# Linear stability of black holes in shift-symmetric Horndeski theories with a time-independent scalar field

Masato Minamitsuji,[1] Kazufumi Takahashi,[2] and Shinji Tsujikawa[3]

[1]*Centro de Astrofísica e Gravitação—CENTRA, Departamento de Física, Instituto Superior Técnico—IST, Universidade de Lisboa—UL, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal*

[2]*Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502 Kyoto, Japan*

[3]*Department of Physics, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*

(Received 1 March 2022; accepted 12 April 2022; published 2 May 2022)

We study linear perturbations about static and spherically symmetric black holes with a time-independent background scalar field in shift-symmetric Horndeski theories, whose Lagrangian is characterized by coupling functions depending only on the kinetic term of the scalar field $X$. We clarify conditions for the absence of ghosts and Laplacian instabilities along the radial and angular directions in both odd- and even-parity perturbations. For reflection-symmetric theories described by a k-essence Lagrangian and a nonminimal derivative coupling with the Ricci scalar, we show that black holes endowed with nontrivial scalar hair are unstable around the horizon in general. This includes nonasymptotically flat black holes known to exist when the nonminimal derivative coupling to the Ricci scalar is a linear function of $X$. We also investigate several black hole solutions in nonreflection-symmetric theories. For cubic Galileons with the Einstein-Hilbert term, there exists a nonasymptotically flat hairy black hole with no ghosts/Laplacian instabilities. Also, for the scalar field linearly coupled to the Gauss-Bonnet term, asymptotically flat black hole solutions constructed perturbatively with respect to a small coupling are free of ghosts/Laplacian instabilities.

DOI: 10.1103/PhysRevD.105.104001

## I. INTRODUCTION

Black holes (BHs) are fundamental objects whose existence is theoretically predicted by general relativity (GR) and other gravitational theories. With the dawn of gravitational-wave astronomy [1], we can now probe physics of BHs and possible deviations from GR at strong gravity regimes [2–5]. The discovery of BH shadows [6] also opened up a new window for exploring the properties of BHs. Under this observational status, it is important to classify what kinds of BHs exist in the presence of additional degree(s) of freedom like a scalar field or in gravitational theories beyond GR.

In GR with an electromagnetic field, a uniqueness theorem states that asymptotically flat and stationary BH solutions are characterized only by mass, angular momentum, and electric charge [7–9]. This "no-hair" property of BHs also holds for a minimally coupled canonical scalar field $\phi$ [10,11], a minimally coupled k-essence [12], as well as a scalar field nonminimally coupled to the Ricci scalar $R$ in the form $F(\phi)R$ [13–16]. The no-hair theorem does not persist in scalar-tensor theories containing derivative couplings like $G_4(X)R$ in the Lagrangian, where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ is the kinetic term of the scalar field. Such derivative couplings can be accommodated in a framework of so-called Horndeski theories, which form the most general class of scalar-tensor theories with second-order Euler-Lagrange equations [17–20].

The Lagrangian of Horndeski theories contains four coupling functions $G_{2,3,4,5}$ depending on both $\phi$ and $X$. If we impose the invariance under the constant shift $\phi \to \phi + c$, the functions $G_{2,3,4,5}$ depend only on $X$. In such shift-symmetric Horndeski theories, Hui and Nicolis [21] argued that a no-hair result of BHs holds under the following three hypotheses [22]:

(i) The background geometry is static and spherically symmetric and the scalar field is also static [see the ansatz (2.3)], i.e., the character of the scalar field is spacelike ($X < 0$).

(ii) The spacetime is asymptotically flat with a vanishing radial field derivative $\phi'(r) \to 0$ at spatial infinity ($r \to \infty$) and the norm of the Noether current associated with the shift symmetry is finite on the BH horizon.

(iii) A canonical kinetic term $X$ is present in the Lagrangian and the $X$-derivatives of $G_{2,3,4,5}$ contain only positive or zero powers of $X$.

Namely, under these assumptions, we end up with the no-hair BH solution, i.e., $\phi'(r) = 0$ everywhere.

If we violate at least one of the conditions given above, it is possible to realize hairy BH solutions endowed with nontrivial scalar hair. For a scalar field with the linear dependence on time $t$ of the form $\phi = qt + \Phi(r)$, which evades the hypothesis (i), there exists a stealth Schwarzschild solution [23].[1] If the asymptotic flatness of spacetime is not imposed, the linear quartic derivative coupling $X$ in $G_4$ gives rise to exact hairy BH solutions with an asymptotic geometry mimicking the Schwarzschild–(anti-)de Sitter [(A)dS] spacetime [26–29] (see also Refs. [30–35]). This is an outcome of the violation of the hypothesis (ii). If we consider a quintic-order derivative coupling of the form $G_5 \propto \alpha_{\rm GB} \ln|X|$, which is equivalent to the Gauss-Bonnet term $R^2_{\rm GB}$ linearly coupled to the scalar field [19], there exists an asymptotically flat hairy BH solution whose metric components are corrected by the Gauss-Bonnet coupling $\alpha_{\rm GB}$ [36,37]. This arises from the violation of the hypothesis (iii).[2] Another asymptotically flat BH solution violating the hypothesis (iii) exists in the model where $G_4(X)$ contains $(-X)^{1/2}$ [44].

The linear stability of BHs with a time-dependent scalar field has been extensively studied in the literature [45–53]. On the other hand, it is yet unclear whether the nonasymptotically flat BHs arising from the violation of the hypothesis (ii) or (iii) are stable against perturbations about the static and spherically symmetric background. The perturbations of static and spherically symmetric BHs in full Horndeski theories in the presence of a time-independent background scalar field were investigated for both odd-parity [54] and even-parity [55] sectors. In these references, the authors obtained conditions for the absence of ghosts and Laplacian instabilities for high-momentum modes, except the angular stability condition of even-parity perturbations. Recently, the authors of Ref. [56] generalized the results of Refs. [54,55] by taking into account a perfect fluid, in which the propagation speeds of gravity and scalar-field sectors along the angular directions were also derived. The linear stability conditions given in Ref. [56] can be applied not only to BHs but also to neutron stars with nontrivial scalar hair [57,58].

In this paper, we study the linear stability of static and spherically symmetric BHs in shift-symmetric Horndeski theories arising from the violation of the hypothesis (ii) or (iii). We keep the hypothesis (i), so that the background scalar field has a static configuration $\phi(r)$. We show that the BH solutions in reflection-symmetric subclass of shift-symmetric Horndeski theories possessing only two coupling functions $G_2(X)$ and $G_4(X)$ are generically prone to the Laplacian instability of even-parity perturbations around the horizon. In particular, this instability shows up for an exact nonasymptotically flat BH present for theories with $G_4 \supset X$ [26–29] as well as for an asymptotically flat BH arising in theories with $G_4 \supset (-X)^{1/2}$ [44].[3] We also study several examples of BHs in shift-symmetric Horndeski theories in the presence of the coupling functions $G_3(X)$ and $G_5(X)$, that break the reflection symmetry. As a first example, we consider a nonasymptotically flat BH in GR with a cubic Galileon ($G_3 \propto X$) and show that the solution satisfies all the stability conditions of linear perturbations. Thus, there exists a stable hairy BH arising from the violation of the hypothesis (ii). As a second example, we study the case with the quintic coupling $G_5$ containing positive powers of $X$, for which it is difficult to realize stable BHs with nontrivial scalar hair. Finally, we investigate the case of a scalar field linearly coupled to the Gauss-Bonnet curvature invariant, which corresponds to $G_5 \propto \alpha_{\rm GB} \ln|X|$. In this case, there is an asymptotically flat BH with a finite field derivative $\phi'(r)$ on the horizon [36,37]. In the regime of small couplings $\alpha_{\rm GB}$, we show that all the linear stability conditions are consistently satisfied for this solution.

The rest of this paper is constructed as follows. In Sec. II, we review the shift-symmetric Horndeski theories and the properties of static and spherically symmetric solutions in vacuum. In Sec. III, we revisit linear stability conditions for the odd- and even-parity perturbations derived in Refs. [54–56]. In Sec. IV, we study the stability of GR BH solutions with a trivial scalar-field profile $\phi'(r) = 0$. In Sec. V, we show that the hairy BHs with nonvanishing scalar-field derivatives appearing in the reflection-symmetric theories are generically unstable. In Sec. VI, we investigate the linear stability of hairy BHs arising in nonreflection-symmetric theories. The last Sec. VII is devoted to giving a brief summary and conclusion.

## II. SHIFT-SYMMETRIC HORNDESKI THEORIES

The action of shift-symmetric Horndeski theories is given by [17–20]

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_H, \tag{2.1}$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and

$$\begin{aligned}\mathcal{L}_H = {}& G_2(X) - G_3(X)\Box\phi + G_4(X)R + G_{4,X}(X)[(\Box\phi)^2 \\ & - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)] \\ & + G_5(X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}(X)[(\Box\phi)^3 \\ & - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) \\ & + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)],\end{aligned} \tag{2.2}$$

[1]The existence conditions for stealth solutions with constant $X$ in higher-order scalar-tensor theories were specified in Refs. [24,25].

[2]We note that there exist hairy BH solutions also for nonshift-symmetric Gauss-Bonnet couplings $\xi(\phi)R^2_{\rm GB}$ such as $\xi(\phi) \propto \phi^n$ ($n > 1$) and $\xi(\phi) \propto \mathrm{e}^{-\phi}$ [38–43].

[3]Here and in what follows, by $G_4 \supset (-X)^p$, we mean that the nonminimal derivative coupling $G_4$ contains a term proportional to $(-X)^p$ on top of the constant term corresponding to the Einstein-Hilbert term.

with $R$ and $G_{\mu\nu}$ being the Ricci scalar and Einstein tensor, respectively. The four functions $G_j$'s ($j = 2, 3, 4, 5$) depend only on the kinetic term $X = -g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi/2$, with the covariant derivative operator $\nabla_\mu$. We also use the notations $\Box\phi \equiv \nabla^\mu\nabla_\mu\phi$, and $G_{j,X} \equiv \mathrm{d}G_j/\mathrm{d}X$, $G_{j,XX} \equiv \mathrm{d}^2G_j/\mathrm{d}X^2$, etc.

We study static and spherically symmetric solutions in shift-symmetric Horndeski theories. The metric and scalar field are assumed to be of the following form:

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + h^{-1}(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \\ \phi &= \phi(r), \end{aligned} \tag{2.3}$$

where $t$, $r$, $(\theta, \varphi)$ are the temporal, radial, and angular coordinates, respectively. The background configuration is characterized by the three functions of $r$, i.e., $f(r)$, $h(r)$, and $\phi(r)$. Note that, on the background (2.3), the kinetic term of the scalar field can be written as

$$X = -\frac{1}{2}h\phi'^2. \tag{2.4}$$

We note that our ansatz (2.3) corresponds to the hypothesis (i) mentioned in Sec. I. The independent equations are the $tt$-, $rr$-, and $\theta\theta$-components of the equations of motion for $g_{\mu\nu}$, which are respectively given by [54–56]

$$\begin{aligned} \mathcal{E}_{tt} \equiv {} & \left(A_1 + \frac{A_2}{r} + \frac{A_3}{r^2}\right)\phi'' + \left(\frac{\phi'}{2h}A_1 + \frac{A_4}{r} + \frac{A_5}{r^2}\right)h' + G_2 \\ & - \frac{2G_{4,X}h^2\phi'^2 + 2G_4(h-1)}{r^2} = 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \mathcal{E}_{rr} \equiv {} & -\left(\frac{\phi'}{2h}A_1 + \frac{A_4}{r} + \frac{A_5}{r^2}\right)\frac{hf'}{f} - \frac{2\phi'}{r}A_1 \\ & - \frac{1}{r^2}\left[\frac{\phi'}{2h}A_2 + (h-1)A_4\right] - G_2 - hG_{2,X}\phi'^2 = 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathcal{E}_{\theta\theta} \equiv {} & \left\{\left[A_2 + \frac{(2h-1)\phi' A_3 + 2hA_5}{h\phi' r}\right]\frac{f'}{4f} + A_1 + \frac{A_2}{2r}\right\}\phi'' \\ & + \frac{1}{4f}\left(2hA_4 - \phi' A_2 + \frac{2hA_5 - \phi' A_3}{r}\right)\left(f'' - \frac{f'^2}{2f}\right) \\ & + \left[A_4 + \frac{2h(2h+1)A_5 - \phi' A_3}{2h^2 r}\right]\frac{f'h'}{4f} \\ & - \frac{h^2G_{4,X}\phi'^2 + hG_4}{r}\frac{f'}{f} + \left(\frac{\phi'}{h}A_1 + \frac{A_4}{r}\right)\frac{h'}{2} + G_2 \\ = {} & 0, \end{aligned} \tag{2.7}$$

where a prime represents the derivative with respect to $r$, and we have defined the quantities $A_1, \ldots, A_5$ in Eq. (A1). These equations are combined to give

$$\frac{f'}{2f}\mathcal{E}_{tt} + \mathcal{E}'_{rr} + \left(\frac{f'}{2f} + \frac{2}{r}\right)\mathcal{E}_{rr} + \frac{2}{r}\mathcal{E}_{\theta\theta} = 0, \tag{2.8}$$

which corresponds to the scalar-field equation of motion obtained by varying the action (2.1) with respect to $\phi$. This is due to the Noether identity associated with diffeomorphism invariance [59]. More explicitly, Eq. (2.8) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sqrt{\frac{f}{h}}J^r\right) = 0, \tag{2.9}$$

where $J^r$ is a radial component of the Noether current $J^\mu$ associated with shift symmetry, given by

$$J^r = h\phi'\mathcal{J}, \tag{2.10}$$

with

$$\begin{aligned} \mathcal{J} \equiv {} & G_{2,X} - \left(\frac{2}{r} + \frac{f'}{2f}\right)h\phi' G_{3,X} + 2\left(\frac{1-h}{r^2} - \frac{hf'}{rf}\right)G_{4,X} \\ & + 2h\phi'^2\left(\frac{h}{r^2} + \frac{hf'}{rf}\right)G_{4,XX} - \frac{f'}{2r^2 f}(1-3h)h\phi' G_{5,X} \\ & - \frac{f'h^3\phi'^3}{2r^2 f}G_{5,XX}. \end{aligned} \tag{2.11}$$

We note that with the ansatz (2.3) $J^r$ is the only nonvanishing component of $J^\mu$. The solution to Eq. (2.9) is expressed in the form

$$J^r(r) = \frac{Q}{r^2}\sqrt{\frac{h}{f}}, \tag{2.12}$$

where $Q$ is an integration constant corresponding to a scalar charge. Then, the current strength squared reads

$$J^2 \equiv g_{\mu\nu}J^\mu J^\nu = g_{rr}(J^r)^2 = \frac{Q^2}{r^4 f}. \tag{2.13}$$

Requiring that $J^2$ is finite on the horizon ($f = 0$), which is the hypothesis (ii) mentioned in Sec. I, the constant $Q$ should vanish. In this case, we have

$$J^r(r) = h\phi'\mathcal{J} = 0, \tag{2.14}$$

for any value of $r$. We mainly study solutions with $Q = 0$ satisfying Eq. (2.14). Note that, for the Gauss-Bonnet term linearly coupled to a scalar field [36,37], the divergence of $J^2$ does not necessarily invoke unphysical properties of the BH solution [60], and hence a nonvanishing $Q$ is allowed in this particular case (see Sec. VI B).

Provided that $\mathcal{J}$ is finite in the limit of $\phi' \to 0$, which is the case when all $G_j$'s are analytic functions of $X$, namely, all $G_j$'s contain only the zero or positive integer powers of $X$ [the hypothesis (iii) in Sec. I], there are two branches of solutions to Eq. (2.14). One is the branch $\phi' = 0$, while the other is a nontrivial branch satisfying

$$\mathcal{J} = 0. \tag{2.15}$$

If we impose the asymptotic flatness ($f \to 1, h \to 1, f' \to 0, h' \to 0$ as $r \to \infty$) with a vanishing field derivative $\phi'(\infty) = 0$ and that the contribution of the ordinary kinetic term in $G_2$, namely $G_{2,X}(0)X$ with $G_{2,X}(0) \neq 0$, is dominant in $\mathcal{J}$ in the large-$r$ limit, which is the hypothesis (iii) mentioned in Sec. I, theories with analytic coupling functions lead to $\mathcal{J} \to G_{2,X}(0)$ at spatial infinity. Hence, $\mathcal{J}$ approaches a nonvanishing constant, meaning that the branch $\mathcal{J} = 0$ is not present. In this case, we end up with the no-hair branch with $\phi'(r) = 0$ [21].

On the other hand, if the asymptotic flatness is not imposed the derivative $f'(r)$ can be a growing function of $r$. Then, it is possible that terms arising from the derivative couplings $G_3, G_4, G_5$ in Eq. (2.11) balance the term $G_{2,X}$ to realize $\mathcal{J} = 0$. The BH solution present for a quartic-order linear derivative coupling $X$ in $G_4$ is such an example [26–29]. The other possibilities for realizing BH solutions with $\phi' \neq 0$ are that $G_2$, $G_3$, $G_4$, and $G_5$ contain $\ln|X|$, or fractional/inverse powers of $X$ [36,37,44], where the derivatives $G_{2,X}$, $G_{3,X}$, $G_{4,X}$, $G_{4,XX}$, $G_{5,X}$, and $G_{5,XX}$ have inverse powers of $\phi'(r)$ and their contributions to $\mathcal{J}$ balance that of the canonical kinetic term in the limit $\phi'(r) \to 0$.

## III. BLACK HOLE LINEAR STABILITY CONDITIONS

In order to discuss the linear stability of BHs on the background (2.3), it is useful to separate perturbations into the odd- and even-parity sectors depending on the transformation properties under the rotation in two-dimensional plane $(\theta, \varphi)$ [61,62]. In full Horndeski theories with a time-independent scalar field, the stability conditions against linear perturbations in the odd- and even-parity sectors were derived for BHs [54,55] and relativistic stars [56] (see also Refs. [57,63–65]). The angular propagation speed of even-parity perturbations was not obtained in Ref. [55], but this issue was addressed in Ref. [56]. It should be noted that the linear stability conditions in Ref. [56] were derived in the presence of a perfect fluid with density $\rho$ and pressure $P$ to model static and spherically symmetric stars, and the stability conditions for BHs follow by taking the limits $\rho \to 0$ and $P \to 0$. In the following, we summarize the linear stability conditions for both odd- and even-parity perturbations.

In the odd-parity sector, the quadratic action for higher multipoles $\ell \geq 2$ can be written in the form [54]

$$\mathcal{S}_{\rm odd} = \int {\rm d}t{\rm d}r\left[\frac{1}{2}K_\chi\dot{\chi}^2 - \frac{1}{2}G_\chi\chi'^2 - \frac{\ell(\ell+1)}{2}W_\chi\chi^2 - \frac{1}{2}M_\chi\chi^2\right], \tag{3.1}$$

where $\chi$ is the master variable and a dot denotes the derivative with respect to $t$, and we have performed the integration over the angular variables. Here, the coefficients $K_\chi$, $G_\chi$, $W_\chi$, and $M_\chi$ are determined by the coupling functions in Eq. (2.2) and the background solution. Although we do not present the explicit form of coefficients, we summarize below the stability conditions which can be read off from the quadratic action. Note that the boundedness of a Hamiltonian is a coordinate-dependent concept [50], and there is a subtlety when the action contains a cross term of time and spatial derivatives [51], which happens for a time-dependent scalar profile $\phi = qt + \Phi(r)$ with $q \neq 0$. In such a case, one should change the coordinate system to remove the cross term. For the time-independent scalar field we are considering now, the cross term is absent from the outset, and hence there is no such an ambiguity. The ghost-free condition is given by $K_\chi > 0$, which reads

$$\mathcal{G} \equiv 2G_4 + 2h\phi'^2G_{4,X} - \frac{f'h^2\phi'^3G_{5,X}}{2f} > 0. \tag{3.2}$$

For high-momentum modes, the squared propagation speeds of odd-parity perturbations along the radial and angular directions are given, respectively, by

$$c_{r,\rm odd}^2 = \frac{g_{rr}}{|g_{tt}|}\frac{G_\chi}{K_\chi} = \frac{\mathcal{G}}{\mathcal{F}}, \qquad c_{\Omega,\rm odd}^2 = \frac{g_{\theta\theta}}{|g_{tt}|}\frac{W_\chi}{K_\chi} = \frac{\mathcal{G}}{\mathcal{H}}, \tag{3.3}$$

where

$$\mathcal{H} \equiv 2G_4 + 2h\phi'^2G_{4,X} - \frac{h^2\phi'^3G_{5,X}}{r}, \tag{3.4}$$

$$\mathcal{F} \equiv 2G_4 - h\phi'^2\left(\frac{1}{2}h'\phi' + h\phi''\right)G_{5,X}. \tag{3.5}$$

Note that the factor $\ell(\ell+1)$ in the quadratic action (3.1) originates from the spherical Laplacian, and hence the coefficient $W_\chi$ is associated with the angular propagation speed. It should also be noted that the squared sound speeds defined in this way are independent of the choice of coordinates. Under the no-ghost condition (3.2), the Laplacian instabilities can be avoided for

$$\mathcal{H} > 0, \tag{3.6}$$

$$\mathcal{F} > 0. \tag{3.7}$$

In the even-parity sector, the quadratic action for higher multipoles $\ell \geq 2$ can be written in the form [55]

$$\mathcal{S}_{\rm even} = \int \mathrm{d}t\mathrm{d}r \sum_{I,J=1}^{2}\left(\frac{1}{2}\boldsymbol{K}_{IJ}\dot{v}^I\dot{v}^J - \frac{1}{2}\boldsymbol{G}_{IJ}v^{I\prime}v^{J\prime} - \boldsymbol{Q}_{IJ}v^I v^{J\prime} - \frac{1}{2}\boldsymbol{M}_{IJ}v^I v^J\right). \tag{3.8}$$

Here, $v^I = (\psi, \delta\phi)$ are the master variables, with $\psi$ and $\delta\phi$ corresponding to gravitational field and scalar-field perturbations, respectively. The coefficient matrices $\boldsymbol{K}$, $\boldsymbol{G}$, $\boldsymbol{Q}$, and $\boldsymbol{M}$ are determined by the coupling functions in Eq. (2.2) and the background solution, and we choose the overall numerical factor so that the components of $\boldsymbol{K}$ are finite in the limit $\ell \to \infty$. Note that the matrices $\boldsymbol{K}$, $\boldsymbol{G}$, and $\boldsymbol{M}$ are symmetric and $\boldsymbol{Q}$ is antisymmetric. Although we do not present the explicit form of coefficient matrices, we summarize below the stability conditions which can be read off from the quadratic action. Ghost instabilities can be avoided if both the eigenvalues of $\boldsymbol{K}$ are positive, i.e.,

$$\boldsymbol{K}_{11} > 0 \quad \text{and} \quad \det \boldsymbol{K} > 0. \tag{3.9}$$

Provided that the condition (3.7) holds, these conditions can be satisfied if

$$\mathcal{K} \equiv 2\mathcal{P}_1 - \mathcal{F} > 0, \tag{3.10}$$

where

$$\mathcal{P}_1 = \frac{h\mu}{2fr^2\mathcal{H}^2}\left(\frac{fr^4\mathcal{H}^4}{\mu^2 h}\right)', \qquad \mu = \frac{2(\phi' a_1 + r\sqrt{fh}\mathcal{H})}{\sqrt{fh}}, \tag{3.11}$$

and $a_1$ is defined in Eq. (A2). In the limit of high frequencies, the squared radial propagation speeds of $\psi$ and $\delta\phi$ are given as eigenvalues of the matrix

$$\boldsymbol{c}_{r,\rm even}^2 \equiv \frac{g_{rr}}{|g_{tt}|}\boldsymbol{K}^{-1}\boldsymbol{G}. \tag{3.12}$$

Written explicitly, we have [54–56]

$$c_{r1,\rm even}^2 = \frac{\mathcal{G}}{\mathcal{F}}, \tag{3.13}$$

$$c_{r2,\rm even}^2 = \frac{2\phi'[4r^2(fh)^{3/2}\mathcal{H}c_4(2\phi' a_1 + r\sqrt{fh}\mathcal{H}) - 2a_1^2 f^{3/2}\sqrt{h}\phi'\mathcal{G} + (a_1 f' + 2c_2 f)r^2 fh\mathcal{H}^2]}{f^{5/2}h^{3/2}(2\mathcal{P}_1 - \mathcal{F})\mu^2}, \tag{3.14}$$

where $c_2$ and $c_4$ are defined in Eqs. (A3) and (A4). Since $c_{r1,\rm even}^2$ is identical to $c_{r,\rm odd}^2$ in Eq. (3.3), the conditions (3.2) and (3.7) ensure that $c_{r1,\rm even}^2 > 0$. In order to avoid the Laplacian instability of $\delta\phi$ along the radial direction, we require that

$$c_{r2,\rm even}^2 > 0. \tag{3.15}$$

For the monopole mode ($\ell = 0$), there is no propagation for the gravitational perturbation $\psi$, while the scalar-field perturbation $\delta\phi$ propagates with the same radial velocity as Eq. (3.14). For the dipole mode ($\ell = 1$), there is a gauge degree of freedom for fixing $\delta\phi = 0$, under which the perturbation $\psi$ propagates with the same radial speed squared as Eq. (3.14). The squared angular propagation speeds of $\psi$ and $\delta\phi$ in the large-$\ell$ limit are obtained as eigenvalues of the matrix

$$\boldsymbol{c}_{\Omega,\rm even}^2 \equiv \lim_{\ell\to\infty}\frac{1}{\ell(\ell+1)}\frac{g_{\theta\theta}}{|g_{tt}|}\boldsymbol{K}^{-1}\boldsymbol{M}. \tag{3.16}$$

This gives the following biquadratic equation [56]:

$$c_\Omega^4 + 2B_1 c_\Omega^2 + B_2 = 0, \tag{3.17}$$

namely,

$$c_{\Omega\pm}^2 = -B_1 \pm \sqrt{B_1^2 - B_2}, \tag{3.18}$$

where $B_1$ and $B_2$ are defined in Eqs. (A5) and (A6). The branch of the square root in Eq. (3.18) is chosen so that $c_{\Omega\pm}^2$ are smooth functions of $r$ [see also the comment below Eq. (6.17)]. Note that the above squared sound speeds of even-parity perturbations along the radial and angular directions are scalar quantities independent of the choices of gauges and coordinates. The Laplacian instabilities along the angular direction can be avoided for

$$c_{\Omega+}^2 > 0, \tag{3.19}$$

$$c_{\Omega-}^2 > 0. \tag{3.20}$$

From Eq. (3.18), these conditions are realized if

$$B_1^2 \geq B_2 > 0 \quad \text{and} \quad B_1 < 0. \tag{3.21}$$

In summary, there are neither ghosts nor Laplacian instabilities under the conditions (3.2), (3.6), (3.7), (3.10), (3.15), and (3.21). In Table I, we summarize these stability conditions for convenience.

TABLE I. Summary of the linear stability conditions.

| | No ghost | $c_r^2 > 0$ | $c_\Omega^2 > 0$ |
|---|---|---|---|
| Odd modes | $\mathcal{G} > 0$ | $\mathcal{F} > 0$ | $\mathcal{H} > 0$ |
| Even modes | $\mathcal{K} > 0$ | $c_{r2,\mathrm{even}}^2 > 0$ | $B_1^2 \geq B_2 > 0$ and $B_1 < 0$ |

## IV. LINEAR STABILITY OF GENERAL RELATIVITY SOLUTIONS

First of all, we study the linear stability of BH solutions with a trivial scalar-field profile, i.e.,

$$\phi' = 0. \tag{4.1}$$

As mentioned in Sec. II, such a solution exists as long as $\mathcal{J}$ is finite in the limit that $\phi' \to 0$. We shall discuss the other branch of hairy BH solution satisfying $\mathcal{J} = 0$ in the next sections. Note that the functions $G_j$'s and their derivatives are evaluated at $X = 0$ throughout this section. In this case, the background equations (2.5)–(2.7) reduce to the following two independent equations:

$$hf' - fh' = 0, \tag{4.2}$$

$$2G_4 h \frac{(rf)'}{r^2 f} - G_2 - \frac{2G_4}{r^2} = 0. \tag{4.3}$$

They can be solved to yield

$$h = C_0 f = 1 - \frac{r_0}{r} + \frac{G_2}{6G_4} r^2, \tag{4.4}$$

with $C_0$ and $r_0$ being integration constants. The integration constant $C_0$ can be absorbed into a rescaling of $t$, and hence we obtain the Schwarzschild-(A)dS metric, i.e., the BH solution in GR. Nevertheless, due to the existence of a dynamical scalar field, the stability under linear perturbations is rather nontrivial, as we shall see below.

Let us first discuss the linear stability of no-hair BHs against odd-parity perturbations. For $\phi' = 0$, the quantities relevant to stability of odd modes are simply given by

$$\mathcal{F} = \mathcal{G} = \mathcal{H} = 2G_4. \tag{4.5}$$

Therefore, the no-ghost condition (3.2) yields

$$G_4 > 0. \tag{4.6}$$

Also, we have

$$c_{r,\mathrm{odd}}^2 = c_{\Omega,\mathrm{odd}}^2 = 1, \tag{4.7}$$

meaning that there is no Laplacian instability.

Next, we study the BH stability against even-parity perturbations. A caveat here is that the quantities associated with stability of even modes contain $\phi'$ in their denominators, though one can obtain finite $\phi' \to 0$ limits. In order to remove vanishing $\phi'$ in the denominator, one should redo the computation of Refs. [55,56] with $\phi'$ set to zero from the outset, but this reproduces the $\phi' \to 0$ limits of the final results. Hence, one can safely take the $\phi' \to 0$ limit. Another point to note is that the quantity $\mathcal{K}$, which is associated with the no-ghost condition of even-parity perturbations, is vanishing in the limit $\phi' \to 0$. This could be a problem because it may imply the degeneracy of the kinetic matrix, which leads to strong coupling. However, as pointed out in Ref. [55], the determinant of the kinetic matrix is proportional to $\mathcal{K}/\phi'^2$, which is finite in the limit $\phi' \to 0$, meaning that the strong coupling problem is actually absent. To be more concrete, we solve Eqs. (2.5)–(2.7) for $h'$, $f'$, $f''$, substitute them into $\mathcal{K}$, and finally take the limits $\phi' \to 0$ and $\phi'' \to 0$. Then, the no-ghost condition is read off from

$$\lim_{\phi' \to 0} \frac{\mathcal{K}}{\phi'^2} = \frac{G_{2,X}G_4 - G_2 G_{4,X}}{2G_4} r^2 > 0. \tag{4.8}$$

On using the inequality (4.6), this condition translates to

$$G_{2,X}G_4 - G_2 G_{4,X} > 0. \tag{4.9}$$

If we consider the Schwarzschild BH solution, we have $G_2 = 0$ in Eq. (4.4). In this case, the condition (4.9) reduces to $G_{2,X}G_4 > 0$, which is consistent with the one derived in Ref. [55]. The squared sound speeds in the radial direction reduce to

$$c_{r1,\mathrm{even}}^2 = c_{r2,\mathrm{even}}^2 = 1. \tag{4.10}$$

Moreover, we have $B_1 = -1$ and $B_2 = 1$, so that

$$c_{\Omega+}^2 = c_{\Omega-}^2 = 1. \tag{4.11}$$

In summary, the Schwarzschild-AdS solution is free of ghosts/Laplacian instabilities if the conditions (4.6) and (4.9) are satisfied, with the propagation speeds equivalent to that of light.

## V. GENERIC INSTABILITY FOR REFLECTION-SYMMETRIC THEORIES

We investigate the linear stability of BH solutions in shift- and reflection-symmetric Horndeski theories, for which $G_3 = G_5 = 0$. Namely, we consider shift-symmetric Horndeski theories containing two arbitrary functions $G_2(X)$ and $G_4(X)$ with the Lagrangian

$$\mathcal{L}_H = G_2(X) + G_4(X)R + G_{4,X}(X)[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)]. \tag{5.1}$$

We focus on BH solutions with a nontrivial scalar-field profile $\phi' \neq 0$ satisfying Eq. (2.15), so that

$$\mathcal{J} = G_{2,X} + 2\left(\frac{1-h}{r^2} - \frac{hf'}{rf}\right)G_{4,X} + 2h\phi'^2\left(\frac{h}{r^2} + \frac{hf'}{rf}\right)G_{4,XX} = 0. \tag{5.2}$$

An explicit example of BH solutions of this type is present for $G_2(X) = \eta X - \Lambda$ and $G_4(X) = M_{\rm Pl}^2/2 - \alpha_1 X/2$ [26–28]. As long as $G_{2,X}$, $G_{4,X}$, and $G_{4,XX}$ do not contain fractional or negative powers of $\phi'$, there is the trivial GR solution $\phi' = 0$ besides the branch (5.2). The linear stability of BH solutions for the GR branch was already studied in Sec. IV.

For the branch satisfying Eq. (5.2), the background equations of motion can be simply expressed as [66]

$$8X(G_{4,X}^2 + G_4 G_{4,XX}) = r^2(\mathcal{G}G_2)_{,X}, \tag{5.3}$$

$$(r\mathcal{G}^2 h)' = \mathcal{G}(2G_4 + r^2 G_2), \tag{5.4}$$

$$\left(\frac{f}{\mathcal{G}^2 h}\right)' = 0. \tag{5.5}$$

We recall that $\mathcal{G}$ is given by Eq. (3.2), which in the present case reads

$$\mathcal{G} = 2(G_4 - 2XG_{4,X}) > 0. \tag{5.6}$$

We can solve Eq. (5.3) to yield $X$ algebraically as a function of $r$. Then, from Eqs. (5.4) and (5.5), we obtain $h$ and $f$ as functions of $r$. More concretely,

$$f = C_1 \mathcal{G}^2 h, \tag{5.7}$$

$$h = \frac{1}{r\mathcal{G}^2}\int_{r_s}^{r} {\rm d}r \mathcal{G}(2G_4 + r^2 G_2), \tag{5.8}$$

where $C_1(\neq 0)$ and $r_s(>0)$ are integration constants. Provided that the integrand in Eq. (5.8) approaches to a constant as $r \to r_s$, we have the following expansions:

$$f = f_1(r - r_s) + f_2(r - r_s)^2 + \cdots, \tag{5.9}$$

$$h = h_1(r - r_s) + h_2(r - r_s)^2 + \cdots, \tag{5.10}$$

where $f_j$ and $h_j$ $(j = 1, 2, \ldots)$ are constants. For $f_1 > 0$ and $h_1 > 0$, the coordinate distance $r_s$ can be identified as the position of BH horizon. If there exists some finite coordinate distance $r_c(> r_s)$ such that $f > 0$ and $h > 0$ for $r_s < r < r_c$ and $f(r_c) = h(r_c) = 0$, the radius $r_c$ can be identified as a cosmological horizon. The expansion of $X$ around the BH horizon is given by

$$X = X_s + X'(r_s)(r - r_s) + \mathcal{O}((r - r_s)^2), \tag{5.11}$$

where $X'(r_s)$ can be evaluated by taking the $r$-derivative of Eq. (5.2). Note that the value of $X_s$ is obtained algebraically from Eq. (5.3) with $r = r_s$. Unless $G_2$ and $G_4$ are fine-tuned, we have $X_s \neq 0$. The functions $G_2(X)$, $G_4(X)$, and their $X$-derivatives appearing in quantities relevant to the BH stability conditions are also expanded around the value of $X = X_s$ at the BH horizon, e.g.,

$$G_4(X) = G_4(X_s) + G_{4,X}(X_s)(X - X_s) + \mathcal{O}((X - X_s)^2). \tag{5.12}$$

In order to avoid ghosts/Laplacian instabilities, we require that all the linear conditions listed in Table I are satisfied from the BH horizon to spatial infinity (or the cosmological horizon, if it exists). In the present case, however, it is impossible to satisfy all these conditions. Taking the product $\mathcal{F}\mathcal{K}B_2$ in the vicinity of $r = r_s$, we obtain

$$\mathcal{F}\mathcal{K}B_2 = -\frac{4X_s^4(G_{4,X}^2 + G_4 G_{4,XX})^2}{(G_4 - 4X_s G_{4,X} - 4X_s^2 G_{4,XX})^2}\frac{r_s^2}{(r - r_s)^2} + \mathcal{O}((r - r_s)^{-1}), \tag{5.13}$$

where $G_4$ and its $X$-derivatives on the right-hand side are evaluated at $X = X_s$. Provided that

$$X_s \neq 0 \quad \text{and} \quad G_{4,X}^2 + G_4 G_{4,XX} \neq 0, \tag{5.14}$$

the leading-order contribution to Eq. (5.13) is negative, i.e.,

$$\mathcal{F}\mathcal{K}B_2 < 0, \quad \text{for } r \to r_s. \tag{5.15}$$

This shows that the quantities $\mathcal{F}$, $\mathcal{K}$, and $B_2$ cannot be positive at the same time near the BH horizon in general, leading to instability. For instance, even if the two conditions $\mathcal{F} > 0$ and $\mathcal{K} > 0$ are satisfied, we have $B_2 < 0$, and hence

$$c_{\Omega-}^2 = -B_1 - \sqrt{B_1^2 - B_2} < 0. \tag{5.16}$$

Thus, the angular Laplacian instability of even-parity perturbations is unavoidable around the BH horizon. We stress that the knowledge of the angular propagation speeds is essential to recognize the instability of this kind. A similar instability was found for stealth Schwarzschild-dS solutions with linearly time-dependent scalar hair in degenerate higher-order scalar-tensor theories [53].

We note that, from Eq. (2.4), $X < 0$ outside the BH horizon $[h(r) > 0]$ and $X > 0$ inside the BH horizon $[h(r) < 0]$, and hence the character of the scalar field is spacelike outside the horizon and timelike inside the horizon, respectively. If a BH solution with $X_s < 0$ could be extended to the interior of the BH horizon, the coordinate invariant $X$ would have a sudden change of the sign across the horizon, indicating that the horizon would become a singular hypersurface. Thus, a BH solution with $X_s < 0$ cannot be extended to the interior of the BH horizon and can be defined only in the domain outside the horizon where $h(r) > 0$ and the character of the scalar field is spacelike. Our result (5.15) suggests that BH solutions with $X_s \neq 0$ generically suffer from instabilities in the domain where the solution can exist, and hence such solutions could not be realistic. In other words, only the physically acceptable BH solution defined in both the exterior and interior of the horizon should have $X_s = 0$. We note that a similar argument could be applied to the cosmological horizon (if it exists), and a static and spherically symmetric solution with $X(r_c) < 0$ could not be extended to the exterior of the cosmological horizon.

The above instability is generic for the theories satisfying the condition (5.14). In Secs. V A–V C, we apply the above results to theories containing positive power-law functions $(-X)^p$ with $p > 0$ in $G_4(X)$ and $G_2(X) = \eta X - \Lambda$, with $\eta$ and $\Lambda$ being constant. Note that the above discussion does not apply if the conditions in Eq. (5.14) are not satisfied, in which case a further analysis is required. Such a situation occurs when, e.g., the nonminimal derivative coupling to the Ricci scalar is absent ($G_4 =$ constant). In this case, as we shall see in Sec. V D, the perturbations would be strongly coupled.

### A. $G_4 \supset (-X)^p$ with $p > 1$

As a demonstration of the generic instability, let us study theories given by the coupling functions

$$G_2 = \eta X - \Lambda, \qquad G_4 = \frac{M_{\rm Pl}^2}{2} + \frac{\alpha_p}{2}(-X)^p, \tag{5.17}$$

where $M_{\rm Pl}$ is the reduced Planck mass, and $\eta$, $\Lambda$, $\alpha_p$, and $p$ are constants. Note that we put a minus sign in $(-X)^p$ because $X = -h\phi'^2/2$ is negative for $h > 0$.

For the branch satisfying Eq. (5.2), the kinetic term of the scalar field is expressed as

$$(-X)^{p-1} = -\frac{\eta r^2 f}{p\alpha_p[(2p-1)(rf'+f)h - f]}. \tag{5.18}$$

The linear derivative coupling ($p = 1$) is a special case in which the left-hand side of Eq. (5.18) is constant. In this section, we study the power-law models with

$$p > 1, \tag{5.19}$$

which accommodate the quartic Galileons ($p = 2$). We shall discuss the $p = 1$ and $p = 1/2$ cases separately in Secs. V B and V C, respectively.

Around the BH horizon $r = r_s$, the leading-order contributions to $f$ and $h$ are $f_1(r - r_s)$ and $h_1(r - r_s)$, respectively. This mean that, as $r \to r_s$, the kinetic term $X$ approaches a constant $X_s$, satisfying

$$(-X_s)^{p-1} = -\frac{\eta r_s^2}{p\alpha_p[(2p-1)h_1 r_s - 1]}, \tag{5.20}$$

and hence $X_s \neq 0$. As a result, $\phi'^2$ diverges on the BH horizon. Due to the property $X_s \neq 0$, the term $G_{4,X}^2 + G_4 G_{4,XX}$ appearing in the numerator of Eq. (5.13) does not generally vanish. Unless the coupling $\alpha_p$ is fine-tuned to satisfy $G_{4,X}^2 + G_4 G_{4,XX} \propto (2p-1)\alpha_p(-X_s)^p + (p-1)M_{\rm Pl}^2 = 0$, the product $\mathcal{F}\mathcal{K}\mathcal{B}_2$ around $r = r_s$ yields

$$\mathcal{F}\mathcal{K}\mathcal{B}_2 = -\frac{p^2\alpha_p^2(-X_s)^{2p}[(2p-1)\alpha_p(-X_s)^p + (p-1)M_{\rm Pl}^2]^2}{[(4p^2-1)\alpha_p(-X_s)^p - M_{\rm Pl}^2]^2} \times \frac{r_s^2}{(r-r_s)^2} + \mathcal{O}((r-r_s)^{-1}). \tag{5.21}$$

The leading-order contribution to Eq. (5.21) is negative outside the BH horizon, and hence the corresponding BH solutions are unstable in general.

### B. $G_4 \supset X$

Having discussed general power-law quartic derivative couplings $G_4 \supset (-X)^p$ with $p > 1$, we now study the special case $p = 1$, for which the coupling functions read

$$G_2 = \eta X - \Lambda, \qquad G_4 = \frac{M_{\rm Pl}^2}{2} - \frac{\alpha_1}{2}X. \tag{5.22}$$

In this case, the field equations (5.3)–(5.5) yield the following exact solution [26–29]:

$$\begin{aligned} f &= -\frac{r_0}{r} + \frac{\sqrt{-\alpha_1\eta}(M_{\rm Pl}^2\eta - \alpha_1\Lambda)^2}{4M_{\rm Pl}^4\eta^3 r}\arctan\left(-\frac{\sqrt{-\alpha_1\eta}}{\alpha_1}r\right) \\ &\quad - \frac{(M_{\rm Pl}^2\eta + \alpha_1\Lambda)^2}{12M_{\rm Pl}^4\alpha_1\eta}r^2 + \frac{(M_{\rm Pl}^2\eta + \alpha_1\Lambda)(3M_{\rm Pl}^2\eta - \alpha_1\Lambda)}{4M_{\rm Pl}^4\eta^2}, \\ h &= \frac{4M_{\rm Pl}^4(\eta r^2 - \alpha_1)^2}{[(M_{\rm Pl}^2\eta + \alpha_1\Lambda)r^2 - 2M_{\rm Pl}^2\alpha_1]^2}f, \\ X &= \frac{(M_{\rm Pl}^2\eta - \alpha_1\Lambda)r^2}{2\alpha_1(\alpha_1 - \eta r^2)}, \end{aligned} \tag{5.23}$$

where we have chosen the integration constants $r_s$ and $C_1$ in Eqs. (5.4) and (5.5) so that $f \simeq 1 - r_0/r$ for small $r$. For the existence of this solution, we require

$$\alpha_1\eta < 0. \tag{5.24}$$

Provided that $M_{\rm Pl}^2\eta - \alpha_1\Lambda \neq 0$, there exists a nontrivial branch with $X \neq 0$. Also, since the character of the scalar field is spacelike and $X$ should be negative outside the BH horizon, we have

$$M_{\rm Pl}^2\eta - \alpha_1\Lambda < 0. \tag{5.25}$$

Let us denote by $X_s(<0)$ the value of $X$ at the BH horizon $r = r_s$. As noted previously, the solution (5.23) could exist only outside the BH horizon, as the scalar field becomes imaginary inside the horizon. On the contrary, if a solution similar to Eq. (5.23) exists inside the BH horizon where $h(r) < 0$ and the scalar field is timelike (i.e., $X > 0$), it could not be extended to the exterior of the BH horizon. Then, the product $\mathcal{F}\mathcal{K}B_2$ yields

$$\mathcal{F}\mathcal{K}B_2 = -\frac{\alpha_1^4 X_s^4}{(M_{\rm Pl}^2 + 3\alpha_1 X_s)^2}\frac{r_s^2}{(r-r_s)^2} + \mathcal{O}((r-r_s)^{-1}), \tag{5.26}$$

whose leading-order term is negative. Thus, the exact BH solution (5.23) is excluded by the instability problem around the BH horizon. The instability of the solution (5.23) is one of our main results. We note that, in generalized Proca theories [67–69] with a vector field $A_\mu$, there is an exact Schwarzschild solution with a nonvanishing longitudinal vector component in the presence of a quartic coupling $G_4(Y)$ containing a linear function of $Y = -A^\mu A_\mu/2$ [67,70,71]. Such BH solutions are also prone to a similar instability problem of vector-field perturbations in the odd-parity sector around the horizon [72].

For the theory (5.22), there exists a GR branch of the vanishing field profile ($\phi' = 0$). This branch is free from the ghost instability under the condition (4.9), i.e., $M_{\rm Pl}^2\eta - \alpha_1\Lambda > 0$, which is an opposite inequality to Eq. (5.25). Thus, under the inequality $M_{\rm Pl}^2\eta - \alpha_1\Lambda < 0$, neither the branch $\phi' \neq 0$ nor the other branch $\phi' = 0$ is stable.

### C. $G_4 \supset (-X)^{1/2}$

Let us study another special case with $p = 1/2$, i.e.,

$$G_2 = \eta X - \Lambda, \qquad G_4 = \frac{M_{\rm Pl}^2}{2} + \frac{\alpha_{1/2}}{2}(-X)^{1/2}. \tag{5.27}$$

Since $\mathcal{J} = (2\eta r^2 X + \alpha_{1/2}\sqrt{-X})/(2r^2X) = 0$, the kinetic term of the scalar field corresponding to the branch $\phi' \neq 0$ is given by

$$X = -\frac{\alpha_{1/2}^2}{4\eta^2 r^4}, \tag{5.28}$$

where we have assumed $\eta\alpha_{1/2} > 0$. A positive power of $\phi'$ is present in the denominator of $\mathcal{J}$, so the radial current equation, $J^r = h\phi'\mathcal{J} = 0$, does not allow the existence of a branch of vanishing field derivative ($\phi' = 0$).

Integrating Eqs. (5.4) and (5.5) with Eq. (5.28), we obtain the following exact solution [44]:

$$f = h = 1 - \frac{r_0}{r} - \frac{\alpha_{1/2}^2}{4M_{\rm Pl}^2\eta r^2} - \frac{\Lambda}{3M_{\rm Pl}^2}r^2, \tag{5.29}$$

where the integration constants have been chosen to have the behavior $f = h \simeq 1 - r_0/r$ for small $r$. We note that, because of the dependence $(-X)^{1/2}$ in $G_4$, the solution (5.29) is defined only in the domain where the scalar field is spacelike (i.e., $X < 0$), which corresponds to the domain outside the BH horizon and inside the cosmological horizon (for $\Lambda > 0$). In the case of $\Lambda = 0$, the solution (5.29) becomes an asymptotically flat spacetime, i.e., $f = h \to 1$ as $r \to \infty$. On using Eq. (5.28), in the vicinity of the BH horizon $r = r_s$, the product $\mathcal{F}\mathcal{K}B_2$ reduces to

$$\mathcal{F}\mathcal{K}B_2 = -\frac{\alpha_{1/2}^4}{64\eta^2 r_s^4}\frac{r_s^2}{(r-r_s)^2} + \mathcal{O}((r-r_s)^{-1}). \tag{5.30}$$

Since the leading-order contribution to $\mathcal{F}\mathcal{K}B_2$ is negative for $\alpha_{1/2} \neq 0$, the exact solution (5.29) with (5.28) inevitably suffers from the instability around the BH horizon.

### D. Strong coupling for k-essence

In this subsection, we study the case of k-essence given by the action

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[G_2(X) + \frac{M_{\rm Pl}^2}{2}R\right]. \tag{5.31}$$

From Eq. (5.2), the branch of a nonvanishing field derivative obeys

$$G_{2,X} = 0, \tag{5.32}$$

implying that $X$ is constant everywhere. Also, the value of $X$ is determined as a solution to the (algebraic) equation (5.32). From the background Eqs. (2.5) and (2.6), we obtain

$$G_2 = \frac{(rhf' + fh - f)M_{\rm Pl}^2}{r^2 f}, \qquad hf' = fh'. \tag{5.33}$$

The quantities associated with the linear stability of BHs yield

$$\mathcal{K} = 0, \qquad c_{r2,\rm even}^2 = \infty, \tag{5.34}$$

with $\mathcal{F} = \mathcal{G} = \mathcal{H} = M_{\rm Pl}^2$ and $c_{\Omega\pm}^2 = 1$. Such a diverging sound speed typically implies an infinitely strong coupling, and hence the perturbative treatment would no longer be viable in this case.

## VI. NONREFLECTION-SYMMETRIC THEORIES: CASE STUDIES

In the previous section, we showed that BH solutions with nontrivial scalar hair in the reflection-symmetric subclass of shift-symmetric Horndeski theories are linearly unstable in general. In this section, we study the linear stability for several examples of BH solutions in non-reflection-symmetric theories containing the coupling functions $G_3(X)$ or $G_5(X)$ in the Lagrangian. In this case, the scalar-field profile around the BH horizon is quite different from the one in the reflection-symmetric case. Indeed, substituting the expansions (5.9)–(5.11) around the BH horizon into the background equations, one finds that the left-hand side of Eq. (2.6) behaves as

$$\sqrt{2h_1}(-X_s)^{3/2}\left[G_{3,X}(X_s)+\frac{1}{r_s^2}G_{5,X}(X_s)\right](r-r_s)^{-1/2} + \mathcal{O}((r-r_s)^0), \tag{6.1}$$

which in general does not vanish unless

$$X_s=0 \quad \text{or} \quad G_{3,X}(X_s)+\frac{1}{r_s^2}G_{5,X}(X_s)=0. \tag{6.2}$$

Therefore, unless $G_3$ and $G_5$ are fine-tuned to satisfy the latter of Eq. (6.2), the kinetic term of the scalar field is vanishing on the BH horizon. In what follows, we study BHs having $X=0$ on the horizon in the presence of the coupling functions $G_3$ and $G_5$. In Sec. VI A, we first consider cubic-order power-law couplings $G_3(X)\propto(-X)^p$ with $p>0$ and focus on the case of cubic Galileons ($p=1$). In Sec. VI B, we discuss quintic-order power-law couplings $G_5(X)\propto(-X)^p$ with $p>0$. In Sec. VI C, we investigate the case with the scalar field linearly coupled to the Gauss-Bonnet curvature invariant, which amounts to $G_5(X)\propto\ln|X|$.

### A. Cubic Galileons

We consider theories characterized by the coupling functions

$$G_2=\eta X-\Lambda, \qquad G_3=\gamma_p(-X)^p,$$
$$G_4=\frac{M_{\rm Pl}^2}{2}, \qquad G_5=0, \tag{6.3}$$

where $\eta$, $\Lambda$, $\gamma_p$, and $p(>0)$ are constants. The branch with $\phi'(r)\neq 0$ obeys Eq. (2.15), i.e.,

$$\eta+p\gamma_p\left(\frac{2}{r}+\frac{f'}{2f}\right)h\phi'(-X)^{p-1}=0, \tag{6.4}$$

where $X=-h\phi'^2/2$. Around the BH horizon $r=r_s$, we expand the metric components $f$ and $h$ as in Eqs. (5.9) and (5.10). At leading order, the scalar-field derivative $\phi'$ around $r=r_s$ is expressed as

$$\phi'^{2p-1}=-\frac{\eta}{p\gamma_p}\left(\frac{h_1}{2}\right)^{-p}(r-r_s)^{-(p-1)}, \tag{6.5}$$

and hence

$$X\propto(r-r_s)^{1/(2p-1)}. \tag{6.6}$$

For $p>1/2$, the kinetic term $X$ vanishes on the horizon. This property is different from that in the case of $G_4(X)\supset(-X)^p$ studied in Sec. V A, for which $X(r_s)<0$.

From Eq. (6.5), we find that the cubic Galileons ($p=1$) correspond to a special case in which $\phi'$ approaches a nonvanishing constant as $r\to r_s$. For $p>1$, the scalar-field derivative diverges as $\phi'\propto(r-r_s)^{-(p-1)/(2p-1)}$ on the horizon. In the following, we study the linear stability of BHs in cubic Galileons, i.e.,

$$G_2=\eta X, \quad G_3=\gamma_1(-X), \quad G_4=\frac{M_{\rm Pl}^2}{2}, \quad G_5=0, \tag{6.7}$$

where the bare cosmological constant has been set to zero for simplicity. Since the action has a symmetry under the simultaneous change $\gamma_1\to-\gamma_1$ and $\phi\to-\phi$, we assume

$$\gamma_1>0, \tag{6.8}$$

without loss of generality. We note that the BH solutions for cubic Galileons with the cosmological constant were discussed in Ref. [73] by assuming a time-dependent scalar field of the form $\phi=qt+\Phi(r)$, where $q$ is a nonvanishing constant. Here, we are considering a time-independent scalar field ($q=0$) and addressing the linear stability of BHs unexplored in Ref. [73].

We exploit the scalar-field equation (6.4) together with the equations of motion (2.5) and (2.6) for the metric to obtain the asymptotic forms of $f$, $h$, and $\phi'$ around the BH horizon and at spatial infinity. The solutions expanded around $r=r_s$ are given by

$$\frac{f}{f_1}=(r-r_s)+\frac{1}{r_s}\left(\frac{\eta^3r_s^4}{\gamma_1^2M_{\rm Pl}^2}-1\right)(r-r_s)^2+\mathcal{O}((r-r_s)^3), \tag{6.9}$$

$$h=\frac{1}{r_s}(r-r_s)-\frac{1}{r_s^2}\left(\frac{3\eta^3r_s^4}{\gamma_1^2M_{\rm Pl}^2}+1\right)(r-r_s)^2+\mathcal{O}((r-r_s)^3), \tag{6.10}$$

$$\phi'=-\frac{2\eta r_s}{\gamma_1}-\frac{4\eta}{\gamma_1}\left(\frac{\eta^3r_s^4}{\gamma_1^2M_{\rm Pl}^2}-1\right)(r-r_s)+\mathcal{O}((r-r_s)^2), \tag{6.11}$$

where $f_1>0$. At spatial infinity, we obtain the following expansion:

$$\frac{f}{f_0} = \frac{\sqrt{6}(-\eta)^{3/2}}{18\gamma_1 M_{\rm Pl}} r^2 + 1 - \frac{r_1}{r} - \frac{3\sqrt{6}\gamma_1 M_{\rm Pl} r_1}{20(-\eta)^{3/2} r^3} + \mathcal{O}(r^{-4}), \tag{6.12}$$

$$h = \frac{\sqrt{6}(-\eta)^{3/2}}{18\gamma_1 M_{\rm Pl}} r^2 + \frac{5}{6} - \frac{r_1}{r} - \frac{11\sqrt{6}\gamma_1 M_{\rm Pl}}{24(-\eta)^{3/2} r^2} + \mathcal{O}(r^{-3}), \tag{6.13}$$

$$\phi' = \frac{\sqrt{6}M_{\rm Pl}}{\sqrt{-\eta} r} - \frac{9M_{\rm Pl}^2\gamma_1}{\eta^2 r^3} + \frac{9M_{\rm Pl}^2\gamma_1 r_1}{\eta^2 r^4} + \frac{75\sqrt{6}M_{\rm Pl}^3\gamma_1^2}{4(-\eta)^{7/2} r^5} + \mathcal{O}(r^{-6}), \tag{6.14}$$

where $f_0(>0)$ and $r_1$ are constants. The constant $f_0$ can be chosen freely due to the time reparametrization invariance. For the existence of this solution, we require that

$$\eta < 0. \tag{6.15}$$

Note that, as in the case of Schwarzschild-AdS BHs, the metric components $f$ and $h$ are growing functions of $r$ in the region $r \gg r_s$. It should also be noted that there exists another branch of solutions where the coefficients of $r^2$ in $f$ and $h$ are negative. We are not interested in this other branch, because a numerical integration from the BH horizon yields the branch of solution with (6.12)–(6.14), as we shall see below.

In what follows, we verify that the above solution satisfies all the stability conditions for both odd- and even-parity perturbations. From Eqs. (3.2), (3.4), and (3.5), we have

$$\mathcal{G} = \mathcal{H} = \mathcal{F} = M_{\rm Pl}^2 > 0, \tag{6.16}$$

so that the stability conditions of odd-parity perturbations are satisfied. For even-parity modes, the squared propagation speeds of gravitational perturbations along the radial and angular directions reduce to

$$c_{r1,\rm even}^2 = 1, \qquad c_{\Omega 1}^2 = 1. \tag{6.17}$$

The latter follows from the fact that the term $B_1^2 - B_2$ in Eq. (3.18) can be factored out in the form $B_1^2 - B_2 = B_3^2$, where $B_3$ can change its sign depending on the coordinate distance $r$. Then, there are the two solutions $c_{\Omega 1}^2 = -B_1 + B_3$ and $c_{\Omega 2}^2 = -B_1 - B_3$. One of them, which is equivalent to unity, corresponds to the propagation speed squared in the gravity sector, while the other to that of the scalar field. Hence, on the static and spherically symmetric background the cubic Galileon does not modify the propagation speed of gravitational perturbations in comparison to GR (analogous to the speed of tensor perturbations on an isotropic cosmological background [19,74]). Around $r = r_s$, the no-ghost condition for the even-parity perturbations translates to

$$\mathcal{K} = -\frac{2\eta^3 r_s^4}{\gamma_1^2} + \mathcal{O}(r - r_s), \tag{6.18}$$

where we have used the inequality (6.15). Since the leading-order contribution to Eq. (6.18) is positive, the ghost is absent. In the vicinity of the horizon, the squared radial and angular propagation speeds of the scalar field are given, respectively, by

$$c_{r2,\rm even}^2 = 1 + \mathcal{O}(r - r_s), \tag{6.19}$$

$$c_{\Omega 2}^2 = 3 + \mathcal{O}(r - r_s), \tag{6.20}$$

and hence there are no Laplacian instabilities around $r = r_s$. At spatial infinity, the quantities $\mathcal{K}$ and $c_{r2,\rm even}^2$ can be estimated as

$$\mathcal{K} = \frac{\sqrt{6}(-\eta)^{3/2} M_{\rm Pl}}{4\gamma_1} r^2 + \mathcal{O}(r), \tag{6.21}$$

$$c_{r2,\rm even}^2 = \frac{(-\eta)^{3/2}}{\sqrt{6}\gamma_1 M_{\rm Pl}} r^2 + \mathcal{O}(r). \tag{6.22}$$

Under the condition (6.15), the leading-order contributions to $\mathcal{K}$ and $c_{r2,\rm even}^2$ are positive. For $r \gg r_s$, the quantities $B_1$ and $B_2$ have the following asymptotic behavior:

$$B_1 = -\frac{1}{2} - \frac{r_1}{4r} + \mathcal{O}(r^{-2}),$$
$$B_2 = \frac{r_1}{2r} - \left[\frac{r_1^2}{2} + \frac{5\sqrt{6}\gamma_1 M_{\rm Pl}}{4(-\eta)^{3/2}}\right] \frac{1}{r^2} + \mathcal{O}(r^{-3}). \tag{6.23}$$

Then, the second angular propagation speed squared yields

$$c_{\Omega 2}^2 = \frac{r_1}{2r} + \left[\frac{r_1^2}{2} + \frac{5\sqrt{6}\gamma_1 M_{\rm Pl}}{4(-\eta)^{3/2}}\right] \frac{1}{r^2} + \mathcal{O}(r^{-3}). \tag{6.24}$$

Hence, the Laplacian stability along the angular directions is ensured for

$$r_1 > 0. \tag{6.25}$$

The above discussion shows that, under the three conditions $\eta < 0$, $\gamma_1 > 0$, and $r_1 > 0$, there are neither ghost nor Laplacian instabilities both around the BH horizon and at spatial infinity. In order to show the existence of stable BH solutions, we solve the background equations of motion outwards from the vicinity of the BH horizon. For this purpose, we introduce a dimensionless parameter $\hat{\gamma}_1 = \gamma_1 M_{\rm Pl}/r_s^2$. In the left panel of Fig. 1, we plot $f$, $h$, and $r_s\phi'/M_{\rm Pl}$ versus the coordinate distance from the

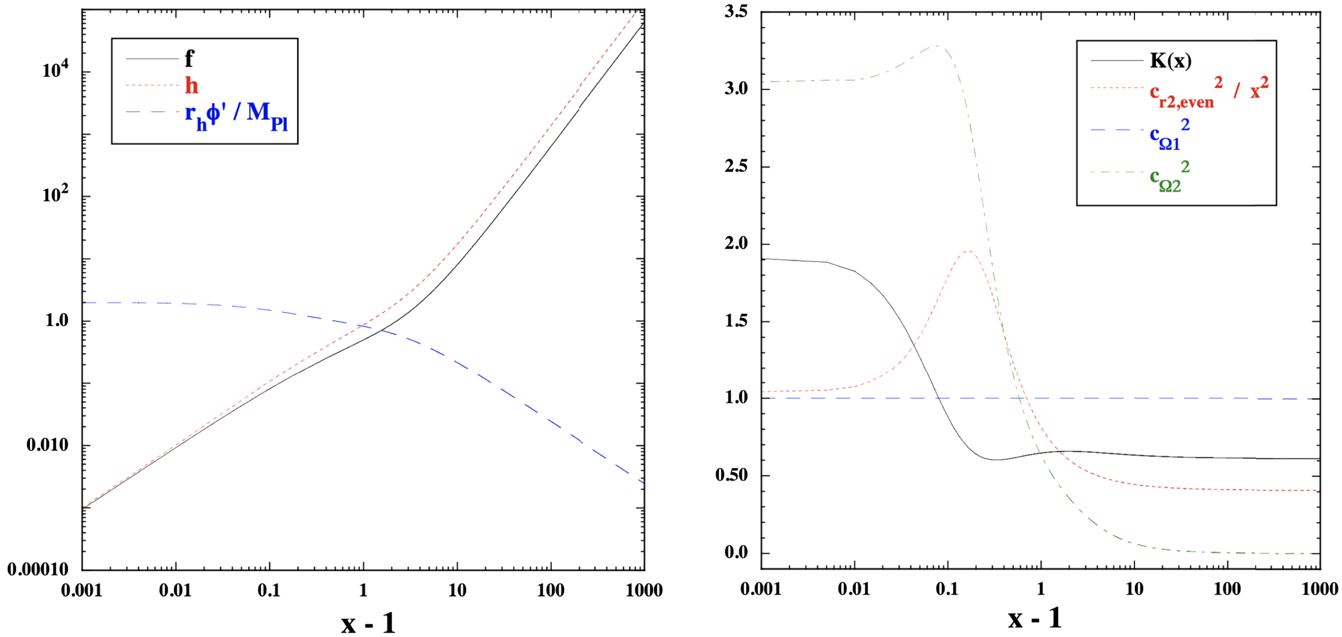


FIG. 1. Left: background metric components $f$, $h$, and the scalar-field derivative $r_s\phi'/M_{\rm Pl}$ versus $x-1$ for $\hat{\gamma}_1 = 1$ and $\eta = -1$, where $x = r/r_s$. The background equations of motion are integrated outwards from the distance $x = 1 + 10^{-6}$. As the boundary conditions around $r = r_s$, we adopt the expanded solutions (6.9)–(6.11), with $f_1 = 1/r_s$. Right: $K(x) = \mathcal{K}/(M_{\rm Pl}^2 x^2)$, $c_{r2,\rm even}^2/x^2$, $c_{\Omega 1}^2$, and $c_{\Omega 2}^2$ versus $x-1$ for the same model parameters and boundary conditions as those used in the left.

BH horizon $x-1$ (where $x = r/r_s$) for the coupling $\hat{\gamma}_1 = 1$ with $\eta = -1$. As estimated from Eqs. (6.9)–(6.11), the metric components $f$ and $h$ around the horizon linearly grow in $r$, with $\phi' \simeq$ constant. For $x - 1 \gtrsim 10$, $f$ and $h$ increase as $f \propto r^2$ and $h \propto r^2$ according to the large-distance solutions (6.12) and (6.13), with $\phi' \propto 1/r$. As $r \to \infty$, the kinetic term $\eta X$ approaches a constant

$$\eta X \to \frac{(-\eta)^{3/2} M_{\rm Pl}}{\sqrt{6}\gamma_1}, \tag{6.26}$$

which is positive. This positive asymptotic value works as a negative cosmological constant, so the leading-order metric components at spatial infinity are similar to those of Schwarzschild-AdS spacetime. As we observe in the left panel of Fig. 1, the solutions in two asymptotic regimes ($r \simeq r_s$ and $r \gg r_s$) are smoothly joined with each other.

In the right panel of Fig. 1, we plot the quantities associated with the stability conditions of even-parity perturbations. The variable $K(x) = \mathcal{K}/(M_{\rm Pl}^2 x^2)$ is positive throughout the horizon exterior, with the asymptotic values of $\mathcal{K}$ given by Eqs. (6.18) and (6.21). The radial propagation speed squared $c_{r2,\rm even}^2$ is also positive for $r > r_s$, with asymptotic behaviors (6.19) and (6.22). The angular propagation speed squared $c_{\Omega 1}^2$ in the gravity sector is unity [see Eq. (6.17)]. Also, Fig. 1 clearly shows that the positivity of $c_{\Omega 2}^2$ holds outside the BH horizon.

While we have shown that the cubic Galileons with $\gamma_1 > 0$ and $\eta < 0$ allow the existence of static, spherically symmetric, and asymptotically AdS BH solutions being compatible with all the linear stability conditions listed in Table, they are not sufficient to conclude that the BH solutions are physically sensible, and further studies would be required. As $r \to \infty$, $c_{\Omega 2}^2$ is vanishing, while $c_{r2,\rm even}^2$ grows toward infinity, which might imply the presence of a strong coupling problem at the timelike AdS boundary. In order to see whether this is the case or not, we need to study whether higher-order nonlinear operators dominate over those of linear perturbations in the Lagrangian. Moreover, the BH stability at the nonlinear level should also be studied, which is beyond the scope of this paper.

### B. Quintic-order positive power-law couplings

We proceed to study the linear stability of BHs in the presence of quintic-order couplings $G_5(X)$. Let us first consider models with a positive power-law coupling given by the functions

$$G_2 = \eta X, \quad G_3 = 0, \quad G_4 = \frac{M_{\rm Pl}^2}{2}, \quad G_5 = \lambda_p (-X)^p, \tag{6.27}$$

where $\lambda_p$ and $p (> 0)$ are constants. In this case, the scalar-field equation (2.15) yields

$$\phi'^{2p-1} = \frac{2^p \eta r^2 f}{\lambda_p p f' h^p [(2p+1)h - 1]}. \tag{6.28}$$

Then, the scalar-field derivative diverges at the coordinate distance $r_d$ satisfying

$$h(r_d) = \frac{1}{2p+1}, \tag{6.29}$$

provided it exists. If we impose asymptotic flatness of the spacetime and construct solutions around the BH horizon ($h \simeq 0$) and at large distances ($h \simeq 1$), then the two solutions cannot be smoothly connected without hitting the singular point (6.29). This problem also persists if $h$ is a growing function of $r$ toward spatial infinity as in the Schwarzschild-AdS spacetime. A possible way out is to make the range of $h$ finite as in the Schwarzschild-dS spacetime so that it does not reach the singular point (6.29).

Let us first consider the model (6.27) with

$$p = 1, \tag{6.30}$$

for concreteness. On using the background Eqs. (2.5)–(2.7), the expanded solution at spatial infinity is given by

$$\frac{f}{f_0} = \pm\left(-\frac{\eta^3}{55566 M_{\rm Pl}^2 \lambda_1^2}\right)^{1/4} r^2 + 1 + \mathcal{O}(r^{-1}), \tag{6.31}$$

$$h = \pm\left(-\frac{7\eta^3}{162 M_{\rm Pl}^2 \lambda_1^2}\right)^{1/4} r^2 + \frac{139}{28} + \mathcal{O}(r^{-1}), \tag{6.32}$$

where $f_0 > 0$ and the double signs are in the same order. The existence of this solution requires that

$$\eta < 0. \tag{6.33}$$

For $p = 1$, the scalar-field derivative on the BH horizon ($r = r_s$) has a nonvanishing value $\phi'(r_s) = -2\eta r_s^3/\lambda_1$ and hence $X(r_s) = 0$, where we have used the expansions (5.9) and (5.10). Around $r = r_s$, the metric components are given by

$$\frac{f}{f_1} = (r - r_s) + \frac{\eta^3 r_s^8 - M_{\rm Pl}^2 \lambda_1^2}{r_s M_{\rm Pl}^2 \lambda_1^2}(r - r_s)^2 + \mathcal{O}((r - r_s)^3), \tag{6.34}$$

$$h = \frac{1}{r_s}(r - r_s) - \frac{3\eta^3 r_s^8 + M_{\rm Pl}^2 \lambda_1^2}{r_s^2 M_{\rm Pl}^2 \lambda_1^2}(r - r_s)^2 + \mathcal{O}((r - r_s)^3), \tag{6.35}$$

where $f_1 > 0$. If we connect the above expanded solutions and require that $h$ does not reach the singular point (6.29), then we should at least choose the minus sign in Eqs. (6.31) and (6.32). On using the expanded solution around $r = r_s$, the quantity associated with the no-ghost condition of even-parity perturbations reads

$$\mathcal{K} = -\frac{2\eta^3 r_s^8}{\lambda_1^2} + \mathcal{O}(r - r_s), \tag{6.36}$$

whose leading-order term is positive under the inequality (6.33). However, we have

$$B_2 = -3 + \frac{2\eta^3 r_s^8}{M_{\rm Pl}^2 \lambda_1^2}, \tag{6.37}$$

which is negative. This means that, even if the two solutions around $r = r_s$ and $r \to \infty$ are connected without reaching the singular point (6.29), the Laplacian instability of even-parity perturbations is present around the BH horizon.

Next, let us consider the case with $p > 1$. Assuming the existence of a BH horizon, from Eq. (6.28), the leading term of the scalar-field derivative around $r = r_s$ is given by $\phi' \propto \eta^{1/(2p-1)}(r - r_s)^{-(p-1)/(2p-1)}$. Then, one can verify that the background Eqs. (2.5) and (2.6) are consistently satisfied around $r = r_s$ only when $\eta = 0$, for which we have $\phi' = 0$ everywhere. Hence, for $p > 1$, a nontrivial scalar-field profile is not present even at the background level.

### C. Gauss-Bonnet couplings

Finally, we consider the case with the scalar field linearly coupled to the Gauss-Bonnet curvature invariant $R_{\rm GB}^2 \equiv R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$, which is described by the action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + \eta X + \alpha_{\rm GB}\phi R_{\rm GB}^2\right], \tag{6.38}$$

where $\alpha_{\rm GB}$ is a coupling constant. This theory can be accommodated in the framework of shift-symmetric Horndeski theories with the following choice of the coupling functions [19]:

$$G_2 = \eta X, \qquad G_3 = 0,$$
$$G_4 = \frac{M_{\rm Pl}^2}{2}, \qquad G_5 = -4\alpha_{\rm GB}\ln|X|. \tag{6.39}$$

In this theory, the background equations (2.5) and (2.6) yield

$$2M_{\rm Pl}^2 - 2h'(M_{\rm Pl}^2 r + 4\alpha_{\rm GB}\phi') + 16\alpha_{\rm GB}h^2\phi'' - h(2M_{\rm Pl}^2 - 24\alpha_{\rm GB}h'\phi' + \eta r^2\phi'^2 + 16\alpha_{\rm GB}\phi'') = 0, \tag{6.40}$$

$$2hf'[M_{\rm Pl}^2 r + 4\alpha_{\rm GB}(1 - 3h)\phi'] - f[2M_{\rm Pl}^2 - h(2M_{\rm Pl}^2 - \eta r^2\phi'^2)] = 0. \tag{6.41}$$

The radial component of the current $J^\mu$ reduces to

$$J^r = h\left[\eta\phi' + 4\alpha_{\rm GB}\frac{(h-1)f'}{r^2 f}\right], \tag{6.42}$$

which obeys Eq. (2.9). The general solution to Eq. (2.9) is given by Eq. (2.12), so that

$$h\left[\eta\phi' + 4\alpha_{\rm GB}\frac{(h-1)f'}{r^2 f}\right] = \frac{Q}{r^2}\sqrt{\frac{h}{f}}. \tag{6.43}$$

As we discussed in Sec. II, the finiteness of the current squared $J^2$ requires that $Q=0$, in which case $J^r=0$. For $Q\neq 0$, $J^2$ diverges on the horizon. In this latter case, it was argued that hairy BH solutions with a nonvanishing scalar-field derivative are present. In spite of the divergence of $J^2$, the components of the energy-momentum tensor and curvature invariants remain finite. In Ref. [60], it was argued that asymptotically flat BHs in shift-symmetric scalar-tensor theories without ghost degrees of freedom can have nontrivial scalar hair only in the presence of the Gauss-Bonnet coupling $\alpha_{\rm GB}\phi R_{\rm GB}^2$.

In the following, we will consider the two cases $Q=0$ and $Q\neq 0$ in turn.

#### 1. $Q=0$

When $Q=0$, Eq. (6.43) gives

$$\phi' = -\frac{4\alpha_{\rm GB}(h-1)f'}{\eta r^2 f}. \tag{6.44}$$

On using the expansions (5.9) and (5.10) around the BH horizon $r=r_s$, the scalar-field derivative has the following dependence:

$$\phi' = \frac{4\alpha_{\rm GB}}{\eta r_s^2(r-r_s)} + \mathcal{O}((r-r_s)^0). \tag{6.45}$$

This means that, in the vicinity of the BH horizon, the left-hand side of Eq. (6.40) behaves as

$$\frac{16h_1\alpha_{\rm GB}^2}{\eta r_s^2(r-r_s)} + \mathcal{O}((r-r_s)^0), \tag{6.46}$$

which does not vanish. Hence, for $Q=0$, we do not have a BH solution endowed with scalar hair. This conclusion agrees with the one reached in Ref. [44].

#### 2. $Q\neq 0$

For $Q\neq 0$, it is possible to realize a solution with a finite value of $\phi'$ on the BH horizon. Applying the expansions (5.9) and (5.10) of $f$ and $h$ to Eq. (6.43), the divergence of $\phi'$ can be avoided for

$$Q = -4\alpha_{\rm GB}\sqrt{f_1 h_1}. \tag{6.47}$$

Then, the scalar-field derivative at $r=r_s$ takes a finite constant value,

$$\phi'(r_s) = -\frac{2\alpha_{\rm GB}[(2h_1^2-h_2)f_1 - 3h_1 f_2]}{\eta f_1 h_1 r_s^2}, \tag{6.48}$$

so that $X(r_s)=0$. The hairy BH solution in Refs. [36,37], whose existence was studied in both perturbative and numerical approaches, corresponds to the nonvanishing value of $Q$ given by Eq. (6.47). Here, we follow the perturbative approach valid in the regime of small Gauss-Bonnet couplings. Substituting Eq. (6.47) into Eq. (6.43) and taking the limit $\alpha_{\rm GB}\to 0$, we obtain the no-hair solution $\phi'=0$. In this limit, Eqs. (6.40) and (6.41) show that the corresponding background geometry is the Schwarzschild spacetime. For small Gauss-Bonnet couplings, we expand $f$, $h$, and $\phi'$ in terms of the dimensionless constant $\hat{\alpha}_{\rm GB}\equiv \alpha_{\rm GB}/(m^2 M_{\rm Pl})$, as

$$f(r) = \left(1-\frac{2m}{r}\right)\left[1+\sum_{j=1}^{\infty}\hat{f}_j(r)(\hat{\alpha}_{\rm GB})^j\right]^2,$$
$$h(r) = \left(1-\frac{2m}{r}\right)\left[1+\sum_{j=1}^{\infty}\hat{h}_j(r)(\hat{\alpha}_{\rm GB})^j\right]^{-2},$$
$$\phi'(r) = \sum_{j=1}^{\infty}\phi'_j(r)(\hat{\alpha}_{\rm GB})^j, \tag{6.49}$$

where $m$ is constant and $\hat{f}_j$, $\hat{h}_j$, $\phi'_j$ are functions of $r$. We substitute the ansatz (6.49) into Eqs. (6.40), (6.41), (2.9) with (6.42) and solve them at each order of $\hat{\alpha}_{\rm GB}$ up to the second order ($j=2$). Although the position of the BH horizon corresponds to $r=2m$, as we will see below, the Arnowitt-Deser-Misner (ADM) mass is different from $2m$ due to the contribution of Gauss-Bonnet couplings.

At first order in $\hat{\alpha}_{\rm GB}$, the resulting solutions are expressed in the forms

$$\hat{f}_1(r) = -\frac{C_2}{r-2m} + C_3, \qquad \hat{h}_1(r) = \frac{C_2}{r-2m},$$
$$\phi'_1(r) = \frac{16m^4 M_{\rm Pl} + r^3\eta C_4}{r^4(r-2m)\eta}, \tag{6.50}$$

where $C_2$, $C_3$, and $C_4$ are integration constants. We set $C_3=0$ by a suitable time reparametrization. We also impose the regularity of perturbative solutions at $r=2m$, which yields $C_2=0$ and $C_4=-2mM_{\rm Pl}/\eta$. Thus, the first-order solutions are given by

$$\hat{f}_1(r) = \hat{h}_1(r) = 0,$$
$$\phi'_1(r) = -\frac{2mM_{\rm Pl}(r^2+2mr+4m^2)}{\eta r^4}. \tag{6.51}$$

Up to this order, the field derivative on the horizon is $\phi'(r_s) = -3\alpha_{\rm GB}/(2\eta m^3)$. Indeed, substituting the zeroth-order metric components $f_1=h_1=1/(2m)$ and $f_2=h_2=-1/(4m^2)$ with $r_s=2m$ into Eq. (6.48), we obtain the same value of $\phi'(r_s)$ at first order in $\alpha_{\rm GB}$.

At the second order in $\alpha_{\rm GB}$, the solution is given by

$$\hat{f}_2(r) = \frac{m(1600m^5 + 416m^4 r - 56m^3 r^2 - 548m^2 r^3 - 294mr^4 - 147r^5)}{120\eta r^6},$$
$$\hat{h}_2(r) = -\frac{m(7360m^5 + 3488m^4 r + 1624m^3 r^2 - 228m^2 r^3 - 174mr^4 - 147r^5)}{120\eta r^6},$$
$$\phi_2'(r) = 0, \tag{6.52}$$

where the integration constants have been fixed by using the time reparametrization invariance for $\hat{f}_2(r)$ and by imposing the regularities of $\hat{h}_2(r)$ and $\phi_2'(r)$ at $r = 2m$. At large distances ($r \gg 2m$), $f$, $h$, and $\phi'$ up to $\mathcal{O}(\alpha_{\rm GB}^2)$ are expressed in the forms

$$f(r) = 1 - \frac{2M}{r} + \mathcal{O}(r^{-2}),$$
$$h(r) = 1 - \frac{2M}{r} + \mathcal{O}(r^{-2}),$$
$$\phi'(r) = -\frac{C}{r^2} + \mathcal{O}(r^{-3}), \tag{6.53}$$

where

$$M = m + \frac{49\alpha_{\rm GB}^2}{40m^3 M_{\rm Pl}^2 \eta}, \qquad C = \frac{2\alpha_{\rm GB}}{m\eta}. \tag{6.54}$$

Here, $M$ and $C$ correspond to the ADM mass and scalar charge, respectively [36,37]. Since both $M$ and $C$ are determined solely by $m$, the scalar charge $C$ is of secondary type. Substituting the leading-order metric components $f_1 = h_1 = 1/(2m)$ into Eq. (6.47), we obtain $Q = -2\alpha_{\rm GB}/m$ and hence $Q = -\eta C$.

In Ref. [37], it was shown that the perturbative solutions (6.49) with Eqs. (6.51) and (6.52) exhibit very good agreement with full numerical results. Moreover, since the numerical BH solution could be constructed only for smaller couplings $|\hat{\alpha}_{\rm GB}| \ll \mathcal{O}(0.1)$ [36,37], the perturbative solutions (6.49) with Eqs. (6.51) and (6.52) should be valid for all the coupling regimes in which the BH solutions exist. Also, by repeating the above procedure, one can compute the perturbative expansion of the solution to an arbitrary order. One can verify that only the even-order terms ($j = 2, 4, 6, \ldots$) of metric components and the odd-order ($j = 1, 3, 5, \ldots$) terms of scalar field are nontrivial. On using these solutions, the radial component of $J^\mu$, up to the order of $\hat{\alpha}_{\rm GB}$, is given by

$$r^2 \sqrt{\frac{f}{h}} J^r = -2mM_{\rm Pl}\hat{\alpha}_{\rm GB} + \mathcal{O}(\hat{\alpha}_{\rm GB}^3). \tag{6.55}$$

Indeed, the leading-order term on the right-hand side of Eq. (6.55) also follows by substituting $Q = -2\alpha_{\rm GB}/m$ into Eq. (2.12). The nonvanishing value of $J^r$ gives rise to a divergent norm of the Noether current $J^2 = (J^r)^2/h$ on the horizon ($h = 0$). As we already mentioned, the components of the energy-momentum tensor and curvature invariants remain finite. As stated in Ref. [60], the divergence of $J^2$ does not give rise to pathologies on the BH properties at least in this case. According to these arguments, the hairy BHs discussed above should be dealt as physical solutions.

We then study stability of the hairy solution (6.49) with Eqs. (6.51) and (6.52) against odd- and even-parity perturbations. The quantities relevant to stability in the odd-parity sector yield

$$\mathcal{F} = M_{\rm Pl}^2 + M_{\rm Pl}^2 \frac{16(36 - 2\hat{r} - \hat{r}^2 - 2\hat{r}^3)}{\eta \hat{r}^6} \hat{\alpha}_{\rm GB}^2 + \mathcal{O}(\hat{\alpha}_{\rm GB}^4), \tag{6.56}$$

$$\mathcal{G} = M_{\rm Pl}^2 + M_{\rm Pl}^2 \frac{16(4 + 2\hat{r} + \hat{r}^2)}{\eta \hat{r}^6} \hat{\alpha}_{\rm GB}^2 + \mathcal{O}(\hat{\alpha}_{\rm GB}^4), \tag{6.57}$$

$$\mathcal{H} = M_{\rm Pl}^2 + M_{\rm Pl}^2 \frac{16(\hat{r}^3 - 8)}{\eta \hat{r}^6} \hat{\alpha}_{\rm GB}^2 + \mathcal{O}(\hat{\alpha}_{\rm GB}^4), \tag{6.58}$$

where we have introduced the dimensionless radial coordinate $\hat{r} \equiv r/m$. Provided that $|\hat{\alpha}_{\rm GB}|/\sqrt{|\eta|} \ll 1$, the terms of order $\hat{\alpha}_{\rm GB}^2$ in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are suppressed relative to the leading-order contribution $M_{\rm Pl}^2$ throughout the horizon exterior ($2 < \hat{r} < \infty$). This means that, under the validity of the expansion with respect to $\hat{\alpha}_{\rm GB}$, i.e.,

$$\frac{|\alpha_{\rm GB}|}{m^2 M_{\rm Pl}\sqrt{|\eta|}} \ll 1, \tag{6.59}$$

the stability conditions against odd-parity perturbations ($\mathcal{F} > 0$, $\mathcal{G} > 0$, and $\mathcal{H} > 0$) are satisfied.

The quantity associated with the no-ghost condition of even-parity perturbations yields

$$\mathcal{K} = M_{\rm Pl}^2 \frac{2(4 + 2\hat{r} + \hat{r}^2)^2}{\eta \hat{r}^6} \hat{\alpha}_{\rm GB}^2 + \mathcal{O}(\hat{\alpha}_{\rm GB}^4). \tag{6.60}$$

Thus, $\mathcal{K} > 0$ is satisfied as long as

$$\eta > 0. \tag{6.61}$$

The squared sound speeds of even-parity perturbations along the radial direction are given by

$$c_{r1,\text{even}}^2 = 1 + \frac{32(\hat{r}-2)(\hat{r}^2+3\hat{r}+8)}{\eta \hat{r}^6}\hat{\alpha}_{\text{GB}}^2 + \mathcal{O}(\hat{\alpha}_{\text{GB}}^4), \tag{6.62}$$

$$c_{r2,\text{even}}^2 = 1 + \mathcal{O}(\hat{\alpha}_{\text{GB}}^4). \tag{6.63}$$

We note that, in order to obtain Eq. (6.62), one has to solve the background metric components and scalar field (6.49) up to the quartic and cubic orders of $\hat{\alpha}_{\text{GB}}$ (i.e., $j=4$ and $j=3$) respectively, though we do not show the explicit forms of these higher-order corrections. The squared sound speeds of even-parity perturbations along the angular directions are

$$c_{\Omega,\pm}^2 = 1 \pm \frac{24}{\hat{r}^3}\sqrt{\frac{2}{\eta}}|\hat{\alpha}_{\text{GB}}| + \frac{8(84+2\hat{r}+\hat{r}^2-\hat{r}^3)}{\eta \hat{r}^6}\hat{\alpha}_{\text{GB}}^2 + \mathcal{O}(\hat{\alpha}_{\text{GB}}^3). \tag{6.64}$$

Under the condition (6.59), the corrections to the sound speeds arising from the Gauss-Bonnet coupling are much smaller than unity throughout the horizon exterior, so that $c_{r1,\text{even}}^2 \simeq 1$, $c_{r2,\text{even}}^2 \simeq 1$, and $c_{\Omega,\pm}^2 \simeq 1$. Thus, in the small-coupling regime, the hairy BH solutions discussed above suffer from neither ghost nor Laplacian instabilities.

## VII. CONCLUSIONS

In this paper, we addressed the stability of hairy BHs in shift-symmetric Horndeski theories against linear perturbations on a static and spherically symmetric background. We assumed that the background scalar field is time independent, but did not necessarily impose the asymptotic flatness at spatial infinity. Moreover, we allowed the possibility that the coupling functions are nonanalytic functions of $X$. In such cases, the no-hair theorem of BHs in shift-symmetric Horndeski theories established in Ref. [21] can be avoided, so that there are several classes of hairy BH solutions. If we require that the norm of the Noether current $J^\mu$ associated with the shift symmetry of the scalar field is finite on the horizon, the radial current component $J^r = h\phi' \mathcal{J}$ vanishes everywhere. Provided $\mathcal{J}$ is finite in the limit of $\phi' \to 0$, there is a branch of nonvanishing field derivative ($\phi' \neq 0$) besides a trivial GR solution ($\phi' = 0$).

The linear perturbations about the static and spherically symmetric background can be decomposed into those of the odd- and even-parity sectors. We clarified the conditions for the absence of ghosts and Laplacian instabilities along the radial and angular directions, which are summarized in Table I at the end of Sec. III. In Sec. IV, we applied these conditions to the GR branch $\phi' = 0$ and showed that there are no ghost/Laplacian instabilities under the conditions (4.6) and (4.9).

In Sec. V, we studied the linear stability of hairy BH solutions in reflection-symmetric theories containing two arbitrary functions $G_2(X)$ and $G_4(X)$. We employed the expansions (5.9)–(5.11) around the BH horizon to see whether all the conditions in Table I can be consistently satisfied. As we see in Eq. (5.13), the product of three quantities $\mathcal{F}$, $\mathcal{K}$, and $B_2$ is negative around the horizon so long as the conditions in Eq. (5.14) are satisfied. Therefore, three of the stability conditions, i.e., $\mathcal{F} > 0$, $\mathcal{K} > 0$, and $B_2 > 0$ cannot be satisfied at the same time in general. For instance, even if we require the absence of Laplacian instabilities along the radial direction in the odd modes ($\mathcal{F} > 0$) and of ghosts in the even modes ($\mathcal{K} > 0$), we have $B_2 < 0$, and correspondingly there would be Laplacian instabilities along the angular directions in the even modes. In this sense, the instability found here is generic and almost all hairy BHs in the shift- and reflection-symmetric Horndeski theories are shown to be linearly unstable. We note that a BH solution with $X < 0$ at the BH horizon cannot be extended to the interior of the BH horizon and can be defined only in the domain outside the horizon where the character of the scalar field is spacelike, as otherwise the coordinate invariant $X$ would have an unphysical jump across the horizon. Thus, BH solutions with $X \neq 0$ at the horizon suffer from instabilities only in the domain outside the BH horizon where the character of the scalar field is spacelike. Similarly, a static and spherically symmetric solution with $X < 0$ at the cosmological horizon (if it exists) could not be extended to the exterior of the cosmological horizon. As specific examples, this generic instability is present for exact BH solutions in theories with $G_4 \supset X$ and in those with $G_4 \supset (-X)^{1/2}$. We also showed that, in k-essence theories given by the Lagrangian $\mathcal{L} = G_2(X) + (M_{\text{Pl}}^2/2)R$, the corresponding BH solution is plagued by a strong coupling problem.

In Sec. VI, we investigated the linear stability of BH solutions in nonreflection-symmetric theories containing either $G_3(X)$ or $G_5(X)$. For cubic Galileons characterized by the function $G_3 \propto X$, we found the existence of non-asymptotically flat solutions satisfying all the conditions for the absence of ghosts/Laplacian instabilities. In this case, the angular propagation speed squared of scalar-field perturbations is vanishing at spatial infinity. In order to see whether this induces a strong coupling problem or not, the analysis of nonlinear perturbations is required. For quintic power-law couplings $G_5 \propto (-X)^p$ with $p \geq 1$, it turned out to be difficult to realize stable BH solutions with nontrivial scalar hair. In the case of the scalar field linearly coupled to the Gauss-Bonnet curvature invariant, which is described by the coupling $G_5 \propto \alpha_{\text{GB}} \ln |X|$, we showed that asymptotically flat BH solutions constructed perturbatively with respect to a small coupling $\alpha_{\rm GB}$ are free of ghosts/Laplacian instabilities. This is a specific example in which asymptotically flat BH solutions with nontrivial scalar hair can be realized in the framework of shift-symmetric Horndeski theories.

We thus narrowed down the range of viable BH solutions by scrutinizing the linear stability conditions including the angular propagation of even-parity perturbations. It will be of interest to study further whether what kinds of hairy BHs survive as stable solutions in full Horndeski theories or in a broader framework of degenerate higher-order scalar-tensor theories [75–80]. This issue is left for future work.

While we have focused on the linear stability of static and spherically symmetric solutions, we expect that our results should still be valid for more general BH solutions in Horndeski theories where the deviation from staticity and/or spherical symmetry is small, for instance for slowly rotating solutions (see e.g., [32,36,81–84]) or for static BHs with the small deviation from spherical symmetry (if they exist). In such cases, we can treat the deviation from staticity and/or spherical symmetry as small perturbations to our case, which would not modify our main results significantly. On the other hand, the perturbative treatment no longer applies to BH solutions with the large deviation from staticity and/or spherical symmetry, e.g., rapidly rotating BHs which have been explored in the context of Einstein-scalar-Gauss-Bonnet theory (see e.g., [85–90]) but (to our knowledge) not fully yet in other Horndeski classes. In this case, we have to develop a new theoretical scheme of BH perturbations to clarify whether results similar to the case of static and spherically symmetric BHs hold. Nevertheless, we speculate that, irrespective of the presence of rotation and/or deviation from the spherical symmetry, BH solutions with the nonvanishing constant kinetic term on the (either BH or cosmological) horizon cannot be extended across the horizon, as otherwise they admit an unphysical jump of $X$ and suffer from similar ghost/Laplacian instabilities discussed in this paper. We hope to come back to these issues in future.

## ACKNOWLEDGMENTS

M. M. was supported by the Portuguese national fund through the Fundação para a Ciência e a Tecnologia (FCT) in the scope of the framework of the Decree-Law 57/2016 of August 29, changed by Law 57/2017 of July 19, and the Centro de Astrofísica e Gravitação (CENTRA) through the Project No. UIDB/00099/2020. K. T. was supported by JSPS (Japan Society for the Promotion of Science) KAKENHI Grant No. JP21J00695. S. T. was supported by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 19K03854.

## APPENDIX: EXPLICIT FORM OF THE COEFFICIENTS

Here, we define the functions appearing in the main text. For the background, the quantities $A_1, \ldots, A_5$ in Eqs. (2.5)–(2.7) are given by

$$\begin{aligned}
A_1 &= -h^2 G_{3,X}\phi'^2, \\
A_2 &= 4h^2\phi'(hG_{4,XX}\phi'^2 - G_{4,X}), \\
A_3 &= h^2 G_{5,X}(3h-1)\phi'^2 - h^4 G_{5,XX}\phi'^4, \\
A_4 &= 2h^2 G_{4,XX}\phi'^4 - 4hG_{4,X}\phi'^2 - 2G_4, \\
A_5 &= -\frac{1}{2}[G_{5,XX}h^3\phi'^5 - h(5h-1)G_{5,X}\phi'^3].
\end{aligned} \tag{A1}$$

For the perturbations, the quantity $a_1$ in Eq. (3.11) is defined by

$$\begin{aligned}
a_1 = \frac{\sqrt{fh}}{2}[&G_{3,X}h\phi'^2 r^2 + 4(G_{4,X} - G_{4,XX}h\phi'^2)h\phi' r \\
&+ G_{5,XX}h^3\phi'^4 - G_{5,X}h(3h-1)\phi'^2].
\end{aligned} \tag{A2}$$

The definitions of $c_2$ and $c_4$ in Eq. (3.14) are

$$\begin{aligned}
c_2 = \phi'\sqrt{fh}\bigg\{&\frac{1}{2}(G_{2,X} - G_{2,XX}h\phi'^2)r^2 - \frac{(rf'+4f)hr\phi'}{4f}(3G_{3,X} - G_{3,XX}h\phi'^2) \\
&- \frac{h(rf'+f)}{f}(3G_{4,X} - 6G_{4,XX}h\phi'^2 + G_{4,XXX}h^2\phi'^4) + G_{4,X} - G_{4,XX}h\phi'^2 \\
&+ \frac{f'h\phi'}{4f}[3G_{5,X}(5h-1) - G_{5,XX}h(10h-1)\phi'^2 + G_{5,XXX}h^3\phi'^4]\bigg\},
\end{aligned} \tag{A3}$$

$$c_4 = \frac{\phi'}{4}\sqrt{\frac{h}{f}}\left[2G_{3,X}f\phi' + \frac{2(rf'+2f)}{r}(G_{4,X} - G_{4,XX}h\phi'^2) - \frac{f'h\phi'}{r}(3G_{5,X} - G_{5,XX}h\phi'^2)\right]. \tag{A4}$$

The functions $B_1$ and $B_2$ in Eq. (3.18) are given by

$$B_1 = \frac{r^3\sqrt{fh}\mathcal{H}[4h(\phi' a_1 + r\sqrt{fh}\mathcal{H})\beta_1 + \beta_2 - 4\phi' a_1\beta_3] - 2fh\mathcal{G}[r\sqrt{fh}(2\mathcal{P}_1 - \mathcal{F})\mathcal{H}(2\phi' a_1 + r\sqrt{fh}\mathcal{H}) + 2\phi'^2 a_1^2\mathcal{P}_1]}{4fh(2\mathcal{P}_1 - \mathcal{F})\mathcal{H}(\phi' a_1 + r\sqrt{fh}\mathcal{H})^2}, \tag{A5}$$

$$B_2 = -r^2\frac{r^2h\beta_1[2fh\mathcal{F}\mathcal{G}(\phi' a_1 + r\sqrt{fh}\mathcal{H}) + r^2\beta_2] - r^4\beta_2\beta_3 - fh\mathcal{F}\mathcal{G}(\phi' fh\mathcal{F}\mathcal{G}a_1 + 2r^3\sqrt{fh}\mathcal{H}\beta_3)}{fh\phi' a_1(2\mathcal{P}_1 - \mathcal{F})\mathcal{F}(\phi' a_1 + r\sqrt{fh}\mathcal{H})^2}, \tag{A6}$$

with

$$\beta_1 = \frac{1}{2}\phi'^2\sqrt{fh}\mathcal{H}e_4 - \phi'(\sqrt{fh}\mathcal{H})'c_4 + \frac{\sqrt{fh}}{2}\left[\left(\frac{f'}{f} + \frac{h'}{h} - \frac{2}{r}\right)\mathcal{H} + \frac{2\mathcal{F}}{r}\right]\phi' c_4 + \frac{f\mathcal{F}\mathcal{G}}{2r^2}, \tag{A7}$$

$$\beta_2 = \left[\frac{\sqrt{fh}\mathcal{F}}{r^2}\left(2hr\phi'^2c_4 + \frac{r\phi' f'\sqrt{h}}{2\sqrt{f}}\mathcal{H} - \phi'\sqrt{fh}\mathcal{G}\right) - \frac{\phi' fh\mathcal{G}\mathcal{H}}{r}\left(\frac{\mathcal{G}'}{\mathcal{G}} - \frac{\mathcal{H}'}{\mathcal{H}} + \frac{f'}{2f} - \frac{1}{r}\right)\right]a_1 - \frac{2}{r}(fh)^{3/2}\mathcal{F}\mathcal{G}\mathcal{H}, \tag{A8}$$

$$\begin{aligned}\beta_3 = {}& \frac{\sqrt{fh}\mathcal{H}}{2}\phi'\left(hc_4' + \frac{1}{2}h'c_4 - \frac{d_3}{2}\right) - \frac{\sqrt{fh}}{2}\left(\frac{\mathcal{H}}{r} + \mathcal{H}'\right)\left(2h\phi' c_4 + \frac{\sqrt{fh}\mathcal{G}}{2r} + \frac{f'\sqrt{h}\mathcal{H}}{4\sqrt{f}}\right)\\ &+ \frac{\sqrt{fh}\mathcal{F}}{4r}\left(2h\phi' c_4 + \frac{3\sqrt{fh}\mathcal{G}}{r} + \frac{f'\sqrt{h}\mathcal{H}}{2\sqrt{f}}\right),\end{aligned} \tag{A9}$$

and

$$\begin{aligned}e_4 = {}& \frac{1}{\phi'}c_4' - \frac{f'}{4fh\phi'^2}(\sqrt{fh}\mathcal{H})' - \frac{\sqrt{f}}{2\phi'^2\sqrt{hr}}\mathcal{G}' + \frac{1}{h\phi' r^2}\left(\frac{\phi''}{\phi'} + \frac{1}{2}\frac{h'}{h}\right)a_1\\ &+ \frac{\sqrt{f}}{8\sqrt{h}\phi'^2}\left[\frac{(f'r - 6f)f'}{f^2r} + \frac{h'(f'r + 4f)}{fhr} - \frac{4f(2\phi''h + h'\phi')}{\phi' h^2r(f'r - 2f)}\right]\mathcal{H} + \frac{h'}{2h\phi'}c_4\\ &+ \frac{f'hr - f}{2r^2\sqrt{f}h^{3/2}\phi'^2}\mathcal{F} + \frac{\sqrt{f}}{2r\phi'^2h^{3/2}}\left[\frac{f(2\phi''h + h'\phi')}{h\phi'(f'r - 2f)} + \frac{2f - f'hr}{2fr}\right]\mathcal{G},\end{aligned} \tag{A10}$$

$$\begin{aligned}d_3 = {}& -\frac{1}{r^2}\left(\frac{2\phi''}{\phi'} + \frac{h'}{h}\right)a_1 + \frac{f^{3/2}h^{1/2}}{(f'r - 2f)\phi'}\left(\frac{2\phi''}{h\phi' r} + \frac{f'^2}{f^2} - \frac{f'h'}{fh} - \frac{2f'}{fr} + \frac{2h'}{hr} + \frac{h'}{h^2r}\right)\mathcal{H}\\ &+ \frac{\sqrt{f}}{\phi'\sqrt{h}r^2}\mathcal{F} - \frac{f^{3/2}}{\sqrt{h}(f'r - 2f)\phi'}\left(\frac{f'}{fr} + \frac{2\phi''}{\phi' r} + \frac{h'}{hr} - \frac{2}{r^2}\right)\mathcal{G}.\end{aligned} \tag{A11}$$

---


[1] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), Phys. Rev. Lett. **116**, 061102 (2016).
[2] E. Berti *et al.*, Classical Quantum Gravity **32**, 243001 (2015).
[3] L. Barack *et al.*, Classical Quantum Gravity **36**, 143001 (2019).
[4] E. Berti, K. Yagi, and N. Yunes, Gen. Relativ. Gravit. **50**, 46 (2018).
[5] E. Berti, K. Yagi, H. Yang, and N. Yunes, Gen. Relativ. Gravit. **50**, 49 (2018).
[6] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), Astrophys. J. Lett. **875**, L1 (2019).
[7] W. Israel, Phys. Rev. **164**, 1776 (1967).
[8] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).
[9] R. Ruffini and J. A. Wheeler, Phys. Today **24**, No. 1, 30 (1971).
[10] S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972).
[11] J. D. Bekenstein, Phys. Rev. Lett. **28**, 452 (1972).
[12] A. A. H. Graham and R. Jha, Phys. Rev. D **89**, 084056 (2014); **92**, 069901(E) (2015).
[13] S. W. Hawking, Commun. Math. Phys. **25**, 167 (1972).
[14] J. D. Bekenstein, Phys. Rev. D **51**, R6608 (1995).
[15] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. **108**, 081103 (2012).
[16] V. Faraoni, Phys. Rev. D **95**, 124013 (2017).
[17] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[18] C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, Phys. Rev. D **84**, 064039 (2011).

[19] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. **126**, 511 (2011).
[20] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, Phys. Rev. Lett. **108**, 051101 (2012).
[21] L. Hui and A. Nicolis, Phys. Rev. Lett. **110**, 241104 (2013).
[22] E. Babichev, C. Charmousis, and A. Lehébel, Classical Quantum Gravity **33**, 154002 (2016).
[23] E. Babichev and C. Charmousis, J. High Energy Phys. 08 (2014) 106.
[24] H. Motohashi and M. Minamitsuji, Phys. Rev. D **99**, 064040 (2019).
[25] K. Takahashi and H. Motohashi, J. Cosmol. Astropart. Phys. 06 (2020) 034.
[26] M. Rinaldi, Phys. Rev. D **86**, 084048 (2012).
[27] A. Anabalon, A. Cisterna, and J. Oliva, Phys. Rev. D **89**, 084050 (2014).
[28] M. Minamitsuji, Phys. Rev. D **89**, 064017 (2014).
[29] A. Cisterna and C. Erices, Phys. Rev. D **89**, 084038 (2014).
[30] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, Classical Quantum Gravity **29**, 205011 (2012).
[31] M. Minamitsuji, Gen. Relativ. Gravit. **46**, 1785 (2014).
[32] A. Cisterna, M. Cruz, T. Delsate, and J. Saavedra, Phys. Rev. D **92**, 104018 (2015).
[33] R. Dong, J. Sakstein, and D. Stojkovic, Phys. Rev. D **96**, 064048 (2017).
[34] J. L. Blázquez-Salcedo and K. Eickhoff, Phys. Rev. D **97**, 104002 (2018).
[35] N. Chatzifotis, C. Vlachos, K. Destounis, and E. Papantonopoulos, arXiv:2109.02678.
[36] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. Lett. **112**, 251102 (2014).
[37] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. D **90**, 124063 (2014).
[38] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D **54**, 5049 (1996).
[39] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D **57**, 6255 (1998).
[40] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. **120**, 131103 (2018).
[41] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Phys. Rev. Lett. **120**, 131104 (2018).
[42] G. Antoniou, A. Bakopoulos, and P. Kanti, Phys. Rev. Lett. **120**, 131102 (2018).
[43] M. Minamitsuji and T. Ikeda, Phys. Rev. D **99**, 044017 (2019).
[44] E. Babichev, C. Charmousis, and A. Lehébel, J. Cosmol. Astropart. Phys. 04 (2017) 027.
[45] H. Ogawa, T. Kobayashi, and T. Suyama, Phys. Rev. D **93**, 064078 (2016).
[46] K. Takahashi, T. Suyama, and T. Kobayashi, Phys. Rev. D **93**, 064068 (2016).
[47] K. Takahashi and T. Suyama, Phys. Rev. D **95**, 024034 (2017).
[48] E. Babichev, C. Charmousis, G. Esposito-Farèse, and A. Lehébel, Phys. Rev. Lett. **120**, 241101 (2018).
[49] D. A. Tretyakova and K. Takahashi, Classical Quantum Gravity **34**, 175007 (2017).
[50] E. Babichev, C. Charmousis, G. Esposito-Farèse, and A. Lehébel, Phys. Rev. D **98**, 104050 (2018).
[51] K. Takahashi, H. Motohashi, and M. Minamitsuji, Phys. Rev. D **100**, 024041 (2019).
[52] J. Khoury, M. Trodden, and S. S. C. Wong, J. Cosmol. Astropart. Phys. 11 (2020) 044.
[53] K. Takahashi and H. Motohashi, J. Cosmol. Astropart. Phys. 08 (2021) 013.
[54] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. D **85**, 084025 (2012); **96**, 109903(E) (2017).
[55] T. Kobayashi, H. Motohashi, and T. Suyama, Phys. Rev. D **89**, 084042 (2014).
[56] R. Kase and S. Tsujikawa, Phys. Rev. D **105**, 024059 (2022).
[57] R. Kase, R. Kimura, S. Sato, and S. Tsujikawa, Phys. Rev. D **102**, 084037 (2020).
[58] R. Kase and S. Tsujikawa, J. Cosmol. Astropart. Phys. 01 (2021) 008.
[59] H. Motohashi, T. Suyama, and K. Takahashi, Phys. Rev. D **94**, 124021 (2016).
[60] P. Creminelli, N. Loayza, F. Serra, E. Trincherini, and L. G. Trombetta, J. High Energy Phys. 08 (2020) 045.
[61] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[62] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[63] A. De Felice, T. Suyama, and T. Tanaka, Phys. Rev. D **83**, 104035 (2011).
[64] H. Motohashi and T. Suyama, Phys. Rev. D **84**, 084041 (2011).
[65] R. Kase, L. A. Gergely, and S. Tsujikawa, Phys. Rev. D **90**, 124019 (2014).
[66] T. Kobayashi and N. Tanahashi, Prog. Theor. Exp. Phys. **2014**, 073E02 (2014).
[67] J. Chagoya, G. Niz, and G. Tasinato, Classical Quantum Gravity **33**, 175007 (2016).
[68] M. Minamitsuji, Phys. Rev. D **94**, 084039 (2016).
[69] L. Heisenberg, J. Cosmol. Astropart. Phys. 05 (2014) 015.
[70] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, Phys. Rev. D **96**, 084049 (2017).
[71] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 08 (2017) 024.
[72] R. Kase, M. Minamitsuji, S. Tsujikawa, and Y.-L. Zhang, J. Cosmol. Astropart. Phys. 02 (**2018**) 048.
[73] E. Babichev, C. Charmousis, A. Lehébel, and T. Moskalets, J. Cosmol. Astropart. Phys. 09 (2016) 011.
[74] A. De Felice and S. Tsujikawa, J. Cosmol. Astropart. Phys. 02 (2012) 007.
[75] D. Langlois and K. Noui, J. Cosmol. Astropart. Phys. 02 (2016) 034.
[76] M. Crisostomi, K. Koyama, and G. Tasinato, J. Cosmol. Astropart. Phys. 04 (2016) 044.
[77] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, J. High Energy Phys. 12 (2016) 100.
[78] K. Takahashi and T. Kobayashi, J. Cosmol. Astropart. Phys. 11 (2017) 038.
[79] D. Langlois, M. Mancarella, K. Noui, and F. Vernizzi, J. Cosmol. Astropart. Phys. 02 (**2019**) 036.
[80] K. Takahashi, H. Motohashi, and M. Minamitsuji, Phys. Rev. D **105**, 024015 (2022).
[81] P. Pani and V. Cardoso, Phys. Rev. D **79**, 084031 (2009).
[82] P. Pani, C. F. B. Macedo, L. C. B. Crispino, and V. Cardoso, Phys. Rev. D **84**, 087501 (2011).

[83] A. Maselli, P. Pani, L. Gualtieri, and V. Ferrari, Phys. Rev. D **92**, 083014 (2015).

[84] A. Maselli, H. O. Silva, M. Minamitsuji, and E. Berti, Phys. Rev. D **92**, 104049 (2015).

[85] B. Kleihaus, J. Kunz, and E. Radu, Phys. Rev. Lett. **106**, 151104 (2011).

[86] B. Kleihaus, J. Kunz, S. Mojica, and E. Radu, Phys. Rev. D **93**, 044047 (2016).

[87] L. G. Collodel, B. Kleihaus, J. Kunz, and E. Berti, Classical Quantum Gravity **37**, 075018 (2020).

[88] J. F. M. Delgado, C. A. R. Herdeiro, and E. Radu, J. High Energy Phys. 04 (2020) 180.

[89] C. A. R. Herdeiro, E. Radu, H. O. Silva, T. P. Sotiriou, and N. Yunes, Phys. Rev. Lett. **126**, 011103 (2021).

[90] E. Berti, L. G. Collodel, B. Kleihaus, and J. Kunz, Phys. Rev. Lett. **126**, 011104 (2021).