# Classification of static black holes in Einstein phantom-dilaton Maxwell–anti-Maxwell gravity systems

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The uniqueness theorem for static, spherically symmetric, asymptotically flat, and higher dimensional phantom black holes, with nondegenerate event horizon, being the solutions of Einstein phantom-dilaton Maxwell–anti-Maxwell gravity systems is considered. Conformal positive energy theorem and conformal transformations authorize the crucial tools for exploiting the boundary conditions and conformal flatness.

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## I. INTRODUCTION

Recent astronomical and astrophysical observations provide sustenance for the fact that our Universe consists of a significant amount of nonbaryonic "dark matter" and, the other mysterious ingredient of the Universe mass causing its acceleration, "dark energy." Due to the observations of cosmic microwave background radiation conducted by the *Planck* satellite and the experiments measuring the supernovae type 1A distances, it is revealed that the "dark sector" constitutes, respectively, for dark energy and dark matter, almost 68% and 27% of its total mass [1]. The expansion of the Universe can be mimicked by scalar fields with negative pressure, the so-called "phantom" fields. The model comprises the case of kinetic terms of the fields, with the "wrong sign," in comparison to the ordinary ones, which gives repulsive coupling to gravity. One should emphasize that the phantom fields violating null energy conditions is one of the possibilities explaining the Universe expansion. In general, the violation of the strong energy condition constitutes the enough factor. The future observations have to decide about the true nature of dark energy.

As was revealed in [2–6], the presence of the additional scalar field in gravity systems modified the known black hole solutions in highly nontrivial ways. On the other hand, in Ref. [7] the problem of static spherically symmetric black holes in Einstein-Maxwell-dilaton gravity with a phantom coupling was elaborated. These new classes of black hole solutions, with single or multiple event horizons, have also unusual causal structure.

Further, the generalization of the aforementioned studies was presented [8] and the precise classification of black hole solutions in the theory in question was given. Among all black hole spacetimes, an infinite series of regular event horizons was found. In the studies the geometrically complete black hole solution was also revealed.

In [9] the static multicenter solutions in phantom Einstein-Maxwell gravity were paid attention to and the regular black hole solutions without spatial symmetry for certain discrete values of dilaton coupling were discovered. The three-dimensional gravitating sigma model, being the result of dimensional reduction of phantom Einstein-Maxwell, phantom Kaluza-Klein, and phantom Einstein-Maxwell-dilaton-axion theories, was discussed.

Both analytical and numerical studies revealed the importance of the influence of dark energy-phantom fields in gravitational collapse [10–15]. The various possible scenarios are possible when one takes into account dark sector component coupling to the electrically charged scalar field. For instance, the gravitational collapse subject to the presence of dark energy ensues the emergence of dynamical wormholes and naked singularities [14,15].

As far as the higher-dimensional spacetime is concerned, static spherical solutions of Einstein and Einstein-Maxwell-dilaton equations with massless phantom fields for $n \geq 4$ dimensional manifolds have been considered in [16–17]. It was found that they could be classified in three groups, i.e., the Fisher, the Ellis-Gibbons, and the Ellis-Bronikov one. It happens that they constitute seeds for generating asymptotically anti–de Sitter solutions [18].

The possible justification of the existence of phantom fields can be sought in the string theory, where they arise quite naturally in the studies of the so-called "negative tension branes," e.g., the symmetry like $SU(N/M)$ can be realized like two stacks of branes, $N$ ordinary and $M$ of the negative tensions [19]. Furthermore, in string theories, we also encounter the so-called ghost condensations, which in turn can precede to the phantomlike fields [20]. On the other hand, such kinds of fields may, in principle, support travers ability of wormhole solutions in four and higher dimensional theories of gravity [21–32].

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Their classification in four-dimensional wormholes and higher dimensional cases have been studied recently in several works [33–37].

Having in mind the interesting features of the phantom black holes and phantom fields, as well as the recent astrophysical estimations of the abundance of dark energy in our Universe, it will be not amiss to ask a question concerning the uniqueness of such kinds of black hole solutions.

The motivation of our work is to explore the problem of black hole classification (uniqueness theorem) for phantom theories with a $U(1)$-gauge field, with nontrivial coupling among them. We shall elaborate the gravity theory provided by the following action:

$$S = \int d^n x \sqrt{-g}(R - 2\eta_1 \nabla_\mu \phi \nabla^\mu \phi + \eta_2 e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu}), \qquad (1)$$

where $R$ stands for the Ricci scalar of the $n$-dimensional manifold, and $\nabla_\alpha$ denotes the Levi-Civita connection in the spacetime in question. On the other hand, $F_{\mu\nu}$ represents the $U(1)$-gauge field strength tensor, while $\phi$ is connected with dilaton one. $\lambda$ depicts the coupling constant between gauge and scalar fields. The action in question is a string theory inspired one and has been widely studied from the point of view of the possible black hole/wormhole solutions [8,9]. The two parameters $\eta_1$ and $\eta_2$ are equal to $\pm 1$, respectively. They enable one to study the case of Einstein dilaton ($\eta_1 = 1$), phantom ($\eta_1 = -1$), Maxwell ($\eta_2 = -1$), and anti-Maxwell ($\eta_2 = 1$) systems.

## II. CLASSIFICATION-UNIQUENESS THEOREM FOR PHANTOM BLACK HOLES

In this section we shall provide the uniqueness theorem for static, spherically symmetric, and asymptotically flat black objects with nondegenerate event horizons in higher-dimensional gravity systems described by the action (1). For the ordinary Einstein-Maxwell-dilaton gravity, being the low-energy limit of the heterotic string theory, the uniqueness for asymptotically flat black hole objects comprises rather complicated mathematical challenges [38–45], in which the proof of the conformal positive energy theorem plays a key role [46] and enables adequate conformal transformations. The conformal transformations enable one to examine the boundary conditions and the conformal flatness of the spacetime under inspection.

To commence with let us suppose that the spacetime under inspection is static in the strict sense, having a timelike Killing vector field $\xi_\alpha = (\partial/\partial t)_\alpha$ defined at each point of the manifold in question. The definition of staticity yields that the timelike Killing vector field is orthogonal to the $(n-1)$-dimensional hypersurface. It implies that the line element of the considered spacetime is provided by

$$ds^2 = -V^2(x_i)dt^2 + g_{ij}dx^i dx^j, \qquad (2)$$

where we set $g_{ij}$ for the metric tensor of $(n-1)$-dimensional Riemannian manifold. Moreover, one also imposes the staticity conditions for the fields appearing in the considered gravity theory. Thus, for the Maxwell field and phantom scalar they are written in the forms as follows:

$$\mathcal{L}_\xi F_{\mu\nu} = 0, \qquad \mathcal{L}_\xi \phi = 0, \qquad (3)$$

where $\mathcal{L}_\xi$ stands for the Lie derivative with respect to the Killing vector field $\xi$.

For the system in question, the dimensionally reduced equations of motion are given by

$$\begin{aligned} &{}^{(n-1)}R_{ij} - \frac{1}{V}{}^{(g)}\nabla_i {}^{(g)}\nabla_j V \\ &\quad = 2\eta_1 {}^{(g)}\nabla_i\phi {}^{(g)}\nabla_j\phi \\ &\qquad + 2\eta_2 e^{2\lambda\phi}\left[\frac{{}^{(g)}\nabla_i\psi {}^{(g)}\nabla^i\psi}{V^2} + g_{ij}\frac{{}^{(g)}\nabla_k\psi {}^{(g)}\nabla^k\psi}{(2-n)V^2}\right], \end{aligned} \qquad (4)$$

$$\begin{aligned} &{}^{(g)}\nabla_i {}^{(g)}\nabla^i\phi + \frac{{}^{(g)}\nabla_i V {}^{(g)}\nabla_i\phi}{V} \\ &\qquad + \frac{\lambda\eta_2}{\eta_1}e^{2\lambda\phi}\frac{{}^{(g)}\nabla_k\psi {}^{(g)}\nabla^k\psi}{V^2} = 0, \end{aligned} \qquad (5)$$

$${}^{(g)}\nabla_i\left(\frac{e^{2\lambda\phi (g)}\nabla^i\psi}{V}\right) = 0, \qquad (6)$$

$${}^{(g)}\nabla_i {}^{(g)}\nabla^i V + 2\eta_2\frac{e^{2\lambda\phi}(n-3){}^{(g)}\nabla_k\psi {}^{(g)}\nabla^k\psi}{(n-2)V} = 0, \qquad (7)$$

whereby ${}^{(n-1)}R_{ij}$ and ${}^{(g)}\nabla_i$ we have denoted the Ricci scalar curvature and the covariant derivative existing in $(n-1)$-dimensional manifold. $\psi$ describes the electrostatic potential.

Let us assume further that in asymptotically flat spacetime, for a compact subset $\mathcal{K} \subset {}^{(n-1)}\Sigma$, which is diffeomorphic to $R^{n-1}/B^{n-1}$, where $B^{n-1}$ is a closed unit ball situated at the origin of $R^{n-1}$. It implies that one has a standard coordinate system enabling the expansion as follows:

$$g_{ij} = \left(1 + \frac{2}{n-3}\frac{M}{r^{n-3}}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{r^{n-2}}\right), \qquad (8)$$

$$V = \left(1 - \frac{M}{r^{n-3}}\right) + \mathcal{O}\left(\frac{1}{r^{n-2}}\right), \qquad (9)$$

$$\psi = \frac{Q}{r^{n-3}} + \mathcal{O}\left(\frac{1}{r^{n-2}}\right), \qquad (10)$$

$$\phi = \phi - \frac{q}{(n-3)r^{n-3}} + \mathcal{O}\left(\frac{1}{r^{n-2}}\right), \qquad (11)$$

where $\phi, M, Q, q$ are constant. $M$ and $q$ represent the Arnowitt-Deser-Misner (ADM) masses and charges $Q$, defined up to a constant factor, while $r^2 = x_m x^m$. The standard notions of asymptotically flat regions are provided by relations (8)–(11).

The conformal positive energy theorem, derived in Ref. [46], will constitute the key role in the subsequent proof of the black hole uniqueness. In order to apply it to the considerations, we should satisfy its assumptions, i.e., one has to have two asymptotically flat Riemannian $(n-1)$-dimensional manifolds, $(\Sigma^{(\Phi)}, {}^{(\Phi)}g_{ij})$ and $(\Sigma^{(\Psi)}, {}^{(\Psi)}g_{ij})$, which metric tensors are connected by the conformal transformation of the form

$$ {}^{(\Psi)}g_{ij} = \Omega^{2\,(\Phi)}g_{ij}, \tag{12} $$

where $\Omega$ is a conformal factor. It turns out that the masses of the above manifolds fulfil the relation ${}^{(\Phi)}m + \beta^{(\Psi)}m \geq 0$, under the additional requirement imposed on the Ricci scalar tensor ${}^{(\Phi)}R + \beta\Omega^{2\,(\Psi)}R \geq 0$, where ${}^{(\Phi)}R$ and ${}^{(\Psi)}R$ are the Ricci scalars, with respect to the adequate metric tensors, defined on the two manifolds. $\beta$ is a positive constant. The inequalities in question are satisfied if $(n-1)$-dimensional manifolds are flat [46]. The conformal positive energy theorem was widely used in proofs of the uniqueness of four and higher-dimensional black objects [41–45], [47–50], as well as wormhole solutions [36,37].

To proceed to the uniqueness proof, let us define the $(n-1)$-dimensional metric tensor which yields

$$ {}^{(n-1)}\tilde{g}_{ij} = V^{\frac{2}{n-3}} g_{ij}. \tag{13} $$

The conformally rescaled metric tensor (13) implies that the Ricci curvature tensor has the form as follows:

$$ \begin{aligned} {}^{(n-1)}\tilde{R}(\tilde{g})_{ij} = &\frac{1}{V^2}\left(\frac{n-2}{n-3}\right){}^{(n-1)}\tilde{\nabla}_i V\, {}^{(n-1)}\tilde{\nabla}_j V \\ &+ 2\eta_1\, {}^{(n-1)}\tilde{\nabla}_i \phi\, {}^{(n-1)}\tilde{\nabla}_j \phi \\ &+ 2\eta_2 e^{\lambda\phi} \frac{{}^{(n-1)}\tilde{\nabla}_i \psi\, {}^{(n-1)}\tilde{\nabla}_j \psi}{V^2}. \end{aligned} \tag{14} $$

In the next step, we define the quantities provided by the relations

$$ \Phi_{\pm 1} = \frac{1}{2}\left[e^{C\phi} V \pm \frac{e^{-C\phi}}{V} - \frac{D^2 e^{-C\phi}\psi^2}{V}\right], \tag{15} $$

$$ \Phi_0 = \frac{D e^{-C\phi}\psi}{V}, \tag{16} $$

$$ \Psi_{\pm 1} = \frac{1}{2}\left[e^{-A\phi} \pm \frac{e^{A\phi}}{V}\right], \tag{17} $$

where $A = C/(n-3)$ and the constants $C$, $D$, and $\lambda$ are bounded with the adequate values of $\eta_1$ and $\eta_2$ appearing in the action (1). For the brevity of notation we depict their exact values in Table I.

TABLE I. Values of the constants for $\eta_i, i = 1, 2$.

| $\eta_i$ | $\eta_1 = 1$ | $\eta_1 = -1$ |
|---|---|---|
| $\eta_2 = 1$ | $C = \sqrt{\frac{2\eta_1}{n-2}}(n-3)$ | $C = \sqrt{\frac{-2\eta_1}{n-2}}(n-3)$ |
| | $D = i\sqrt{2\eta_2(n-3)}$ | $D = i\sqrt{2\eta_2(n-3)}$ |
| | $\lambda = -\sqrt{\frac{2\eta_1}{n-2}}(n-3)$ | $\lambda = -\sqrt{\frac{-2\eta_1}{n-2}}(n-3)$ |
| $\eta_2 = -1$ | $C = \sqrt{\frac{2\eta_1}{n-2}}(n-3)$ | $C = \sqrt{\frac{-2\eta_1}{n-2}}(n-3)$ |
| | $D = \sqrt{2\eta_2(n-3)}$ | $D = \sqrt{2\eta_2(n-3)}$ |
| | $\lambda = -\sqrt{\frac{2\eta_1}{n-2}}(n-3)$ | $\lambda = -\sqrt{\frac{-2\eta_1}{n-2}}(n-3)$ |

Then, the following symmetric tensors can be constructed on the aforementioned manifolds:

$$ \begin{aligned} {}^{(\Phi)}\tilde{R}_{ij} = &{}^{(n-1)}\tilde{\nabla}_i \Phi_{-1}\, {}^{(n-1)}\tilde{\nabla}_j \Phi_{-1} - {}^{(n-1)}\tilde{\nabla}_i \Phi_0\, {}^{(n-1)}\tilde{\nabla}_j \Phi_0 \\ &- {}^{(n-1)}\tilde{\nabla}_i \Phi_1\, {}^{(n-1)}\tilde{\nabla}_j \Phi_1, \\ {}^{(\Psi)}\tilde{R}_{ij} = &{}^{(n-1)}\tilde{\nabla}_i \Psi_{-1}\, {}^{(n-1)}\tilde{\nabla}_j \Psi_{-1} - {}^{(n-1)}\tilde{\nabla}_i \Psi_1\, {}^{(n-1)}\tilde{\nabla}_j \Psi_1. \end{aligned} \tag{18} $$

Setting the metric in the given form, $\eta_{AB} = \mathrm{diag}(1, -1, -1)$, enables one to find that $\Psi_A \Psi^A = \Phi_A \Phi^A = -1$, where $A = (0, 1, -1)$. Next, by virtue of the relation (18), one obtains

$$ {}^{(n-1)}\tilde{\nabla}_m\, {}^{(n-1)}\tilde{\nabla}^m \Psi_B = {}^{(\Psi)}\tilde{R}_i{}^i \Psi_B, \tag{19} $$

$$ {}^{(n-1)}\tilde{\nabla}_m\, {}^{(n-1)}\tilde{\nabla}^m \Phi_B = {}^{(\Phi)}\tilde{R}_i{}^i \Phi_B. \tag{20} $$

It can be also proved that the Ricci curvature tensor of the conformally rescaled metric ${}^{(n-1)}\tilde{g}_{ij}$ implies

$$ \tilde{R}_{ij} = \left({}^{(\Phi)}\tilde{R}_{ij} + \frac{1}{n-3}\, {}^{(\Psi)}\tilde{R}_{ij}\right). \tag{21} $$

Consequently, in order to meet the requirements of the conformal positive energy theorem, one defines the conformal transformations given by

$$ {}^{(\Phi)}g^{\pm}_{ij} = {}^{(\Phi)}\omega_{\pm}^{\frac{2}{n-3}} \tilde{g}_{ij}, \qquad {}^{(\Psi)}g^{\pm}_{ij} = {}^{(\Psi)}\omega_{\pm}^{\frac{2}{n-3}} \tilde{g}_{ij}, \tag{22} $$

where the conformal factors yield

$$ {}^{(\Phi)}\omega_{\pm} = \frac{\Phi_1 \pm 1}{2}, \qquad {}^{(\Psi)}\omega_{\pm} = \frac{\Psi_1 \pm 1}{2}. \tag{23} $$

The careful scrutiny of the metric tensors defined by the relation (22) envisages that we obtain four $(n-1)$-dimensional manifolds denoted, respectively, as $(\Sigma^{+(\Phi)}, {}^{(\Phi)}g^{+}_{ij})$, $(\Sigma^{-(\Phi)}, {}^{(\Phi)}g^{-}_{ij})$, $(\Sigma^{+(\Psi)}, {}^{(\Psi)}g^{+}_{ij})$, and $(\Sigma^{-(\Psi)}, {}^{(\Psi)}g^{-}_{ij})$.

Pasting them together [43,44], across the surface $V = 0$, one achieves complete regular hypersurfaces $\Sigma^{(\Phi)} = \Sigma^{+(\Phi)} \cup \Sigma^{-(\Phi)}$ and $\Sigma^{(\Psi)} = \Sigma^{+(\Psi)} \cup \Sigma^{-(\Psi)}$, and moreover, ${}^{(\Phi)}g^{\pm}_{ij}$ and ${}^{(\Phi)}g^{\pm}_{ij}$ metrics are complete.

The asymptotic conditions imposed on $g_{ij}$, electric potential $\psi$, and the scalar field, show their explicit asymptotical behavior.

The resulting manifolds $\Sigma^{(\Phi)}$ and $\Sigma^{(\Psi)}$ are geodesically complete. If $(\Sigma^{(m)}, {}^{(m)}g_{ij}, \Phi_A, \Psi_A)$, where $m = \Phi, \Psi$, are asymptotically flat solutions of Eqs. (19) and (20), with nondegenerate black hole event horizons, then in the next step one ought to check if the gravitational mass on them is equal to zero.

In the next step it remains to show that the static slice is conformally flat. In order to achieve this goal, we implement the conformal positive energy theorem [46]. Let us define the other conformal transformation described by

$$\hat{g}^{\pm}_{ij} = [({}^{(\Phi)}\omega_{\pm})^{\frac{2}{n-3}}({}^{(\Psi)}\omega_{\pm})^2]^{\frac{1}{n-2}}\tilde{g}_{ij}. \tag{24}$$

On the other hand, the Ricci curvature tensor on the defined space may be written as

$$\begin{aligned}[({}^{(\Phi)}\omega_{\pm})^{\frac{2}{n-3}}({}^{(\Psi)}\omega_{\pm})^2]^{\frac{1}{n-2}}\hat{R}^{\pm} \\ = \left({}^{(\Phi)}\omega_{\pm}^{\frac{2}{n-3}}{}^{(\Phi)}R^{\pm} + \frac{1}{n-3}{}^{(\Psi)}\omega_{\pm}^{\frac{2}{n-3}}{}^{(\Psi)}R^{\pm}\right) \\ + \frac{1}{n-3}(\tilde{\nabla}_i \ln {}^{(\Phi)}\omega_{\pm} - \tilde{\nabla}_i \ln {}^{(\Psi)}\omega_{\pm})^2. \end{aligned} \tag{25}$$

Consequently, by the direct calculations it can be verified that the first term in the brackets in the relation (25) may be rewritten as follows:

$$\begin{aligned}{}^{(\Phi)}\omega_{\pm}^{\frac{2}{n-3}}{}^{(\Phi)}R^{\pm} + \frac{1}{n-3}{}^{(\Psi)}\omega_{\pm}^{\frac{2}{n-3}}{}^{(\Psi)}R^{\pm} \\ = \frac{|\Phi_0\tilde{\nabla}_i\Phi_{-1} - \Phi_{-1}\tilde{\nabla}_i\Phi_0|^2}{(\Phi_1 \pm 1)^2} \\ + \frac{1}{n-3}\frac{|\Psi_1\tilde{\nabla}_i\Psi_{-1} - \Psi_{-1}\tilde{\nabla}_i\Psi_1|^2}{(\Psi_1 \pm 1)^2}. \end{aligned} \tag{26}$$

The relations (25) and (26) reveal the conclusion that $\hat{R}^{\pm}$ is greater or equal to zero. Then, by virtue of the application of the conformal positive energy theorem, one has that $(\Sigma^{(\Phi)}, {}^{(\Phi)}g_{ij})$, $(\Sigma^{(\Psi)}, {}^{(\Psi)}g_{ij})$, and $(\hat{\Sigma}, \hat{g}_{ij})$ are flat, and these facts entail that the conformal factors ${}^{(\Phi)}\omega = {}^{(\Psi)}\omega$ and $\Phi_1 = \Psi_1$, as well as $\Phi_0 = \text{const}\Phi_{-1}$ and $\Psi_0 = \text{const}\Psi_{-1}$.

Finally, one can conclude that the manifold $({}^{(n-1)}\Sigma, g_{ij})$ is conformally flat, and the metric tensor $\hat{g}_{ij}$ may be depicted in a conformally flat form. Namely, we define a function provided by the following:

$$\hat{g}_{ij} = \mathcal{U}^{\frac{4}{n-3}}{}^{(\Phi)}g_{ij}, \tag{27}$$

where we set $\mathcal{U} = ({}^{(\Phi)}\omega_{\pm}V)^{-1/2}$. The considered equations of motion of the studied gravity system reduce now to the Laplace equation on the three-dimensional Euclidean manifold. Namely, we have

$$\nabla_i\nabla^i\mathcal{U} = 0, \tag{28}$$

where $\nabla$ is the connection on a flat manifold. It follows from the fact that the Ricci scalar for the metric $\hat{g}_{ij}$ is equal to zero. Moreover, one has that the following metric for the flat base space is valid:

$${}^{(\Phi)}g_{ij} = \tilde{\rho}^2 d\mathcal{U}^2 + \tilde{h}_{AB}dx^A dx^B, \tag{29}$$

where $\tilde{\rho}^2 = \nabla_b\mathcal{U}\nabla^b\mathcal{U}$.

In the next step we shall demonstrate that the conformally transformed event horizon constitutes a geometric sphere. In order to proceed, let us consider how the event horizon is embedded into the base space $(\hat{\Sigma}, \hat{g}_{ij})$. Namely, we can define a local coordinate in the neighborhood $\mathcal{M} \in \hat{\Sigma}$ for the flat base space

$$\hat{g}_{ij}dx^i dx^j = \delta_{ij}dx^i dx^j = \rho^2 d\mathcal{U}^2 + h_{AB}dx^A dx^B. \tag{30}$$

The manifold in question is totally geodesic, which means that any of its submanifold geodesic is a geodesic in the considered manifold. The event horizon is located at some $\mathcal{U} = \text{const}$, and the other important fact is that the embedding of $\hat{\Sigma}$ into Euclidean $(n-1)$ manifold is totally umbilical [51], which results that each connected component of $\hat{\Sigma}$ constitutes a geometric sphere of a certain radius.

The studied embedding is also rigid [51], which yields that one is always able to locate one connected component of the event horizon $\mathcal{H}$, of a certain radius $\rho$, at $r = r_0$ surface on $\hat{\Sigma}$. Thus, the above mathematical construction leads us to a boundary value problem for the Laplace equation on the base space $\Theta = E^{n-1}/B^{n-1}$, with a Dirichlet boundary condition.

The system in question is characterized by a parameter which fixes the radius of the inner boundary and authorizes a black hole of a specific radius $\rho_{|_{\mathcal{H}}}$ in the gravity theory described by the action (1). We have also the following limit condition for $\mathcal{U}$, i.e., it tends to $\mathcal{U} \sim 1 + \mathcal{O}(r^{n-3})$, when $r \to \infty$.

Let us assume further, that we have two solutions being subject to the same boundary value problem, i.e., $\mathcal{U}_1$ and $\mathcal{U}_2$. By means of Green identity and integrating over the volume $\Theta$, one has that

$$\left(\int_{r\to\infty} - \int_{\mathcal{H}}\right)(\mathcal{U}_1 - \mathcal{U}_2)\frac{\partial}{\partial r}(\mathcal{U}_1 - \mathcal{U}_2)dS = \int_{\Theta} |\nabla(\mathcal{U}_1 - \mathcal{U}_2)|^2 d\Theta. \quad (31)$$

The surface integrals on the left-hand side of the above relation vanish due to the imposed boundary conditions. This fact provides that the volume integral has to be identically equal to zero. We conclude that the aforementioned two solutions of the Laplace equation with the Dirichlet boundary conditions are identical. It accounts for the main conclusion of our considerations.

*Theorem 1.*—Let us assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ constitute the two solutions of the Laplace equation on the base space $\Theta = E^{n-1}/B^{n-1}$, as defined above. They authorize solutions of Einstein phantom-dilaton Maxwell–anti-Maxwell gravity systems (depending on the special choice of $\eta_i$, where $i = 1, 2$) of equations of motion, describing static, spherically symmetric, asymptotically flat, and adequate black holes with nondegenerate event horizons. The solutions of the equations of motion of the theory under inspection are subject to the same boundary and regularity conditions. Then, $\mathcal{U}_1 = \mathcal{U}_2$ in all of the region of the base space $\Theta$, provided that $\mathcal{U}_1(p) = \mathcal{U}_2(p)$ for at least one point belonging to the aforementioned region.

## III. CONCLUSIONS

The recent astrophysical and astronomical observations reveal that dark energy is a possible ingredient of our Universe, which implicates that phantom fields and phantom black objects should be subjects of interest and careful scrutiny. In our paper we have constructed the uniqueness theorem for Einstein phantom-dilaton Maxwell–anti-Maxwell gravity systems, depending on the special choice on $\eta_i$, where $i = 1, 2$, black hole solutions.

We have paid attention to $n$-dimensional spherically symmetric, asymptotically flat, and static black holes with nondegenerate event horizons, being the solutions of the aforementioned theory. The key role in the proof was played by the conformal positive energy theorem and conformal transformations helping us to examine the boundary conditions and the conformal flatness of the examined spacetime.


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