=0$ and $w_r=\frac{1}{3}$, respectively.

As we show in Sec. II, the first equation for the scalar field ϕ is derived from the Einstein equations. In what follows, we rewrite the above basic equations to solve them.

Introducing new Hubble parameter by

$$\bar{H}\equiv H+\alpha_3M_{\rm PL}^{-1}\dot{\phi},$$

we rewrite the above three equations of motion as

$$6\bar{H}^2+2\dot{\bar{H}}-6\alpha_3M_{\rm PL}^{-1}\bar{H}\dot{\phi}+\alpha_2|\dot{\phi}|-\frac{1}{3\alpha_3M_{\rm PL}}V_{,\phi} -\frac{\alpha_2}{\alpha_3}M_{\rm PL}\bar{H}\epsilon_{\dot{\phi}}=0, \tag{3.5}$$

$$3M_{\rm PL}^2\bar{H}^2=\rho+V(\phi), \tag{3.6}$$

$$3\bar{H}^2+2\dot{\bar{H}}-6\alpha_3M_{\rm PL}^{-1}\bar{H}\dot{\phi}+\alpha_2|\dot{\phi}|+M_{\rm PL}^{-2}(P-V(\phi))=0. \tag{3.7}$$

From Eqs. (3.5) and (3.7), we find

$$\bar{H}^2=\frac{1}{3M_{\rm PL}^2}(P-V(\phi))+\frac{1}{9\alpha_3M_{\rm PL}}V_{,\phi}+\frac{\alpha_2}{3\alpha_3}M_{\rm PL}\bar{H}\epsilon_{\dot{\phi}}.$$

With Eq. (3.6), we obtain the following equation

$$\bar{H}\epsilon_{\dot{\phi}}=\frac{\alpha_3}{\alpha_2M_{\rm PL}^3}\left[\rho-P+2V(\phi)-\frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}\right], \tag{3.8}$$

or

$$\bar{H}^2=\frac{\alpha_3^2}{\alpha_2^2M_{\rm PL}^6}\left[\rho-P+2V(\phi)-\frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}\right]^2. \tag{3.9}$$

From Eqs. (3.6) and (3.9), we obtain one constraint equation

$$\frac{1}{3}(\rho+V(\phi))=\frac{\alpha_3^2}{\alpha_2^2M_{\rm PL}^4}\left[\rho-P+2V(\phi)-\frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}\right]^2. \tag{3.10}$$

This constraint equation gives the relation between the scalar field ϕ and matter and radiation, once we assume the potential $V(\phi)$. The scalar field ϕ is no longer dynamical, but it is fixed by matter fluid (ρ, P).

For the perfect fluids, we find the time evolution of their densities as

$$\rho_i\propto a^{-3(1+w_i)}, \qquad P_i=w_i\rho_i,$$

from the energy conservation equation. Hence ρ and P are given by some known function of the e-folding number $N\equiv\ln(a/a_0)$ as $\rho(N)$ and $P(N)$, where a_0 is the present value of the scale factor.

Solving the constraint equation (3.10) for the scalar field ϕ in terms of the e-folding number N, we find

$$\phi=\phi(N).$$

Since

$$\bar{H}=H+\frac{\alpha_3}{M_{\rm PL}}\dot{\phi}=HZ(N),$$

where

$$Z(N)\equiv 1+\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN},$$

we obtain the effective Friedmann equation from Eq. (3.6) as

$$H^2=\frac{1}{3M_{\rm PL}^2Z^2(N)}[\rho(N)+V(\phi(N))]. \tag{3.11}$$

This equation gives the solution of the scale factor, $a=a(t)$. The prefactor Z^{-2} modifies the Friedmann equation from the general relativistic one. Note that there is no kinetic term of a scalar field.

IV. EXPONENTIAL POTENTIAL

In order to analyze the cosmological evolution, we have to give a concrete form of the potential $V(\phi)$. Here we shall assume an exponential potential,

$$V = \epsilon_V M_{\rm PL}^4 \exp(\lambda\alpha_3 M_{\rm PL}^{-1}\phi), \tag{4.1}$$

where λ is a coupling constant. Without loss of generality, we can normalize the coefficient of the potential as $\epsilon_V = \pm 1$ because of rescaling of a scalar field ϕ.

Assuming there exist matter and radiation as matter components, the constraint equation (3.10) is

$$\frac{1}{3}(\rho_m + \rho_r + V) = \frac{\alpha_3^2}{\alpha_2^2 M_{\rm PL}^4}\left[\rho_m + \frac{2}{3}\rho_r + \left(2 - \frac{\lambda}{3}\right)V\right]^2,$$

which is rewritten as

$$\left(2-\frac{\lambda}{3}\right)^2 V^2 + \left[2\left(2-\frac{\lambda}{3}\right)\left(\rho_m + \frac{2}{3}\rho_r\right) - \frac{\alpha_2^2 M_{\rm PL}^4}{3\alpha_3^2}\right]V + \left(\rho_m + \frac{2}{3}\rho_r\right)^2 - \frac{\alpha_2^2 M_{\rm PL}^4}{3\alpha_3^2}(\rho_m + \rho_r) = 0. \tag{4.2}$$

This must have a real solution for V. If $\lambda = 6$, we always have a simple solution

$$V = \frac{3\alpha_3^2}{\alpha_2^2 M_{\rm PL}^4}\left(\rho_m + \frac{2}{3}\rho_r\right)^2 - (\rho_m + \rho_r).$$

For the case of $\lambda \neq 6$, we have a quadratic equation. Before solving it, we shall take the limit of $a \to \infty$ (or equivalently $\rho_m, \rho_r \to 0$). Eq. (4.2) gives

$$V(V - V_\infty) = 0,$$

where

$$V_\infty \equiv \frac{3\alpha_2^2}{(\lambda-6)^2\alpha_3^2} M_{\rm PL}^4. \tag{4.3}$$

TABLE I. The existence range of $\tilde{\rho}_m$ and $\tilde{\rho}_r$ for the solution of Eq. (4.4) for $\tilde{V}$. For $\lambda \geq 6$ or $\lambda \leq 3$ ($\lambda \neq 0$), we find the full range of the densities. In the case of $4 \leq \lambda < 6$, there exists some upper bound on densities (or lower bound for a scale factor). For the case of $3 < \lambda < 4$, depending on the parameters, there are two possibilities (see the detail in the text and Appendix).

	Exponent	Existence range
(a)	$\lambda > 6$	$0 \leq \tilde{\rho}_m, \tilde{\rho}_r < \infty$
(b)	$\lambda = 6$	$0 \leq \tilde{\rho}_m, \tilde{\rho}_r < \infty$
(c)	$4 < \lambda < 6$	$(\lambda-3)\tilde{\rho}_m + (\lambda-4)\tilde{\rho}_r \leq \frac{1}{4}(6-\lambda)$
(d)	$\lambda = 4$	$\tilde{\rho}_m \leq \frac{1}{2}$
(e)	$3 < \lambda < 4$	$(\lambda-3)\tilde{\rho}_m \leq (4-\lambda)\tilde{\rho}_r + \frac{1}{4}(6-\lambda)$
(f)	$\lambda = 3$	$0 \leq \tilde{\rho}_m, \tilde{\rho}_r < \infty$
(g)	$0 < \lambda < 3$	$0 \leq \tilde{\rho}_m, \tilde{\rho}_r < \infty$
(h)	$\lambda < 0$	$0 \leq \tilde{\rho}_m, \tilde{\rho}_r < \infty$

We then normalize the variables and parameters by V_∞, which are described by those with a tilde. The quadratic equation for V is now

$$\tilde{V}^2 - \left[\frac{2}{\lambda-6}(3\tilde{\rho}_m + 2\tilde{\rho}_r) + 1\right]\tilde{V} + \frac{1}{(\lambda-6)^2}(3\tilde{\rho}_m + 2\tilde{\rho}_r)^2 - (\tilde{\rho}_m + \tilde{\rho}_r) = 0, \tag{4.4}$$

where

$$\tilde{V} \equiv \frac{V}{V_\infty}, \qquad \tilde{\rho}_m \equiv \frac{\rho_m}{V_\infty}, \quad \text{and} \quad \tilde{\rho}_r \equiv \frac{\rho_r}{V_\infty}.$$

In order to have a real solution for $\tilde{V}$, the following condition should be satisfied:

$$D \equiv 1 + \frac{4}{\lambda-6}[(\lambda-3)\tilde{\rho}_m + (\lambda-4)\tilde{\rho}_r] \geq 0.$$

This condition gives the constraint on ρ_m and ρ_r. We can classify the possible cases by the exponent λ of the potential. We summarize the classification in Table I, in which we show the range of $\tilde{\rho}_m$ and $\tilde{\rho}_r$ for existence of a real solution $\tilde{V}$.

For $\lambda \geq 6$ or $\lambda \leq 3$ ($\lambda \neq 0$), we find the full range of the densities. In the case of $4 \leq \lambda < 6$, there exists some upper bound on densities (or lower bound for a scale factor). For the case of $3 < \lambda < 4$, depending on the parameters, there are two possibilities: Either the full range of the densities is possible or two separated finite ranges of the scale factor are possible, i.e., $a \leq a_1$ or $a_2 \leq a$ $(a_2 < a_1)$. The latter case happens either when λ is close to 4 or matter density is large enough.

The solution $\phi_\pm$ is given by

$$\phi_\pm = \frac{M_{\rm PL}}{\lambda\alpha_3}\ln\left[\frac{3\alpha_2^2}{\epsilon_V(\lambda-6)^2\alpha_3^2}\tilde{V}_\pm\right], \tag{4.5}$$

where

$$\tilde{V}_\pm \equiv \tilde{V}(\phi_\pm) = \frac{1}{2}\left[1 + \frac{2}{\lambda-6}(3\tilde{\rho}_m + 2\tilde{\rho}_r) \pm \sqrt{D}\right].$$

We call them $\pm$ branches, respectively. In order to exist the real solution, we have the constraint such that

$$\epsilon_V \tilde{V}_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) \geq 0,$$

which means that the potential is positive definite ($\epsilon_V = 1$) for the case of $\tilde{V}_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) > 0$, otherwise it is negative definite ($\epsilon_V = -1$). $\epsilon_{\dot{\phi}}$ is determined from Eq. (4.5). For example, for $\lambda > 6$, ϕ_+ decreases as a increases (or densities decrease), which gives $\epsilon_{\dot{\phi}} = -1$.

In order to derive the effective Friedmann equation (3.11), we have to evaluate the prefactor Z^{-2}. Using the relation

$$\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi_\pm}{d\ln a}=\frac{1}{\lambda}\frac{d\ln V_\pm}{d\ln a}$$
$$=\frac{1}{\lambda}\left[\frac{\partial\ln\tilde{V}_\pm}{\partial\tilde{\rho}_m}\frac{d\tilde{\rho}_m}{d\ln a}+\frac{\partial\ln\tilde{V}_\pm}{\partial\tilde{\rho}_r}\frac{d\tilde{\rho}_r}{d\ln a}\right]$$
$$=-\frac{1}{\lambda}\left[3\frac{\tilde{\rho}_m}{\tilde{V}_\pm}\frac{\partial\tilde{V}_\pm}{\partial\tilde{\rho}_m}+4\frac{\tilde{\rho}_r}{\tilde{V}_\pm}\frac{\partial\tilde{V}_\pm}{\partial\tilde{\rho}_r}\right],$$

we find

$$Z_\pm=1+\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi_\pm}{d\ln a}=\frac{F_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)}{S_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)\sqrt{D}},$$

where

$$S_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)\equiv 2\tilde{V}_\pm=1+\frac{2}{\lambda-6}(3\tilde{\rho}_m+2\tilde{\rho}_r)\pm\sqrt{D},$$
$$F_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)\equiv\left[1+\frac{2}{\lambda(\lambda-6)}(3(\lambda-3)\tilde{\rho}_m+2(\lambda-4)\tilde{\rho}_r)\right]\sqrt{D}\pm\left[1+\frac{2}{\lambda(\lambda-6)}((\lambda-3)(2\lambda-3)\tilde{\rho}_m+2(\lambda-2)(\lambda-4)\tilde{\rho}_r)\right].$$

Since

$$\rho_m+\rho_r+V(\phi_\pm)=\rho_m+\rho_r+V_\pm=\frac{V_\infty}{2}R_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda),$$

where

$$R_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)\equiv 1+\frac{2}{\lambda-6}((\lambda-3)\tilde{\rho}_m+(\lambda-4)\tilde{\rho}_r)\pm\sqrt{D},$$

we obtain the effective Friedmann equation as

$$H^2=\frac{1}{3M_{\rm PL}^2}\frac{V_\infty D(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)S_\pm^2(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)R_\pm(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)}{2F_\pm^2(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)}. \tag{4.6}$$

A. Two limiting stages

We first consider two limiting stages ($a\to\infty$ and $a\to 0$), assuming their existence. Those correspond to $\rho_m,\rho_r\to 0$ and $\rho_m,\rho_r\to\infty$, respectively.

1. $a\to\infty$ ($\rho_m,\rho_r\to 0$)

In this limit, the potentials for two branch solutions are approximated as

$$\tilde{V}(\phi_+)=\tilde{V}_+\approx 1+\frac{1}{\lambda-6}[\lambda\tilde{\rho}_m+(\lambda-2)\tilde{\rho}_r]$$
$$\tilde{V}(\phi_-)=\tilde{V}_-\approx-(\tilde{\rho}_m+\tilde{\rho}_r)+\frac{[(\lambda-3)\tilde{\rho}_m+(\lambda-4)\tilde{\rho}_r]^2}{(\lambda-6)^2}.$$

We find $\epsilon_V=+1$ for $+$ branch ($\tilde{V}_+>0$), while $\epsilon_V=-1$ for $-$ branch ($\tilde{V}_-<0$).

For the $+$ branch solution $\tilde{V}_+$, we find

$$M_{\rm PL}^2H^2\approx\frac{V_\infty}{3},$$

which gives the de-Sitter type accelerating universe as

$$a(t)\propto\exp[H_\infty t], \tag{4.7}$$

where

$$H_\infty\equiv M_{\rm PL}^{-1}\sqrt{\frac{V_\infty}{3}}=\frac{|\alpha_2|}{(\lambda-6)\alpha_3}M_{\rm PL}.$$

In order to explain the present acceleration of the universe in this model, we have a strong constraint on the coupling constant as

$$\frac{|\alpha_2|}{(\lambda-6)\alpha_3}\sim O(10^{-60})\ll 1.$$

The scalar field approaches as

$$\phi\to\phi_\infty\equiv\frac{M_{\rm PL}}{\lambda\alpha_3}\ln\left[\frac{3\alpha_2^2}{(\lambda-6)^2\alpha_3^2}\right].$$

For the other branch solution ϕ_-, we find

$$M_{\rm PL}^2H^2=\frac{1}{3Z^2}(\rho_m+\rho_r+V_-)\approx\frac{\lambda^2}{3(\lambda-6)^2}V_\infty\rho_m^2\propto\frac{1}{a^6},$$

because $\rho_m\gg\rho_r$ as $a\to\infty$. It gives the asymptotic behavior as

$$a(t)\propto t^{1/3},$$

which is the expansion law for the stiff matter ($P=\rho$) in general relativity (GR), although matter density dominates the universe. The scalar field approaches as

$$\phi\to-\infty.$$

2. $a \to 0$ ($\rho_m, \rho_r \to \infty$)

The asymptotic behaviors of the two branch solutions ($V_\pm$) and the Friedmann equation become the same forms as

$$\tilde{V}_\pm \approx \frac{1}{\lambda-6}(3\tilde{\rho}_m + 2\tilde{\rho}_r),$$

and

$$M_{\rm PL}^2 H^2 \approx \frac{\lambda^2}{3(\lambda-6)} \frac{(3\rho_m + 2\rho_r)^2[(\lambda-3)\rho_m + (\lambda-4)\rho_r]}{[3(\lambda-3)\rho_m + 2(\lambda-4)\rho_r]^2}.$$

If $\lambda \neq 3, 4$,

$$M_{\rm PL}^2 H^2 \approx \begin{cases} \frac{\lambda^2}{3(\lambda-3)(\lambda-6)}\rho_m & \text{for } \rho_m \gg \rho_r \text{ (MD)} \\ \frac{\lambda^2}{3(\lambda-4)(\lambda-6)}\rho_r & \text{for } \rho_m \ll \rho_r \text{ (RD)} \end{cases},$$

where MD and RD denote matter dominant stage and radiation dominant stage, respectively.

This gives

$$a(t) \propto \begin{cases} t^{\frac{2}{3}} & \text{for } \rho_m \gg \rho_r \text{ (MD)} \\ t^{\frac{1}{2}} & \text{for } \rho_m \ll \rho_r \text{ (RD)} \end{cases},$$

which is the same as the evolution history in the standard big-bang model. However the effective gravitational constant in the Friedmann equation $G_{\rm F}$ is different from the Newtonian gravitational constant $G_{\rm N} \equiv (8\pi M_{\rm PL}^2)^{-1}$. Note that the scalar field approaches in this limit as

$$\phi_\pm \to \infty,$$

for both branches.

$G_{\rm F}$ shows a gap between the values at radiation dominant stage and at the matter dominant stage. In fact, we find

$$G_{\rm F} = \begin{cases} \frac{\lambda^2}{(\lambda-3)(\lambda-6)} G_{\rm N} & \text{for } \rho_m \gg \rho_r \text{ (MD)} \\ \frac{\lambda^2}{(\lambda-4)(\lambda-6)} G_{\rm N} & \text{for } \rho_m \ll \rho_r \text{ (RD)} \end{cases}.$$

One may wonder what happens if $3 \le \lambda \le 6$, when $G_{\rm F} < 0$. As we show in Appendix C, in such a case, there is no limit of $a \to 0$. The scale factor a is bounded from below, that is $a \ge a_{\rm min} (> 0)$.

In the cases of $\lambda = 3$ and $\lambda = 4$, we find strange behaviors in the Friedmann equation as follows: For $\lambda = 3$,

$$M_{\rm PL}^2 H^2 \approx \begin{cases} \frac{9}{4}\frac{\rho_m^2}{\rho_r} & \text{for } \rho_m \gg \rho_r \text{ (MD)} \\ \rho_r & \text{for } \rho_m \ll \rho_r \text{ (RD)} \end{cases},$$

for which the expansion law becomes

$$a(t) \propto \begin{cases} t & \text{for } \rho_m \gg \rho_r \text{ (MD)} \\ t^{\frac{1}{2}} & \text{for } \rho_m \ll \rho_r \text{ (RD)} \end{cases}.$$

On the other hand, for $\lambda = 4$, there exists no solution in this limit.

B. Whole history

In the two liming stages, we may find an appropriate evolution of the universe, i.e., radiation/matter dominance in the early stage ($a \to 0$), and de Sitter expansion for $+$ branch in the early stage ($a \to \infty$). However the above two limiting stages can be disconnected if there exists some finite scale factor at which the Hubble parameter H vanishes or diverges, or H^2 becomes negative. It may happen when one of the following conditions is satisfied

$$\text{(i) } D(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) \le 0,$$
$$\text{(ii) } S_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) = 0,$$
$$\text{(iii) } R_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) \le 0,$$
$$\text{(iv) } F_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) = 0.$$

In fact H vanishes when $S_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) = 0$, while it diverges when $F_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) = 0$. In those cases, the above two limits are disconnected at that point. On the other hand, when $D(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) < 0$ or $R_\pm(\tilde{\rho}_m, \tilde{\rho}_r; \lambda) < 0$, no solution exists in such a range of densities $\tilde{\rho}_m$, $\tilde{\rho}_r$ (or a scale factor a).

In what follows, we just discuss one simple case ($\lambda > 6$). For the other cases, we show them in Appendix C.

1. Exponential potential with $\lambda > 6$

In this case, we find $D > 0$, which guarantees the solution exists for full range of densities, i.e., $0 \le \rho_m$, $\rho_r < \infty$. For the ϕ_- branch, there exists one point where H vanishes, that is, it happens when

$$(3\tilde{\rho}_m + 2\tilde{\rho}_r)^2 = (\lambda-6)^2(\tilde{\rho}_m + \tilde{\rho}_r).$$

which is obtained from the condition (ii). We find the corresponding scale factor $a_{\rm cr}$ as

$$\rho_r(a_{\rm cr}) = \frac{(\lambda-6)^2}{4} V_\infty \quad \text{if it happens in RD}$$
$$\rho_m(a_{\rm cr}) = \frac{(\lambda-6)^2}{9} V_\infty \quad \text{if it happens in MD.}$$

Since we find

$$H^2 \propto (a - a_{\rm cr})^2,$$

near $a = a_{\rm cr}$, the universe approaches $a_{\rm cr}$ exponentially with respect time as

$$a(t) \approx a_{\rm cr} \mp a_* \exp(\mp K_* t) \quad \text{as } t \to \pm\infty,$$

where a_* and K_* are positive constants. As a result, we have two histories of the universe ($a_1(t)$ and $a_2(t)$) as

$$a_1(t) \propto \begin{cases} t^{\frac{1}{2}} & \text{in} \quad t \to 0 \\ \text{(RD)} & \\ a_{\rm cr} & \text{as} \quad t \to \infty \end{cases},$$

or

$$a_1(t) \propto \begin{cases} t^{\frac{1}{2}} & \to & t^{\frac{2}{3}} & \text{in the early stage} \\ \text{(RD)} & & \text{(MD)} & \\ & a_{\rm cr} & & \text{as} \quad t \to \infty \end{cases},$$

and

$$a_2(t) \propto \begin{cases} a_{\rm cr} & \text{as } t \to -\infty \\ t^{\frac{1}{3}} & \text{as } t \to \infty \end{cases}.$$

For the + branch, both denominator and numerator in the right hand side of the Friedmann equation (4.6) do not vanish for any values of ρ_m, ρ_r. Hence the above two limits are connected. We find radiation dominant era and matter dominant era in the early stage of the universe, which is followed by de Sitter accelerating expansion.

$$a(t) \propto \begin{cases} t^{\frac{1}{2}} & \to & t^{\frac{2}{3}} & \text{in the early stage} \\ \text{(RD)} & & \text{(MD)} & \\ & \exp(H_\infty t) & & \text{as} \quad t \to \infty \end{cases}.$$

TABLE II. The classification of cosmic evolution of the universe with the positive exponential potential ($\epsilon_V = 1$). RD/MD denotes the Friedmann universe of radiation dominant stage, possibly followed by matter dominant stage. dS means de Sitter accelerating universe, while P[p] gives the power-law expanding universe with the power-exponent p ($a \propto t^p$). M[a] shows Minkowski spacetime with the scale factor a, while S[a] means a singularity at finite scale factor a.

	Exponent	+ branch	− branch
(a)	$\lambda > 6$	RD/MD → dS	RD/MD → M[$a_{\rm cr}$]
(b)	$\lambda = 6$	P[1/4] → M[$a_{\rm cr}$] when $\alpha_2/\alpha_3 \ll O(1)$	
(c)	$4 < \lambda < 6$	M[$a_{\rm min}$] → dS	M[$a_{\rm min}$] → M[$a_{\rm cr}$]
(d)	$\lambda = 4$	M[$a_{\rm cr}$] → dS	⋯
(e)	$3 < \lambda < 4$	M[$a_{\rm cr}$] → dS	⋯
(f)	$\lambda = 3$	M[$a_{\rm cr}$] → dS	⋯
(g)	$0 < \lambda < 3$	M[$a_{\rm cr}$] → dS	⋯
(h)	$\lambda < 0$	M[$a_{\rm cr}^{(S)}$] → S[$a_{\rm cr}^{(F)}$] → dS	⋯

TABLE III. The classification of cosmic evolution of the universe with the negative exponential potential ($\epsilon_V = -1$). The notations are the same as those in Table II.

	Exponent	+branch	−branch
(a)	$\lambda > 6$	⋯	M[$a_{\rm cr}$] → P[1/3]]
(b)	$\lambda = 6$	P[1/4] → P[1/3] when $\alpha_2/\alpha_3 \gtrsim O(1)$	
		M[$a_{\rm cr}$] → P[1/3] when $\alpha_2/\alpha_3 \ll O(1)$	
(c)	$4 < \lambda < 6$	⋯	M[$a_{\rm cr}$] → P[1/3]
(d)	$\lambda = 4$	M[$a_{\rm min}$] → M[$a_{\rm cr}$]	M[$a_{\rm min}$] → P[1/3]
(e)	$3 < \lambda < 4$	M[$a_{\rm min}$] → M[$a_{\rm cr}$]	M[$a_{\rm min}$] → P[1/3]
(f)	$\lambda = 3$	RD/MD → M[$a_{\rm cr}$]	RD/MD → P[1/3]
(g)	$0 < \lambda < 3$	RD/MD → M[$a_{\rm cr}$]	RD/MD → P[1/3]
(h)	$\lambda < 0$	RD/MD → M[$a_{\rm cr}^{(S)}$]	RD/MD → P[1/3]

2. Summary of exponential potential

Here we summarize the results on the cosmic evolution in Tables II, III and Figs. 1, 2. The details for the case of $\lambda \le 6$ are given in Appendix C.

As we show in the tables and schematic figures, the acceleration of the universe is obtained only for the + branch solutions with a positive definite potential ($\epsilon_V = 1$). For the case with $\lambda < 6$, we may not have radiation/matter

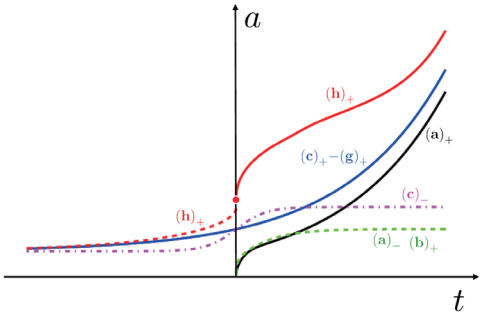

FIG. 1. The schematic evolution curves of the universe with positive exponential potential. (a), (b), ⋯, (h) correspond to the classification in Table II and the suffixes ± denote the branches.

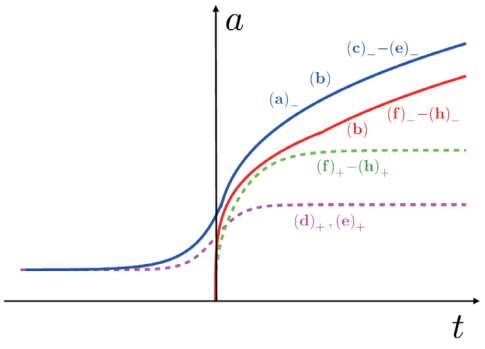

FIG. 2. The schematic evolution curves of the universe with negative exponential potential classified in Table III. The notations are the same as those in Fig. 1.

dominant era in the early stage, which is inconsistent with the big-bang nucleosynthesis.

C. Gravitational "constant" in effective Friedmann equation and Hubble constant

Since we are interested in the accelerating universe, we discuss the detail of the cosmological evolution for + branch.

Using the redshift z, which is defined by $1+z=a_0/a$, the densities of matter and radiation are given by

$$\rho_m = 3\Omega_{m,0}M_{\rm PL}^2H_0^2(1+z)^3, \quad \rho_r = 3\Omega_{r,0}M_{\rm PL}^2H_0^2(1+z)^4.$$

We then have

$$\tilde{\rho}_m = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}(1+z)^3, \qquad \tilde{\rho}_r = \frac{\Omega_{r,0}}{\Omega_{\Lambda,0}}(1+z)^4, \tag{4.8}$$

where

$$\Omega_{\Lambda,0} \equiv \frac{V_\infty}{3M_{\rm PL}^2H_0^2}.$$

Note that the H_0 here is based on the ΛCDM model. It is not the present value of the Hubble parameter in our model.

Inserting Eq. (4.8) into the Friedmann equation (4.6), we find the Hubble parameter H in terms of the redshift z.

We show the result in Fig. 3.

We rewrite the Friedmann equation (4.6) as follows:

$$H^2 = \frac{8\pi G_{\rm F}(z)}{3}(\rho_m+\rho_r+V_\infty),$$

where $G_{\rm F}(z)$ is defined by

$$G_{\rm F}(z) = \frac{1}{16\pi M_{\rm PL}^2}\frac{D(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)S_+^2(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)R_+(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)}{(1+\tilde{\rho}_m+\tilde{\rho}_r)F_+^2(\tilde{\rho}_m,\tilde{\rho}_r;\lambda)}.$$

If $G_{\rm F}=G_{\rm N}$, it gives the Friedmann equation in general relativity. Hence we can interpret the effect on the Friedmann equation by the cuscuton ϕ as modification

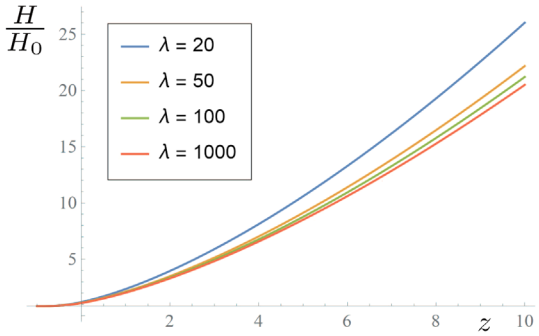

FIG. 3. Evolutions of H in terms of the redshift z (left figure). We set $\lambda = 20, 50, 100$, $\Omega_{\Lambda,0}=0.7$, $\Omega_{m,0}=0.3$, and $\Omega_{r,0}=0.0001$.

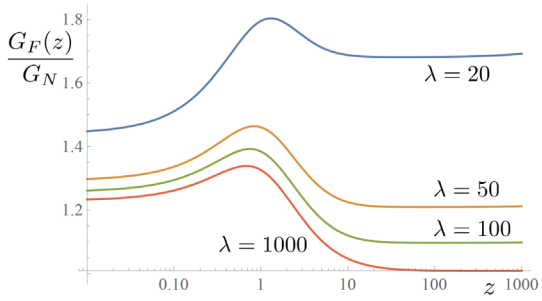

FIG. 4. Evolutions of $G_{\rm F}$ in terms of the redshift z. We choose the same parameter values as those in Fig. 3.

of the gravitational "constant" $G_{\rm F}$, which depends on z. The asymptotic behavior of $G_{\rm F}$ is given as

$$G_{\rm F}(z) \approx \begin{cases} \frac{\lambda^2}{(\lambda-4)(\lambda-6)}G_{\rm N} & \text{in RD } (\rho_r \gg \rho_m, V_\infty) \\ \frac{\lambda^2}{(\lambda-3)(\lambda-6)}G_{\rm N} & \text{in MD } (\rho_m \gg \rho_r, V_\infty) \\ G_{\rm N} & \text{in DED } (V_\infty \gg \rho_m, \rho_r) \end{cases}, \tag{4.9}$$

where DED denotes dark energy dominant stage.

We show some example of time evolution of $G_{\rm F}$ in Fig. 4.

Since this gravitational "constant" $G_{\rm F}$ depends on time and it deviates from $G_{\rm N}$, we have the observational constraints by the big-bang nucleosynthesis [41] such that

$$\frac{G_{\rm BBN}}{G_{\rm N}} = 0.99^{+0.06}_{-0.05}. \tag{4.10}$$

In the present model, $G_{\rm F}$ in the radiation dominant era is given by Eq. (4.9), which gives the constraint on λ as

$$\lambda \gtrsim 208.$$

We now present the comparison with the ΛCDM model. In Fig. 5, we present the evolution of the ratio of our Hubble expansion parameter to that in the ΛCDM model, $H/H_{\Lambda\rm CDM}(z)$, which is normalized at $z=1100$, i.e., $H/H_{\Lambda\rm CDM}(z=1100)=1$.

This figure shows that for $\lambda > 200$, the Hubble expansion rate at $z \le 1$ is about 10% larger than the value of the ΛCDM model, which tendency might explain the Hubble tension [42–46]. We shall discuss about it in Sec. VII.

V. CONSTRUCTION OF APPROPRIATE POTENTIAL

Although the exponential potential may provide the interesting feature in the Cuscuta-Galileon gravity theory, it may not explain the observational data precisely. Hence we shall discuss how to construct an appropriate potential $V(\phi)$ in our present model when the better evolution of the Hubble parameter is known. Once we

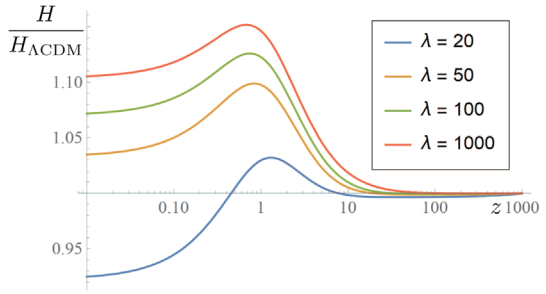

FIG. 5. Comparison with ΛCDM model. The parameters are chosen as the same as Fig. 3.

can phenomenologically construct an appropriate potential from observational data, we might be able to find a fundamental theory behind it.

The basic equations are

$$HZ\epsilon_{\dot{\phi}} = \frac{\alpha_3}{\alpha_2 M_{\rm PL}^3}\left(\rho - P + 2V - \frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}\right), \tag{5.1}$$

$$H^2Z^2 = \frac{1}{3M_{\rm PL}^2}(\rho + V). \tag{5.2}$$

The e-folding number $N \equiv \ln(a/a_0)$ measured from the present time is related to the redshift z as

$$N = -\ln(1+z).$$

Since

$$\frac{dV}{dN} = V_{,\phi}\frac{d\phi}{dN},$$

using the above basic equations, we find

$$\frac{d}{dN}\left[3M_{\rm PL}^2H^2\left(1+\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN}\right)^2 - \rho\right] = \frac{d\phi}{dN}\left[\frac{\alpha_2 M_{\rm PL}^2}{\alpha_3}H\left(1+\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN}\right)\epsilon_{\dot{\phi}} - (\rho - P) - 2V\right].$$

Eliminating V and using the energy conservation

$$\frac{d\rho}{dN} + 3(P+\rho) = 0,$$

we obtain the second-order differential equation for ϕ, which can be rewritten as

$$\frac{dZ}{dN} + Q_1(N)Z - 3Z^2 = Q_2(N), \tag{5.3}$$

where

$$Z \equiv 1 + \frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN} \tag{5.4}$$

$$Q_1 \equiv \left(3 + \frac{1}{H}\frac{dH}{dN} + \frac{\alpha_2 M_{\rm PL}\epsilon_{\dot{\phi}}}{2\alpha_3 H}\right) \tag{5.5}$$

$$Q_2 \equiv -\frac{1}{2M_{\rm PL}^2}\left(\frac{P+\rho}{H^2} - \frac{\alpha_2 M_{\rm PL}^3\epsilon_{\dot{\phi}}}{\alpha_3 H}\right). \tag{5.6}$$

Since Eq. (5.3) is the Riccati equation for Z, once we can find a special solution $Z_*(N)$, we obtain a general solution as follows: Setting $Z = Z_* + Y$, we find the Bernoulli equation as

$$\frac{dY}{dN} + (Q_1 - 6Z_*))Y = 3Y^2,$$

which can be linearized by setting $Y = 1/X$ as

$$\frac{dX}{dN} - (Q_1 - 6Z_*)X = -3.$$

First we solve the homogeneous solution X_H, which satisfies

$$\frac{dX_H}{dN} - (Q_1 - 6Z_*)X_H = 0.$$

Using this homogenous solution, we obtain general solution as

$$X(N) = -3X_H(N)\left[\int dN' \frac{1}{X_H(N')}\right], \tag{5.7}$$

where

$$X_H(N) = \exp\left[\int^N dN'(Q_1(N') - 6Z_*(N'))\right]. \tag{5.8}$$

As a result, we obtain a general solution for Z as

$$Z(N) = Z_*(N) + \frac{1}{X(N)}. \tag{5.9}$$

Integrating Eq. (5.9), we find the scalar field in terms of N as

$$\phi = \phi_0 + \frac{M_{\rm PL}}{\alpha_3}\int_0^N dN' Z(N'), \tag{5.10}$$

ϕ_0 is the present value of the scalar field.

Solving the inverse problem given by Eq. (5.10), we find the e-folding N in terms of ϕ, i.e., $N = N(\phi)$. As a result, inserting it in Eq. (5.2), we obtain the potential as

$$V(\phi) = -\rho(N(\phi)) + 3M_{\rm PL}^2H^2(N(\phi))Z(N(\phi))^2.$$

A. Potential for ΛCDM model

Now assuming matter dominant stage ($\rho = \rho_m$), we shall show the potential form for ΛCDM model, which is given by

$$H^2 = \frac{1}{3M_{\rm F}^2}(\rho_m + \rho_{\rm vac}),$$

where $M_{\rm F}$ and $\rho_{\rm vac}$ are positive constants representing the modified Planck mass and the vacuum energy density, respectively.

To perform the integrations, we change the variable N to ξ, which is defined by

$$\xi \equiv \sqrt{1 + \frac{\rho_m}{\rho_{\rm vac}}}.$$

Since the energy density is given by

$$\rho_m = \rho_{m,0} e^{-3N},$$

we find

$$d\xi = -\frac{3(\xi^2 - 1)}{2\xi} dN. \tag{5.11}$$

Using Eq. (5.11) and

$$H = \frac{\sqrt{\rho_{\rm vac}}}{\sqrt{3}M_{\rm F}}\xi,$$

we also find

$$Q_1 = \frac{3}{2\xi^2}(\xi^2 + 2p\xi + 1)$$
$$Q_2 = -\frac{3M_{\rm F}^2}{2M_{\rm PL}^2}\frac{\xi^2 - 1}{\xi^2} + \frac{3p}{\xi},$$

where

$$p \equiv \frac{1}{2\sqrt{3}}\frac{\alpha_2 M_{\rm PL} M_{\rm F} \epsilon_{\dot\phi}}{\alpha_3 \sqrt{\rho_{\rm vac}}}.$$

The differential equation is

$$\frac{dZ}{d\xi} - \frac{(\xi^2 + 2p\xi + 1)}{\xi(\xi^2 - 1)} Z + \frac{2\xi}{\xi^2 - 1} Z^2 - \frac{M_{\rm F}^2}{M_{\rm PL}^2}\frac{1}{\xi} + \frac{2p}{\xi^2 - 1} = 0,$$

which is still the Riccati equation.

The equation for the scalar field and the potential are given by

$$\frac{d\phi}{d\xi} = -\frac{2M_{\rm PL}}{3\alpha_3}\frac{\xi}{\xi^2 - 1}(Z(\xi) - 1),$$
$$V = 3M_{\rm PL}^2 H^2 Z^2 - \rho_m$$
$$= V_0\left[\xi^2 Z^2(\xi) - \frac{M_{\rm F}^2}{M_{\rm PL}^2}(\xi^2 - 1)\right],$$

where

$$V_0 \equiv \frac{M_{\rm PL}^2}{M_{\rm F}^2}\rho_{\rm vac}.$$

In order to find the analytic solution, we have to find a special solution Z_*. It can be obtained by the hypergeometric functions. However, since it is quite complicated, we may solve it numerically.

As for the initial condition, we shall consider the limit of $\xi \to 1$ ($\rho_m \to 0$). In this limit, ΛCDM model gives de Sitter expanding universe with $H =$ constant. If the potential V is finite, Z is also finite. As a result, $d\phi/dN$ must vanish in this limit. It gives $Z \to 1$ as $\xi \to 1$. In fact, we find the approximate solution by the power-series expansion near $\xi = 1$ as

$$Z(\xi) \approx 1 + z_1(\xi - 1) + z_2(\xi - 1)^2 + \cdots,$$
$$\phi(\xi) \approx \phi_1(\xi - 1) + \phi_2(\xi - 1)^2 + \cdots,$$

where

$$z_1 = \frac{1 - r^2}{p - 2}, \qquad z_2 = \frac{(r^2 - 1)[p^2 - 9p + 2(r^2 + 6)]}{2(p - 3)(p - 2)^2}, \ldots,$$
$$\phi_1 = -\frac{z_1}{3}\frac{M_{\rm PL}}{\alpha_3}, \qquad \phi_2 = -\frac{z_1 + 2z_2}{12}\frac{M_{\rm PL}}{\alpha_3}, \cdots.$$

Here we define r by

$$r \equiv \frac{M_{\rm F}}{M_{\rm PL}}.$$

We then find the potential near $\phi = 0$ as

$$V(\phi) = V_0[1 - 6(p - 1)\phi + \cdots].$$

We can also find the asymptotic solution in the limit of $\xi \to \infty$ as

$$Z(\xi) \to Z_\infty + c_Z \xi^{-\sqrt{1+8r^2}} + \cdots,$$
$$\phi(\xi) \to -\frac{2M_{\rm PL}}{3\alpha_3}(Z_\infty - 1)\ln\xi + \cdots,$$

where

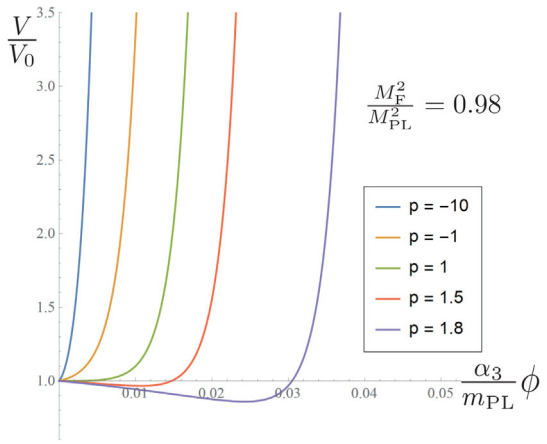

FIG. 6. Potentials for ΛCDM model. We set $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$, and $\frac{M_F^2}{M_{PL}^2} = 0.98$.

$$Z_\infty \equiv \frac{1+\sqrt{1+8r^2}}{4},$$

and c_z is some constant.

Since the potential is given in this limit as

$$V \to V_0(Z_\infty^2 - r^2)\xi^2,$$

we find the asymptotic form of the potential as

$$V \approx V_0(Z_\infty^2 - r^2)\exp\left[-\frac{3\alpha_3}{(Z_\infty - 1)M_{PL}}\phi\right],$$

which is the exponential potential (4.1) with the exponent λ given by

$$\lambda = -\frac{3(\sqrt{1+8r^2}+3)}{2(r^2-1)}.$$

We show some numerical examples in Figs. 6 and 7. Here we assume that $\frac{M_F^2}{M_{PL}^2} = 0.98$ or 1.02, because the "modified Planck" mass M_F in the Friedmann equation should be close to the Planck mass M_{PL}.

Since $Z_\infty^2 - r^2 > 0$ and $Z_\infty - 1 < 0$ for $r < 1$, while $Z_\infty^2 - r^2 < 0$ and $Z_\infty - 1 > 0$ for $r > 1$, we understand the above potential form with the fact that $dV/d\phi = -6(p-1)V_0$ at $\phi = 0$.

We cannot construct numerically any appropriate potential for the parameter $p \geq 2$.

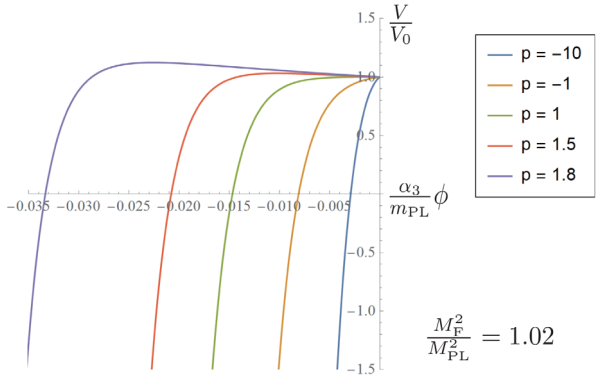

FIG. 7. Potentials for ΛCDM model with $\frac{M_F^2}{M_{PL}^2} = 1.02$. The other parameters are the same as those in Fig. 6.

VI. DENSITY PERTURBATIONS AND EFFECTIVE GRAVITATIONAL CONSTANT

A. Basic equations for density perturbations

According to Refs. [37,47,48] we consider the perturbed metric on the flat FLRW background as

$$ds^2 = -(1+2\Psi)dt^2 + 2\partial_i\psi dx^i dt + a(t)^2(1+2\Phi)\delta_{ij}dx^i dx^j, \tag{6.1}$$

when $\psi = 0$, it corresponds to the Newtonian gauge. The energy-momentum tensor with perturbations are defined as

$$T_0^0 = -(\rho_m + \delta\rho_m), \qquad T_i^0 = -\rho_m \partial_i v_m, \qquad T_j^i = 0, \tag{6.2}$$

where v_m is a velocity potential of the perfect fluid. Note that we are considering only perturbations of nonrelativistic matter.

Expanding the following action up to second order

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{PL}^2 R + \alpha_2 M_{PL}^2\sqrt{-X} + \alpha_3 M_{PL}\ln\left(-\frac{X}{\Lambda^4}\right)\Box\phi - V(\phi) + 3\alpha_3^2 X\right] + S_M(g_{\mu\nu}, \psi_M). \tag{6.3}$$

Varying with respect to Ψ, Φ, ψ, and $\delta\phi$, we find a set of equations in Fourier space as follows

$$E_\Psi:\ A_1\dot{\Phi} + A_2\delta\dot{\phi} + A_3\frac{k^2}{a^2}\Phi + A_4\Psi + A_5\frac{k^2}{a^2}\psi + \left(A_6\frac{k^2}{a^2} - \mu\right)\delta\phi - \delta\rho_m = 0, \tag{6.4}$$

$$E_\Phi:\ B_1\ddot{\Phi} + B_2\delta\ddot{\phi} + B_3\dot{\Phi} + B_4\delta\dot{\phi} + B_5\dot{\Psi} + B_6\frac{k^2}{a^2}\Phi + 3\nu\delta\phi + \left(B_8\frac{k^2}{a^2} + B_9\right)\Psi + B_{10}\frac{k^2}{a^2}\dot{\psi} + B_{11}\frac{k^2}{a^2}\psi = 0, \tag{6.5}$$

$$E_\psi:\ C_1\dot{\Phi} + C_2\delta\dot{\phi} + C_3\Psi + C_4\delta\phi + \rho_m v_m = 0, \tag{6.6}$$

$$E_{\delta\phi}:\ D_1\dot{\Phi} + D_2\ddot{\delta\phi} + D_3\dot{\Phi} + D_4\dot{\delta\phi} + D_5\dot{\Psi} + D_6\frac{k^2}{a^2}\dot{\psi} + D_8\Phi + \left(D_9\frac{k^2}{a^2} - M^2\right)\delta\phi + \left(D_{10}\frac{k^2}{a^2} + D_{11}\right)\Psi + D_{12}\frac{k^2}{a^2}\psi = 0. \tag{6.7}$$

Components of the set of equations are

$$A_1 = 6M_{\rm PL}^2 H + 6M_{\rm PL}\alpha_3\dot{\phi}, \qquad A_2 = 6\alpha_3 M_{\rm PL}H + 6\alpha_3^2\dot{\phi}, \qquad A_3 = 2M_{\rm PL}^2,$$
$$A_4 = -6M_{\rm PL}^2H^2 - 12\alpha_3 M_{\rm PL}H\dot{\phi} - \rho_m - 6\alpha_3^2\dot{\phi}^2, \qquad A_5 = 2M_{\rm PL}^2H + 2\alpha_3 M_{\rm PL}\dot{\phi},$$
$$A_6 = 2\alpha_3 M_{\rm PL}, \qquad \mu = V_{,\phi},$$
$$B_1 = 6M_{\rm PL}^2, \qquad B_2 = 6\alpha_3 M_{\rm PL}, \qquad B_3 = 18M_{\rm PL}^2H,$$
$$B_4 = \frac{3\alpha_2 M_{\rm PL}^2|\dot{\phi}|}{\dot{\phi}} - 18\alpha_3^2\dot{\phi}, \qquad B_5 = -6M_{\rm PL}^2H - 6\alpha_3 M_{\rm PL}\dot{\phi}, \qquad B_6 = 2M_{\rm PL}^2,$$
$$B_8 = 2M_{\rm PL}^2, \qquad B_9 = -6M_{\rm PL}^2\dot{H} - 18M_{\rm PL}^2H^2 - 18\alpha_3 M_{\rm PL}H\dot{\phi} - 6\alpha_3 M_{\rm PL}\ddot{\phi} + 3\rho_m,$$
$$B_{10} = 2M_{\rm PL}^2, \qquad B_{11} = 2M_{\rm PL}^2H, \qquad \nu = -V_{,\phi},$$
$$C_1 = 2M_{\rm PL}^2, \qquad C_2 = 2\alpha_3 M_{\rm PL}, \qquad C_3 = -2M_{\rm PL}^2H - 2\alpha_3 M_{\rm PL}\dot{\phi}, \qquad C_4 = -6\alpha_3 M_{\rm PL}H + \frac{\alpha_2 M_{\rm PL}^2|\dot{\phi}|}{\dot{\phi}} - 6\alpha_3^2\dot{\phi},$$
$$D_1 = 6\alpha_3 M_{\rm PL}, \qquad D_2 = 6\alpha_3^2, \qquad D_3 = 36\alpha_3 M_{\rm PL}H - \frac{3\alpha_2 M_{\rm PL}^2|\dot{\phi}|}{\dot{\phi}} + 18\alpha_3^2\dot{\phi},$$
$$D_4 = 18\alpha_3^2 H, \qquad D_5 = -6\alpha_3 M_{\rm PL}H - 6\alpha_3^2\dot{\phi}, \qquad D_6 = 2\alpha_3 M_{\rm PL},$$
$$D_8 = 18\alpha_3 M_{\rm PL}\dot{H} - \frac{9\alpha_2 M_{\rm PL}^2 H|\dot{\phi}|}{\dot{\phi}} + 54\alpha_3 M_{\rm PL}H^2 + 54\alpha_3^2 H\dot{\phi} + 18\alpha_3^2\ddot{\phi} - 3V_{,\phi},$$
$$D_9 = 6\alpha_3^2 + \frac{8\alpha_3 M_{\rm PL}H}{\dot{\phi}} - \frac{\alpha_2 M_{\rm PL}^2}{|\dot{\phi}|}, \qquad D_{10} = 2\alpha_3 M_{\rm PL},$$
$$D_{11} = -6\alpha_3 M_{\rm PL}\dot{H} - 18\alpha_3 M_{\rm PL}H^2 - 18\alpha_3^2 H\dot{\phi} - 6\alpha_3^2\ddot{\phi} - V_{,\phi},$$
$$D_{12} = 8\alpha_3 M_{\rm PL}H - \frac{\alpha_2 M_{\rm PL}^2|\dot{\phi}|}{\dot{\phi}} + 6\alpha_3^2\dot{\phi}, \qquad M^2 = V_{,\phi\phi}.$$

Note that $B_7 = D_7 = 0$. Since the matter is conserved, the perturbed energy-momentum tensor is satisfied

$$\delta(\nabla_\mu T^\mu_\nu) = 0. \tag{6.8}$$

From these perturbation equations, we can also confirm that this theory has two degrees of freedom. Although the perturbation equations contain $\delta\dot{\phi}$ and $\delta\ddot{\phi}$ as well as $\delta\phi$, we can eliminate those derivative terms by combining the perturbation equations, and obtain $\delta\phi$ in terms of the perturbation variables of matter fluid and metric components ($\delta\rho_m, v, \Phi, \Psi$, and ψ) and those time derivatives. Hence the perturbation of the scalar field is algebraically determined by the other perturbation variables. There is no additional degree of freedom coming from the scalar field.

Choosing the Newtonian gauge, the components $\nu = 0$ and $\nu = i$ lead to

$$\delta\dot{\rho}_m + 3H\delta\rho_m + \frac{k^2}{a^2}\rho_m v_m + 3\rho_m\dot{\Phi} = 0, \tag{6.9}$$

$$\dot{v}_m = \Psi, \tag{6.10}$$

respectively. The useful combination is

$$3(\dot{E}_\psi + 3HE_\psi) - E_\Phi = 0, \tag{6.11}$$

with the basic equations of the flat FLRW background and Eq. (6.10), the above relation becomes

$$B_6\Phi + B_8\Psi = 0. \tag{6.12}$$

We are interested in the subhorizon regime, $k^2/a^2 \gg H^2$, and using the quasistatic approximation, i.e., the dominant contributions terms are k^2/a^2, $\delta\rho_m$, and M^2. We also neglect the oscillating term of $\delta\phi$ and assume that the variations on gravitational potentials are small. Thereby, the E_Ψ and the $E_{\delta\phi}$ become

$$A_3\frac{k^2}{a^2}\Phi + A_6\frac{k^2}{a^2}\delta\phi - \delta\rho_m \simeq 0, \tag{6.13}$$

$$\left(D_9\frac{k^2}{a^2} - M^2\right)\delta\phi + D_{10}\frac{k^2}{a^2}\Psi \simeq 0. \tag{6.14}$$

Solving Eqs. (6.12)–(6.14) we find

$$\frac{k^2}{a^2}\Psi \simeq -\frac{(B_6D_9\frac{k^2}{a^2} - B_6M^2)\delta\rho_m}{(A_6^2B_6 + B_8^2D_9)\frac{k^2}{a^2} - B_8^2M^2}, \tag{6.15}$$

$$\Phi = -\Psi. \tag{6.16}$$

Under the above approximations, taking time derivative on Eq. (6.9) and using Eq. (6.10) and the conservation of matter density equation, equation of motion of the density contrast is given by

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi = 0, \tag{6.17}$$

where the density contrast is defined as $\delta_m = \delta\rho_m/\rho_m$.

Substituting Eq. (6.15) into above equation we find

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\rm eff}\rho_m\delta_m \simeq 0, \tag{6.18}$$

where the effective gravitational constant is

$$G_{\rm eff} = \frac{2M_{\rm PL}^2(B_6D_9\frac{k^2}{a^2} - B_6M^2)}{(A_6^2B_6 + B_8^2D_9)\frac{k^2}{a^2} - B_8^2M^2}G_{\rm N} = \frac{\left(6\alpha_3^2 + \frac{8\alpha_3M_{\rm PL}H}{\dot\phi} - \frac{\alpha_2M_{\rm PL}^2}{|\dot\phi|} - M^2\frac{a^2}{k^2}\right)}{\left(8\alpha_3^2 + \frac{8\alpha_3M_{\rm PL}H}{\dot\phi} - \frac{\alpha_2M_{\rm PL}^2}{|\dot\phi|} - M^2\frac{a^2}{k^2}\right)}G_{\rm N}.$$

Here we set $G_{\rm N} \equiv \frac{1}{8\pi M_{\rm PL}^2}$.

In the subhorizon limit the term M^2a^2/k^2 is very small compared to other terms, then we can neglect this term. Using the new Hubble parameter and Eq. (3.8) the effective gravitational constant becomes

$$G_{\rm eff} = \left\{1 - \frac{2\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN}(\rho_m + 2V - \frac{M_{\rm PL}}{3\alpha_3}V_{,\phi})}{(1+\frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN})[8(\rho_m + 2V - \frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}) - \frac{\alpha_2^2}{\alpha_3^2}M_{\rm PL}^4]}\right\}G_{\rm N}, \tag{6.19}$$

where we neglect contributions from radiation.

The gravitational slip parameter is defined as

$$\eta \equiv -\frac{\Phi}{\Psi}, \tag{6.20}$$

which is always equal to one in this model.

B. Effective gravitational constant and observational constraints

Since the effective gravitational "constant" is time-dependent, we have to take into account the observational constraints. The lunar-laser ranging experiment [49,50] gives the constraint such that

$$\frac{\dot G}{G} = (-5.0 \pm 9.6)\times 10^{-15}\ {\rm yr}^{-1}, \tag{6.21}$$

$$\frac{\ddot G}{G} = (1.6 \pm 2.0)\times 10^{-16}\ {\rm yr}^{-2}. \tag{6.22}$$

If the gravitational constant evolves due to the cosmic expansion, we expect that $\dot G_{\rm N}/G_{\rm N} \sim O(H_0) \sim 7\times 10^{-11}\ {\rm yr}^{-1}$ and $\ddot G_{\rm N}/G_{\rm N} \sim O(H_0^2) \sim 5\times 10^{-21}\ {\rm yr}^{-1}$. As a result, the condition (6.21) will give a strong constraint, but the constraint (6.22) may be much weaker.

1. Exponential potential with $\lambda > 6$

Assuming the exponential potential with $\lambda > 6$ discussed in Sec. IV B 1, we show the behavior of $G_{\rm eff}(z)$. We consider only + branch solution with $\epsilon_V = 1$. We then find the effective gravitational "constant" is given by

$$\frac{G_{\rm eff}}{G_{\rm N}} = 1 - \frac{2(Z_+ - 1)H_+\epsilon_{\dot\phi}}{8H_+Z_+\epsilon_{\dot\phi} - \frac{\alpha_2}{\alpha_3}M_{\rm PL}},$$

where

$$Z_+ = \frac{F_+}{S_+\sqrt{D}}$$

$$H_+^2 = \frac{V_\infty}{6M_{\rm PL}^2}\frac{DS_+^2R_+}{F_+^2}.$$

In Fig. 8, we depict the evolution of $G_{\rm eff}/G_{\rm N}$. Taking the time derivative of $G_{\rm eff}$, we show the behavior of $\dot G_{\rm eff}/G_{\rm N}$ in terms of the redshift z in Fig. 9.

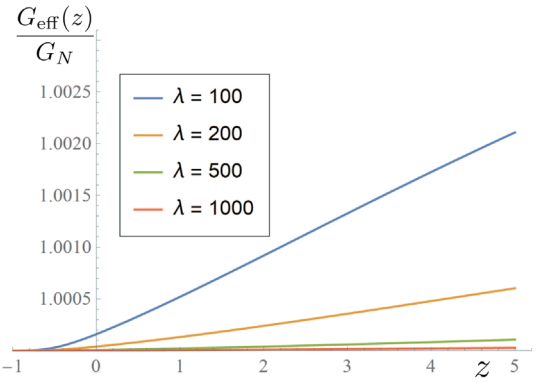

FIG. 8. Evolutions of $G_{\rm eff}/G_{\rm N}$ for the cases of $\lambda = 100$, 200, 500 and 1000 in terms of the redshift z. We choose the same parameter values as those in Fig. 3.

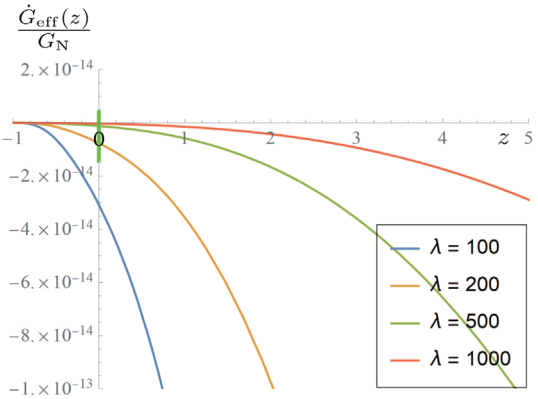

FIG. 9. Evolutions of $\dot{G}_{\rm eff}$ in terms of the redshift z. We choose the same parameter values as those in Fig. 8. The constraint from lunar-laser ranging experiment, Eq. (6.21), is given by the green line segment at $z = 0$.

In order to satisfy the constraint (6.21), we find $\lambda \geq 145$, which corresponds to

$$0 > \left.\frac{\dot{G}_{\rm eff}}{G_{\rm N}}\right|_0 \geq -1.458 \times 10^{-14}\,{\rm yr}^{-1},$$

where $\dot{G}_{\rm eff}/G_{\rm N}|_0$ is the present value. Hence the constraint obtained from the big-bang nucleosynthesis ($\lambda \geq 208$) gives the sufficient condition.

The constraint (6.22) on $\ddot{G}/G$ is always satisfied for any values of λ as we expected.

2. *The potential for* ΛCDM *background universe*

If the potential is given by one discussed in Sec. V A, we recover ΛCDM model for the background dynamics. However the effective gravitational constant is no longer constant. It depends on time as

$$\frac{G_{\rm eff}}{G_{\rm N}} = 1 - \frac{2(Z-1)H\epsilon_{\dot{\phi}}}{8HZ\epsilon_{\dot{\phi}} - \frac{\alpha_2}{\alpha_3}M_{\rm PL}}.$$

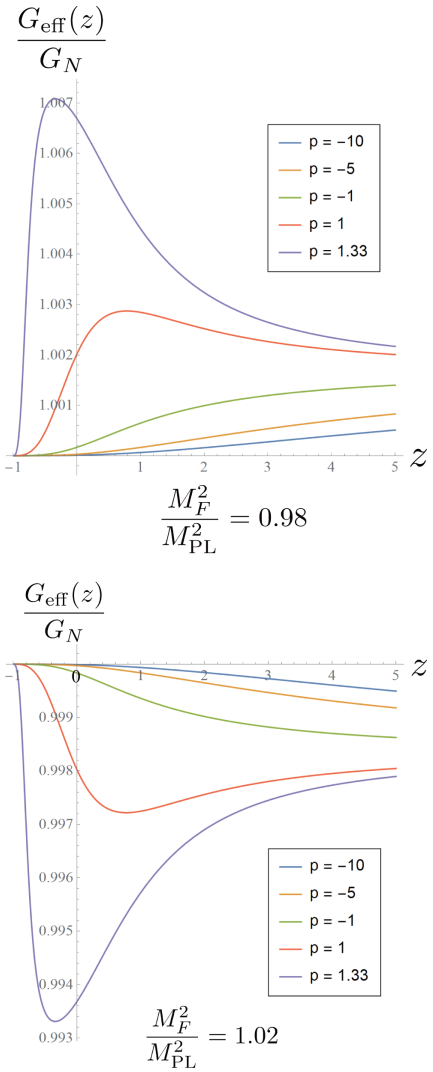

FIG. 10. Evolutions of $G_{\rm eff}$ for the cases of $p = 1.33, 1-1, -5$ and -10 in terms of the redshift z. The top and bottom figures correspond to $\frac{M_{\rm F}^2}{M_{\rm PL}^2} = 0.98$ and $\frac{M_{\rm F}^2}{M_{\rm PL}^2} = 1.02$, respectively.

We show the evolution of $G_{\rm eff}$ in terms of the redshift z in Fig. 10. We consider only the cases of $p \leq 1.33$ because $G_{\rm eff}$ will diverges at some value of z when $p \gtrsim 1.34$.

We can also discuss the time evolution of $\dot{G}_{\rm eff}$, which plots are given in Fig. 11.

These figures show that when we decrease the value of p, the present value of $\dot{G}_{\rm eff}$ becomes smaller. From the observational constraint (6.21), we find

$$p \leq \begin{cases} -2.4 & \text{for } M_{\rm F}^2/M_{\rm PL}^2 = 0.98 \\ -6.0 & \text{for } M_{\rm F}^2/M_{\rm PL}^2 = 1.02. \end{cases} \tag{6.23}$$

The constraint (6.22) is automatically satisfied for any values of p.

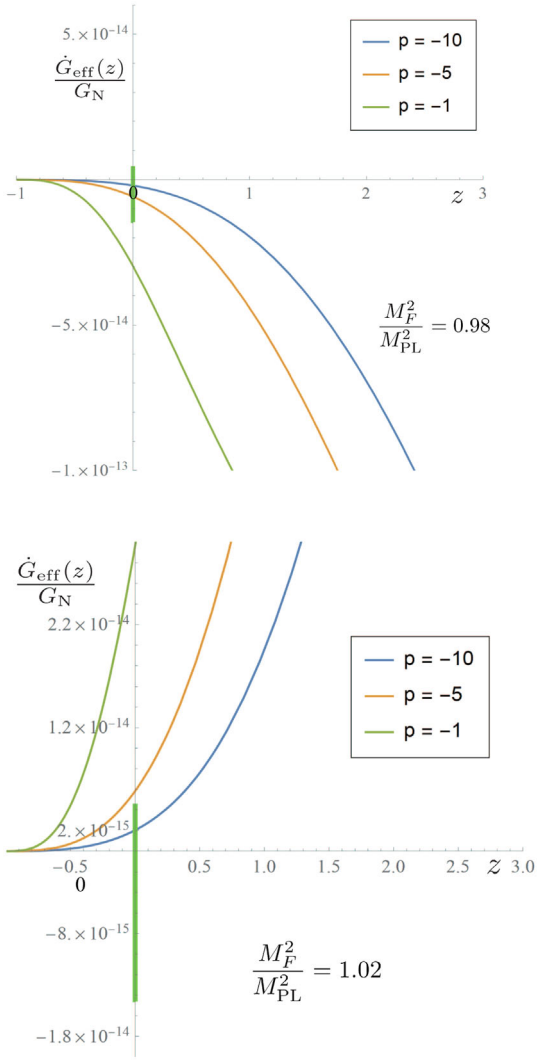

FIG. 11. Evolutions of $\dot{G}_{\rm eff}$ for the cases of $p = -1, -5$ and -10 in terms of the redshift z. The constraint from lunar-laser ranging experiment, Eq. (6.21), is given by the green line segment at $z = 0$. The top and bottom figures correspond to $\frac{M_{\rm F}^2}{M_{\rm PL}^2} = 0.98$ and $\frac{M_{\rm F}^2}{M_{\rm PL}^2} = 1.02$, respectively.

The above constraints (6.23) correspond to

$$\frac{\alpha_2}{\alpha_3} > 8.4\sqrt{\frac{\rho_{\rm vac}}{M_{\rm PL}^4}}$$

for $M_{\rm F}^2/M_{\rm PL}^2 = 0.98$, and

$$\frac{\alpha_2}{\alpha_3} > 20.6\sqrt{\frac{\rho_{\rm vac}}{M_{\rm PL}^4}}$$

for $M_{\rm F}^2/M_{\rm PL}^2 = 1.02$.

VII. DISCUSSION AND REMARKS

We discuss a Cuscuta-Galileon gravity theory, which is one simple extension of a cuscuton gravity theory and still preserves two degrees of freedom. We apply it to cosmological model and present the effective Friedmann equation assuming the flat FLRW metric. Although there exists no additional degrees of freedom, introduction of a potential of a scalar field changes the dynamics. The scalar field is completely determined by matter fields.

Giving an exponential potential as an example, we discuss the evolution of the Hubble expansion parameter. Since the gravitational constant $G_{\rm F}$ in the effective Friedmann equation becomes time-dependent. we restrict the parameters in our models with the constraint by the big-bang nucleosynthesis.

We also present how to construct a potential once we know the evolution of the Hubble parameter. As an example, we present the potential form to obtain the ΛCDM cosmology for the background evolution.

We then analyze the density perturbations, which equation is characterized only by a change of the gravitational constant $G_{\rm eff}$. Note that $G_{\rm eff}$ in the above ΛCDM model is also time-dependent. Hence it is not exactly the same as the ΛCDM cosmology in GR. We then restrict the parameters in our models using the observational constraints by the lunar-laser-ranging experiment.

In the case of exponential potential, there appears the time-dependence of the gravitational constant in the effective Friedmann equation, which may give a chance to explain the Hubble tension problem [42–46].

As shown in Fig. 5, the Hubble expansion rate at $z \leq 1$ is about 10% larger than the value of the ΛCDM model. We then plot the present value of the Hubble expansion rate in terms of λ in Fig. 12. For the reference, we also show the

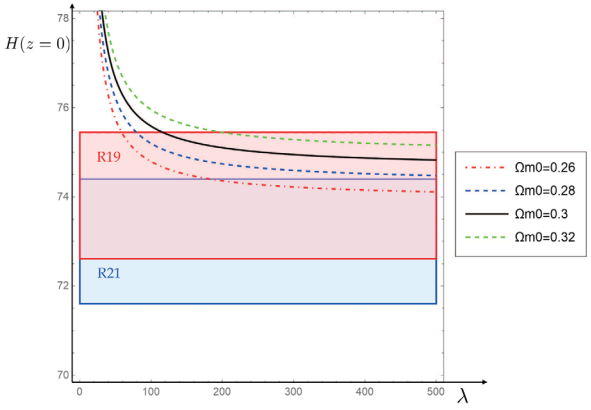

FIG. 12. The present value of the Hubble expansion rate in terms of λ. The dashed green line, black solid line, dashed blue line, and dot-dashed red line correspond to $\Omega_{\rm m,0} = 0.32, 0.3, 0.28$ and 0.26, respectively. Two observational data by R19 [44] and by R21 [45] are given by the red shaded and blue shaded regions, respectively.

observational data R19 of the Hubble expansion rate near $z = 0$, which is obtained from observations of 70 long-period Cepheids in the Large Magellanic Cloud [44]. This figure shows that our model with $\Omega_{\mathrm{m},0} = 0.3$ is consistent with the observational data R19 if $\lambda > 117$, which should be satisfied from the constraint by nucleosynthesis ($\lambda > 208$). If we take the observational data R21, which is determined from observations of 75 Milky Way Cepheids [45], it strays from the allowed range. However the result depends on the density parameter $\Omega_{\mathrm{m},0}$. If $\Omega_{\mathrm{m},0} \lesssim 0.28$, our model with large λ is still consistent with R21 as well as R19.

Our model will be improved when we add a negative vacuum energy ρ_{vac} as well as matter and radiation densities, ρ_m and ρ_r. The effective Friedmann equation is given in Appendix D. Assuming $\Omega_{\mathrm{m},0} = 0.3$, we plot the present value of the Hubble expansion rate in terms of λ in Fig. 13. The case with $\rho_{\mathrm{vac}} = -0.05 V_\infty$ fits well both for R19 and R21, where $V_\infty = \frac{3\alpha_2^2}{(\lambda-6)^2\alpha_3^2} M_{\mathrm{PL}}^4$. Such a small negative vacuum energy might be obtained in the context of string theory [51].

When we take the limit of $\lambda \to \infty$ and $\alpha_3 \to 0$ with keeping V_∞ finite, we obtain the same results as those in the original cuscuton theory with an exponential potential, which Friedmann equation is given by Eq. (B7). Since our model could be successful to explain the history of our universe when λ is large, the original cuscuton theory with an exponential potential may also have the possibility to solve the Hubble tension problem. In fact, the present Hubble constant becomes $H_0 = 74.65$ km/s/Mpc when we normalize the Hubble parameter at $z = 1100$ by use of the CMB data based on the ΛCDM universe model. This is quite close to the value in our model with large λ. One difference is that two "gravitational constants," G_{F} and G_{eff}, are exactly the same as G_{N} in the original cuscuton theory.

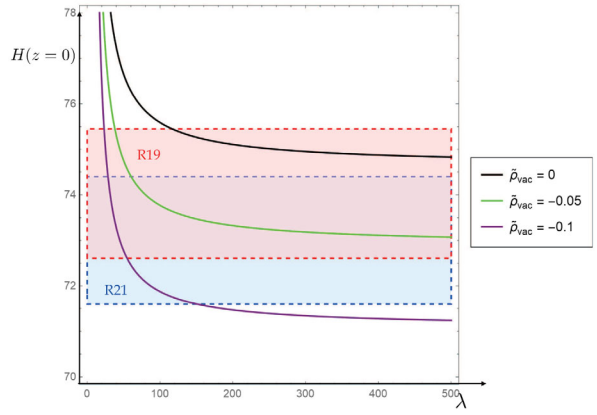

FIG. 13. The present value of the Hubble expansion rate in terms of λ when we add a vacuum energy ρ_{vac}. We assume $\Omega_{\mathrm{m},0} = 0.3$. The black line, green line, and purple line correspond to $\tilde{\rho}_{\mathrm{vac}} = 0, -0.05$ and -0.1, respectively, where $\tilde{\rho}_{\mathrm{vac}} = \rho_{\mathrm{vac}}/V_\infty$. Two observational data by R19 [44] and by R21 [45] are given by the red shaded and blue shaded regions, respectively.

In the case of the potential for the ΛCDM universe discussed in Sec. V A, we also find the cosmological model in the cuscuton theory as the limiting case of our Cuscuta-Galileon theory. In fact, if we take the limit of $p \to -\infty$ as well as $\alpha_3 \to 0$ keeping $p\alpha_3$ finite, the constructed potential in Sec. V A becomes a quadratic function of the scalar field ϕ (see Appendix B 1).

The above two examples suggest that our cosmological model includes that in the original cuscuton theory as the limiting case. The difference is G_{eff}, which is time-dependent in our model, while that in the original cuscuton theory is constant (G_{N}).

Although we may explain the present large Hubble constant by the observation of nearby SNe Ia as well as small value obtained from CMB data assuming ΛCDM model, we may have to analyze our model more carefully from the observational view points. Even if it turns out that the present model with the exponential potential is not consistent with observational data, we still have many possibilities. We may find a better model by tuning the potential as shown in the construction method (Sec. V). We can also extend our Cuscuta-Galileon gravity theory [32,33] because our model is the simplest one. We may obtain a better theory for observations. We shall leave these analyses as future works.

ACKNOWLEDGMENTS

K. M. would like to thank Antonio De Felice, Shinji Mukohyama, and Masroor C. Pookkillath for useful comments and fruitful discussions. K. M. also acknowledges the Yukawa Institute for Theoretical Physics at Kyoto University, where most of the present work was completed during the Visitors Program of FY2021. This work as supported in part by JSPS KAKENHI Grants No. JP17H06359 and No. JP19K03857 and by a Waseda University Grant for Special Research Project (No. 2021C-569).

APPENDIX A: RESCALING OF SCALAR FIELD

In the present Cuscuta-Galileon model defined by the action (2.1), without loss of generality, unless $\alpha_3 = 0$, we can always set $\alpha_3 = 1$ by rescaling the scalar field ϕ as $\tilde{\phi} = \alpha_3\phi$. In fact, defining

$$\tilde{X} \equiv g^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi} = \alpha_3^2 X,$$

we find that the above action S is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\mathrm{PL}}^2 R + \frac{\alpha_2}{\alpha_3}M_{\mathrm{PL}}^2\sqrt{-\tilde{X}} + \ln\left(-\frac{\tilde{X}}{\alpha_3^2\Lambda^4}\right)\Box\tilde{\phi} - V(\tilde{\phi}/\alpha_3) + 3\tilde{X}\right] + S_M(g_{\mu\nu}, \psi_M),$$

Introducing the scaled parameters as

$$\tilde{\alpha}_2 = \frac{\alpha_2}{\alpha_3}, \qquad \tilde{\Lambda}^4 = \alpha_3^2 \Lambda^4,$$

and redefining the potential as

$$\tilde{V}(\tilde{\phi}) = V(\tilde{\phi}/\alpha_3),$$

we find

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm PL}^2 R + \tilde{\alpha}_2 M_{\rm PL}^2\sqrt{-\tilde{X}} + \ln\left(-\frac{\tilde{X}}{\tilde{\Lambda}^4}\right)\Box\tilde{\phi} - \tilde{V}(\tilde{\phi}) + 3\tilde{X}\right] + S_M(g_{\mu\nu}, \psi_M), \tag{A1}$$

which is the action (2.1) with $\alpha_3 = 1$.

APPENDIX B: ORIGINAL CUSCUTON GRAVITY ($\alpha_3 = 0$)

Here we reanalyze cosmological dynamics in the original cuscuton gravity with a potential ($\alpha_3 = 0$). The basic equations are given by

$$H{\rm sgn}(\dot{\phi}) = -\frac{1}{3\alpha_2 M_{\rm PL}^2} V_{,\phi}, \tag{B1}$$

$$H^2 = \frac{1}{3M_{\rm PL}^2}(\rho + V), \tag{B2}$$

where $\rho = \rho_m + \rho_r$.

We then discuss two potentials, the quadratic potential and the exponential potential as analyzed in [30,31].

1. Quadratic potential

We first assume the potential is given by

$$V = V_0 + \frac{1}{2}m^2\phi^2.$$

In this case, since $V_{,\phi} = m^2\phi$, we have a constraint such that

$$\frac{1}{3M_{\rm PL}^2}\left(\rho + V_0 + \frac{1}{m^2}\phi^2\right) = \frac{1}{9\alpha_2^2 M_{\rm PL}^4} m^4\phi^2 (= H^2),$$

which gives

$$\phi^2 = \frac{6\alpha_2^2 M_{\rm PL}^2(\rho + V_0)}{m^2(2m^2 - 3\alpha_2^2 M_{\rm PL}^2)}.$$

Using this relation, we find the Friedmann equation as

$$H^2 = \frac{1}{3M_{\rm F}^2}(\rho + V_0), \tag{B3}$$

where

$$M_{\rm F}^2 \equiv \left(1 - \frac{3\alpha_2^2 M_{\rm PL}^2}{2m^2}\right) M_{\rm PL}^2. \tag{B4}$$

Equation (B3) describes the ΛCDM model with new gravitational constant

$$G_{\rm F} \equiv \frac{G_{\rm N}}{\left(1 - \frac{3\alpha_2^2 M_{\rm PL}^2}{2m^2}\right)} (> G_{\rm N}). \tag{B5}$$

Since the gravitational constant in the Friedmann equation must be close to the Newtonian gravitational constant $G_{\rm N}$, we have a constraint

$$m^2 \gg \frac{3\alpha_2^2}{2} M_{\rm PL}^2.$$

2. Exponential potential

Next we consider the exponential potential

$$V = \epsilon_V M_{\rm PL}^4 \exp(\lambda\phi/M_{\rm PL}).$$

The constraint equation (3.10) with $\alpha_3 = 0$ is

$$\frac{1}{3}[\rho + \epsilon_V M_{\rm PL}^4 \exp(\lambda\phi/M_{\rm PL})] = \frac{1}{9\alpha_2^2 M_{\rm PL}^2} V_{,\phi}^2 = \frac{\lambda^2 M_{\rm PL}^4}{9\alpha_2^2}\exp(2\lambda\phi/M_{\rm PL}).$$

By setting $\chi \equiv \exp(\lambda\phi/M_{\rm PL})$, we find the quadratic equation for χ as

$$\chi^2 - \frac{3\epsilon_V\alpha_2^2}{\lambda^2}\chi - \frac{3\alpha_2^2}{\lambda^2 M_{\rm PL}^4}\rho = 0. \tag{B6}$$

In order to have real positive roots for this equation, we find the condition such that

$$\left(\frac{3\epsilon_V\alpha_2^2}{\lambda^2}\right)^2 + \frac{12\alpha_2^2}{\lambda^2 M_{\rm PL}^4}\rho \geq 0,$$

which is always satisfied because $\rho \geq 0$.

The solution for Eq. (B6) is

$$\chi = \chi_+(\rho) \equiv \frac{3\alpha_2^2}{2\lambda^2}\left(\epsilon_V + \sqrt{1 + \frac{4\lambda^2}{3\alpha_2^2 M_{\rm PL}^4}\rho}\right).$$

Only a + branch of solutions is possible because χ should be positive. Note that $\epsilon_V = \pm 1$.

We then find the scalar field ϕ in terms of ρ as

$$\phi = \phi_+ \equiv \frac{M_{\rm PL}}{\lambda}\ln\left[\frac{3\alpha_2^2}{2\lambda^2}\left(\epsilon_V + \sqrt{1+\frac{4\lambda^2}{3\alpha_2^2 M_{\rm PL}^4}\rho}\right)\right].$$

As a result, the Friedmann equation (B2) is given by

$$H_+^2 = \frac{1}{3M_{\rm PL}^2}\left[\rho + \frac{3\alpha_2^2 M_{\rm PL}^4}{2\lambda^2}\left(1+\epsilon_V\sqrt{1+\frac{4\lambda^2}{3\alpha_2^2 M_{\rm PL}^4}\rho}\right)\right]. \tag{B7}$$

In the early stage ($\rho \to \infty$), the universe starts from the radiation dominant stage and follows by the matter dominant stage both for $\epsilon_V = \pm 1$.

For the late stage, we discuss the cosmic evolution for two cases ($\epsilon_V = \pm 1$) separately.

a. $\epsilon_V = +1$ (positive potential)

In the limit of $\rho \to 0$, we obtain

$$3M_{\rm PL}^2 H_+^2 = \frac{3\alpha_2^2}{\lambda^2}M_{\rm PL}^4 \equiv \rho_{\rm DE}(>0),$$

which gives de Sitter expansion with the Hubble expansion rate $H_{\rm DE} = |\alpha_2|M_{\rm PL}/|\lambda|$. For the present acceleration, we have to impose the condition such that

$$\frac{|\alpha_2|}{|\lambda|} \sim O(10^{-60}) \ll 1. \tag{B8}$$

b. $\epsilon_V = -1$ (negative potential)

In this case, in the limit of $\rho \to 0$, we find the Friedmann equation as

$$3M_{\rm PL}^2 H_+^2 = \frac{\lambda^2\rho^2}{3\alpha_2^2 M_{\rm PL}^4}, \quad \text{and} \quad \rho \sim \rho_m,$$

which gives

$$a(t) \propto t^{\frac{1}{3}}.$$

This is the expansion law for the stiff matter ($P = \rho$) in GR.

Consequently, only the case of $\epsilon_V = +1$ (positive exponential potential) provides the big-bang universe followed by an accelerating expansion.

3. Construction of appropriate potential

We may construct an appropriate potential once we know the expansion of the universe from observation. Here we provide how to construct the potential giving the Hubble expansion parameter H in terms of the redshift z.

From basic equations we find

$$V_{,\phi}^2 = 9\alpha_2 M_{\rm PL}^4 H^2, \tag{B9}$$

$$V = 3M_{\rm PL}^2 H^2 - \rho. \tag{B10}$$

We rewrite Eq. (B9) in terms of z as

$$\left(\frac{d\phi}{dz}\right)^2 = \left(\frac{dV/dz}{V_{,\phi}}\right)^2 = \frac{1}{9\alpha_2^2 M_{\rm PL}^4 H^2}\left(\frac{dV}{dz}\right)^2.$$

From Eq. (B10), we obtain

$$\frac{dV}{dz} = 6M_{\rm PL}^2 H\frac{dH}{dz} - \frac{d\rho}{dz},$$

then

$$\frac{d\phi}{dz} = \pm\frac{1}{3|\alpha_2|M_{\rm PL}^2 H(z)}\frac{dV}{dz} = \pm\frac{1}{|\alpha_2|}\left(2\frac{dH}{dz} - \frac{1}{3M_{\rm PL}^2 H}\frac{d\rho}{dz}\right).$$

Integrating this equation, we find $\phi = \phi(z)$. Solving $z = z(\phi)$ as the inverse problem, and inserting it into Eq. (B10), we find the potential $V(\phi)$.

In order to show it more explicitly, in what follows, we assume $\rho = \rho_m$. Since

$$\frac{d\rho_m}{dz} = \frac{3}{1+z}\rho_m,$$

we find

$$\frac{d\phi}{dz} = \pm\frac{1}{|\alpha_2|}\left(2\frac{dH}{dz} - \frac{\rho_m}{M_{\rm PL}^2(1+z)H}\right).$$

Using $\rho_m = 3\Omega_{m,0}M_{\rm PL}^2 H_0^2(1+z)^3$, we obtain

$$\phi = \phi_0 \pm \frac{1}{|\alpha_2|}\left[2(H(z)-H_0) - 3\Omega_{m,0}H_0^2\int_0^z dz\frac{(1+z)^2}{H(z)}\right].$$

Once we know $H(z)$, we can integrate this equation, which gives the relation between ϕ and z. Solving the inverse problem, we find the appropriate potential $V(\phi)$.

We can easily check it by assuming ΛCDM model

$$H^2 = \frac{1}{3M_{\rm F}^2}(\rho_m + V_0).$$

Since $\rho_m = 3\Omega_{m,0}M_{\rm F}^2 H_0^2(1+z)^3$,

$$H^2 = \frac{V_0}{3M_{\rm F}^2}\left(1 + \frac{3\Omega_{m,0}M_{\rm F}^2 H_0^2}{V_0}(1+z)^3\right).$$

We then find the solution as

$$\phi(z) = \phi_0 \pm \frac{2}{|\alpha_2|}\left(1 - \frac{M_{\rm F}^2}{M_{\rm PL}^2}\right)(H(z) - H_0).$$

The potential is then given as

$$\begin{aligned} V &= 3M_{\rm PL}^2H^2 - \rho_m = 3M_{\rm PL}^2H^2 - (3M_{\rm F}^2H^2 - V_0) \\ &= 3(M_{\rm PL}^2 - M_{\rm F}^2)H^2 + V_0 \\ &= 3(M_{\rm PL}^2 - M_{\rm F}^2)\left[H_0 \pm \frac{|\alpha_2|}{2(1-\frac{M_{\rm F}^2}{M_{\rm PL}^2})}(\phi - \phi_0)\right]^2 + V_0 \\ &= \frac{3\alpha_2^2 M_{\rm PL}^4}{4(M_{\rm PL}^2 - M_{\rm F}^2)}(\phi - \phi_*)^2 + V_0, \end{aligned} \tag{B11}$$

where

$$\phi_* \equiv \phi_0 \mp \frac{2H_0(1-\frac{M_{\rm F}^2}{M_{\rm PL}^2})}{|\alpha_2|}.$$

This is just a quadratic potential of ϕ with

$$m^2 \equiv \frac{3\alpha_2^2 M_{\rm PL}^4}{2(M_{\rm PL}^2 - M_{\rm F}^2)},$$

which is consistent with Eq. (B4).

APPENDIX C: EXPONENTIAL POTENTIAL WITH $\lambda \le 6$

In Sec. IV B 2, we give only the summary of the cosmic evolution for the exponential potential (4.1) with $\lambda \le 6$. In this Appendix, we shall give the details of calculation. The cosmic evolution can be easily understood by analyzing the behaviors of the functions D, $S_\pm$, $R_\pm$ and $F_\pm$ in the effective Friedmann equation (4.6).

1. $0 < \lambda < 3$

In this case, we find $S_+ = 0$ at $a = a_{\rm cr}$ for $+$ branch, while $S_- < 0$ for $-$ branch. As a result we find the following cosmic evolution: For $+$ branch, since $S_+ < 0$ for $a < a_{\rm cr}$ while $S_+ > 0$ for $a > a_{\rm cr}$, we find for the negative potential ($\epsilon_V = -1$),

$$a_{-+}(t) \propto \begin{cases} t^{\frac{1}{2}} & \to & t^{\frac{2}{3}} & \text{in the early stage} \\ (\text{RD}) & & (\text{MD}) & \\ & a_{\rm cr} & & \text{as} \quad t \to \infty \end{cases},$$

and for the positive potential ($\epsilon_V = 1$),

$$a_{++}(t) \propto \begin{cases} a_{\rm cr} & \text{as } t \to -\infty \\ \exp[H_\infty t] & \text{as } t \to \infty \end{cases}.$$

For $-$ branch, the potential must be negative ($\epsilon_V = -1$). We then find

$$a_{--}(t) \propto \begin{cases} t^{\frac{1}{2}} & \to & t^{\frac{2}{3}} & \text{in the early stage} \\ (\text{RD}) & & (\text{MD}) & \\ & t^{\frac{1}{3}} & & \text{as} \quad t \to \infty \end{cases}.$$

Here we have used the notation for the scale factor such that $a_{\epsilon_V,\text{branch}}$.

2. $\lambda < 0$

In this case, for $+$ branch, we find two vanishing points such that $F_+ = 0$ at $a = a_{\rm cr}^{(F)}$ and $S_+ = 0$ at $a = a_{\rm cr}^{(S)}$, where $a_{\rm cr}^{(F)} > a_{\rm cr}^{(S)}$. When F_+ vanishes, we find the Friedmann equation near $a_{\rm cr}^{(F)}$ as

$$H^2 \propto (a - a_{\rm cr}^{(F)})^{-2},$$

which gives

$$a(t) - a_{\rm cr}^{(F)} \propto (t - t_{\rm cr}^{(F)})^{1/2},$$

where $t_{\rm cr}^{(F)}$ is a positive constant. We find a singularity at $t_{\rm cr}^{(F)}$ although the scale factor $a_{\rm cr}^{(F)}$ is finite.

As a result we find three histories of the universe ($a_{-+}(t)$, $a_{++}^{(1)}(t)$, and $a_{++}^{(2)}(t)$) as

$$a_{-+}(t) \propto \begin{cases} t^{\frac{1}{2}} & (\text{or } t^{\frac{1}{2}} \to t^{\frac{2}{3}}) & \text{as} \quad t \to 0 \\ (\text{RD}) & (\text{or RD} \to \text{MD}) & \\ a_{\rm cr}^{(S)} & & \text{as} \quad t \to \infty \end{cases},$$

$$a_{++}^{(1)}(t) \propto \begin{cases} a_{\rm cr}^{(S)} & \text{as } t \to -\infty \\ a_{\rm cr}^{(F)} & \text{as } t \to t_{\rm cr}^{(F)} \end{cases},$$

and

$$a_{++}^{(2)}(t) \propto \begin{cases} a_{\rm cr}^{(F)} & \text{as } t \to t_{\rm cr}^{(F)} \\ \exp[H_\infty t] & \text{as } t \to \infty \end{cases}.$$

For $-$ branch, no terms vanish nor become negative, and $S_< 0$. As a result, for the negative potential ($\epsilon_V = -1$), we find

$$a_{--}(t) \propto \begin{cases} t^{\frac{1}{2}} & \to & t^{\frac{2}{3}} & \text{in the early stage} \\ (\text{RD}) & & (\text{MD}) & \\ & t^{\frac{1}{3}} & & \text{as} \quad t \to \infty \end{cases}.$$

3. $3 < \lambda < 4$

From the condition of $D \geq 0$, we find the lower bound on the scale factor as

$$a \geq a_{\rm min}.$$

For $+$ branch, we also find $a_{\rm cr}(> a_{\rm min})$ from $S_+ = 0$, while for $-$ branch, no additional vanishing point appears.

Near $a = a_{\rm min}$, we find the Friedmann equation as

$$H^2 \propto (a - a_{\rm min}),$$

which gives

$$a(t) - a_{\rm min} \propto (t - t_{\rm min})^2.$$

We then find the following cosmic evolution: For $+$ branch, we have two histories ($a_{-+}(t)$ and $a_{++}(t)$) as

$$a_{-+}(t) \propto \begin{cases} a_{\rm min} & \text{as } t \to t_{\rm min} \\ a_{\rm cr} & \text{as } t \to \infty \end{cases},$$
$$a_{++}(t) \propto \begin{cases} a_{\rm cr} & \text{as } t \to -\infty \\ \exp[H_\infty t] & \text{as } t \to \infty \end{cases},$$

while for $-$ branch,

$$a_{--}(t) \propto \begin{cases} a_{\rm min} & \text{as } t \to t_{\rm min} \\ t^{\frac{1}{3}} & \text{as } t \to \infty \end{cases}.$$

4. $4 < \lambda < 6$

In this case, we also find the lower bound on the scale factor from the condition of $D \geq 0$ as

$$a \geq a_{\rm min}.$$

For $-$ branch, we also find $a_{\rm cr}(> a_{\rm min})$ if $\lambda \geq 5$ from $S_- = 0$, while the vanishing point $S_+ = 0$ becomes larger than $a_{\rm min}$, which means S_- is always negative for $a \leq a_{\rm min}$ if $\lambda \leq \frac{9}{2}$. In the case of $\frac{9}{2} < \lambda < 5$, the behavior depends on the parameters. For $+$ branch, no additional vanishing point appears.

We then find the following cosmic evolution: For $+$ branch,

$$a_{++}(t) \propto \begin{cases} a_{\rm min} & \text{as } t \to t_{\rm min} \\ \exp[H_\infty t] & \text{as } t \to \infty \end{cases}.$$

While for $-$ branch, we have two histories ($a_{+-}(t)$ and $a_{--}(t)$) as

$$a_{+-}(t) \propto \begin{cases} a_{\rm min} & \text{as } t \to t_{\rm min} \\ a_{\rm cr} & \text{as } t \to \infty \end{cases},$$
$$a_{--}(t) \propto \begin{cases} a_{\rm cr} & \text{as } t \to -\infty \\ t^{\frac{1}{3}} & \text{as } t \to \infty \end{cases}.$$

In this case, however, we have a constraint such that $2\rho_m + \rho_r \leq V_\infty$ from $D \geq 0$. If V_∞ is the present vacuum energy, this constraint cannot explain the big bang universe.

5. Exponential potential with $\lambda = 6$

In this case, Eq. (4.2) is a linear equation for V. Since

$$V \equiv \epsilon_V M_{\rm PL}^4 e^{6\alpha_3 M_{\rm PL}^{-1}\phi},$$

we obtain the scalar field ϕ as

$$\phi = \frac{M_{\rm PL}}{6\alpha_3} \ln\left[\frac{-(\rho_m + \rho_r) + \frac{3}{a_2^2}(\rho_m + \frac{2}{3}\rho_r)^2}{\epsilon_V M_{\rm PL}^4}\right], \tag{C1}$$

which gives

$$\begin{aligned} Z &= 1 + \frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN} \\ &= 1 - \frac{\alpha_3}{M_{\rm PL}}\left(3\rho_m \frac{d\phi}{d\rho_m} + 4\rho_r \frac{d\phi}{d\rho_r}\right) \\ &= \frac{(\rho_m + \frac{2}{3}\rho_r)(3 + \frac{4}{a_2^2}\rho_r)}{6\left[(\rho_m + \rho_r) - \frac{3}{a_2^2}(\rho_m + \frac{2}{3}\rho_r)^2\right]}. \end{aligned}$$

Here we define

$$a_2 \equiv \frac{\alpha_2}{\alpha_3} M_{\rm PL}^2.$$

We then find the Friedmann equation as

$$\begin{aligned} M_{\rm PL}^2 H^2 &= Z^{-2}\frac{(\rho_m + \rho_r + V(\phi))}{3} \\ &= \frac{4\left[(\rho_m + \rho_r) - \frac{3}{a_2^2}(\rho_m + \frac{2}{3}\rho_r)^2\right]^2}{a_2^2\left[1 + \frac{4}{3a_2^2}\rho_r\right]^2}. \end{aligned} \tag{C2}$$

If $\frac{\alpha_2}{\alpha_3} \gtrsim O(1)$, $\rho_m, \rho_r \ll a_2^2$ because $\rho_m, \rho_r \ll M_{\rm PL}^4$. In this case ϵ_V must be -1 from Eq. (C1), and we find

$$M_{\rm PL}^2 H^2 = \frac{4(\rho_m + \rho_r)^2}{a_2^2}, \tag{C3}$$

which gives

$$a(t) \sim \begin{cases} t^{\frac{1}{4}} & \text{RD} \\ t^{\frac{1}{3}} & \text{MD} \end{cases}.$$

The former expansion law is obtained by the equation of state $P = \frac{5}{3}\rho$ in GR, which is quite strange matter, while the latter one corresponds to the equation of state of stiff matter.

On the other hand, if $\frac{\alpha_2}{\alpha_3} \ll 1$ such that $\rho_m, \rho_r \gg a_2^2$, we find

$$M_{\rm PL}^2 H^2 \approx \frac{81}{4a_2^2}\frac{(\rho_m + \frac{2}{3}\rho_r)^4}{\rho_r^2},$$

which gives

$$a(t) \sim \begin{cases} t^{\frac{1}{4}} & \text{radiation dominant} \\ t^{\frac{1}{2}} & \text{matter dominant} \end{cases}.$$

The exists an intermediate parameter region such that $\rho_m, \rho_r \sim a_2^2 \ll M_{\rm PL}^4$. In this case, the Hubble expansion rate H vanishes at some scale factor $a_{\rm cr}$ where $a_{\rm cr}$ is given by

$$\rho_m(a_{\rm cr}) + \rho_r(a_{\rm cr}) = \frac{3}{a_2^2}\left(\rho_m(a_{\rm cr}) + \frac{2}{3}\rho_r(a_{\rm cr})\right)^2.$$

In this case, the universe expands as follows: If $\epsilon_V = -1$, we find $a \geq a_{\rm cr}$, and

$$a(t) \sim \begin{cases} a_{\rm cr} & t \to -\infty \\ t^{\frac{1}{3}} & t \to \infty \end{cases},$$

while when $\epsilon_V = 1$, we find $a \leq a_{\rm cr}$ and

$$a(t) \sim \begin{cases} t^{\frac{1}{4}} & t \to 0 \\ a_{\rm cr} & t \to \infty \end{cases}.$$

APPENDIX D: A NEGATIVE VACUUM ENERGY

As one of matter fluid in Eq. (3.4), we may add a vacuum energy $\rho_{\rm vac}$. Here we shall discuss such a case. The effective Friedmann equation, when $\lambda \neq 6$, is now:

$$H^2 = \frac{1}{3M_{\rm PL}^2}\frac{V_\infty D(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda) S_\pm^2(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda) R_\pm(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda)}{2F_\pm^2(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda)}, \tag{D1}$$

where

$$D(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda) \equiv 1 + \frac{4}{\lambda - 6}[(\lambda - 3)\tilde\rho_m + (\lambda - 4)\tilde\rho_r + \lambda\tilde\rho_{\rm vac}], \tag{D2}$$

$$S_\pm(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda) \equiv 1 + \frac{2}{\lambda - 6}(3\tilde\rho_m + 2\tilde\rho_r + 6\tilde\rho_{\rm vac}) \pm \sqrt{D}, \tag{D3}$$

$$R_\pm(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda) \equiv 1 + \frac{2}{\lambda - 6}[(\lambda - 3)\tilde\rho_m + (\lambda - 4)\tilde\rho_r + \lambda\tilde\rho_{\rm vac}] \pm \sqrt{D}, \tag{D4}$$

$$F_\pm(\tilde\rho_m, \tilde\rho_r, \tilde\rho_{\rm vac}; \lambda) \equiv \left\{1 + \frac{2}{\lambda(\lambda - 6)}[3(\lambda - 3)\tilde\rho_m + 2(\lambda - 4)\tilde\rho_r + 6\lambda\tilde\rho_{\rm vac}]\right\}\sqrt{D} \pm \left\{1 + \frac{2}{\lambda(\lambda - 6)}[(\lambda - 3)(2\lambda - 3)\tilde\rho_m + 2(\lambda - 2)(\lambda - 4)\tilde\rho_r + 2\lambda^2\tilde\rho_{\rm vac}]\right\}. \tag{D5}$$

Here we define

$$V_\infty \equiv \frac{3\alpha_2^2}{(\lambda - 6)^2\alpha_3^2}M_{\rm PL}^4,$$

and introduce the variables normalized by V_∞ as

$$\tilde\rho_m = \frac{\rho_m}{V_\infty}, \qquad \tilde\rho_r = \frac{\rho_r}{V_\infty}, \qquad \tilde\rho_{\rm vac} = \frac{\rho_{\rm vac}}{V_\infty}.$$

In order to find an accelerating universe in the limit of $\rho_m, \rho_r \to 0$, we find

$$\begin{aligned} \tilde\rho_{\rm vac} &> -\frac{\lambda - 6}{4\lambda} \quad \text{for } \lambda > 6 \quad \text{or} \quad \lambda < 0 \\ &< -\frac{\lambda - 6}{4\lambda} \quad \text{for } 0 < \lambda < 6. \end{aligned}$$

The observed dark energy density is given by

$$\rho_{\rm DE} \equiv 3M_{\rm PL}^2 H_\infty^2 = \frac{V_\infty}{2}\left[1 + \frac{2\lambda}{\lambda - 6}\tilde\rho_{\rm vac} + \sqrt{1 + \frac{4\lambda}{\lambda - 6}\tilde\rho_{\rm vac}}\right],$$

where

$$H_\infty \equiv H(a \to \infty).$$

As discussed in the text, $\lambda > 6$ may provide a consistent cosmological history, that is, starting from radiation era, the universe evolves into matter dominant stage, and eventually transits to dark energy dominant phase. In that case, we find

$$\frac{1}{4}V_\infty \le \rho_{\rm DE} \le V_\infty,$$

for

$$-\frac{\lambda-6}{4\lambda}V_\infty \le \rho_{\rm vac} \le 0.$$

A negative vacuum energy reduces dark energy density maximally to one quarter of the case without a negative vacuum energy. Such a small negative cosmological constant might be obtained in the context of string theory [51].

APPENDIX E: PECULIARITY OF VACUUM CASE

If we consider there exist no matter fluid, we find some peculiarity. In the case of the vacuum state, we have the constraint

$$\frac{1}{3}V(\phi) = \frac{\alpha_3^2}{\alpha_2^2 M_{\rm PL}^4}\left[2V(\phi) - \frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}\right]^2. \tag{E1}$$

Once we specify the potential form, this constraint fixes the value of the scalar field $\phi = \phi_{\rm vac} =$ constant. Since the scalar field must be time-dependent such that $X > 0$, such a solution is not allowed. There is no vacuum solution in the cuscuton gravity theory.[1]

However there is one exceptional case, i.e., if the potential V satisfies the constraint (E1) for any value of ϕ, it does not fix the value of ϕ. Instead we find a very peculiar behavior of the cosmic evolution or dynamics of the scalar field as shown below.

1. Ordinary cuscuton theory ($\alpha_3 = 0$)

In this case, the constraint (E1) is now

$$V = \frac{1}{3\alpha_2^2 M_{\rm PL}^2}(V_{,\phi})^2,$$

which gives

$$\frac{dV}{d\phi} = \pm\sqrt{3}|\alpha_2|M_{\rm PL}V^{1/2}.$$

Solving this differential equation, we find the potential form as

$$V = \frac{3}{4}\alpha_2^2 M_{\rm PL}^2(\phi-\phi_0)^2. \tag{E2}$$

This looks very similar to the potential for ΛCDM model given by Eq. (B11). But in this case, $M_{\rm F} = 0$ and $V_0 = 0$.

The evolution of the scalar field is given by

$$\phi = \phi_0 \pm \frac{2}{|\alpha_2|}(H - H_0), \tag{E3}$$

and the Friedman equation is

$$H^2 = \frac{1}{3M_{\rm PL}^2}V(\phi). \tag{E4}$$

Since these two equations are not independent when the potential is given by Eq. (E2), we cannot fix the scalar field ϕ or the Hubble parameter H. When H is given by some function of the e-folding number N, the scalar field evolves as Eq. (E3), while if we assume the evolution of ϕ, we find the cosmic evolution H by Eq. (E4). The theory cannot determine the evolution of the universe.

What is the origin of this ambiguity or freedom? It may be related to a choice of the time slicing. When we have matter fluid in the FLRW spacetime, we have a natural choice of time coordinate, by which the energy density becomes homogeneous. However, if we do not have such a reference object, we may have a freedom to choose time coordinate, which corresponds to the above ambiguity.

2. Cuscuta-Galileon theory ($\alpha_3 \neq 0$)

We also find the similar problem for the Cuscuta-Galileon theory. If the constraint (E1) is satisfied for any value of ϕ, it gives the differential equation for $V(\phi)$ in terms of ϕ, i.e.,

$$\frac{dV}{d\phi} = 6\alpha_3 M_{\rm PL}^{-1}V \pm \sqrt{3}\alpha_2 M_{\rm PL}V^{1/2}. \tag{E5}$$

This can be easily integrated as

$$V(\phi) = V_0\left[1 - \frac{C}{\sqrt{3}\alpha_2}\exp\left(\frac{3\alpha_3}{M_{\rm PL}}\phi\right)\right]^2,$$

where

$$V_0 \equiv \frac{\alpha_2^2 M_{\rm PL}^4}{12\alpha_3^2}, \tag{E6}$$

[1]It is not the case if the 3-space has a curvature. In fact, we find de Sitter solution or Minkowski spacetime for the open or closed FLRW metric ansatz. ϕ becomes time-dependent.

and C is a positive integration constant. We shall rewrite the potential as

$$V = V_0\left[1 - \exp\left(\frac{3\alpha_3}{M_{\rm PL}}(\phi - \phi_0)\right)\right]^2. \tag{E7}$$

This is quite similar to the potential appeared in the Starobinsky inflation model [52] or the Higgs inflation model [53–56] after conformal transformation [57–59], although the present scalar field is not dynamical. The potential approaches a positive constant as $\phi \to -\infty$, and vanishes at $\phi = \phi_0$, and then it increases and diverges as $\phi \to \infty$.

In this case, we also find one independent equation for two unknown variables ϕ and H, which is

$$\epsilon_\phi H = \frac{\alpha_2 M_{\rm PL}}{6\alpha_3}\frac{(1 - \exp[\frac{3\alpha_3}{M_{\rm PL}}(\phi - \phi_0)])}{(1 + \frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN})}.$$

For given arbitrary function of $\phi(N)$, we find the evolution of the universe given by this Hubble parameter H, or vice versa.

3. Case with matter field

In the case of the original cuscuton gravity, if the potential V is given by Eq. (E2), we cannot introduce matter fluid. The basic equations force matter density to 0.

On the other hand, for the Cuscuta-Galileon gravity, the situation changes. We can add matter fluid in the Cuscuta-Galileon theory with the potential (E7). We shall discuss its cosmic evolution.

If we assume the potential V is given by Eq. (E7), the constraint (3.10) becomes

$$\frac{\rho}{3} = \frac{(\rho - P)}{a_2^2}\left[(\rho - P) + 2\left(2V - \frac{M_{\rm PL}}{3\alpha_3}V_{,\phi}\right)\right].$$

In this case, there are two branches: One is vacuum ($\rho = P = 0$), and the other gives

$$V = V_0\frac{[\rho - \frac{1}{4V_0}(\rho - P)^2]^2}{(\rho - P)^2}. \tag{E8}$$

Here we use the condition (E5) and the definition (E6), i.e.,

$$2V - \frac{M_{\rm PL}}{3\alpha_3}V_{,\phi} = \pm 2\sqrt{V_0 V},$$

to eliminate $V_{,\phi}$.

Assuming $\rho = \rho_m$, we find

$$\frac{V}{V_0} = \left(1 - \frac{\rho_m}{4V_0}\right)^2.$$

Since the potential V is given by the scalar field ϕ as Eq. (E7), this equation determines the behavior of ϕ in terms of ρ_m, i.e.,

$$\exp\left[\frac{3\alpha_3}{M_{\rm PL}}(\phi - \phi_0)\right] = 1 \mp \left(1 - \frac{\rho_m}{4V_0}\right) = \begin{cases} \frac{\rho_m}{4V_0} \\ 2 - \frac{\rho_m}{4V_0} \end{cases}. \tag{E9}$$

We find two solution for ϕ as

$$\phi = \phi_\mp \equiv \begin{cases} \phi_0 + \frac{M_{\rm PL}}{3\alpha_3}\ln\frac{\rho_m}{4V_0} \\ \phi_0 + \frac{M_{\rm PL}}{3\alpha_3}\ln\left(2 - \frac{\rho_m}{4V_0}\right) \end{cases}. \tag{E10}$$

The Friedmann equation is now

$$\begin{aligned}\bar{H}^2 = H^2Z^2 &= \frac{1}{3M_{\rm PL}^2}(\rho_m + V) \\ &= \frac{1}{3M_{\rm PL}^2}\left[\rho_m + V_0\left(1 - \frac{\rho_m}{4V_0}\right)^2\right] \\ &= \frac{V_0}{3M_{\rm PL}^2}\left(1 + \frac{\rho_m}{4V_0}\right)^2,\end{aligned} \tag{E11}$$

where

$$Z = 1 + \frac{\alpha_3}{M_{\rm PL}}\frac{d\phi}{dN}.$$

Since $\rho_m \propto e^{-3N}$, we find

$$Z(\phi_-) = 1 + \frac{\alpha_3}{M_{\rm PL}}\frac{d}{dN}\left(\frac{M_{\rm PL}}{3\alpha_3}\ln\frac{\rho_m}{4V_0}\right) = 0,$$

which is an irrelevant solution. For ϕ_+, we find

$$\begin{aligned}Z(\phi_+) &= 1 + \frac{\alpha_3}{M_{\rm PL}}\frac{d}{dN}\left(\frac{M_{\rm PL}}{3\alpha_3}\ln\left(2 - \frac{\rho_m}{4V_0}\right)\right) \\ &= \frac{1}{1 - \frac{\rho_m}{8V_0}}.\end{aligned}$$

We obtain the Friedmann equation (E11) as

$$H = H_{\rm vac}\left(1 - \frac{\rho_m}{8V_0}\right)\left(1 + \frac{\rho_m}{4V_0}\right),$$

where

$$H_{\rm vac} \equiv \sqrt{\frac{V_0}{3}}\frac{1}{M_{\rm PL}} = \frac{|\alpha_2|}{6\alpha_3}M_{\rm PL}. \tag{E12}$$

Note that $\rho_m \le 8V_0$, which strongly restricts matter density.

Introducing

$$\eta \equiv \frac{\rho_m}{8V_0},$$

which is proportional to e^{-3N}, we find

$$H = \frac{dN}{dt} = -\frac{1}{3}\frac{d\ln\eta}{dt}.$$

The Friedmann equation is now

$$-\frac{1}{3}\frac{d\ln\eta}{dt} = H_{\rm vac}(1-\eta)(1+2\eta).$$

We can easily integrate this equation as

$$\ln\frac{\eta}{(1-\eta)^{1/3}(1+2\eta)^{2/3}} = -3H_{\rm vac}(t-t_*),$$

or

$$\frac{\eta}{(1-\eta)^{1/3}(1+2\eta)^{2/3}} = \exp\left[-3H_{\rm vac}(t-t_*)\right], \quad \text{(E13)}$$

where t_* is an integration constant. This solution gives the time evolution of matter density as

$$\rho_m = 8V_0\eta(t),$$

and the behavior of the scale factor as

$$a = a_0\left(\frac{\eta}{\eta_0}\right)^{-\frac{1}{3}}.$$

In order to find the explicit form, we have to solve the cubic equation (E13) for η.

We consider some limiting cases as follows:

(1) $\eta \to 0$

This limit corresponds to $\rho_m \to 0$ or $a \to \infty$. We find from Eq. (E13)

$$\frac{1}{\eta} \propto \exp[3H_{\rm vac}t],$$

and

$$a \propto \exp[H_{\rm vac}t].$$

The scalar field approaches some constant as

$$\phi \to \phi_0 + \frac{M_{\rm PL}}{3\alpha_3}\ln 2.$$

The potential value approaches as

$$V \to V_0.$$

We find de Sitter accelerating universe.

(2) $\eta \to 1$

In this limit, which corresponds to $\rho_m \to 8V_0$ and $a \to$ constant, we find

$$\eta \to 1 - \frac{1}{9}\exp[9H_{\rm vac}(t-t_*)]$$

as

$$t \to -\infty.$$

The scalar field behaves as

$$\phi \to -\infty.$$

(3) Whole history We then find the evolution of the universe as follows:

$$a(t) \propto \begin{cases} \text{constant} & \text{as } t \to -\infty \\ \exp(H_{\rm vac}t) & \text{as } t \to \infty \end{cases}, \quad \text{(E14)}$$

$$\phi \propto \begin{cases} -\infty & \text{as } t \to -\infty \\ \phi_0 + \frac{M_{\rm PL}}{3\alpha_3}\ln 2 & \text{as } t \to \infty \end{cases}, \quad \text{(E15)}$$

$$V = \begin{cases} V_0 & \text{as } t \to -\infty \\ V_0 & \text{as } t \to \infty \end{cases}. \quad \text{(E16)}$$

There is no matter/radiation dominant stage. This can be easily understood from the fact that

$$\rho_m \leq 8V_0.$$

[1] A. G. Riess *et al.* (Supernova Search Team), Astron. J. **116**, 1009 (1998).
[2] S. Perlmutter *et al.* (Supernova Cosmology Project), Astrophys. J. **517**, 565 (1999).
[3] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[4] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
[5] E. J. Copeland, A. R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).
[6] L. Heisenberg, J. Cosmol. Astropart. Phys. 05 (2014) 015.

[7] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa, and Y. l. Zhang, J. Cosmol. Astropart. Phys. 06 (2016) 048.
[8] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010).
[9] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011).
[10] C. de Rham, G. Gabadadze, L. Heisenberg, and D. Pirtskhalava, Phys. Rev. D **83**, 103516 (2011).
[11] A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).
[12] C. M. Will, Living Rev. Relativity **17**, 4 (2014).
[13] J. Khoury and A. Weltman, Phys. Rev. Lett. **93**, 171104 (2004).
[14] J. Khoury and A. Weltman, Phys. Rev. D **69**, 044026 (2004).
[15] K. Hinterbichler and J. Khoury, Phys. Rev. Lett. **104**, 231301 (2010).
[16] A. I. Vainshtein, Phys. Lett. **39B**, 393 (1972).
[17] A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D **79**, 064036 (2009).
[18] P. Brax, C. van de Bruck, A. C. Davis, and D. Shaw, Phys. Rev. D **82**, 063519 (2010).
[19] C. Burrage and J. Khoury, Phys. Rev. D **90**, 024001 (2014).
[20] S. Panpanich, S. Ponglertsakul, and K. Maeda, Phys. Rev. D **100**, 044038 (2019).
[21] P. Brax, C. Burrage, and A. C. Davis, J. Cosmol. Astropart. Phys. 01 (2013) 020.
[22] E. Babichev, C. Deffayet, and R. Ziour, Int. J. Mod. Phys. D **18**, 2147 (2009).
[23] E. Babichev and C. Deffayet, Classical Quantum Gravity **30**, 184001 (2013).
[24] C. Lin and S. Mukohyama, J. Cosmol. Astropart. Phys. 10 (2017) 033.
[25] K. Aoki, C. Lin, and S. Mukohyama, Phys. Rev. D **98**, 044022 (2018).
[26] K. Aoki, A. De Felice, C. Lin, S. Mukohyama, and M. Oliosi, J. Cosmol. Astropart. Phys. 01 (2019) 017.
[27] S. Mukohyama and K. Noui, J. Cosmol. Astropart. Phys. 07 (2019) 049.
[28] A. De Felice, A. Doll, and S. Mukohyama, J. Cosmol. Astropart. Phys. 09 (2020) 034.
[29] K. Aoki, A. De Felice, S. Mukohyama, K. Noui, M. Oliosi, and M. C. Pookkillath, Eur. Phys. J. C **80**, 708 (2020).
[30] N. Afshordi, D. J. H. Chung, and G. Geshnizjani, Phys. Rev. D **75**, 083513 (2007).
[31] N. Afshordi, D. J. H. Chung, M. Doran, and G. Geshnizjani, Phys. Rev. D **75**, 123509 (2007).
[32] A. Iyonaga, K. Takahashi, and T. Kobayashi, J. Cosmol. Astropart. Phys. 12 (2018) 002.
[33] A. Iyonaga, K. Takahashi, and T. Kobayashi, J. Cosmol. Astropart. Phys. 07 (2020) 004.
[34] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, Phys. Rev. Lett. **114**, 211101 (2015).
[35] C. de Rham and H. Motohashi, Phys. Rev. D **95**, 064008 (2017).
[36] S. Panpanich and K. Maeda, arXiv:2109.12288.
[37] S. Tsujikawa, Lect. Notes Phys. **892**, 97 (2015).
[38] R. Kase and S. Tsujikawa, Phys. Rev. D **90**, 044073 (2014).
[39] R. Kase and S. Tsujikawa, Int. J. Mod. Phys. D **23**, 1443008 (2014).
[40] C. Lin, S. Mukohyama, R. Namba, and R. Saitou, J. Cosmol. Astropart. Phys. 10 (2014) 071.
[41] J. Alvey, N. Sabti, M. Escudero, and M. Fairbairn, Eur. Phys. J. C **80**, 148 (2020).
[42] P. A. R. Ade *et al.* (Planck Collaboration), Astron. Astrophys. **571**, A16 (2014).
[43] N. Aghanim *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A6 (2020); **652**, C4 (2021).
[44] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri, and D. Scolnic, Astrophys. J. **876**, 85 (2019).
[45] A. G. Riess, S. Casertano, W. Yuan, J. B. Bowers, L. Macri, J. C. Zinn, and D. Scolnic, Astrophys. J. Lett. **908**, L6 (2021).
[46] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D. F. Mota, A. G. Riess, and J. Silk, Classical Quantum Gravity **38**, 153001 (2021).
[47] A. De Felice, R. Kase, and S. Tsujikawa, Phys. Rev. D **83**, 043515 (2011).
[48] A. De Felice, T. Kobayashi, and S. Tsujikawa, Phys. Lett. B **706**, 123 (2011).
[49] F. Hofmann and J. Müller, Classical Quantum Gravity **35**, 035015 (2018).
[50] L. Biskupek, J. Müller, and J. M. Torre, Universe **7**, 34 (2021).
[51] M. Demirtas, M. Kim, L. McAllister, J. Moritz, and A. Rios-Tascon, Phys. Rev. Lett. **128**, 011602 (2022).
[52] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
[53] F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B **659**, 703 (2008).
[54] F. L. Bezrukov, A. Magnin, and M. Shaposhnikov, Phys. Lett. B **675**, 88 (2009).
[55] F. Bezrukov and M. Shaposhnikov, J. High Energy Phys. 07 (2009) 089.
[56] A. De Simone, M. P. Hertzberg, and F. Wilczek, Phys. Lett. B **678**, 1 (2009).
[57] K. Maeda, Phys. Rev. D **37**, 858 (1988).
[58] T. Futamase and K. Maeda, Phys. Rev. D **39**, 399 (1989).
[59] K. Maeda, J. A. Stein-Schabes, and T. Futamase, Phys. Rev. D **39**, 2848 (1989).