Birefringence of wave packets in gravity

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In most experiments to test the weak equivalence principle, the test objects coupled with gravity are treated as point particles. To study whether the weak equivalence principle is obeyed by nonlocal objects, especially wave packets with interesting physical configurations, one must first properly define the trajectory of such objects. A phenomenon called the geometric spin Hall effect suggests that one can directly use the energy-momentum tensor to describe the motion of wave packets. In this paper, we construct spin-polarized free-falling electromagnetic wave packets in gravity, and calculate the evolution of the center of the energy-momentum tensor. We find that the electromagnetic wave packets with opposite helicity are separated in the direction perpendicular to spin and gravity. This behavior is thus a kind of gravitational birefringence, and it means that motions of these spin-polarized wave packets in gravity do violate the weak equivalence principle. Furthermore, we find that the trajectories defined by different components and expressions of the energy-momentum tensor are not the same, and they are all different from the trajectory given by the Mathisson-Papapetrou-Dixon equations with the constraint $p_\mu S^{\mu\rho} = 0$. This suggests that the trajectory of spin-polarized wave packets in gravity, and its confrontation with the weak equivalence principle, depend not only on the gravitational interaction but also strongly on how the wave packets are measured and analyzed.

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I. INTRODUCTION

There are many experiments aiming to test the weak equivalence principle (WEP) by using classical objects, such as the GP-B [1–3] and GINGERino [4–6] experiments. It has also been proposed to use a quantum object to test the WEP [7–9]. However, the test objects coupled with gravity in these experiments are treated as point particles. It means that these experiments cannot tell us whether a nonlocal object obeys the weak equivalence principle or not. There are also several approaches aiming to describe the motion of a nonlocal object in the gravitational field. Using a multipole expansion of the energy-momentum tensor (EMT), the dynamics of the spinning test objects have been studied in the form of the Mathisson-Papapetrou-Dixon (MPD) equations [10–13]. But it is impossible to find a unique trajectory by solving the MPD equations directly unless we artificially add a constraint to the angular momentum $S^{\mu\nu}$. So the different artificial constraints would lead to different trajectories. With the constraint $S^{0\rho} = 0$, Papapetrou finds that the massive test particles with different spins are separated in the transverse direction [14], and this phenomenon is called gravitational birefringence. However, the constraint $S^{0\rho} = 0$ does not make sense for the massless particles and it cannot determine a unique world line [15,16]. By using the constraint $p_\mu S^{\mu\rho} = 0$, Duval finds a different trajectory of massless particles [17]. He also finds that the angle between the trajectories of test particles with opposite helicity is $\theta = -\lambda/\pi r_0 + 4GM\lambda/\pi r_0^2$ when they come from a finite distance r_0 to infinity [17]. But it is unreasonable that there is a gravity-independent term $-\lambda/\pi r_0$ in the angle θ given by Duval. Oancea finds a third different trajectory from the Maxwell equations in gravity by taking the Wentzel-Kramers-Brillouin approximation, and there is no gravity-independent term in the angle θ [18]. There is also another example called the Frenkel-Pirani constraint $u_\mu S^{\mu\nu} = 0$ [19–21]. These three constraints show different advantages in describing the motion of objects, and more importantly, the differences between the trajectories given by them are not negligible [15]. But it is unclear how these different constraints and trajectories are to be correlated with actual experiments.

The energy flux center of a polarized light beam will have a transverse shift when it is observed from a reference frame tilted with respect to the direction of propagation of the beam, and this phenomenon is called the geometric spin Hall effect (GSHE) [22]. It suggests that we can directly use

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EMT to describe the motion of wave packets in gravity. So, in this paper, we construct a spin-polarized free-falling electromagnetic wave packet in gravity. Then we calculate the center of EMT. We find there is an intersection angle θ between the propagations of the two wave packets with opposite helicity when they come from a finite distance r_0. However, in our results, the angle is $\theta \simeq \alpha GM\lambda/\pi r_0^2$, where the factor α depends on using which component and expression of the EMT to describe the motion of the wave packets. We find $\alpha = 2$ when using the energy density of the symmetric EMT to describe the motion of the wave packets. In this case, the angle θ given by us is the same as the gravity-dependent term $2GM\lambda/\pi r_0^2$ in the angle given by Duval. But the trajectory given by the energy density of the symmetric EMT is different from the trajectory given by Duval because of the gravity-independent term $-\lambda/\pi r_0$. This phenomenon means that the motions of spin-polarized wave packets in gravity do violate the WEP.

We also find $\alpha = 1$ in the angle θ when using the center of energy flux of the symmetric EMT to describe the motion of the same spin-polarized wave packet. So, the trajectories defined by different components of the EMT are different from each other. When using the center of energy flux of the canonical EMT to describe, we find $\alpha = 1/2$. This phenomenon means there are also differences between the trajectories given by different expressions of the EMT. The former difference is mainly due to the GSHE induced by gravity, while the latter difference is caused by the evolution of spin angular momentum in the gravitational field. So, we can also know which expression of EMT would effectively describe the interaction between matter and gravity by observing the gravitational birefringence of spin-polarized wave packets.

The birefringence of light in materials has been studied using the Fresnel equation and corresponding Kummer surfaces [23–25]. Propagation of light in the gravitational field can also be described by an effective optical material [26,27]. By comparing the Maxwell equations in the gravitational field and materials, we find an effective Fresnel equation of light propagating in gravity. The so-obtained angle between the two light beams with opposite helicity agrees with the one given by the energy density of the symmetric EMT.

Our paper is organized as follows. In Sec. II, we construct a spin-polarized free-falling wave packet in Schwarzschild spacetime and solve the Maxwell equations of the wave packet. In Sec. III, we numerically calculate the center of the EMT, and find that the wave packets with opposite helicity will be separated in the transverse direction near the gravity source. We also discuss the differences between the trajectories defined by different EMTs. In Sec. IV, we discuss the birefringence of wave packets in Schwarzschild spacetime when they move to infinity. We also discuss the differences between our results and those given by the MPD equations. In Sec. V, we study the propagation of light in gravity described by an effective optical material. A summary of the main results is given in Sec. VI.

II. A FREE-FALLING SPIN-POLARIZED WAVE PACKET IN GRAVITY

We construct a spin-polarized free-falling electromagnetic wave packet in gravity. Its initial position and velocity are $\vec{X} = (0, 0, 0)$ and $\vec{v} = (0, 0, 1)$. In this paper, we set $\hbar = c = 1$. The gravity is along the negative direction of the x axis. Figure 1 is a simple diagram of this physical system.

In general relativity, the motion of a electromagnetic wave packet is determined by the Maxwell equations:

$$\nabla_\mu F^{\mu\nu} = 0. \tag{1}$$

By applying the covariant Lorentz gauge $\nabla_\lambda A^\lambda = 0$, it becomes

$$\nabla_\mu \nabla^\mu A^\nu = 0. \tag{2}$$

When we adopt the harmonic coordinates, the Schwarzschild metric reads

$$d\tau^2 = \frac{1 - GM/r}{1 + GM/r} dt^2 - \left(1 + \frac{GM}{r}\right)^2 d\vec{x}^2 - \left(\frac{1 + GM/r}{1 - GM/r}\right) \frac{G^2 M^2}{r^4} (\vec{x} \cdot d\vec{x})^2. \tag{3}$$

In a weak gravitational field, we can expand the metric and keep it to the first order of GM/r:

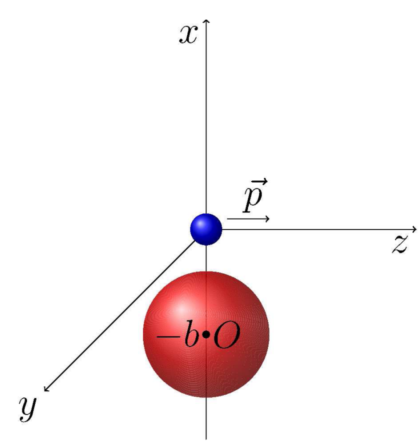


FIG. 1. The big red ball represents the source of gravity, and O is its center. The coordinates of O are $(-b, 0, 0)$. The small blue ball is a free-falling wave packet in gravity, and its initial momentum $\vec{p}$ is along the z axis.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} = \frac{2GM}{r}\delta_{\mu\nu}. \tag{4}$$

Furthermore, we treat the influence of gravity on the wave packet as a perturbation term $\tilde{A}^\nu$. Then we have $A^\nu = \bar{A}^\nu + \tilde{A}^\nu$. $\bar{A}^\nu$ is the zeroth order of the wave packet and is given by

$$\eta^{\mu\lambda}\partial_\mu\partial_\lambda\bar{A}^\nu = 0. \tag{5}$$

The perturbation term $\tilde{A}^\nu$ is determined by the first-order terms of GM/r in the Maxwell equations:

$$\partial_\lambda\partial^\lambda\tilde{A}^\rho = h\delta^{\lambda\alpha}\partial_\lambda\partial_\alpha\bar{A}^\rho + \eta^{\lambda\alpha}\Gamma^\beta_{\lambda\alpha}\partial_\beta\bar{A}^\rho - \partial^\alpha\Gamma^\rho_{\alpha\beta}\bar{A}^\beta - 2\Gamma^\rho_{\alpha\beta}\partial^\alpha\bar{A}^\beta, \tag{6}$$

where $\Gamma^\alpha_{\rho\lambda} = \frac{1}{2}\eta^{\alpha\gamma}(\partial_\rho h\delta_{\lambda\gamma} + \partial_\lambda h\delta_{\rho\gamma} - \partial_\gamma h\delta_{\lambda\rho})$. Then Eq. (6) is simplified to

$$(-\partial_t^2 + \nabla^2)\tilde{A}^\rho = 2h\nabla^2\bar{A}^\rho + 2\partial^\rho h\partial_i\bar{A}^i - \delta^{\rho\alpha}\partial_i\bar{A}_\alpha\partial_i h - \delta^{\rho\alpha}\partial_\alpha\bar{A}^i\partial_i h, \tag{7}$$

where $h = 2GM/r$. We have taken $\bar{A}^0 = 0$ in Eq. (7).

Equation (7) is much simpler than the Maxwell equations in gravity, but it is still too difficult to solve. To simplify the calculation, we assume that the wave packet's radius R is much smaller than the impact parameter b, and that for the studied duration the wave packet travels a distance much smaller than b. Therefore, we can expand $h_{\mu\nu}$ at the initial position $\vec{X} = (-b, 0, 0)$. When we keep $h_{\mu\nu}$ to the first order of x^i/b, h and $\partial_i h$ become

$$h \simeq \frac{2GM}{b}\left(1 - \frac{x}{b}\right), \qquad \partial_x h \simeq -\frac{2GM}{b^2} + \frac{4GMx}{b^3}, \qquad \partial_y h \simeq -\frac{2GMy}{b^3}, \qquad \partial_z h \simeq -\frac{2GMz}{b^3}. \tag{8}$$

Therefore, the evolution of $\tilde{A}^0$ in gravity is determined by

$$(-\partial_t^2 + \nabla^2)\tilde{A}^0 = \left(\frac{2GM}{b^2} - \frac{4GMx}{b^3}\right)\partial_t\bar{A}^1 + \frac{2GMy}{b^3}\partial_t\bar{A}^2 + \frac{2GMz}{b^3}\partial_t\bar{A}^3. \tag{9}$$

We can apply a Fourier transform to Eq. (9), which then becomes

$$(\partial_t^2 + \omega^2)\tilde{A}^0_f = \left(\frac{2GM}{b^2} - \frac{4GMi}{b^3}\frac{\partial}{\partial k_x}\right)(i\omega\bar{A}^1_f) - \frac{2GM}{b^3}\frac{\partial}{\partial k_y}(\omega\bar{A}^2_f) - \frac{2GM}{b^3}\frac{\partial}{\partial k_z}(\omega\bar{A}^3_f), \tag{10}$$

where $\bar{A}^i_f$ and $\tilde{A}^\rho_f$ are

$$\bar{A}^i_f = \frac{1}{(2\pi)^3}\int \bar{A}^i \exp(-i\vec{k}\cdot\vec{x})d^3x, \qquad \tilde{A}^\rho_f = \frac{1}{(2\pi)^3}\int \tilde{A}^\rho \exp(-i\vec{k}\cdot\vec{x})d^3x. \tag{11}$$

By using the same method, we find that $\tilde{A}^i_f$ is given by

$$\begin{cases} (\partial_t^2 + \omega^2)\tilde{A}^1_f = \frac{4GM}{b}\left(1 - \frac{i}{b}\frac{\partial}{\partial k_x}\right)(\omega^2\bar{A}^1_f) - \frac{4GM}{b^2}\left(1 - \frac{2i}{b}\frac{\partial}{\partial k_x}\right)(ik_x\bar{A}^1_f) - \frac{2GMi}{b^3}\frac{\partial}{\partial k_y}(ik_y\bar{A}^1_f + ik_x\bar{A}^2_f) \\ \qquad - \frac{2GMi}{b^3}\frac{\partial}{\partial k_z}(ik_z\bar{A}^1_f + ik_x\bar{A}^3_f), \\ (\partial_t^2 + \omega^2)\tilde{A}^2_f = \frac{4GM}{b}\left(1 - \frac{i}{b}\frac{\partial}{\partial k_x}\right)(\omega^2\bar{A}^2_f) - \frac{2GM}{b^2}\left(1 - \frac{2i}{b}\frac{\partial}{\partial k_x}\right)(ik_x\bar{A}^2_f + ik_y\bar{A}^1_f) - \frac{4GMi}{b^3}\frac{\partial}{\partial k_y}(ik_y\bar{A}^2_f) \\ \qquad - \frac{2GMi}{b^3}\frac{\partial}{\partial k_z}(ik_z\bar{A}^2_f + ik_y\bar{A}^3_f), \\ (\partial_t^2 + \omega^2)\tilde{A}^3_f = \frac{4GM}{b}\left(1 - \frac{i}{b}\frac{\partial}{\partial k_x}\right)(\omega^2\bar{A}^3_f) - \frac{2GM}{b^2}\left(1 - \frac{2i}{b}\frac{\partial}{\partial k_x}\right)(ik_x\bar{A}^3_f + ik_z\bar{A}^1_f) - \frac{2GMi}{b^3}\frac{\partial}{\partial k_y}(ik_y\bar{A}^3_f + ik_z\bar{A}^2_f) \\ \qquad - \frac{3GMi}{b^3}\frac{\partial}{\partial k_z}(ik_z\bar{A}^3_f) \end{cases}. \tag{12}$$

To find the perturbation term $\tilde{A}^\nu_f$ by solving Eqs. (10) and (12), we need to construct a zeroth-order term $\bar{A}^i_f$. A spin-polarized electromagnetic wave packet $\bar{A}^i$ in the Minkowski spacetime is

$$\bar{A}^i = \int \frac{1}{\omega^2}\begin{pmatrix} k_z^2 + k_y^2 - i\sigma k_x k_y \\ i\sigma k_z^2 + i\sigma k_x^2 - k_x k_y \\ -i\sigma k_y k_z - k_x k_z \end{pmatrix} \exp\left(-\frac{(\vec{k} - \vec{k}_0)^2}{2\Delta k^2} - i\omega t\right)\exp(i\vec{k}\cdot\vec{x})d^3k, \tag{13}$$

where $\sigma = \pm 1$ is the helicity of the wave packet $\bar{A}^i$, and $\omega = |\vec{k}|$. When we take $\vec{k} = (0, 0, k_0)$, the momentum of the wave packet $\bar{A}^i$ is $\vec{p} = (0, 0, k_0)$. This means the wave packet will move along the third direction if the gravity vanishes. According to Eq. (11), the $\bar{A}^i_f$ is given by

$$\bar{A}^i_f = \frac{1}{\omega^2} \begin{pmatrix} k_z^2 + k_y^2 - i\sigma k_x k_y \\ i\sigma k_z^2 + i\sigma k_x^2 - k_x k_y \\ -i\sigma k_y k_z - k_x k_z \end{pmatrix} \exp\left(-\frac{(\vec{k} - \vec{k}_0)^2}{2\Delta k^2} - i\omega t\right). \tag{14}$$

After straightforward calculations, we find that the equations of $\tilde{A}^\rho_f$ have the following form:

$$(\partial_t^2 + \omega^2)\tilde{A}^\rho_f = c^\rho(k)\exp(-i\omega t) + d^\rho(k)\exp(-i\omega t)t, \tag{15}$$

where factors $c^\rho(k)$ and $d^\rho(k)$ are determined by Eqs. (10)–(14). When we take the following initial conditions:

$$\tilde{A}^\rho_f|_{t=0} = 0 \quad \text{and} \quad \partial_t \tilde{A}^\rho_f|_{t=0} = 0; \tag{16}$$

Eq. (15) has a solution:

$$\tilde{A}^\rho_f = \frac{c^\rho(k)(1 - \exp(2i\omega t) + 2i\omega t)\exp(-i\omega t)}{4\omega^2} + \frac{d^\rho(k)(2\omega t - i + 2i\omega^2 t^2 + i\exp(2i\omega t))\exp(-i\omega t)}{8\omega^3}. \tag{17}$$

Finally, a spin-polarized electromagnetic wave packet A^ρ in gravity is

$$A^\rho = \bar{A}^\rho + \tilde{A}^\rho, \\ = \int \bar{A}^\rho_f \exp(i\vec{k}\cdot\vec{x})d^3k + \int \tilde{A}^\rho_f \exp(i\vec{k}\cdot\vec{x})d^3k. \tag{18}$$

We can see that the perturbation term $\tilde{A}^\nu$ is determined by the factors $c^\rho(k)$ and $d^\rho(k)$ through Eqs. (17) and (18). Therefore, the wave packet A^ρ could be calculated in principle. However, the integrals in Eq. (18) are too difficult to calculate analytically. So, we have to do numerical integrations.

III. THE BIREFRINGENCE IN THE NEAR-FIELD REGION

The notion of classical trajectory does not apply naively to a nonlocal object like the wave packet. But we can use the center of EMT to describe the motion of a wave packet. It means we can define the trajectory of a wave packet by the center of its EMT. However, there are many expressions of the EMT. In this paper, we focus on the canonical and symmetric ones.

In gravity, the center of the EMT is

$$\langle x^i \rangle = \frac{\int \sqrt{g} x^i T^{\mu\nu} dV}{\int \sqrt{g} T^{\mu\nu} dV}. \tag{19}$$

In this article, we only care about the motions induced by gravity. Therefore, we can use the first-order terms of GM/b in the EMT to define the center of a wave packet:

$$\langle x^i \rangle \simeq \frac{\int x^i \tilde{T}^{\mu\nu} dV}{\int \bar{T}^{\mu\nu} dV}, \tag{20}$$

where $\bar{T}^{\mu\nu}$ is the zeroth-order terms of GM/b in the EMT of the wave packet. To make Eq. (20) physical, the integration of $\bar{T}^{\mu\nu}$ must satisfy $\int \bar{T}^{\mu\nu} dV \neq 0$. In our settings, we have taken $\vec{k}_0 = (0, 0, k_0)$. After straightforward calculations, we find that only four components of the EMT can be used to define the center of the wave packet by Eq. (20). In this paper, we focus on using the energy density and energy flux to describe the motion of the wave packet in gravity.

We can use the energy density and energy flux of the symmetric EMT to define the center of the wave packet:

$$\langle x^i \rangle_{se} \simeq \frac{\int x^i \tilde{T}^{00}_S dV}{\int \bar{T}^{00}_S dV} \quad \text{and} \quad \langle x^i \rangle_{sk} \simeq \frac{\int x^i \tilde{T}^{30}_S dV}{\int \bar{T}^{30}_S dV}, \tag{21}$$

where $\tilde{T}^{00}_S$ and $\tilde{T}^{30}_S$ are the first-order terms of $h = GM/b$ in the symmetric energy density and energy flux, while $\bar{T}^{00}_S$ and $\bar{T}^{30}_S$ are the zeroth-order terms. We can also use the energy density and energy flux of the canonical EMT to describe the motion of the wave packet:

$$\langle x^i \rangle_{ce} \simeq \frac{\int x^i \tilde{T}^{00}_C dV}{\int \bar{T}^{00}_C dV} \quad \text{and} \quad \langle x^i \rangle_{ck} \simeq \frac{\int x^i \tilde{T}^{30}_C dV}{\int \bar{T}^{30}_C dV}, \tag{22}$$

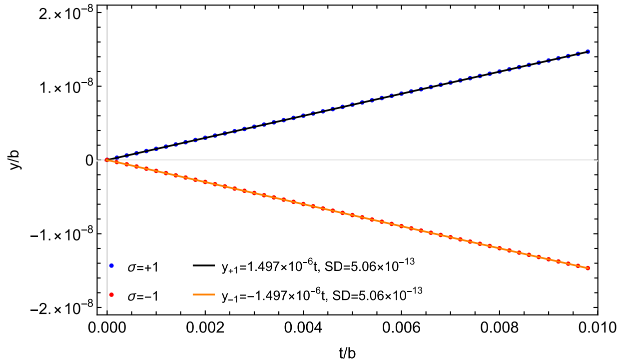
where $\tilde{T}^{00}_C$ and $\tilde{T}^{30}_C$ are the first-order terms of $h = GM/b$ in the canonical energy density and energy flux, while $\bar{T}^{00}_C$ and $\bar{T}^{30}_C$ are the zeroth-order terms. In the following, we will show that these definitions of the center lead to different trajectories.

After numerical calculations, we find that the motion of a spin-polarized electromagnetic wave packet in the direction of gravity can be fitted with

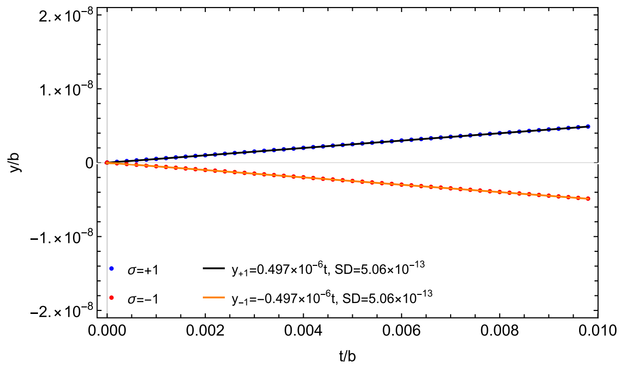
$$\langle x \rangle \simeq -\frac{GM}{b^2} t^2. \tag{23}$$

Equation (23) is independent of the spin of wave packets. We also find that the motions in the direction of gravity described by different components and expressions of the EMT are the same as Eq. (23).

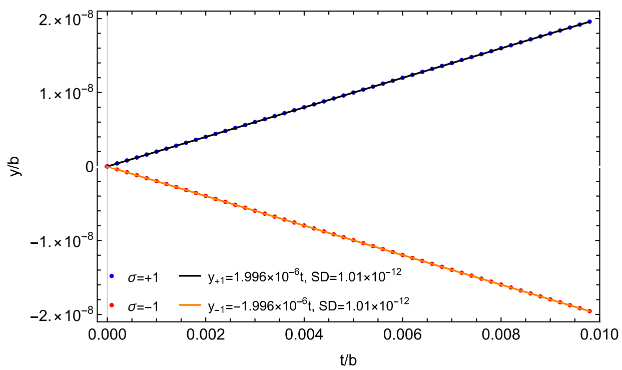
However, a spin-polarized wave packet does have a transverse shift depending on the spin and the definitions of the trajectory. The numerical results are shown in Fig. 2.



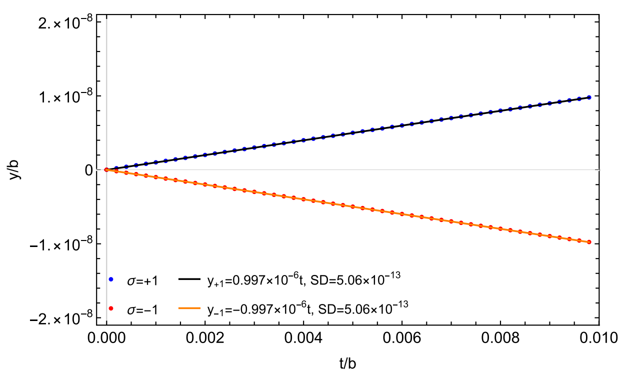
(a) Center of energy density of the canonical EMT



(b) Center of energy flux of the canonical EMT



(c) Center of energy density of the symmetric EMT



(d) Center of energy flux of the symmetric EMT

FIG. 2. The transverse motions of wave packets with the helicity $\sigma = \pm 1$. Here, we have set $GM/b = 1$ and $k_0 b = 10^6$. The blue and orange bullets represent the numerically calculated center coordinates. The black and orange straight lines are the fitted results, and SD means the standard deviation.

TABLE I. The relationship between α and trajectories.

α	Definitions of trajectories
2	Center of energy density of the symmetric EMT
$\frac{3}{2}$	Center of energy density of the canonical EMT
1	Center of energy flux of the symmetric EMT
$\frac{1}{2}$	Center of energy flux of the canonical EMT

The transverse shift of a wave packet with the helicity σ in the transverse direction agrees with the expression:

$$\langle y \rangle \simeq \alpha \frac{GM}{b} \frac{\sigma t}{k_0 b}, \tag{24}$$

where $\sigma = \pm 1$, and $\alpha = 2, 3/2, 1, 1/2$ depends on the definitions of the trajectory. In detail, we find that $\alpha = 2$ when using the energy center of the symmetric EMT to describe the motion of the wave packet, and $\alpha = 1$ when using instead the energy-flux center of the symmetric EMT. If we use the energy density of the canonical EMT, α becomes $3/2$. We also find that $\alpha = 1/2$ if using canonical energy flux to define the center of the wave packet. The relationships between the factor α and trajectories are shown in Table I.

According to Eq. (24), we find that the transverse velocity of the spin-polarized wave packet is $dy/dt \simeq \alpha GM\sigma/k_0 b^2$. So, the transverse velocity dy/dt is independent of the time of motion t. It means that the transverse motion of the wave packet is a kinematic effect when t is much less than the impact parameter b. If we expand the trajectory given by Duval [17] and keep it to the first order of t/b, we can also find such a kinematic effect in the transverse trajectory.

When we use different components and expressions of the EMT to define the center of the same spin-polarized wave packet, the factor α has different values. Therefore, we can determine which expression of the density would effectively describe the interaction between matter and gravity by measuring the transverse velocity dy/dt. However, we will figure out why there are differences between the trajectories first. The transverse trajectories given by different components and expressions of the EMT are shown in Fig. 3 when $\sigma = +1$.

The difference between the center of energy density and energy flux is

$$\delta y \equiv \langle y \rangle_k - \langle y \rangle_e \simeq -\frac{GM}{b} \frac{\sigma t}{k_0 b}. \tag{25}$$

As is known from GSHE, the energy flux center of a polarized light beam will have a transverse shift when it is observed from a reference frame tilted with respect to the direction of propagation of the beam. According to Eq. (23), the angle between the energy flux and the

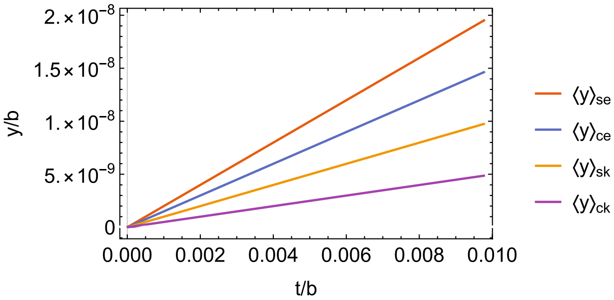


FIG. 3. The transverse motion of a wave packet with the helicity $\sigma = +1$. Here, we have set $GM/b = 1$ and $k_0 b = 10^6$. The $\langle y\rangle_{se}$ and $\langle y\rangle_{ce}$ are the centers of energy densities of the symmetric and canonical EMTs. And the $\langle y\rangle_{sk}$ and $\langle y\rangle_{ck}$ are the centers of symmetric and canonical energy flux.

propagation of the wave packet is $\beta = 2GMt/b^2$. By numerical calculations, the transverse shift of the energy flux center is found to agree with the expression

$$\langle y\rangle_{gs} \simeq -\frac{GM}{b}\frac{\sigma t}{k_0 b}. \tag{26}$$

We see that the transverse shift $\langle y\rangle_{gs}$ is the same as the difference δy. When a wave packet moves in a gravitational field, its propagation is changed by gravity. Therefore, the angle β between the propagation and energy flux of the wave packet is also changed. It means that if the wave packet is observed from a reference frame tilted with respect to its propagation, and if we still use the center of the energy flux to describe the motion of the wave packet, then the energy flux center does have a transverse shift dependent on the spin and angle β because of the GSHE. This transverse shift is just induced by the angle $\beta \neq 0$. If there is no interaction between spin and gravity, then the energy flux center will still have this transverse shift. So, this transverse shift is called the GSHE induced by gravity [28].

When using the center of energy density to describe the motion of a spin-polarized wave packet, the transverse shift is caused by the interaction between spin and gravity. However, if the energy flux is used, the transverse shift is caused by two effects. One is from the interaction between spin and gravity. The other one is the GSHE induced by gravity. Therefore, the differences between the transverse shifts of the wave packets described by different components of the EMT are due to the GSHE induced by gravity.

There are also differences between transverse shifts of the wave packets described by the canonical and symmetric EMTs. According to the Eq. (24), the differences are

$$\delta y_e = \delta y_k \simeq \frac{GM}{2b}\frac{\sigma t}{k_0 b}, \tag{27}$$

where $\delta y_e \equiv \langle y\rangle_{se} - \langle y\rangle_{ce}$ is the difference between the centers of energy densities of the symmetric and canonical EMTs, and $\delta y_k \equiv \langle y\rangle_{sk} - \langle y\rangle_{ck}$ is the difference between the centers of symmetric and canonical energy flux.

As we all know, the differences between the canonical and symmetric EMTs are the Belinfant-Rosenfeld terms:

$$T_s^{\mu\nu} = T_c^{\mu\nu} + \nabla_\lambda B^{\lambda\mu\nu}, \tag{28}$$

where $B^{\lambda\mu\nu} = \frac{1}{2}(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu})$. $S^{\mu\nu\lambda} = F^{\mu\nu}A^\lambda - F^{\mu\lambda}A^\nu$ is the spin angular momentum tensor of electromagnetic wave packets. So, the difference between the energy densities of the canonical and symmetric EMTs is

$$\nabla_\lambda B^{\lambda 00} = \nabla_\lambda S^{00\lambda}. \tag{29}$$

In this paper, we only pay attention to the motions caused by gravity. So, we can use the first-order terms of $h_{\mu\nu}$ in the EMT to define the center of a wave packet. Then the difference between energy densities of canonical and symmetric EMTs is simplified to

$$\nabla_\lambda B^{\lambda 00} = \partial_i \tilde{S}^{00i}, \tag{30}$$

where $\tilde{S}^{00i}$ is the first-order terms of $h_{\mu\nu}$ in the spin angular momentum tensor S^{00i}.

According to Eqs. (21), (22), and (30), the difference δy_s between the centers of energy densities of canonical and symmetric EMTs is

$$\delta y_s = \frac{\int \tilde{S}^{020}(\tilde{A})dV}{\int \bar{T}_C^{00}dV} = \tilde{S}^{02}/E, \tag{31}$$

where E is the total energy, and $\tilde{S}^{02} \equiv \int \tilde{S}^{020}dV$ is the spin angular momentum induced by gravity.

To solve the perturbation term $\tilde{A}^\rho$, we have set the Fourier transform of $\tilde{A}^\rho$ to be $\tilde{A}^\rho_f = 0$ at $t = 0$. According to Eq. (18), the perturbation term becomes $\tilde{A}^\rho = 0$ at $t = 0$. So, the spin angular momentum of the wave packet is $\tilde{S}^{02} = 0$ at $t = 0$. However, $\tilde{S}^{02}$ is changed with time t by the interaction between spin and gravity. After numerical calculations, we find that the difference δy_s agrees with the expression:

$$\delta y_s = \frac{GM}{2b}\frac{\sigma t}{k_0 b}. \tag{32}$$

The difference δy_s is the same as δy_e and δy_k. It means that the differences between the centers of canonical and symmetric EMTs are due to the evolution of spin angular momentum in the gravitational field.

According to Eq. (32), the difference between the transverse shifts given by canonical and symmetric EMTs is $\delta y \sim GMt/k_0^2 b^2$. The gravitational potential on the surface of the Earth is $GM/b \sim 10^{-9}$. So, this difference is $\delta y \sim 10^{-9}/k_0$ when the spin-polarized wave packet moves on the Earth. We find this difference δy is mush less than the wavelength $\lambda \sim 1/k_0$. Therefore, this difference δy is too small to be detected. However, the trajectory of a spin-polarized wave packet emitted from a celestial body should be influenced by the gravity of the celestial body. When the wave packet is detected on the Earth, the time of motion t could be much larger than the radius R of the celestial body. In this case, the difference δy might be larger than the wavelength λ. Therefore, the difference between the transverse shifts given by canonical and symmetric EMTs could possibly be detected. In Sec. IV, we will discuss it in detail.

IV. THE BIREFRINGENCE IN THE FAR-FIELD REGION

According to Eq. (24), the spin-polarized electromagnetic wave packets do have different trajectories in gravity, and this phenomenon violates the weak equivalence principle. The transverse shift of a spin-polarized wave packet near the gravity source is

$$y \simeq \alpha \frac{GM}{b} \frac{\lambda \sigma t}{2\pi b}, \tag{33}$$

where $\lambda = 2\pi/k_0$ is the wavelength and $\sigma = \pm 1$ is the helicity. The transverse velocity is

$$\frac{dy}{dt} \simeq \alpha \frac{GM}{b} \frac{\lambda \sigma}{2\pi b}. \tag{34}$$

According to Eq. (34), we find that the transverse motion of a spin-polarized wave packet in gravity is only a kinematic effect when the time of motion t is much less than the impact parameter b. So, we can assume that when the time t is long, the transverse velocity becomes

$$\frac{dy}{dt} \simeq \frac{\alpha}{2\pi} \frac{GM\sigma\lambda}{b^2 + t^2}. \tag{35}$$

However, when the time t is much less than b, Eq. (35) should return to Eq. (34). So, it is reasonable to set that the factor α in Eq. (35) is equal to the factor α in Eq. (34). The relationships between the factor α and trajectories have been shown in Table I.

When a wave packet moves from $z = -\infty$ to $z = +\infty$, the initial conditions are $y = 0$ and $dy/dt = 0$ at $t = -\infty$. After straightforward calculations, the trajectory in the transverse direction is

$$y \simeq \frac{\alpha}{2\pi} \frac{GM\sigma\lambda}{b} \left[\frac{\pi}{2} + \arctan\left(\frac{t}{b}\right)\right]. \tag{36}$$

The trajectory is shown in Fig. 4 when $\alpha = 1$. The transverse velocity is $dy/dt = 0$ at $t = +\infty$. So, two wave packets with opposite helicity are separated, and there is no intersection angle between them.

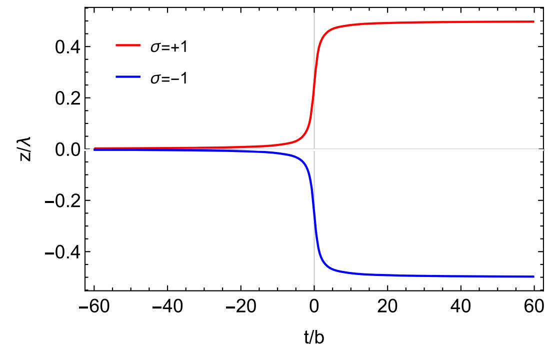


FIG. 4. The transverse trajectories of spin-polarized wave packets moving from $z = -\infty$ to $z = +\infty$ when $\alpha = 1$. The blue and red lines are the trajectories. Here, we have set $GM/b = 1$. λ is the wavelength.

According to the equations of motion of a spin-polarized particle given by Papapetrou [14], the separation distance l_{MPD} of two particles with opposite helicity is

$$l_{MPD} = \frac{4GM\lambda}{\pi b}, \tag{37}$$

where we have set velocity $v \sim 1$. But according to Eq. (36), the separation distance l is

$$l \simeq \frac{\alpha\pi}{4} \frac{4GM\lambda}{\pi b}, \tag{38}$$

where $\alpha = 2, 3/2, 1, 1/2$. We find that the separation distance l given by the canonical and symmetric EMTs cannot be equal to l_{MPD}. Therefore, the trajectory of a spin-polarized wave packet given by the center of EMT is different from the trajectory given by Papapetrou.

When a wave packet moves from a finite distance $z = -a$ to $z = +\infty$, the initial conditions are $y = 0$ and $dy/dt = 0$ at $t = 0$. According to Eq. (35), the transverse trajectory of a spin-polarized wave packet is

$$y \simeq \frac{GM\alpha}{2\pi b} \sigma\lambda \left(\arctan(a/b) - \arctan\left(\frac{a - t}{b}\right)\right) - \frac{\alpha}{2\pi} \frac{GM\sigma\lambda t}{b^2 + (a)^2}. \tag{39}$$

The transverse trajectories are shown in Fig. 5 when $\alpha = 1$. These trajectories are similar to those given by Oancea [18]. The transverse velocity dy/dt at $t = +\infty$ becomes

$$\frac{dy}{dt} \simeq -\frac{\alpha}{2\pi} \frac{GM\sigma\lambda}{b^2 + a^2}. \tag{40}$$

So, the angle θ between the two wave packets with opposite helicity when moving from a finite distance $r_0 = \sqrt{b^2 + a^2}$ to infinity is

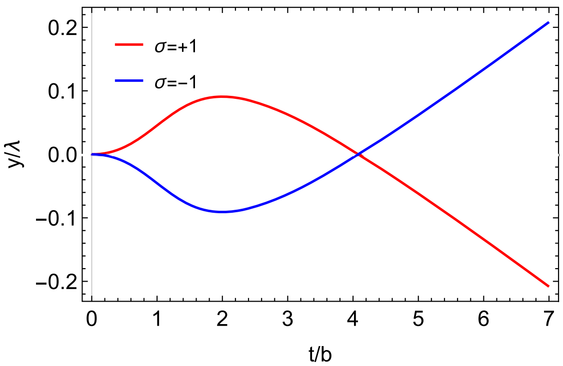


FIG. 5. The transverse trajectories of wave packets with opposite helicity moving from $z = -a$ to $z = +\infty$ when $\alpha = 1$. The blue and red lines are the trajectories of wave packets. λ is the wave length. Here, we have set $GM/b = 1$ and $a = b$.

$$\theta \simeq \frac{\alpha}{2}\frac{2GM\lambda}{\pi r_0^2}. \tag{41}$$

But the angle θ given by Duval [17] with the constraint $p_\alpha S^{\alpha\rho} = 0$ is

$$\theta = -\frac{\lambda}{\pi r_0} + \frac{2GM\lambda}{\pi r_0^2}. \tag{42}$$

There is a gravity-independent term $-\lambda/\pi r_0$ in Eq. (42). This gravity-independent term means that the two spin-polarized wave packets would also be separated when the gravity vanishes. So, this gravity-independent term is unreasonable in physics.

When we take $\alpha = 2$, the angle θ in Eq. (41) is equal to the gravity-dependent term $2GM\lambda/\pi r_0^2$ in Eq. (42) given by Duval [17]. So, it means the trajectory given by Duval with the constraint $p_\alpha S^{\alpha\rho} = 0$ is the same as the trajectory given by the center of energy density of symmetric EMT if we ignore the gravity-independent term in Eq. (42).

If two wave packets with opposite helicity are emitted from a celestial body with radius R, the angle between their propagations should be $\theta \simeq \alpha GM\lambda/\pi R^2$. When the two wave packets reach the Earth, the separation distance between them should be $l \simeq \alpha GML\lambda/\pi R^2$, where $L \gg R$ is the distance from the celestial body to the Earth. If the celestial body is the Sun, the radius is about $R_\odot \sim 7.0 \times 10^5$ km. The distance from the Sun to the Earth is $L_{\mathrm{Sun}} = 1$ Au, and the gravitational potential on the surface of the Sun is about $GM_\odot/R_\odot \sim 2.1 \times 10^{-6}$. So the separation distance between the two wave packets with opposite helicity is about $l_{\mathrm{Sun}} \sim 4.6 \times 10^{-4}\,\alpha\lambda$. We find the separation distance l_{Sun} is much less than the wavelength λ. It means the separation distance between the two wave packets emitted from the Sun is too small to be detected. However, if the celestial body is the Proxima Centauri, the gravitational potential is similar to the Sun. The radius of the Proxima Centauri is about $R \sim 0.15 R_\odot$. The distance from the Proxima Centauri to the Earth is about $L \sim 6.4 \times 10^4$ AU. So the separation distance between the two wave packets emitted from the Proxima Centauri should be about $l \sim 79.7\,\alpha\lambda$. In this case, the separation distance l is about one hundred times the wavelength λ. We can find that the farther away from the Earth, the larger the separation distance is. Therefore, the separation distance between two wave packets with opposite helicity could be large enough to be detected when they are emitted from a celestial body far away from the Earth.

However, the factor α in the separation distance l is dependent on the definitions of the trajectory. If we use the center of energy density of the symmetric EMT to define the trajectory, the factor α is 2. So, the separation distance of the two wave packets with opposite helicity given by the energy density of symmetric EMT is $l_S \simeq 2GML\lambda/\pi R^2$. But when using the center of energy density of the canonical EMT to define the trajectory, we find that the factor α becomes $3/2$. Therefore, the separation distance given by the energy density of canonical EMT should be $l_C \simeq 3GML\lambda/2\pi R^2$, and it is not equal to the separation distance l_S given by the energy density of symmetric EMT. There is a difference $\delta l \simeq GML\lambda/2\pi R^2$ between the separation distances l_S and l_C. It is important that this difference δl could be detected when the two wave packets are emitted from a celestial body. So, there can only be one separation distance that is consistent with a particular experiment. And in this particular experiment, we can find a suitable expression of the EMT to effectively describe the interaction between matter and gravity.

V. BIREFRINGENCE IN GRAVITY DESCRIBED BY AN EFFECTIVE OPTICAL MATERIAL

In general relativity, we have

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu\nu}) \tag{43}$$

and

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{44}$$

So, we could define an excitation tensor $\mathcal{H}^{\mu\nu}$

$$\mathcal{H}^{\mu\nu} \equiv \sqrt{g} g^{\mu\rho} g^{\nu\alpha} F_{\rho\alpha}, \tag{45}$$

and then the Maxwell equations in gravity become

$$\partial_\mu \mathcal{H}^{\mu\nu} = 0. \tag{46}$$

In the absence of sources, the Maxwell equations in materials are given by

$$\begin{cases} \nabla \times \vec{H} = \partial_t \vec{D} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{D} = 0, \end{cases} \tag{47}$$

where $\vec{E}$ and $\vec{B}$ are electric and magnetic field quantities, and $\vec{D}$ and $\vec{H}$ are corresponding derived fields.

By comparing Eqs. (46) and (47), we find

$$D^i = \mathcal{H}^{0i} \quad \text{and} \quad H^i = \epsilon^{ijk}\mathcal{H}^{jk}, \tag{48}$$

where D^i and H^i are the components of $\vec{D}$ and $\vec{H}$, ϵ^{ijk} is the Levi-Civita symbol. By applying the metric given by Eqs. (4) and (48) becomes

$$\vec{D} = (1+h)\vec{E} \quad \text{and} \quad \vec{H} = (1-h)\vec{B}, \tag{49}$$

where $h = 2GM/r$.

If the gravitational field is described by an effective optical material, the effective permittivity ε and magnetic permeability μ of this material are $\varepsilon = 1+h$ and $\mu \simeq 1+h$. As light propagates in this effective material, its velocity is $v = 1/\sqrt{\varepsilon\mu} \simeq 1-h$. When light passes from the vacuum into this effective material and travels near the initial position $\vec{X} = (0,0,0)$, the angle of incidence is about $\theta_i \simeq t/b \ll 1$. So the angle between the incident and refracted light beams is about

$$\beta \simeq -\frac{2GMt}{b^2}, \tag{50}$$

and this angle is equal to the one given by Eq. (23).

According to Eqs. (47) and (49), the propagation of light in this effective optical material is determined by

$$(1+2h)\partial_t^2 E_i - \nabla^2 E_i - 2\partial_j h \partial_i E_j + \partial_j h \partial_j E_i = 0, \tag{51}$$

where we have ignored the term $\partial_i \partial_j h E_j \sim GM/r^3$ because it is much smaller than the others. For convenience, we can assume $\vec{E} \propto \exp(i\vec{k}\cdot\vec{x} - i\omega t)$, and then Eq. (51) becomes

$$[(1+2h)\omega^2\delta_{ij} - k^2\delta_{ij} + 2ik_i\partial_j h - ik_k\partial_k h\delta_{ij}]E_j = 0. \tag{52}$$

In Sec. II, we have set the initial position of wave packet $\vec{X} = (0,0,0)$ and the center of gravity source $\vec{O}_g = (-b,0,0)$. So, h and $\partial_i h$ near the position are given by

$$h = \frac{2GM}{b} \quad \text{and} \quad \partial_i h = -\delta_i^1 \frac{2GM}{b^2}. \tag{53}$$

According to Eq. (47), $\vec{k}\cdot\vec{E}$ could be simplified to

$$\vec{k}\cdot\vec{E} = -i\frac{2GM}{b^2}E_1. \tag{54}$$

When a light beam passes into this effective material from the vacuum along the third direction, we can find the leading terms of k_1 and k_2 are the first order of h according to Eq. (51). If the helicity of this light beam is σ, we can set $E_2/E_1 = i\sigma$ for convenience. According to Eqs. (54) and (52), we find that the angle between the two light beams with opposite helicity is

$$\theta = 2\left|\frac{k_2}{k_3}\right| \simeq \frac{2GM\lambda}{\pi b^2}. \tag{55}$$

When taking $\alpha = 2$ in Eq. (34), we can find the angle θ is equal to $2|v_y/v_z| \simeq 2GM\lambda/\pi b^2$. So, the angle between the two light beams with opposite helicity when propagating into this effective material from the vacuum is the same as the one given by the center of energy density of the symmetric EMT in gravity.

VI. SUMMARY

We construct a spin-polarized free-falling electromagnetic wave packet in gravity, and then we calculate the center of the EMT. We find that the spin-polarized free-falling wave packets are indeed separated in the transverse direction. It means we can use gravity to discriminate observed objects with different spins by measuring the EMT. The trajectories of spin-polarized wave packets are not the same and thus violate the weak equivalence principle.

When a spin-polarized wave packet moves near the gravitational source, the transverse shift given by the center of energy density of the symmetric EMT is $\langle y\rangle_{se} \simeq 2GM\sigma t/k_0 b^2$. But the transverse shift given by the center of energy flux of the symmetric EMT is $\langle y\rangle_{sk} \simeq GM\sigma t/k_0 b^2$, and it is different from $\langle y\rangle_{se}$. So, there are differences between the transverse shifts of a spin-polarized wave packet given by different components of the EMT. By numerical calculations, we find these differences between the different components are caused by the geometric spin Hall effect induced by gravity. However, the transverse shift given by the energy density of canonical EMT is $\langle y\rangle_{ce} \simeq 3GM\sigma t/2k_0 b^2$. We find that $\langle y\rangle_{ce}$ is not equal to the transverse shift $\langle y\rangle_{se}$ given by the energy density of symmetric EMT either. Therefore there are also differences between the transverse shifts of a spin-polarized wave packet given by the canonical and symmetric expressions of the EMT. And these differences between the canonical and symmetric expressions are mainly due to the evolution of spin angular momentum in the gravitational field. Finally, we compare the Maxwell equations in the gravitational field and materials. We find the angle between the light beams with opposite helicity given by the energy density of symmetric EMT is the same as the one when the gravitational field is described by an effective optical material.

We also compare our results with those given by the MPD equations. We find that the trajectories given by the canonical and symmetric EMTs are different from the trajectories given by the MPD equations. If two wave packets with opposite helicity move from a finite distance r_0 to infinity, the angle between their propagations is $\theta \simeq \alpha GM\lambda/\pi r_0^2$. It means that the separation distance of the two spin-polarized wave packets emitted from a celestial body can be large enough to be detected. Therefore, it can be tested experimentally whether the trajectories of spin-polarized wave packets violate the weak equivalence principle or not. Because there are differences between the trajectories given by different expressions of the EMT, we can also determine which expression of the EMT would effectively describe the interaction between matter and gravity.

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