Entropy of Kerr-Newman-AdS black holes with torsion

M. Blagojević and B. Cvetković*

Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

(Received 1 April 2022; accepted 20 April 2022; published 11 May 2022)

The canonical approach to black hole entropy in Poincaré gauge theory without matter is extended to include the Maxwell field as a matter source. The new formalism is used to calculate asymptotic charges and entropy of Kerr-Newmann-AdS black holes with torsion. The result implies that the first law, with a nontrivial contribution of the Maxwell field, takes the same form as in general relativity.

DOI: 10.1103/PhysRevD.105.104014

I. INTRODUCTION

The analysis of black hole spacetimes in general relativity (GR) shows that astrophysically, the most significant among them are those produced by rotating massive bodies [1]. The simplest spacetime of this type is the rotating, asymptotically flat solution found by Kerr [2]. The Kerr metric has been further generalized by including first the electric charge, and then a nonvanishing cosmological constant [3,4]. The final result of these generalizations is the Kerr-Newman-Anti de Sitter (KN-AdS) black hole, which is the most general stationary, asymptotically anti-de Sitter solution of Einstein-Maxwell field equations [5].

Starting from the early 1980s, many well-known exact solutions of GR have been generalized to solutions of the Poincaré gauge theory (PG), a modern gauge theory of gravity in which both the curvature and the torsion have their own dynamical roles [6]. Successful constructions of exact solutions with torsion [7–9] have been followed, *inter alia*, by an intensive investigation of the concept of *conserved charge* [10,11]. In contrast to that, a systematic investigation of *black hole entropy* in PG has long been neglected, although some incomplete attempts could have been noticed in the literature, as noted in [12].

A few years ago, a general canonical approach to black hole entropy in PG was proposed in [12]. The approach is based on a canonical formulation of the idea developed in GR, according to which entropy is just the Noether charge on the horizon [13]. Applying this approach to a number of black holes with or without torsion [14–16], we found a somewhat unexpected result: in spite of many geometric and dynamic differences with respect to GR, entropy of black holes in PG *without matter*, as well as the associated first law, follows essentially the same pattern as in GR, up to a multiplicative constant. In the present paper, we extend our investigation of entropy by introducing the Maxwell field as a matter source for gravity (PG-Maxwell system). The analysis is focused on exploring thermodynamic properties of the generalized KN-AdS black hole, constructed by Baekler *et al.* [8] in the late 1980s.

The paper is organized as follows. In Sec. II, we present a brief account of the general thermodynamic aspects of the PG-Maxwell system. In particular, a new definition of the black hole entropy in the presence of a Maxwell field is introduced as a natural generalization of the earlier definition, valid in vacuum. In Sec. III, we describe geometric aspects of the KN-AdS black hole as a solution of the PG-Maxwell system. Next, in Secs. IV and V, we use these results to calculate energy, angular momentum, and entropy. The thermodynamic role of the Maxwell field and the resulting first law are clarified in Sec. VI. Section VII is devoted to concluding remarks, and appendixes contain some important technical details.

Our conventions are the same as in Ref. [16]. The Latin indices $(i, j, \dots)$ are the local Lorentz indices, the Greek indices $(\mu, \nu, \dots)$ are the coordinate indices, and both run over 0,1,2,3. The orthonormal coframe (tetrad) ϑ^i and the metric compatible (Lorentz) connection $\omega^{ij} = -\omega^{ji}$ are 1-forms, the dual basis (frame) is $e_i = e_i{}^\mu \partial_\mu$, the interior product of e_i with ϑ^j is $e_i \lrcorner \vartheta^j = \delta_i^j$, and A is the electromagnetic potential 1-form. The metric components in the local Lorentz and coordinate basis are $\eta_{ij} = (1, -1, -1, -1)$ and $g_{\mu\nu} = \eta_{ij}\vartheta^i{}_\mu \vartheta^j{}_\nu$, respectively, and ε_{ijmn} is the totally antisymmetric symbol with $\varepsilon_{0123} = 1$. The Hodge dual is marked by a star $\star$, and the wedge product of forms is implicit.

II. PG-MAXWELL SYSTEM

We begin with an overview of the general Lagrangian and thermodynamic aspects of the PG dynamics in the presence of a Maxwell field; for more details, see Refs. [12,17].

A. Lagrangian formalism

In PG, the structure of spacetime is characterized by a Riemann-Cartan geometry, in which the torsion

*mb@ipb.ac.rs, cbranislav@ipb.ac.rs

$T^i = d\vartheta^i + \omega^i{}_k\vartheta^k$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k\omega^{kj}$ (2-forms) are the gravitational field strengths, associated with the Poincaré (translational and Lorentz) gauge potentials, the tetrad ϑ^i, and the Lorentz connection ω^{ij}, respectively. Moreover, our physical system contains also the Maxwell field characterized by the field strength $F = dA$ (2-form), where A is the electromagnetic gauge potential (1-form).

Dynamical properties of the PG-Maxwell system are defined by the total Lagrangian

$$L = L_G + L_M, \tag{2.1}$$

where $L_G = L_G(\vartheta^i, T^i, R^{ij})$ is a parity even PG Lagrangian, assumed to be at most quadratic in the field strengths, and $L_M = L_M(\vartheta^i, F)$ describes the Maxwell field interacting with gravity. The gravitational field equations are obtained by varying L with respect to the gravitational potentials ϑ^i and ω^{ij}. Introducing the gravitational covariant momenta, $H_i \coloneqq \partial L_G/\partial T^i$ and $H_{ij} \coloneqq \partial L_G/\partial R^{ij}$, and the associated energy-momentum and spin currents, $E_i = \partial L_G/\partial b^i$ and $E_{ij} \coloneqq \partial L_G/\partial\omega^{ij}$, these equations can be written in a compact form as

$$\delta b^i\colon\ \nabla H_i + E_i = -\tau_i, \tag{2.2a}$$

$$\delta\omega^{ij}\colon\ \nabla H_{ij} + E_{ij} = 0. \tag{2.2b}$$

The source term on the right-hand side of (2.2a) is the Maxwell energy-momentum current $\tau_i \coloneqq \partial L_M/\partial\vartheta^i$, while the related spin current vanishes, $\sigma_{ij} \coloneqq \partial L_M/\partial\omega^{ij} = 0$. Similarly, the variation of L with respect to the electromagnetic potential A yields the Maxwell equation,

$$\delta A\colon\ dH = 0, \tag{2.2c}$$

where $H \coloneqq \partial L_M/\partial A$ is the electromagnetic covariant momentum.

The PG part of the total Lagrangian (2.1) has the form

$$L_G = -{}^\star(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^\star(a_n{}^{(n)}T_i) + \frac{1}{2}R^{ij}\sum_{n=1}^{6} {}^\star(b_n{}^{(n)}R_{ij}), \tag{2.3a}$$

where (a_0, Λ, a_n, b_n) are the gravitational coupling constants, and ${}^{(n)}T_i, {}^{(n)}R_{ij}$ are irreducible parts of the field strengths. The Maxwell part reads

$$L_M \coloneqq 4a_1\left(-\frac{1}{2}F^\star F\right), \qquad F \coloneqq dA, \tag{2.3b}$$

where $4a_1$ is a suitably normalized coupling constant.

In the analysis of black hole thermodynamics, we need the following explicit formulas:

$$H_i = 2\sum_{m=1}^{2} {}^\star(a_n{}^{(m)}T_i), \tag{2.4a}$$

$$H_{ij} = -2a_0{}^\star(\vartheta_i\vartheta_j) + 2\sum_{n=1}^{6} {}^\star(b_n{}^{(n)}R_{ij}), \tag{2.4b}$$

$$H = -4a_1{}^\star F. \tag{2.4c}$$

B. Thermodynamics

The Hamiltonian approach to black hole entropy in PG [12] is based on the ideas developed originally in GR [13,18], according to which the asymptotic charges (energy and angular momentum) as well as entropy, can be defined by certain boundary terms. Here, we introduce an extended version of that approach, suitable for analyzing nonvacuum solutions of the PG-Maxwell system.

Consider a stationary black hole spacetime whose spatial section Σ has a two-component boundary, one component at infinity and the other at horizon, $\partial\Sigma = S_\infty \cup S_H$. Then, asymptotic charges and entropy of a PG-Maxwell black hole are determined by the boundary integral $\Gamma \coloneqq \Gamma_\infty - \Gamma_H$, determined by the following variational equations:

$$\delta\Gamma_\infty = \oint_{S_\infty}\delta B(\xi), \qquad \delta\Gamma_H = \oint_{S_H}\delta B(\xi), \tag{2.5a}$$

$$\begin{aligned}\delta B(\xi) \coloneqq{}& (\xi \lrcorner \vartheta^i)\delta H_i + \delta\vartheta^i(\xi \lrcorner H_i) + \frac{1}{2}(\xi \lrcorner \omega^{ij})\delta H_{ij} \\ &+ \frac{1}{2}\delta\omega^{ij}(\xi \lrcorner \delta H_{ij}) + (\xi \lrcorner A)\delta H \\ &+ (\delta A)(\xi \lrcorner H).\end{aligned} \tag{2.5b}$$

By construction, δB is obtained from the canonical generator of local translations. It contains not only the gravitational term (upper line), but also the Maxwell term (bottom line), extending thereby the construction adopted in [12] to nonvacuum solutions.[1] Specific forms of the Killing vector ξ ($\xi = \partial_t, \partial_\varphi$ or a linear combination thereof) are chosen so that the boundary integrals $(\Gamma_\infty, \Gamma_H)$ could be physically interpreted in terms of the asymptotic charges, black hole entropy, and an external, Maxwell term. To have a consistent interpretation, we require the operation δ to satisfy the following rules:

[1]The electric charge is not defined by the Maxwell term in (2.5b); it is, by definition, related to the electromagnetic $U(1)$ boundary term; see Sec. VI.

(i) (r1) On S_∞, the variation δ acts on the parameters of a black hole solution, but not on the parameters of the background configuration.
(ii) (r2) On S_H, the variation δ must keep surface gravity constant.

Moreover, mathematical consistency strongly depends on the boundary conditions:

(iii) (r3) When the boundary terms $(\delta\Gamma_\infty, \delta\Gamma_H)$ are δ-integrable and finite, they can be given the usual thermodynamic interpretation.

Finally, note that Γ_∞ and Γ_H are introduced as *a priori* independent objects. However, the analysis of their construction from the canonical gauge generator reveals that the regularity of the generator can be expressed by the relation

$$\delta\Gamma_\infty - \delta\Gamma_H = 0, \tag{2.6}$$

which is equivalent to the *first law* of black hole thermodynamics. The Maxwell contribution to δB is an essential part of the first law.

III. GEOMETRY AND DYNAMICS

In this section, we analyze basic properties of KN-AdS black holes as solutions of the PG-Maxwell system [8].

A. Metric and tetrad

The metric of a KN-AdS black hole in Boyer-Lindquist coordinates has the form [1]

$$\begin{aligned} ds^2 = {}& \frac{\Delta}{\rho^2}\left(dt + \frac{a}{\alpha}\sin^2\theta d\varphi\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \frac{\rho^2}{f}d\theta^2 \\ & - \frac{f}{\rho^2}\sin^2\theta\left[adt + \frac{(r^2+a^2)}{\alpha}d\varphi\right]^2, \end{aligned} \tag{3.1a}$$

where

$$\begin{aligned} &\Delta(r) := (r^2+a^2)(1+\lambda r^2) - 2(mr - q^2), \qquad \alpha := 1-\lambda a^2, \\ &\rho^2(r,\theta) := r^2 + a^2\cos^2\theta, \qquad f(\theta) := 1 - \lambda a^2\cos^2\theta. \end{aligned} \tag{3.1b}$$

Here, m, a. and q are the parameters characterizing energy (mass), angular momentum, and electric charge of the solution, and $\lambda = -\Lambda/3a_0$. The orthonormal tetrad associated with the metric is chosen in the form

$$\begin{aligned} &\vartheta^0 = N\left(dt + \frac{a}{\alpha}\sin^2\theta d\varphi\right), \qquad \vartheta^1 = \frac{dr}{N}, \\ &\vartheta^2 = Pd\theta, \qquad \vartheta^3 = \frac{\sin\theta}{P}\left[adt + \frac{(r^2+a^2)}{\alpha}d\varphi\right], \end{aligned} \tag{3.2a}$$

where

$$N(r,\theta) = \sqrt{\Delta/\rho^2}, \qquad P(r,\theta) = \sqrt{\rho^2/f}. \tag{3.2b}$$

The larger root of $\Delta(r) = 0$ defines the outer horizon,

$$(r_+^2 + a^2)(1+\lambda r_+^2) - 2(mr_+ - q^2) = 0, \tag{3.3}$$

and the angular velocity and surface gravity have the same form as in GR [5,15],

$$\omega_+ = \frac{a\alpha}{r_+^2 + a^2}, \qquad \Omega_+ := \omega_+ + \lambda a = \frac{a(1+\lambda r_+^2)}{r_+^2 + a^2}, \tag{3.4}$$

$$\kappa = \frac{r_+^2 + 3\lambda r_+^4 + \lambda a^2 r_+^2 - a^2 - 2q^2}{2r_+(r_+^2 + a^2)}, \tag{3.5}$$

and the area of the horizon is

$$A_H = \int_{r_+} b^2 b^3 = \frac{4\pi(r_+^2 + a^2)}{\alpha}. \tag{3.6}$$

The Riemannian connection $\tilde\omega^{ij}$ is calculated in Appendix A.

B. Torsion, connection, and curvature

Riemann-Cartan geometry of PG is characterized by a nonvanishing torsion. For KN-AdS black holes, the ansatz for torsion is formally the same as for the Kerr-AdS case [8,14],

$$\begin{aligned} T^0 &= T^1 \\ &= \frac{1}{N}[-V_1\vartheta^0\vartheta^1 - 2V_4\vartheta^2\vartheta^3] + \frac{1}{N^2}[V_2\vartheta^-\vartheta^2 + V_3\vartheta^-\vartheta^3], \\ T^2 &:= \frac{1}{N}[V_5\vartheta^-\vartheta^2 + V_4\vartheta^-\vartheta^3], \\ T^3 &:= \frac{1}{N}[-V_4\vartheta^-\vartheta^2 + V_5\vartheta^-\vartheta^3], \end{aligned} \tag{3.7}$$

where $\vartheta^- = \vartheta^0 - \vartheta^1$, but the metric function N and the torsion functions V_n are modified by the nonvanishing electric charge parameter q^2,

$$\begin{aligned} V_1 &= \frac{1}{\rho^4}[(mr - 2q^2)r - ma^2\cos^2\theta], \\ V_2 &= -\frac{1}{\rho^4 P}(mr - q^2)a^2\sin\theta\cos\theta, \\ V_3 &= \frac{1}{\rho^4 P}(mr - q^2)ra\sin\theta, \\ V_4 &= \frac{1}{\rho^4}(mr - q^2)a\cos\theta, \\ V_5 &= \frac{1}{\rho^4}(mr - q^2)r. \end{aligned} \tag{3.8}$$

Having introduced torsion, the Riemann-Cartan connection can be expressed as

$$\omega^{ij} = \tilde{\omega}^{ij} + K^{ij}, \tag{3.9a}$$

where K^{ij} is the contortion 1-form, implicitly defined by the relation $T^i = K^i{}_k b^k$,

$$\begin{aligned}
K^{01} &= \frac{1}{N} V_1 \vartheta^-, \\
K^{02} &= K^{12} = -\frac{1}{N^2} V_2 \vartheta^- + \frac{1}{N}(V_5 \vartheta^2 - V_4 \vartheta^3), \\
K^{03} &= K^{13} = -\frac{1}{N^2} V_3 \vartheta^- + \frac{1}{N}(V_4 \vartheta^2 + V_5 \vartheta^3), \\
K^{23} &= -\frac{2}{N} V_4 \vartheta^-.
\end{aligned} \tag{3.9b}$$

The curvature 2-form $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$ has only two nonvanishing irreducible parts:

$$^{(6)}R^{ij} = \lambda \vartheta^i \vartheta^j, \qquad ^{(4)}R^{Ac} = \frac{\lambda}{\Delta}(mr - q^2)\vartheta^- \vartheta^c. \tag{3.10}$$

The quadratic invariants (Euler, Pontryagin, and Nieh-Yan) are given by

$$\begin{aligned}
I_E &:= (1/2)\varepsilon_{ijmn} R^{ij} R^{mn} \equiv {}^\star R_{mn} R^{mn} = 12\lambda^2 \hat{\epsilon}, \\
I_P &:= R^{ij} R_{ij} = 0, \qquad I_{NY} = T^i T_i - R_{ij} b^i b^j = 0.
\end{aligned} \tag{3.11}$$

C. PG-Maxwell field equations

Since the only nonvanishing parts of the gravitational field strengths are ${}^{(1)}T^i$, ${}^{(2)}T^i$ and ${}^{(4)}R^{ij}$, ${}^{(6)}R^{ij}$, the "effective" form of the gravitational Lagrangian reads

$$\begin{aligned}
L_G = &-{}^\star(a_0 R + 2\Lambda) + T^i{}^\star(a_1{}^{(1)}T_i + a_2{}^{(2)}T_i) \\
&+ \frac{1}{2} R^{ij}{}^\star(b_4{}^{(4)}R_{ij} + b_6{}^{(6)}R_{ij}).
\end{aligned} \tag{3.12}$$

The covariant momenta H_i and H_{ij}, appearing in the field equations (2.2), are given by

$$\begin{aligned}
H_i &= 2a_1{}^\star({}^{(1)}T_i - 2{}^{(2)}T_i), \\
H_{ij} &= -2A_0'{}^\star(\vartheta_i \vartheta_j) + 2b_4{}^{\star(4)}R_{ij}, \qquad A_0' := a_0 - \lambda b_6,
\end{aligned} \tag{3.13}$$

and the corresponding spin currents are

$$\begin{aligned}
E_i &= e_i \lrcorner L_G - (e_i \lrcorner T^m) H_m - \frac{1}{2}(e_i \lrcorner R^{mn}) H_{mn}, \\
E_{ij} &= -(\vartheta_i E_j - \vartheta_j E_i).
\end{aligned} \tag{3.14}$$

The contribution of the electromagnetic sector to Eqs. (2.2) is described by the Maxwell energy-momentum current [17]

$$\tau_i = e_i \lrcorner L_M - (e_i \lrcorner F) H. \tag{3.15}$$

The form of τ_i depends on the Maxwell potential in a KN-AdS spacetime [19],

$$A := -\frac{q_e r}{\rho\sqrt{\Delta}} \vartheta^0 \equiv -\frac{q_e r}{\rho^2}\left(dt + \frac{a}{\alpha}\sin^2\theta d\varphi\right), \tag{3.16}$$

where q_e is the electromagnetic charge parameter. This expression is a natural generalization of the spherically symmetric form $A = -(q_e/r)dt$. The related field strength and the covariant momentum are

$$F = -\frac{q_e}{\rho^4}[(r^2 - a^2\cos^2\theta)\vartheta^0\vartheta^1 + 2ar\cos\theta\vartheta^2\vartheta^3], \tag{3.17a}$$

$$H = -4a_1 \frac{q_e}{\rho^4}[(r^2 - a^2\cos^2\theta)\vartheta^2\vartheta^3 - 2ar\cos\theta\vartheta^0\vartheta^1]. \tag{3.17b}$$

When all the previous results taken into account, the explicit calculation shows that basic dynamical variables $(\vartheta^i, \omega^{ij}, A)$ of a KN-AdS black hole, which are defined in Eqs. (3.2a), (3.9a) and (3.16), solve the PG-Maxwell field equations (2.2) if the Lagrangian parameters (a_n, b_n, Λ) and the solution parameters (λ, q, q_e) satisfy the relations

$$\begin{aligned}
&2a_1 + a_2 = 0, \qquad a_0 - a_1 - \lambda(b_4 + b_6) = 0, \\
&3\lambda a_0 + \Lambda = 0, \qquad q_e^2 = 2q^2.
\end{aligned} \tag{3.18}$$

Thus, according to our conventions, the electromagnetic charge parameter q_e differs from the metric charge parameter q. However, none of them coincides with the asymptotic Maxwell charge, as will be shown in Sec. VI.

IV. ASYMPTOTIC BOUNDARY TERMS

The asymptotic values of energy and angular momentum are defined by the boundary term $\delta B(\xi)$ in (2.5). Two aspects of explicit calculations deserve special attention.

First, Carter [20] and Henneaux and Teitelboim [21] demonstrated that the asymptotic metric of Kerr-AdS spacetimes cannot be properly described in Boyer-Lindquist coordinates. They found a new set of coordinates in which this deficiency is brought under control. However, our variational approach (2.5) allows a simpler procedure [14,15], in which only the subset (t, φ) of the Boyer-Lindquist coordinates is transformed to the "untwisted" form,

$$T = t, \qquad \phi = \varphi - \lambda a t. \tag{4.1a}$$

Under these transformations, the components (v_t, v_φ) of a 4-vector v_μ transform as

$$v_T = v_t + \lambda a v_\varphi, \qquad v_\phi = v_\varphi \tag{4.1b}$$

In particular,

$$g_{T\varphi} = g_{t\varphi} + g_{\varphi\varphi},$$
$$\Omega_+ := \left(\frac{g_{T\varphi}}{g_{\varphi\varphi}}\right)_{r_+} = \omega_+ + \lambda a = \frac{a(1+\lambda r_+^2)}{r_+^2 + a^2}. \tag{4.1c}$$

And second, the background configuration, defined by $m = q = 0$, depends on the parameter a. To avoid the variation of those a's that are associated with the background, we introduce a more precise formulation of the rule (r1) for the variation δ, given below Eq. (2.5), $(r1')$ In calculating $\delta\Gamma_\infty(\xi)$, first apply δ to all the parameters (m, a, q), then subtract those δa terms that survive the limit $m = q = 0$, as they come from the background.

Before continuing, it is interesting to note that the lower line in the expression $\delta B(\xi)$, Eq. (2.5), which refers to the contribution of the Maxwell field, yields vanishing boundary terms at infinity, but not at horizon. This follows from the asymptotic behavior of the variables A and H, defined by Eqs. (3.16) and (3.17). Hence, nontrivial energy and angular momentum are generated only by the contributions stemming from the gravitational sector.

In the subsequent calculations, we use the following notation:

$$d\Omega := \sin\theta d\theta d\varphi \to 4\pi, \qquad d\Omega' := \sin^3\theta d\theta d\varphi \to \frac{2}{3}4\pi.$$

A. Angular momentum

The angular momentum is defined by $\delta E_\varphi := \delta\Gamma_\infty(\partial_\varphi)$. The calculation is performed by ignoring (m, q)-independent δa terms (background), even when they are divergent, and by omitting asymptotically vanishing terms. The nonvanishing contributions are

$$\omega^{13}{}_\varphi \delta H_{13} + \delta\omega^{13} H_{13\varphi} = 2a_1 \delta\left(\frac{ma}{\alpha^2}\right) d\Omega',$$
$$b^0{}_\varphi \delta H_0 + \delta b^0 H_{0\varphi} = 4a_1 \delta\left(\frac{ma}{\alpha^2}\right) d\Omega'.$$

Summing up the two terms, one obtains

$$\delta E_\varphi = 16\pi a_1 \delta\left(\frac{ma}{\alpha^2}\right). \tag{4.2}$$

B. Energy

Going over to energy, we calculate the nonvanishing contributions to $\delta E_t = \delta\Gamma_\infty(\partial_t)$,

$$\delta\omega^{12} H_{12t} + \delta\omega^{13} H_{13t} = 2a_1 m \delta\left(\frac{1}{\alpha}\right) d\Omega,$$
$$b^0{}_t \delta H_0 = 4a_1 \delta\left(\frac{m}{\alpha}\right) d\Omega.$$

Hence,

$$\delta E_t = 16\pi a_1 \left[\frac{m}{2}\delta\left(\frac{1}{\alpha}\right) + \delta\left(\frac{m}{\alpha}\right)\right].$$

The result is not δ-integrable but, as we mentioned above, it can be corrected by moving to the untwisted coordinates (T, ϕ):

$$\delta E_T = \delta E_t + \lambda a \delta E_\varphi = 16\pi a_1 \delta\left(\frac{m}{\alpha^2}\right). \tag{4.3}$$

The expressions (4.2) and (4.3) are proportional to the corresponding GR values.

V. ENTROPY

In this section, we analyze the PG part of the boundary term at horizon, $\delta\Gamma_H$, where the Killing vector ξ is given by

$$\xi := \partial_T - \Omega_+ \partial_\phi = \partial_t - \omega_+ \partial_\varphi. \tag{5.1}$$

As will be shown, this part defines the black hole entropy. The Maxwell contribution to $\delta\Gamma_H$ will be discussed in the next section.

In what follows, we use the notation $v_\xi := \xi \lrcorner v$ and $A_0' := a_0 - \lambda b_6$.

A. Nonvanishing terms

The calculation entropy is organized in two technical steps.

1. $\delta\Gamma_1 = \frac{1}{2}\omega^{ij}{}_\xi \delta H_{ij} + \frac{1}{2}\delta\omega^{ij} H_{ij\xi}$

The only nonvanishing contributions stemming from the first element of $\delta\Gamma_1$ are

$$\omega^{01}{}_\xi \delta H_{01} [=] \, \omega^{01}{}_\xi \delta H_{01\theta\varphi}$$
$$= 2A_0' \left(\kappa - V_1 \frac{\rho_+^2}{r_+^2 + a^2}\right) \delta\left(\frac{r_+^2 + a^2}{\alpha}\right) \sin\theta, \tag{5.2a}$$

$$\omega^{03}{}_\xi \delta H_{03} + \omega^{13}{}_\xi \delta H_{13}$$
$$[=] \, K^{03}{}_\xi \delta(H_{03\theta\varphi} + H_{13\theta\varphi}) + \tilde\omega^{13}{}_\xi \delta H_{13\theta\varphi}$$
$$= 2A_0' \left(\frac{1}{N} V_3 \frac{\rho_+^2}{r_+^2 + a^2}\right) \cdot \delta\left(PN\frac{a}{\alpha}\right) \sin^3\theta$$
$$+ 2\lambda b_4 \frac{a r_+ N}{P(r_+^2 + a^2)} \delta\left(\frac{m r_+ - q^2}{N\rho_+^2}\frac{Pa}{\alpha}\right) \sin^3\theta. \tag{5.2b}$$

Here, the symbol $[=]$ stands for an equality up to the factor $d\theta d\varphi$.

In the second element of $\delta\Gamma_1$, there are two more nonvanishing contributions,

$$\delta\omega^{02}H_{02\xi}+\delta\omega^{12}H_{12\xi}$$
$$[=]\,\delta K^{02}{}_{\theta}(H_{02\xi\varphi}+H_{12\xi\varphi})+\delta\tilde{\omega}^{12}{}_{\theta}H_{12\xi\varphi}$$
$$=-2A_0'\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^4}\frac{P}{N}\right)\frac{N\rho_+^2}{P\alpha}\sin\theta$$
$$-2\lambda b_4\delta\left(\frac{NPr_+}{\rho_+^2}\right)\frac{mr_+-q^2}{NP\alpha}\sin\theta, \tag{5.3a}$$

and

$$\delta\omega^{03}H_{03\xi}+\delta\omega^{13}H_{13\xi}$$
$$[=]-\delta K^{03}{}_{\varphi}(H_{03\xi\theta}+H_{13\xi\theta})-\delta\tilde{\omega}^{13}{}_{\varphi}H_{13\xi\theta}$$
$$=-2A_0'\delta\left(\frac{(mr_+-q^2)r_+}{NP\rho_+^2\alpha}\right)\frac{NP\rho_+^2}{r_+^2+a^2}\sin\theta$$
$$-2\lambda b_4\delta\left(\frac{Nr_+}{\alpha P}\right)\frac{mr_+-q^2}{N}\frac{P}{r_+^2+a^2}\sin\theta. \tag{5.3b}$$

2. $\delta\Gamma_2=b^i{}_\xi\delta H_i+\delta b^i H_{i\xi}$

In $\delta\Gamma_2$, the nonvanishing contributions are

$$b^0{}_\xi\delta H_0\,[=]\,b^0{}_\xi\delta H_{0\theta\varphi}$$
$$=2a_1N\frac{\rho_+^2}{r_+^2+a^2}\delta\left(\frac{(mr_+-q^2)r_+}{N\alpha\rho_+^4}(r_+^2+a^2+\rho_+^2)\right)$$
$$\times\sin\theta, \tag{5.4a}$$

$$\delta b^0H_{0\xi}\,[=]-\delta b^0{}_\varphi H_{0\xi\theta}$$
$$=-2a_1\delta\left(\frac{Na}{\alpha}\right)\frac{V_3P}{N}\frac{\rho_+^2}{r_+^2+a^2}\sin^2\theta, \tag{5.4b}$$

$$\delta b^2H_{2\xi}\,[=]\,\delta b^2{}_\theta H_{2\xi\varphi}=2a_1(\delta P)(V_1-V_5)\frac{\sin\theta}{P\alpha}\rho_+^2, \tag{5.4c}$$

$$\delta b^3H_{3\xi}\,[=]-\delta b^3{}_\varphi H_{3\xi\theta}$$
$$=2a_1\delta\left(\frac{r_+^2+a^2}{P\alpha}\right)(V_1-V_5)P\frac{\rho_+^2}{r_+^2+a^2}\sin\theta. \tag{5.4d}$$

B. Simplifications

The above contributions can be simplified using the following properties (see Appendix B):

(i) *S1*. The sum of the terms proportional to $\delta N/N$ in (5.2)–(5.4) vanishes.

(ii) *S2*. The sum of the terms proportional to $\delta P/P$ in (5.2)–(5.4) vanishes.

Hence, the original contributions (5.2)–(5.4) can be simplified as follows:

$$(5.2a):\ 2A_0'\left(\kappa-V_1\frac{\rho_+^2}{r_+^2+a^2}\right)\delta\left(\frac{r_+^2+a^2}{\alpha}\right)\sin\theta,$$

$$(5.2b):\ 2A_0'\frac{a(mr_+-q^2)r_+}{\rho_+^2(r_+^2+a^2)}\cdot\delta\left(\frac{a}{\alpha}\right)\sin^3\theta$$
$$+2\lambda b_4\frac{ar_+}{r_+^2+a^2}\delta\left(\frac{mr_+-q^2}{\rho_+^2}\frac{a}{\alpha}\right)\sin^3\theta.$$

$$(5.3a):\ -2A_0'\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^4}\right)\frac{\rho_+^2}{\alpha}\sin\theta$$
$$-2\lambda b_4\delta\left(\frac{r_+}{\rho_+^2}\right)\frac{mr_+-q^2}{\alpha}\sin\theta,$$
$$(5.3b):\ -2A_0'\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^2\alpha}\right)\frac{\rho_+^2}{r_+^2+a^2}\sin\theta$$
$$-2\lambda b_4\delta\left(\frac{r_+}{\alpha}\right)\frac{mr_+-q^2}{r_+^2+a^2}\sin\theta.$$

$$(5.4a):\ 2a_1\frac{\rho_+^2}{r_+^2+a^2}\delta\left(\frac{(mr_+-q^2)r_+}{\alpha\rho_+^4}(r_+^2+a^2+\rho_+^2)\right)\sin\theta$$
$$(5.4b):\ -2a_1\delta\left(\frac{a}{\alpha}\right)\frac{mr_+-q^2}{\rho_+^4}\frac{\rho_+^2}{r_+^2+a^2}r_+a\sin^3\theta,$$

$$(5.4c):\ =0,$$
$$(5.4d):\ 2a_1\delta\left(\frac{r_+^2+a^2}{\alpha}\right)(V_1-V_5)\frac{\rho_+^2}{r_+^2+a^2}\sin\theta.$$

Next, we use the relation $A_0'=\lambda b_4+a_1$ to express these contributions in terms of only two independent constants, λb_4 and a_1. The analysis of the λb_4 part leads to an additional simplification (Appendix B).

(iii) *S3*. When the λb_4 part is integrated over $d\theta\delta\varphi$, it vanishes.

The conclusions **S1**, **S2**, and **S3** are the KN-AdS extensions of the results found for the Kerr-AdS black holes in [14].

C. The terms proportional to a_1

The property **S3** allows us to simply replace A_0' by a_1 in (5.2) and (5.3), ignoring the vanishing λb_4 terms. Then,

$$(5.2a)+(5.2b)_1+(5.3a)_1+(5.3b)_1:$$
$$2a_1\sin\theta\left[\left(\kappa-V_1\frac{\rho_+^2}{r_+^2+a^2}\right)\delta\left(\frac{r_+^2+a^2}{\alpha}\right)\right.$$
$$+\frac{(mr_+-q^2)r_+}{\rho_+^2(r_+^2+a^2)}\cdot\delta\left(\frac{a}{\alpha}\right)a\sin^2\theta$$
$$\left.-\frac{\rho_+^2}{\alpha}\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^4}\right)-\frac{\rho_+^2}{r_+^2+a^2}\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^2\alpha}\right)\right],$$

$(5.4a) + (5.4b) + (5.4d)$:

$$2a_1 \frac{\rho_+^2}{r_+^2 + a^2} \sin\theta \left[V_1 \delta\left(\frac{r_+^2 + a^2}{\alpha}\right) - \frac{(mr_+ - q^2)r_+}{\rho_+^4} \delta\left(\frac{a}{\alpha}\right) a \sin^2\theta + \frac{r_+^2 + a^2}{\alpha} \delta\left(\frac{(mr_+ - q^2)r_+}{\rho_+^4}\right) + \delta\left(\frac{(mr_+ - q^2)r_+}{\alpha\rho_+^2}\right)\right].$$

All terms except the first one (proportional to κ) cancel each other, so that the sum becomes

$$2a_1 \kappa \sin\theta \delta\left(\frac{r_+^2 + a^2}{\alpha}\right). \tag{5.5}$$

Then, the integration over $d\theta d\varphi$ yields

$$(\delta\Gamma_H)^{PG} = 8\pi a_1 \kappa \delta\left(\frac{r_+^2 + a^2}{\alpha}\right) = T\delta S, \qquad S \coloneqq 16\pi a_1 \frac{\pi(r_+^2 + a^2)}{\alpha}, \tag{5.6}$$

where $T \coloneqq \kappa/2\pi$ is the temperature. Thus, entropy is also proportional to the GR value.

VI. MAXWELL BOUNDARY TERM AND THE FIRST LAW

The standard canonical analysis of the Maxwell sector implies that the asymptotic electric charge Q can be defined by the boundary integral

$$Q = -\int_{S_\infty} H = 4a_1 \int_{S_\infty} \frac{q_e}{\rho^4}(r^2 - a^2\cos^2\theta) b^2 b^3 = 16\pi a_1 \frac{q_e}{\alpha}. \tag{6.1}$$

The minus sign is just a matter of convention. Next, following Ref. [19], we define the electric potential Φ by

$$\Phi \coloneqq A_\xi \Big|_{r_+}^{\infty} = -\frac{q_e r_+}{\rho_+^2 N} b^0{}_\xi \Big|_{r_+}^{\infty} = \frac{q_e r_+}{r_+^2 + a^2}. \tag{6.2}$$

Then, the Maxwell contribution on horizon has the form

$$(\delta\Gamma_H)^M = A_\xi \delta H + (\delta A) H_\xi = A_\xi \delta H = \Phi\delta Q. \tag{6.3}$$

Combining this relation with the result obtained in Eqs. (4.2), (4.3) and (5.6), one can immediately conclude that the first law $\delta\Gamma_H = \delta\Gamma_\infty$ takes the form

$$T\delta S + \Phi\delta Q = \delta E_T - \Omega_+ \delta E_\varphi. \tag{6.4}$$

The result is confirmed by the identity (C.2). After removing the common factor $16\pi a_1$, the first law (6.4) becomes identical to its GR form.

In our approach to black hole thermodynamics, all Lagrangian parameters, including the cosmological constant Λ, are treated as constants. However, in recent years, an alternative formalism has been developed in which Λ is promoted to a new thermodynamic variable, the vacuum pressure, and, as a consequence, the first law is modified; for more details see [22]. The consistency of the new formalism in the presence of torsion has not yet been examined.

VII. CONCLUDING REMARKS

The canonical approach to black hole entropy proposed in [12] has been successfully applied to a number of vacuum solutions of PG [14–16]. In the present paper, we introduce its natural extension to *nonvacuum* solutions, by including the Maxwell field as a matter source of gravity. Using this formalism, we study thermodynamic properties of KN-AdS black holes, encoded in the boundary terms at infinity and horizon, $\delta\Gamma_\infty$ and $\delta\Gamma_H$, respectively.

Analyzing energy and angular momentum as the boundary terms at infinity, we found that their KN-AdS values are exactly the same as for the uncharged Kerr-AdS solution [4,14]. This is in agreement with the fact that the asymptotic Maxwell contribution vanishes. Moreover, these asymptotic charges are proportional to the related GR expressions.

The boundary term at horizon produces entropy and an external, Maxwell term, such that both of them are also proportional to the corresponding GR expressions [4,15]. Then, the first law is described by the general relation $\delta\Gamma_\infty = \delta\Gamma_H$, which follows from the way the boundary terms are constructed, see Sec. II B. Apart from this general argument, we give an explicit proof of the first law based on the identity derived in Appendix B. After removing the overall multiplicative factor, the first law becomes identical to its GR form.

Thus, although PG has a rather different dynamical structure from GR, the present description of the KN-AdS thermodynamics is rather close to the GR results. A reason for this "accidental" similarity might be hidden in the identity found in Appendix B.

ACKNOWLEDGMENTS

This work was partially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

APPENDIX A: TECHNICAL FORMULAS

The condition of vanishing torsion, $d\vartheta^i + \omega^i{}_k \vartheta^k = 0$, defines the Riemannian connection:

$$
\begin{aligned}
\tilde\omega^{01} &= -N' b^0 - \frac{ar}{P\rho^2}\sin\theta b^3,\\
\tilde\omega^{02} &= \frac{a^2\sin\theta\cos\theta}{P\rho^2} b^0 - \frac{aN}{\rho^2}\cos\theta b^3,\\
\tilde\omega^{03} &= -\frac{ar}{P\rho^2}\sin\theta b^1 + \frac{aN}{\rho^2}\cos\theta b^2,\\
\tilde\omega^{12} &= \frac{a^2\sin\theta\cos\theta}{\rho^2 P} b^1 + \frac{rN}{\rho^2} b^2,\\
\tilde\omega^{13} &= -\frac{ar}{P\rho^2}\sin\theta b^0 + \frac{Nr}{\rho^2} b^3,\\
\tilde\omega^{23} &= -\frac{aN}{\rho^2}\cos\theta b^0 + \frac{P\cos\theta - \partial_\theta P\sin\theta}{P^2\sin\theta} b^3.
\end{aligned} \tag{A1}
$$

Some general relations:

$$
\begin{aligned}
&N\partial_r N|_{r_+} = \frac{\kappa(r_+^2+a^2)}{\rho_+^2},\\
&(\xi \lrcorner \vartheta^0)|_{r_+} = N\frac{\rho_+^2}{r_+^2+a^2}, \qquad (\xi \lrcorner \vartheta^a)|_{r_+} = 0.
\end{aligned} \tag{A2}
$$

Interior products $\xi \lrcorner \tilde\omega^{ij}$:

$$
\begin{aligned}
&\xi \lrcorner \tilde\omega^{01} = -N'(\xi \lrcorner b^0) = -\kappa, \qquad \xi \lrcorner \tilde\omega^{02} = \frac{Na^2\sin\theta\cos\theta}{P(r_+^2+a^2)},\\
&\xi \lrcorner \tilde\omega^{13} = -\frac{Nar_+}{P(r_+^2+a^2)}\sin\theta,\\
&\xi \lrcorner \tilde\omega^{03} = \xi \lrcorner \tilde\omega^{12} = 0, \qquad \xi \lrcorner \tilde\omega^{23} \sim N^2.
\end{aligned}
$$

The explicit form of the covariant momenta H_i and H_{ij} is given by

$$
\begin{aligned}
H_0 &= \frac{4a_1}{N}[-V_4 b^0 b^1 + V_5 b^2 b^3] + \frac{2a_1}{N^2}[-V_2 b^- b^3 + V_3 b^- b^2)],\\
H_1 &= -H_0,\\
H_2 &= \frac{2a_1}{N}[(V_1 - V_5) b^- b^3 - V_4 b^- b^2],\\
H_3 &= \frac{2a_1}{N}[-(V_1 - V_5) b^- b^2 - V_4 b^- b^3].
\end{aligned} \tag{A3}
$$

and

$$
\begin{aligned}
H_{01} &= -2A_0' b^2 b^3,\\
H_{02} &= 2A_0' b^1 b^3 + 2b_4\frac{\lambda}{\Delta}(mr - q^2) b^- b^3,\\
H_{12} &= -2A_0' b^0 b^3 - 2b_4\frac{\lambda}{\Delta}(mr - q^2) b^- b^3,\\
H_{03} &= -2A_0' b^1 b^2 - 2b_4\frac{\lambda}{\Delta}(mr - q^2) b^- b^2,\\
H_{13} &= 2A_0' b^0 b^2 + 2b_4\frac{\lambda}{\Delta}(mr - q^2) b^- b^2,\\
H_{23} &= -2A_0' b^0 b^1.
\end{aligned} \tag{A4}
$$

APPENDIX B: ON THE EVALUATION OF ENTROPY

In this appendix, we discuss certain technical details of the derivation of entropy.

1. Elimination of $\delta N/N$ and $\delta P/P$ terms

Starting from the basic results on entropy obtained in Eqs. (5.2)–(5.4) in Sec. V A, we are now going to show that both $\delta N/N$ and $\delta P/P$ terms cancel out.

Consider first the coefficients of the $\delta N/N$ terms. By a suitable rearrangement of these coefficients, shown in the following formulas,

$$
\begin{aligned}
&(5.3a)+(5.3b):\ 2(A_0' - \lambda b_4)\frac{r_+(mr_+ - q^2)}{\alpha\rho_+^2}\left(1 + \frac{\rho_+^2}{r_+^2+a^2}\right)\\
&\qquad \times \sin\theta,\\
&(5.4a):\ -2a_1\frac{r_+(mr_+ - q^2)}{\alpha\rho_+^2}\left(1 + \frac{\rho_+^2}{r_+^2+a^2}\right)\sin\theta,
\end{aligned}
$$

one can directly conclude that their sum vanishes, as a consequence of $A_0' \equiv a_1 + \lambda b_4$.There are two more contributions of this type,

$$
\begin{aligned}
&(5.2b):\ 2(A_0' - \lambda b_4)\frac{(mr_+ - q^2)r_+ a^2}{\alpha\rho_+^2(r_+^2+a^2)}\sin^3\theta,\\
&(5.4b):\ -2a_1\frac{(mr_+ - q^2)r_+ a^2}{\alpha\rho_+^2(r_+^2+a^2)}\sin^3\theta,
\end{aligned}
$$

whose sum also vanishes. Hence, all $(\delta N)/N$ terms in entropy can be simply ignored.

After removing $\delta N/N$ terms, one finds that the sum of $\delta P/P$ terms also vanishes:

$$
\begin{aligned}
&(5.2b) + (5.3a) + (5.3b):\ 0,\\
&(5.4c) + (5.4d):\ 0.
\end{aligned}
$$

2. Elimination of λb_4 terms

After eliminating all $\delta N/N$ and $\delta P/P$ terms, one can use the relation $A_0' = a_1 + \lambda b_4$ in Eqs. (5.2) and (5.3), Sec. V B, to express them in terms of only two independent parameters, a_1 and λb_4. Focusing on the λb_4 terms and omitting the overall factor $2\lambda b_4$, the resulting contributions take the form

$$
\begin{aligned}
&(5.2a):\ \left[\kappa - V_1\frac{\rho_+^2}{r_+^2+a^2}\right]\delta\left(\frac{r_+^2+a^2}{\alpha}\right)\sin\theta,\\
&(5.2b):\ \left[\frac{a(mr_+ - q^2)r_+}{\rho_+^2(r_+^2+a^2)}\delta\left(\frac{a}{\alpha}\right) + \frac{ar_+}{r_+^2+a^2}\delta\left(\frac{mr_+ - q^2}{\rho_+^2}\frac{a}{\alpha}\right)\right]\\
&\qquad \times \sin^3\theta,
\end{aligned}
$$

$$(5.3a)\colon\ -\left[\frac{\rho_+^2}{\alpha}\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^4}\right)+\frac{mr_+-q^2}{\alpha}\delta\left(\frac{r_+}{\rho_+^2}\right)\right]\times\sin\theta,$$

$$(5.3b)\colon\ -\left[\frac{\rho_+^2}{r_+^2+a^2}\delta\left(\frac{(mr_+-q^2)r_+}{\rho_+^2\alpha}\right)+\frac{mr_+-q^2}{r_+^2+a^2}\delta\left(\frac{r_+}{\alpha}\right)\right]\times\sin\theta.$$

Step 1. Let us first transform the first term in (5.2a) using the identity (C.2),

$$(5.2a)\colon\ 2\left[\delta\left(\frac{m}{\alpha^2}\right)-\Omega_+\delta\left(\frac{am}{\alpha^2}\right)-\frac{2r_+q}{r_+^2+a^2}\delta\left(\frac{q}{\alpha}\right)-V_1\frac{\rho_+^2}{r_+^2+a^2}\delta\left(\frac{r_+^2+a^2}{\alpha}\right)\right]\sin\theta.$$

The result can be conveniently written as a sum of two parts, proportional to $\delta(mr_+-q^2)$ and (mr_+-q^2),

$$(5.2a)_1\colon\ \frac{2r_+}{\alpha}\frac{\sin\theta}{r_+^2+a^2}\delta(mr_+-q^2),$$

$$(5.2a)_2\colon\ -\frac{2r_+(mr_+-q^2)}{(r_+^2+a^2)\rho_+^2}\left[(r_+^2+a^2-2\rho_+^2)\delta\left(\frac{1}{\alpha}\right)+\frac{\delta(r_+^2+a^2)}{\alpha}\right]\sin\theta,$$

where we used the identities

$$V_1\rho_+^2=\frac{2r_+(mr_+-q^2)}{\rho_+^2}-m,$$

$$\Omega_+=\frac{a\alpha}{r_+^2+a^2}+\lambda a,$$

$$(1-\lambda a^2)\delta\left(\frac{1}{\alpha^2}\right)-\frac{\lambda a}{\alpha^2}\delta a=\frac{3}{2}\delta\left(\frac{1}{\alpha}\right).$$

Step 2. Looking at the remaining contributions in (5.2b), (5.3a) and (5.3b), one again finds two types of terms. The part proportional to $\delta(mr_+-q^2)$ is given by

$$[(5.2b)+(5.3a)+(5.3b)]_1\colon\ -\frac{2r_+}{\alpha}\frac{\sin\theta}{r_+^2+a^2}\delta(mr_+-q^2),$$

and it directly cancels the contribution $(5.2a)_1$ given above, as expected.

As far as the part proportional to (mr_+-q^2) is concerned, we find it convenient to separate the terms proportional to δr_+, $\delta(1/\alpha)$ and the remaining $a\delta a$ terms[2]:

$$[(5.2b)+(5.3a)+(5.3b)]_2\colon$$

$$\delta r_+\colon -\frac{2(mr_+-q^2)(r_+^2+a^2+\rho_+^2)}{\alpha\rho_+^2(r_+^2+a^2)}\left(1-\frac{2r_+^2}{\rho_+^2}\right)\sin\theta,$$

$$\delta\left(\frac{1}{\alpha}\right)\colon \frac{2r_+(mr_+-q^2)}{(r_+^2+a^2)\rho_+^2}(a^2\sin^2\theta-\rho_+^2)\sin\theta,$$

$$a\delta a\colon \frac{2r_+(mr_+-q^2)}{\alpha\rho_+^2(r_+^2+a^2)}\left(\sin^2\theta+\frac{2(r_+^2+a^2+\rho_+^2)}{\rho_+^2}\cos^2\theta\right)\sin\theta.$$

Summing these terms with the corresponding expressions in $(5.2a)_2$, one obtains

$$\delta r_+\colon -\frac{2(mr_+-q^2)}{\alpha}\underline{\left(\frac{1}{\rho_+^2}+\frac{1}{r_+^2+a^2}-\frac{2r_+^2}{\rho_+^4}\right)\sin\theta}_{\times},$$

$$\delta\left(\frac{1}{\alpha}\right)\colon 0,$$

$$a\delta a\colon \frac{2r_+(mr_+-q^2)}{\alpha(r_+^2+a^2)}\underline{\left(-\frac{\sin^2\theta}{\rho_+^2}+\frac{2(r_+^2+a^2)\cos^2\theta}{\rho_+^4}\right)\sin\theta}_{\times}.$$

Since the integrals over θ of the underlined terms vanish, it follows that the total contribution proportional to λb_4 also vanishes.

APPENDIX C: PROOF OF THE FIRST LAW

In this appendix, we derive an identity which is of essential importance for understanding the kinematic origin of the first law.

We start by introducing the notation

$$M\coloneqq\frac{m}{\alpha^2},\qquad J\coloneqq Ma,\qquad \Phi\coloneqq\frac{q_e r_+}{r_+^2+a^2}.$$

After using the horizon equation to express δr_+ in terms of $(\delta m,\delta q_e,\delta a)$, one finds

$$L\coloneqq\frac{\kappa}{2}\delta\left(\frac{r_+^2+a^2}{\alpha}\right)=L_m\delta m+L_a\delta a-\Phi\delta q_e,$$

$$L_m\coloneqq\frac{r_+^2}{\alpha(r_+^2+a^2)},$$

$$L_a=\frac{a(1+\lambda r_+^2)(-1+3\lambda r_+^2)}{2\alpha^2 r_+}-\frac{a(1+\lambda r_+^2)q_e^2}{2\alpha^2 r_+(r_+^2+a^2)}.\quad\text{(C1a)}$$

In an analogous manner, one obtains the relation

[2]The *remaining* $a\delta a$ terms are those that do not stem from $\delta\alpha$.

$$R := \delta M - \Omega_+ \delta J - \Phi\delta\left(\frac{q_e}{\alpha}\right) = R_m \delta m + R_a \delta a - \Phi\delta\left(\frac{q_e}{\alpha}\right),$$

$$R_m := L_m,$$

$$R_a = \frac{a(-1+3\lambda r_+^2)(1+\lambda r_+^2)}{2\alpha^2 r_+} + \frac{a(-1+3\lambda r_+^2)q_e^2}{2\alpha^2 r_+(r_+^2+a^2)}. \qquad \text{(C1b)}$$

Then, a direct comparison shows that the relation $L = R$ is identically satisfied:

$$\frac{\kappa}{2}\delta\left(\frac{r_+^2+a^2}{\alpha}\right) = \delta M - \Omega_+ \delta J - \Phi\delta\left(\frac{q_e}{\alpha}\right). \qquad \text{(C2)}$$

This identity coincides with the first law in GR.

[1] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).

[2] R. P. Kerr, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics, Phys. Rev. Lett. **11**, 237 (1963).

[3] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).

[4] G. W. Gibbons and S. W. Hawking, Cosmological event horizons, thermodynamics, and particle creation, Phys. Rev. D **15**, 2738 (1977).

[5] G. W. Gibbons, M. J. Perry, and C. N. Pope, The first law of thermodynamics for Kerr-anti-de Sitter black holes, Classical Quantum Gravity **22**, 1503 (2005).

[6] *Gauge Theories of Gravitation, A Reader with Commentaries*, edited by M. Blagojević and F. W. Hehl (Imperial College Press, London, 2013). Comments on exact solution of PG are presented in Sec. 16.

[7] P. Baekler, A spherically symmetric vacuum solution of the quadratic Poincaré gauge field theory of gravitation with Newtonian and confinement potentials, Phys. Lett. **99B**, 329 (1981); Ch. H. Lee, A spherically symmetric electro-vacuum solution of the Poincaré gauge field theory of gravitation, Phys. Lett. **130B**, 257 (1983); J. D. McCrea, P. Baekler, and M. Gürses, A Kerr-like solution of the Poincaré gauge field equations, Nuovo Cimento B **99**, 171 (1987).

[8] P. Baekler, M. Gürses, F. W. Hehl, and J. D. McCrea, The exterior gravitational field of a charges spinning source in the Poincaré gauge theory: A Kerr-Newmann metric with dynamic torsion, Phys. Lett. A **128**, 245 (1988).

[9] Yu. N. Obukhov, Exact solutions in Poincaré gauge gravity theory, Universe **5**, 127 (2019).

[10] Ch.-M. Chen, J. M. Nester, and R.-S. Tung, Gravitational energy for GR and Poincaré gauge theory: A covariant Hamiltonian approach, Int. J. Mod. Phys. D **24**, 1530026 (2015).

[11] R. D. Hecht and J. M. Nester, A new evaluation of PGT mass and spin, Phys. Lett. A **180**, 324 (1993); Ch.-M. Chen and J. M. Nester, Quasi local quantities for GR and other gravity theories, Classical Quantum Gravity **16**, 1279 (1999).

[12] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian ap-proach, Phys. Rev. D **99**, 104058 (2019).

[13] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D **48**, R3427 (1993); The thermodynamics of black holes, Living Rev. Relativity **4**, 6 (2001).

[14] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Kerr-AdS solution, Phys. Rev. D **102**, 064034 (2020).

[15] M. Blagojević and B. Cvetković, Entropy in general relativity: Kerr-AdS black hole, Phys. Rev. D **101**, 084023 (2020); Thermodynamics of Riemannian Kerr-AdS black holes in Poincaré gauge theory, Phys. Lett. B **816**, 136242 (2021).

[16] M. Blagojević and B. Cvetković, Entropy of Reissner-Nordström-like black holes, Phys. Lett. B **824**, 136815 (2022).

[17] F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics* (Birkhäuser, Boston, 2003).

[18] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88**, 286 (1974).

[19] M. M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Classical Quantum Gravity **17**, 399 (2000).

[20] B. Carter, Black hole equilibrium states, in *Black Holes, 1972 Les Houches Lectures*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), pp. 58–214.

[21] M. Henneaux and C. Teitelboim, Hamiltonian treatment of asymptotically anti-de Sitter spaces, Phys. Lett. **142B**, 355 (1984); Asymptotically anti-de Sitter spaces, Commun. Math. Phys. **98**, 391 (1985).

[22] J. P. S. Lemos and O. B. Zaslavskii, Black hole thermodynamics with the cosmological constant as independent variable: Bridge between the enthalpy and the Euclidean path integral approaches, Phys. Lett. B **786**, 296 (2018).