Stealth metastable state of scalar-tensor thermodynamics

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(Received 31 March 2022; accepted 21 April 2022; published 5 May 2022)

We investigate a puzzle in the recent thermodynamics of scalar-tensor gravity, in which general relativity is a zero-temperature state of equilibrium and scalar-tensor gravity is the nonequilibrium configuration of an effective dissipative fluid. A stealth solution of Brans-Dicke gravity with constant positive temperature is shown to be analogous to a metastable state for the effective fluid and to suffer from an instability. The stability analysis employs a version of the Bardeen-Ellis-Bruni-Hwang gauge-invariant formalism for cosmological perturbations adapted to modified gravity. The metastable state is destroyed by tensor perturbations.

DOI: 10.1103/PhysRevD.105.104006

I. INTRODUCTION

Gravity is one of only four fundamental forces known but it behaves differently than the electroweak and strong interactions in several respects. For this reason, it has been suggested that gravity may not be fundamental after all, but rather be emergent, an idea that has been formulated in various approaches (see [1–10] for reviews). A particularly influential approach was that of Jacobson's thermodynamics of spacetime in which the Einstein equation was derived with purely thermodynamical considerations [11]. Later on, scalar-tensor gravity [in its incarnation as metric $f(R)$ gravity] was examined and its field equations were again derived from thermodynamics [12]. Moreover, the idea was advanced that general relativity (GR) constitutes a state of equilibrium while modified gravity corresponds to an out-of-equilibrium state ([12], see also [13]). This idea is not outrageous if one thinks that the field content of scalar-tensor gravity consists of the two massless spin two modes of GR plus a (usually massive) scalar mode propagating as well. Exciting this scalar mode corresponds to an excited state with respect to GR.

In spite of a large literature, the order parameter, or equations, describing the approach to equilibrium have never been found. Recently, a different approach was proposed for scalar-tensor gravity, including a definition of "temperature of gravity" quantifying the proximity (or lack thereof) of a theory of gravity to GR and an equation describing the approach to equilibrium [14–18]. We refer to this approach of [14–17] as "thermodynamics of scalar-tensor gravity" and we stress that, in spite of similarities with the basic idea of Refs. [11,12], it is *not* thermodynamics of spacetime (as commonly the work inspired by [11,12] is referred to), but it is a very different approach. Although less insightful in fundamental aspects, at the same time it is minimalistic in its assumptions, which is the reason why it allows progress in finding an effective temperature of gravity and in describing the approach to the GR equilibrium state [14–17].

The key feature of this new formalism is that the field equations of scalar-tensor gravity are written as effective Einstein equations by moving all terms containing ϕ and its first and second derivatives to the right-hand side, where they form an effective stress-energy tensor $T_{ab}^{(\phi)}$. It is then shown that, when the gradient $\nabla^a\phi$ is timelike, $T_{ab}^{(\phi)}$ has the form of the stress-energy tensor of a dissipative fluid [14,17,19,20], to which one can apply the constitutive relations of Eckart's first-order thermodynamics [21]. Within the well-known limitations of Eckart's theory (e.g., [22,23]), these relations lead to the definition and physical interpretation of fluid quantities as effective temperature of gravity $\mathcal{T}$, thermal conductivity $\mathcal{K}$, heat flux density, anisotropic stresses, and shear and bulk viscosity coefficients [14–16]. This formalism has since been extended to Horndeski gravity [17] and applied to Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology [18] (its application to other situations in gravity is in progress).

The thermodynamics of scalar-tensor gravity is consistent (within the limitations of Eckart's theory) in both the general theory and its application to specific analytical solutions but, in this context, a little puzzle remains. There is a stealth solution of Brans-Dicke theory that corresponds to a state of constant nonzero temperature [16], which is

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difficult to interpret. The present paper is devoted to solving this conundrum and learning a lesson that is important for the general theory beyond specific solutions. Here we show that this Brans-Dicke stealth solution can be interpreted as a sort of metastable state of the theory which exists at constant (nonzero) temperature and, therefore, always remains far away from the GR state of equilibrium (which corresponds to zero temperature instead). However, this state is unstable and small perturbations destroy it, as it happens for supercooled water, which freezes as soon as its container is shaken or impurities acting as ice nucleation nuclei are thrown into it. In our case, the perturbations are either scalar or tensor perturbations and the stability analysis is performed using a gauge-invariant formalism designed for modified gravity.

Let us be more specific about the theory, the solution, and the thermodynamical formalism (we follow the notation of Ref. [24]). The context is the Brans-Dicke theory of gravity, described by the action [25]

$$S_{\mathrm{BD}} = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] \tag{1.1}$$

in vacuo, where ϕ is the Brans-Dicke scalar field corresponding, approximately, to the inverse of the effective gravitational coupling G_{eff}, R is the Ricci scalar, ω is a constant ("Brans-Dicke coupling"), $V(\phi)$ is a scalar field potential, and g is the determinant of the metric g_{ab}.

Several analytical solutions of Brans-Dicke gravity are known, including time-dependent ones (see the review [26]). The solution of interest here [27] corresponds to the choice

$$\omega = -1, \qquad V(\phi) = V_0\phi, \tag{1.2}$$

where V_0 is a positive constant.[1] The line element and scalar field in spherical coordinates $(t, r, \vartheta, \varphi)$ are [27]

$$ds^2 = -dt^2 + A^{-\sqrt{2}}(r)dr^2 + A^{1-\sqrt{2}}(r)r^2 d\Omega^2_{(2)}, \tag{1.3}$$

$$\phi(t, r) = \phi_0 \mathrm{e}^{2a_0 t} A^{1/\sqrt{2}}(r), \tag{1.4}$$

where $A(r) = 1 - 2m/r$, m, a_0, ϕ_0 are constants, and $d\Omega^2_{(2)} = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere. This geometry is conformal to that of the Fonarev solution of GR [28] and can be seen as a special case of the Campanelli-Lousto family of solutions [29,30], but the functional form of the scalar field is different from the Campanelli-Lousto one [29,30]. Since $\omega = -1$ Brans-Dicke gravity is the low-energy limit of bosonic string theory [31,32], presumably there is some stringy analog of this solution. Here, however, we are only interested in the special case obtained by the limit $m \to 0$, which produces the Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{(2)} \tag{1.5}$$

with

$$\phi(t) = \phi_0 \mathrm{e}^{2a_0 t}, \tag{1.6}$$

a little-known stealth solution in which the scalar field does not gravitate. In Ref. [16], this analytical solution was examined as an example of the new thermodynamics of scalar-tensor gravity, resulting in a surprise: in general, the product of thermal conductivity and temperature is given by [15,16]

$$\mathcal{K}T = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi} \tag{1.7}$$

which, for the stealth solution (1.5) and (1.6), becomes constant [16]

$$\mathcal{K}T = \frac{|a_0|}{4\pi}. \tag{1.8}$$

Such states are in principle possible because the approach to the GR equilibrium is described by the equation [16]

$$\frac{d(\mathcal{K}T)}{d\tau} = 8\pi(\mathcal{K}T)^2 - \Theta\mathcal{K}T + \frac{\Box\phi}{8\pi\phi}, \tag{1.9}$$

where τ is the proper time of the effective ϕ-fluid, its four-velocity is

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^c\phi\nabla_c\phi}}, \tag{1.10}$$

and Θ is its expansion scalar, given by [14]

$$\begin{aligned}\Theta &= \nabla_a u^a \\ &= \frac{1}{\sqrt{-\nabla_c\phi\nabla^c\phi}}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla_e\phi\nabla^e\phi}\right).\end{aligned} \tag{1.11}$$

In order for u^a to be future oriented for the specific solution (1.5) and (1.6), it must be $a_0 < 0$ [16]. It is clear that states $\mathcal{K}T = \text{const.} \equiv C$ can be obtained if

$$C^2 - C\Theta + \frac{\Box\phi}{8\pi\phi} = 0. \tag{1.12}$$

[1] Since $\phi = G_{\mathrm{eff}}^{-1} > 0$, the potential is then effectively bounded from below.

For the particular solution (1.5) and (1.6) it is $\Theta = 0$, while the equation of motion of the Brans-Dicke field

$$\Box\phi = \frac{1}{2\omega+3}\left(\phi\frac{dV}{d\phi} - 2V\right) \tag{1.13}$$

gives $\Box\phi = -V_0\phi$ for $\omega = -1$, $V(\phi) = V_0\phi$, and $\phi = \phi_0 e^{2a_0\phi}$. As a result, Eq. (1.12) is satisfied if $V_0 = 4a_0^2$, a relation derived also in [16] with independent considerations.

The physical interpretation of this state with constant $\mathcal{K}T > 0$ remained unclear and is the subject of the present paper. We show that this solution can be interpreted as a sort of metastable state of the theory which exists at constant (nonzero) temperature and, therefore, always remains far away from the GR state of equilibrium (which corresponds to $\mathcal{K}T = 0$). However, this state is unstable with respect to tensor perturbations of the spacetime (1.5) and (1.6). The stability analysis with respect to these perturbations is carried out using the Bardeen-Ellis-Bruni-Hwang gauge-invariant formalism developed for cosmological perturbations in [33–37] and adapted to modified gravity in [38–44]. It can be applied because the stealth Minkowski spacetime (1.5) and (1.6) is a trivial FLRW spacetime.

The instability matches the intuition that this stealth state of gravity is analogous to a metastable state in which a fluid (in our case, an effective fluid) can remain hanged for a while, but is destroyed by arbitrarily small perturbations. Section II recalls the needed equations of the gauge-invariant formalism for cosmological perturbations of scalar-tensor gravity, which are solved in Sec. III, establishing the presence of instability. Section IV contains conclusions on the significance of this metastable state for the thermodynamics of scalar-tensor gravity.

II. EQUATIONS OF GAUGE-INVARIANT PERTURBATION THEORY FOR MODIFIED GRAVITY

Gauge-invariant cosmological perturbation theory in modified gravity was developed in [38–44] for a rather general class of theories, i.e., mixed scalar-tensor/$f(R)$ gravity in the metric formalism. The vacuum action is

$$S = \int d^4x\sqrt{-g}\left[\frac{f(\phi,R)}{2} - \frac{\bar{\omega}(\phi)}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\right]. \tag{2.1}$$

Assuming a spatially flat FLRW universe with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2+dy^2+dz^2), \tag{2.2}$$

the corresponding field equations analogous to the Einstein-Friedmann equations of GR read

$$H^2 = \frac{1}{3F}\left(\frac{\bar{\omega}}{2}\dot{\phi}^2 + \frac{RF}{2} - \frac{f}{2} + V - 3H\dot{F}\right), \tag{2.3}$$

$$\dot{H} = -\frac{1}{2F}(\bar{\omega}\dot{\phi}^2 + \ddot{F} - H\dot{F}), \tag{2.4}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\bar{\omega}}\left(\frac{d\bar{\omega}}{d\phi}\dot{\phi}^2 - \frac{\partial f}{\partial\phi} + 2\frac{dV}{d\phi}\right) = 0, \tag{2.5}$$

where an overdot denotes differentiation with respect to the comoving time t, $H \equiv \dot{a}/a$ is the Hubble function, and $F \equiv \partial f/\partial R$. The metric perturbations in the Bardeen-Ellis-Bruni-Hwang formalism [33–37] are completely described by the quantities A, B, H_L, and H_T defined by

$$g_{00} = -a^2(1+2AY), \tag{2.6}$$

$$g_{0i} = -a^2 BY_i, \tag{2.7}$$

$$g_{ij} = a^2[h_{ij}(1+2H_L) + 2H_T Y_{ij}], \tag{2.8}$$

where h_{ij} is the three-dimensional metric of the FLRW background, the scalar harmonics Y are the eigenfunctions of the eigenvalue problem $\bar{\nabla}_i\bar{\nabla}^i Y = -k^2 Y$, k is the corresponding eigenvalue, and $\bar{\nabla}_i$ is the covariant derivative operator of h_{ij}. The vector and tensor harmonics Y_i and Y_{ij} are given by

$$Y_i = -\frac{1}{k}\bar{\nabla}_i Y, \tag{2.9}$$

$$Y_{ij} = \frac{1}{k^2}\bar{\nabla}_i\bar{\nabla}_j Y + \frac{1}{3}Yh_{ij}, \tag{2.10}$$

respectively. The Bardeen gauge-invariant potentials [33] are

$$\Phi_H = H_L + \frac{H_T}{3} + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right), \tag{2.11}$$

$$\Phi_A = A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) + \frac{a}{k}\left[\dot{B} - \frac{1}{k}(a\dot{H}_T)^{\cdot}\right], \tag{2.12}$$

the Ellis-Bruni variable [34] is

$$\Delta\phi = \delta\phi + \frac{a}{k}\dot{\phi}\left(B - \frac{a}{k}\dot{H}_T\right), \tag{2.13}$$

while similar relations define the gauge-invariant variables $\Delta f, \Delta F$, and ΔR.

To first order, the gauge-invariant perturbations satisfy the (redundant) system of equations [38–44]

$$\Delta\ddot{\phi}+\left(3H+\frac{\dot{\phi}}{\bar{\omega}}\frac{d\bar{\omega}}{d\phi}\right)\Delta\dot{\phi}+\left[\frac{k^2}{a^2}+\frac{\dot{\phi}^2}{2}\frac{d}{d\phi}\left(\frac{1}{\bar{\omega}}\frac{d\bar{\omega}}{d\phi}\right)-\frac{d}{d\phi}\left(\frac{1}{2\bar{\omega}}\frac{\partial f}{\partial\phi}-\frac{1}{\bar{\omega}}\frac{dV}{d\phi}\right)\right]\Delta\phi$$
$$=\dot{\phi}(\dot{\Phi}_A-3\dot{\Phi}_H)+\frac{\Phi_A}{\bar{\omega}}\left(\frac{\partial f}{\partial\phi}-2\frac{dV}{d\phi}\right)+\frac{1}{2\bar{\omega}}\frac{\partial^2 f}{\partial\phi\partial R}\Delta R, \tag{2.14}$$

$$\Delta\ddot{F}+3H\Delta\dot{F}+\left(\frac{k^2}{a^2}-\frac{R}{3}\right)\Delta F+\frac{F}{3}\Delta R+\frac{2}{3}\bar{\omega}\,\dot{\phi}\,\Delta\dot{\phi}+\frac{1}{3}\left(\dot{\phi}^2\frac{d\bar{\omega}}{d\phi}+2\frac{\partial f}{\partial\phi}-4\frac{dV}{d\phi}\right)\Delta\phi$$
$$=\dot{F}(\dot{\Phi}_A-3\dot{\Phi}_H)+\frac{2}{3}(FR-2f+4V)\Phi_A, \tag{2.15}$$

$$\ddot{H}_T+\left(3H+\frac{\dot{F}}{F}\right)\dot{H}_T+\frac{k^2}{a^2}H_T=0, \tag{2.16}$$

$$-\dot{\Phi}_H+\left(H+\frac{\dot{F}}{2F}\right)\Phi_A=\frac{1}{2}\left(\frac{\Delta\dot{F}}{F}-H\frac{\Delta F}{F}+\frac{\bar{\omega}}{F}\dot{\phi}\Delta\phi\right), \tag{2.17}$$

$$\left(\frac{k}{a}\right)^2\Phi_H+\frac{1}{2}\left(\frac{\bar{\omega}}{F}\dot{\phi}^2+\frac{3}{2}\frac{\dot{F}^2}{F^2}\right)\Phi_A=\frac{1}{2}\left\{\frac{3}{2}\frac{\dot{F}\Delta\dot{F}}{F^2}+\left(3\dot{H}-\frac{k^2}{a^2}-\frac{3H}{2}\frac{\dot{F}}{F}\right)\frac{\Delta F}{F}\right.$$
$$\left.+\frac{\bar{\omega}}{F}\dot{\phi}\Delta\dot{\phi}+\frac{1}{2F}\left[\dot{\phi}^2\frac{d\bar{\omega}}{d\phi}-\frac{\partial f}{\partial\phi}+2\frac{dV}{d\phi}+6\bar{\omega}\,\dot{\phi}\left(H+\frac{\dot{F}}{2F}\right)\right]\Delta\phi\right\}, \tag{2.18}$$

$$\Phi_A+\Phi_H=-\frac{\Delta F}{F}, \tag{2.19}$$

$$\ddot{\Phi}_H+H\dot{\Phi}_H+\left(H+\frac{\dot{F}}{2F}\right)(2\dot{\Phi}_H-\dot{\Phi}_A)+\frac{1}{2F}(f-2V-RF)\Phi_A$$
$$=-\frac{1}{2}\left[\frac{\Delta\ddot{F}}{F}+2H\frac{\Delta\dot{F}}{F}+(P-\rho)\frac{\Delta F}{2F}+\frac{\bar{\omega}}{F}\dot{\phi}\Delta\dot{\phi}+\frac{1}{2F}\left(\dot{\phi}^2\frac{d\bar{\omega}}{d\phi}+\frac{\partial f}{\partial\phi}-2\frac{dV}{d\phi}\right)\Delta\phi\right], \tag{2.20}$$

and

$$\Delta R=6\left[\ddot{\Phi}_H+4H\dot{\Phi}_H+\frac{2}{3}\frac{k^2}{a^2}\Phi_H-H\dot{\Phi}_A-\left(2\dot{H}+4H^2-\frac{k^2}{3a^2}\right)\Phi_A\right]. \tag{2.21}$$

Although this system is complicated, it simplifies substantially in the case of the Minkowski background associated with the analytical solution of Brans-Dicke gravity under examination with homogeneous, but time-dependent, stealth scalar field. Even with this simplification, solving these equations is nontrivial.

III. STABILITY OF THE CONSTANT $\mathcal{K}T$ STEALTH SOLUTION

Let examine the equations for gauge-invariant perturbations and assess the stability of the stealth solution (1.5) and (1.6). We set

$$H=0,\qquad \dot{H}=0,\qquad a=1,\quad R=0, \tag{3.1}$$

then the comparison of the actions (1.1) and (2.1) with $\omega=-1$ and $V=V_0\phi$ yields

$$f(\phi,R)=2\phi R,\qquad \bar{\omega}(\phi)=-\frac{2}{\phi},\qquad F=2\phi, \tag{3.2}$$

while Eq. (1.6) gives $\dot{\phi}/\phi=2a_0$.

The gauge-invariant equations listed in Sec. II simplify considerably. Equation (2.16) for the tensor modes decouples from the other equations, assuming the form

$$\ddot{H}_T+2a_0\dot{H}_T+k^2H_T=0. \tag{3.3}$$

The term containing $\dot{H}_T$ describes friction if $a_0>0$ and antifriction if $a_0<0$, therefore, tensor modes are stable if

$a_0 \geq 0$ and unstable if $a_0 < 0$. As seen in Sec. II, it must be $a_0 < 0$ in order for the four-velocity of the effective ϕ-fluid to be future oriented, and we conclude that the stealth solution is unstable with respect to tensor modes of short wavelengths.

Proceeding, Eqs. (2.17)–(2.19) give

$$-\dot{\Phi}_H + a_0\Phi_A = \frac{1}{2}\left(\frac{\Delta\dot{\phi}}{\phi} - 2a_0\frac{\Delta\phi}{\phi}\right), \tag{3.4}$$

$$k^2\Phi_H + a_0{}^2\Phi_A = \frac{1}{2}\left[a_0\frac{\Delta\dot{\phi}}{\phi} + \frac{\Delta\phi}{\phi}\left(-k^2 + \frac{V_0 - 8a_0{}^2}{2}\right)\right], \tag{3.5}$$

and

$$\Phi_A + \Phi_H = -\frac{\Delta\phi}{\phi}. \tag{3.6}$$

Equation (2.21) gives

$$\Delta R = 6\left(\ddot{\Phi}_H + \frac{2k^2}{3}\Phi_H + \frac{k^2}{3}\Phi_A\right), \tag{3.7}$$

which, substituted into Eq. (2.15), yields

$$\Delta\ddot{\phi} - \frac{4a_0}{3}\Delta\dot{\phi} + \left(k^2 + \frac{4}{3}a_0{}^2 - \frac{2}{3}V_0\right)\Delta\phi = \phi\left[-2\ddot{\Phi}_H + 2a_0(\dot{\Phi}_A - 3\dot{\Phi}_H) + \frac{4}{3}\left(V_0 - \frac{k^2}{2}\right)\Phi_A - \frac{4k^2}{3}\Phi_H\right]. \tag{3.8}$$

Using Eq. (3.6), we can now eliminate $\ddot{\Phi}_A$, $\dot{\Phi}_A$, and Φ_A; Eqs. (3.4) and (3.5) simplify to

$$\dot{\Phi}_A + a_0\Phi_A + \frac{\Delta\dot{\phi}}{2\phi} - a_0\frac{\Delta\phi}{\phi} = 0, \tag{3.9}$$

$$(k^2 - a_0{}^2)\Phi_A = -\frac{a_0}{2}\frac{\Delta\dot{\phi}}{\phi} - \left(\frac{k^2}{2} + \frac{V_0}{4} - 2a_0^2\right)\frac{\Delta\phi}{\phi}, \tag{3.10}$$

which, in turn, allows one to eliminate $\ddot{\Phi}_A$, $\dot{\Phi}_A$, and Φ_A. Equation (3.8) then becomes

$$(4a_0{}^2 - V_0)\Delta\dot{\phi} + \frac{1}{2a_0}\left[\left(2a_0{}^2 - \frac{V_0}{2}\right)k^2 + a_0^2\left(-26a_0{}^2 + \frac{21}{2}V_0\right) + V_0^2\right]\Delta\phi = 0. \tag{3.11}$$

The relation $V_0 = 4_0^2$ of the unperturbed Minkowski space then implies that $\Delta\phi = 0$. However, the Bardeen potentials $\Phi_{A,H}$ diverge: in fact, Eq. (3.9) yields

$$\dot{\Phi}_A = -a_0\Phi_A, \tag{3.12}$$

with solution $\Phi_A(t) = (\Phi_A)_0 e^{-a_0 t}$ which diverges as $t \to +\infty$ since $a_0 < 0$, as already established. Equation (3.6) gives the other Bardeen potential $\Phi_H = -\Phi_A$, which diverges as well, together with

$$\Delta R = -6\left(a_0^2 + \frac{k^2}{3}\right)\Phi_A, \tag{3.13}$$

which follows from Eq. (3.7).

IV. CONCLUSIONS

We have uncovered an instability of the stealth solution (1.5) and (1.6) with respect to tensor perturbations. The timescale for the instability is $|a_0|^{-1}$. The only exception occurs for the parameter value $a_0 = 0$ which corresponds to Einstein theory, to constant scalar field $\phi = \phi_0 e^{2a_0 t}$, and to stability, in agreement with the fact that GR is the zero-temperature state of equilibrium for scalar-tensor gravity in the thermodynamical formalism [14–17].

The information that a_0 is negative is crucial to establish the instability of the solution (1.5) and (1.6) and comes from the requirement that the four-velocity of the effective ϕ-fluid be future oriented. The correct time orientation is crucial when discussing dissipative phenomena that are irreversible, but no hint on the sign of the parameter a_0 would be available without the point of view of the thermodynamics of scalar-tensor gravity, and the conclusion that the Brans-Dicke stealth spacetime (1.5) and (1.6) is unstable depends crucially on this argument and only acquires physical meaning in this context. Perhaps the only other indication corroborating this point is that, if $a_0 > 0$, then the effective gravitational coupling $G_{\text{eff}} = \phi_0^{-1} e^{2|a_0|t}$ diverges in the far future, which could by itself be taken as signaling instability. The lesson to be learned here is that the Brans-Dicke stealth spacetime (1.5) and (1.6), describing the analog of a metastable state of the effective ϕ-fluid with constant $\mathcal{K}T$, is unstable with respect to tensor perturbations. In retrospect this conclusion could have been expected, but first one needed to understand the role of this analytical solution. It is significant that the thermodynamics of scalar-tensor gravity gives new meaning to phenomena and analytical solutions that would otherwise be unremarkable.

Apart from the specific solution (1.5) and (1.6) and the specific $\omega = -1$ Brans-Dicke theory considered (which is, however, somehow special as it is the low-energy limit of the bosonic string [31,32]), one learns that nontrivial states can in principle exist in the thermodynamics of scalar-tensor gravity. Future work will search for other possible metastable states and, above all, will check further the consistency of the formalism and its consequences for various gravity regimes and physical situations.

Going beyond the particular solution examined here, is known that a variety of stealth solutions are possible in Horndeski and in higher-order scalar-tensor theories [45,46], a much more general framework than Brans-Dicke or "first generation" scalar-tensor gravity. Presumably, the stability of these stealth solutions depends on the details of the scalar fields appearing in them, as is the case for the situation examined in the present manuscript. The extension of our discussion to these theories is not trivial because the Bardeen-Ellis-Bruni-Hwang formalism does not apply directly to cosmological perturbations in these theories. Second, the thermodynamics of scalar-tensor gravity that motivates this work has been extended to "viable" Horndeski gravities (which turn out to be those that admit an Einstein-frame representation) [17], but not to other Horndeski and higher-order theories. Assessing the stability of stealth solutions in these more general scalar-tensor theories will be the subject of future research.

ACKNOWLEDGMENTS

We thank a referee for pointing out an error in a previous version of the manuscript and for suggestions improving the presentation. This work is supported, in part, by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2016-03803 to V. F.).

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