

Fixed particle number constraint in a simple model of a thermal expanding system and pp collisions at the LHC

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Two-boson momentum correlations at fixed particle number constraint are studied in a simple analytically solvable model of a thermal expanding system. We show that the increase of expansion rate, as well as increase of particle multiplicity, enhances the ground-state contribution to particle momentum spectra and leads to suppression of the Bose-Einstein momentum correlations. The relations of these findings to the multiplicity-dependent measurements of the Bose-Einstein momentum correlations in high-multiplicity $p + p$ collision events at the LHC are discussed.

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I. INTRODUCTION

Notwithstanding the evidence of the hydrodynamic expansion in high-multiplicity $p + p$ collision events at the CERN Large Hadron Collider (LHC), for recent reviews see, e.g., Refs. [1,2], robust interpretation of the multiplicity-dependent Bose-Einstein momentum correlations of identical particles created in such collisions is still absent. To a good extent this is due to the fact that some peculiarities of the data, such as saturation effect in the multiplicity dependence of the interferometry correlation radius parameters (so-called HBT radii) for high charged-particle multiplicity [3,4], as well as low values of the correlation strength parameter λ [3,4], are at variance with the expected behavior for emission from hydrodynamically expanding thermalized systems.¹ A systematic and quantitative analysis and theoretical interpretation of the multiplicity-dependent Bose-Einstein momentum correlations may therefore elucidate the nature of the particle emitter in $p + p$ collisions at the LHC in crucial ways.

In a recent paper [6], fixed particle number constraint was applied to a quantum-field thermal state of the non-relativistic ideal gas of bosons at fixed temperature trapped by means of a harmonic chemical potential. It was

demonstrated in this paper that increase with N of the particle number density is accompanied for fairly high N by the noticeable ground-state Bose-Einstein condensation, and that such a condensation leads to suppression of the two-boson momentum correlation function. It is worth noting that this effect takes place at fixed N in the thermal ensemble where averaged over all multiplicities mean particle number density is below the critical one and, therefore, there is no grand-canonical ground-state condensation. In the present work, we further develop the model of Ref. [6] aiming to account for the system's expansion and thereby to bring the model closer to $p + p$ collision experiments. We find an exact analytical solution of the quantum thermal model with the system's expansion and show that suppression of the Bose-Einstein momentum correlations, i.e., decrease of the λ parameter, is increased if intensity of the flow increases. We attribute such a suppression to the ground-state contribution to particle momentum spectra and suggest that certain features of the multiplicity-dependent two-boson momentum correlations at high multiplicities in $p + p$ collisions at the LHC can be interpreted as a signature of the presence of ground-state condensate.

II. QUASIEQUILIBRIUM STATE OF EXPANDING NONRELATIVISTIC BOSON FIELD

We begin with a brief overview of a standard procedure for constructing a relevant statistical operator ρ (see, e.g., Ref. [7]). It has been known for a long time (see Ref. [8]) that for a given set of relevant observables A_n , the actual expectation values of which $\langle A_n \rangle(t)$ are known at some given time, the statistical operator $\rho(t)$, “least biased” as for unmonitored degrees of freedom, can be found from the

¹Let us make a reservation, however, that a semiquantitative prediction of the saturation effect for the interferometric radii at large multiplicities in $p + p$ collisions, assuming thermalization and hydrodynamic expansion of the arising system, was undertaken, in fact, in Ref. [5].

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maximum of the von Neumann entropy, $S = -\text{Tr}[\rho \ln \rho]$, subject to the constraints

$$\langle A_n \rangle(t) = \text{Tr}[A_n \rho(t)], \quad (1)$$

$$\text{Tr}[\rho(t)] = 1. \quad (2)$$

This can be done by varying the functional $S'[\rho']$, where $a_n(t)$ and $(\Phi(t) - 1)$ are Lagrange multipliers,

$$S'[\rho'] = -\text{Tr}[\rho' \ln \rho'] - \sum_n a_n (\text{Tr}[A_n \rho'] - \langle A_n \rangle) - (\Phi - 1)(\text{Tr}[\rho'] - 1), \quad (3)$$

with respect to ρ' and then putting variation $\delta S'[\rho']$ equal to zero,

$$\delta S'[\rho'] = -\text{Tr} \left[\left(\ln \rho' + \Phi + \sum_n a_n A_n \right) \delta \rho' \right] = 0. \quad (4)$$

It yields

$$\rho(t) = \frac{1}{Z} \exp \left(-\sum_n a_n(t) A_n \right), \quad (5)$$

where

$$Z = \exp(\Phi) = \text{Tr} \left[\exp \left(-\sum_n a_n(t) A_n \right) \right] \quad (6)$$

is the normalizing factor making $\text{Tr}[\rho(t)] = 1$. The statistical operator (5) is sometimes called the relevant statistical operator. One can see that it corresponds to the generalized Gibbs state described by some set of observables. If a true state of a system is unknown or very complicated, then one can utilize reduced (incomplete) description (5), characterized by the knowledge of mean values of some observables only, to make a reasonable estimate of any other observable B at that time by means of the equation $\langle B \rangle(t) = \text{Tr}[B \rho(t)]$. The crucial point here is the choice of the set of relevant observables, which is adequate for the reduced description of a system. For example, for a thermalized hydrodynamically expanding system, the relevant observables are the energy-momentum tensor and currents of conserved quantities. The corresponding relevant statistical operator for such a system is called sometimes the ‘‘quasiequilibrium statistical operator.’’

Here, to make the problem tractable, we choose as relevant observables mean values of energy and momentum density, as well as particle number density, of a free nonrelativistic scalar field defined at some moment of time. We begin with the Lagrangian density for a free real relativistic scalar field in Minkowski spacetime $x^\mu = (t, \mathbf{r})$, $\mathbf{r} = (x, y, z)$ [we use the convention $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$],

$$L = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{m^2}{2} \phi^2. \quad (7)$$

The canonical momentum field is then $\pi = \dot{\phi}$, where an overdot denotes a derivative with respect to time, and the Hamiltonian density is given by

$$H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2, \quad (8)$$

where $\nabla = (\partial_1, \partial_2, \partial_3) = (d/dx, d/dy, d/dz)$. The corresponding energy-momentum tensor reads

$$T^{\mu\nu}(x) = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} L. \quad (9)$$

There are different methods to arrive at an effective nonrelativistic description for a real scalar field. Typically these methods start with an appropriate field redefinition. Here, for the sake of convenience, we relate the complex nonrelativistic field Ψ to the real relativistic field ϕ by using the relations (see, e.g., Ref. [9])

$$\phi(t, \mathbf{r}) = \frac{1}{\sqrt{2m}} (e^{-imt} \Psi(t, \mathbf{r}) + e^{imt} \Psi^\dagger(t, \mathbf{r})), \quad (10)$$

$$\pi(t, \mathbf{r}) = -i \sqrt{\frac{m}{2}} (e^{-imt} \Psi(t, \mathbf{r}) - e^{imt} \Psi^\dagger(t, \mathbf{r})). \quad (11)$$

Equations (10) and (11) give a one-to-one mapping between the complex-valued Ψ and the real-valued ϕ and its conjugate momentum π . The quantization prescriptions $[\phi(t, \mathbf{r}), \pi(t, \mathbf{r}')] = i\delta^{(3)}(\mathbf{r} - \mathbf{r}')$, $[\phi(t, \mathbf{r}), \phi(t, \mathbf{r}')] = [\pi(t, \mathbf{r}), \pi(t, \mathbf{r}')] = 0$ result in the commutation relations

$$[\Psi(t, \mathbf{r}), \Psi^\dagger(t, \mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad (12)$$

and

$$[\Psi(t, \mathbf{r}), \Psi(t, \mathbf{r}')] = [\Psi^\dagger(t, \mathbf{r}), \Psi^\dagger(t, \mathbf{r}')] = 0. \quad (13)$$

The Fourier-transformed operators are defined as

$$\Psi(t, \mathbf{p}) = (2\pi)^{-3/2} \int d^3 r e^{-i\mathbf{p}\mathbf{r}} \Psi(t, \mathbf{r}), \quad (14)$$

$$\Psi^\dagger(t, \mathbf{p}) = (2\pi)^{-3/2} \int d^3 r e^{i\mathbf{p}\mathbf{r}} \Psi^\dagger(t, \mathbf{r}). \quad (15)$$

They satisfy the following canonical commutation relations:

$$[\Psi(t, \mathbf{p}), \Psi^\dagger(t, \mathbf{p}')] = \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \quad (16)$$

and

$$[\Psi(t, \mathbf{p}), \Psi(t, \mathbf{p}')] = [\Psi^\dagger(t, \mathbf{p}), \Psi^\dagger(t, \mathbf{p}')] = 0. \quad (17)$$

In order to take the nonrelativistic limit in the Hamiltonian density, one can substitute Eqs. (10) and (11) into Eq. (8). The corresponding expression contains rapidly oscillating terms proportional to $e^{\pm 2imt}$. These terms are usually neglected in the nonrelativistic limit, because in the limit of large m they average to zero on timescales larger than $1/m$, and the remaining terms are expected to be slowly varying compared to the timescale $1/m$. We then obtain the following nonrelativistic Hamiltonian density²:

$$H(t, \mathbf{r}) = T^{00}(t, \mathbf{r}) = \frac{1}{2m} \nabla \Psi \nabla \Psi^\dagger + m \Psi \Psi^\dagger. \quad (18)$$

In a similar way, using Eq. (9) and neglecting terms with fast oscillatory factors $e^{\pm 2imt}$, we obtain that ($j = 1, 2, 3$)

$$T^{0j}(t, \mathbf{r}) = -\frac{i}{2} (\Psi \partial^j \Psi^\dagger - \Psi^\dagger \partial^j \Psi). \quad (19)$$

In the nonrelativistic approximation, the particle number density is given by

$$N(t, \mathbf{r}) = \Psi \Psi^\dagger. \quad (20)$$

The quasiequilibrium statistical operator, associated with the expectation values of the selected observables (18)–(20), reads³

$$\rho = \frac{1}{Z} \hat{\rho}, \quad (21)$$

where Z is the partition function,

$$Z = \text{Tr}[\hat{\rho}], \quad (22)$$

and

$$\hat{\rho} = \exp \left[- \int d^3 r \beta(\mathbf{r}) \left(\frac{1}{\sqrt{1 - \mathbf{u}^2(\mathbf{r})}} T^{00} + \frac{u_j(\mathbf{r})}{\sqrt{1 - \mathbf{u}^2(\mathbf{r})}} T^{0j} - \mu(\mathbf{r}) \Psi \Psi^\dagger \right) \right]. \quad (23)$$

Here $\beta = 1/T$ is inverse temperature, $\mathbf{u} = (u^1, u^2, u^3)$ ($u_j = -u^j$) is collective velocity, μ is chemical potential, $j = 1, 2, 3$, and summation of repeated indices is implied.

Given (18) and (19), we may now consider the effective description for such a model in the nonrelativistic limit

when $\frac{1}{\sqrt{1 - \mathbf{u}^2}} \approx (1 + \frac{1}{2} \mathbf{u}^2)$ and $\frac{\mathbf{u}}{\sqrt{1 - \mathbf{u}^2}} \approx \mathbf{u}$. Then, taking into account that $\mathbf{u}^2 (\frac{1}{2m} \nabla \Psi \nabla \Psi^\dagger + m \Psi \Psi^\dagger) \approx \mathbf{u}^2 m \Psi \Psi^\dagger$ in the nonrelativistic limit of large m , we obtain

$$\left(1 + \frac{1}{2} \mathbf{u}^2(\mathbf{r}) \right) T^{00} \approx \frac{1}{2m} \nabla \Psi \nabla \Psi^\dagger + m \left(1 + \frac{1}{2} \mathbf{u}^2(\mathbf{r}) \right) \Psi \Psi^\dagger. \quad (24)$$

Finally, taking into account Eqs. (18), (19), and (24), one can rewrite (23) in the form

$$\hat{\rho} = \exp \left[- \int d^3 r \beta(\mathbf{r}) \left(\frac{1}{2m} [-i \nabla - m \mathbf{u}(\mathbf{r})] \times \Psi [i \nabla - m \mathbf{u}(\mathbf{r})] \Psi^\dagger - \hat{\mu}(\mathbf{r}) \Psi \Psi^\dagger \right) \right], \quad (25)$$

where

$$\hat{\mu} = \mu - m. \quad (26)$$

If actual expectation values $\langle T^{0j} \rangle = 0$, then $\mathbf{u} = \mathbf{0}$ and the quasiequilibrium statistical operator (25) describes a nonexpanding system of noninteracting nonrelativistic bosons. Below we demonstrate that the quasiequilibrium statistical operator of an expanding system, see Eq. (25), can be rewritten to such a form by means of a simple unitary redefinition of Ψ and Ψ^\dagger fields, if the collective velocity \mathbf{u} belongs to the class of the potential velocity fields,

$$\mathbf{u}(\mathbf{r}) = -\frac{1}{m} \nabla \theta(\mathbf{r}), \quad (27)$$

where θ is a dimensionless flow potential. For this aim let us rewrite the operator-valued fields Ψ and Ψ^\dagger in terms of the fields $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ as follows:

$$\Psi(\mathbf{r}) = e^{-i\theta(\mathbf{r})} \hat{\Psi}(\mathbf{r}), \quad (28)$$

$$\Psi^\dagger(\mathbf{r}) = e^{i\theta(\mathbf{r})} \hat{\Psi}^\dagger(\mathbf{r}). \quad (29)$$

One can see that

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad (30)$$

and

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')] = [\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = 0. \quad (31)$$

Substituting (28) and (29) into (25) and accounting for Eq. (27), we have

²It is noteworthy that this expression can be also obtained as a leading term in a low-energy ($p^2/m^2 \ll 1$) expansion by using a nonlocal field redefinition proposed in Ref. [9] instead of the local one defined in Eqs. (10) and (11).

³For simplicity, we will suppress the dependence on t .

$$\hat{\rho} = \exp \left[- \int d^3r \beta(\mathbf{r}) \left(\frac{1}{2m} [-i\nabla] \hat{\Psi} [i\nabla] \hat{\Psi}^\dagger - \hat{\mu}(\mathbf{r}) \hat{\Psi} \hat{\Psi}^\dagger \right) \right]. \quad (32)$$

Hence, we obtain that the quasiequilibrium statistical operator $\rho = \hat{\rho}/Z$ describes the expanding state of field Ψ , see Eq. (25), and nonexpanding state of the transformed field $\hat{\Psi}$, see Eq. (32).

III. FIXED PARTICLE NUMBER CONSTRAINT IN EXACTLY SOLVABLE MODEL OF THE QUASIEQUILIBRIUM STATE

Calculations of expectation values with statistical operator are significantly simplified if the corresponding statistical operator can be diagonalized in some representation. To make it possible with the quasiequilibrium statistical operator ρ , see Eqs. (21), (25), and (32), below we assume that $\beta = 1/T$ is constant and that the chemical potential $\hat{\mu}(\mathbf{r})$ reads

$$\hat{\mu}(\mathbf{r}) = -\frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) + \mu^*, \quad (33)$$

where $\mu^* = \text{const}$. Such a choice for the chemical potential means a ‘‘harmonic trap’’ distribution of particles. Then Eq. (32) takes the form

$$\hat{\rho} = e^{-\beta K}, \quad (34)$$

where K is defined by

$$K = \int d^3r \hat{\Psi}^\dagger(\mathbf{r}) \left(-\frac{1}{2m} \nabla^2 - \hat{\mu}(\mathbf{r}) \right) \hat{\Psi}(\mathbf{r}). \quad (35)$$

It is worth noting that K does not commute with the Hamiltonian $H = \int d^3r H(t, \mathbf{r})$, see Eq. (18). Therefore, the zero-temperature ground state of the statistical operator is an eigenstate of K but not of H .

It is well known that K , see Eqs. (33) and (35), can be diagonalized in the oscillator representation

$$\hat{\Psi}(\mathbf{r}) = \sum_{n,k,l=0}^{\infty} \alpha(n, k, l) \phi_n(x) \phi_k(y) \phi_l(z), \quad (36)$$

where the creation $\alpha^\dagger(n, k, l)$ and annihilation $\alpha(n, k, l)$ operators satisfy the commutation relations

$$[\alpha(n, k, l), \alpha^\dagger(n', k', l')] = \delta_{nn'} \delta_{kk'} \delta_{ll'}, \quad (37)$$

and

$$[\alpha(n, k, l), \alpha(n', k', l')] = [\alpha^\dagger(n, k, l), \alpha^\dagger(n', k', l')] = 0. \quad (38)$$

Functions $\phi_n(x)$, $\phi_k(y)$, and $\phi_l(z)$ are the harmonic oscillator eigenfunctions, for example,

$$\phi_n(x) = (2^n n! \pi^{1/2} b_x)^{-1/2} H_n \left(\frac{x}{b_x} \right) \exp \left(-\frac{1}{2} \left(\frac{x}{b_x} \right)^2 \right), \quad (39)$$

where $H_n(x/b_x)$ is the Hermite polynomial, and

$$\epsilon_n = \omega_x \left(n + \frac{1}{2} \right), \quad (40)$$

$$b_x = (m\omega_x)^{-1/2}. \quad (41)$$

In such a basis, the K reads

$$K = \sum_{n,k,l=0}^{\infty} (\epsilon_n + \epsilon_k + \epsilon_l - \mu^*) \alpha^\dagger(n, k, l) \alpha(n, k, l). \quad (42)$$

This implies that statistical operator $\rho = \hat{\rho}/Z$ involves states with a various number of particles N and describes a grand-canonical ensemble.⁴ To consider the canonical subensemble, where the number of particles is fixed, one needs to make the corresponding projection and define the canonical statistical operator,

$$\rho_N = \frac{1}{Z_N} \hat{\rho}_N, \quad (43)$$

which corresponds to subensemble of events with fixed particle number constraint. Here $\hat{\rho}_N = \mathcal{P}_N \hat{\rho} \mathcal{P}_N$, where \mathcal{P}_N is the projection operator that automatically invokes the corresponding constraint, and Z_N is the corresponding canonical partition function that is needed to insure the probability interpretation of the ensemble obtained in the result of this projection, $Z_N = \text{Tr}[\hat{\rho}_N]$. Corresponding formalism was developed in Ref. [6] and the reader is referred to this paper for details of the calculations. Taking into account that ρ_N does not depend on μ^* [this dependence is factored out in Eq. (43)], one can rewrite Eq. (43) as

$$\rho_N = \frac{1}{Z_N^0} \hat{\rho}_N^0. \quad (44)$$

Here we denote $\hat{\rho}_N$ and Z_N associated with $\mu^* = 0$ as $\hat{\rho}_N^0$ and Z_N^0 , respectively. The canonical partition functions satisfy the recursive formula [10]

⁴Note that the mean particle number $\langle N \rangle$, defined by the grand-canonical ensemble, as well as the particle number N , are the same for Ψ , $\hat{\Psi}$, and α particles because transformations (28) and (36) do not mix creation and annihilation operators and preserve the standard commutation relations.

$$nZ_n^0 = \sum_{s=1}^n \sum_{\mathbf{j}} e^{-s\beta\epsilon_{\mathbf{j}}} Z_{n-s}^0, \quad (45)$$

where $\epsilon_{\mathbf{j}} = \epsilon_{n,k,l} = \epsilon_n + \epsilon_k + \epsilon_l$, $Z_0^0 = \langle 0|0 \rangle = 1$, and $n = 1, \dots, N$.

The goal of this study is to evaluate the two-boson momentum correlation functions at fixed multiplicities for an expanding system. Such a correlation function is defined as the ratio of the two-particle momentum spectrum to one-particle ones and can be written at fixed multiplicities as

$$C_N(\mathbf{k}, \mathbf{q}) = G_N \frac{\langle \Psi^\dagger(\mathbf{p}_1) \Psi^\dagger(\mathbf{p}_2) \Psi(\mathbf{p}_1) \Psi(\mathbf{p}_2) \rangle_N}{\langle \Psi^\dagger(\mathbf{p}_1) \Psi(\mathbf{p}_1) \rangle_N \langle \Psi^\dagger(\mathbf{p}_2) \Psi(\mathbf{p}_2) \rangle_N}. \quad (46)$$

Here and below $\langle \dots \rangle_N \equiv \text{Tr}[\rho_N \dots]$, $\mathbf{k} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$, and G_N is the normalization constant. The latter is needed to normalize the theoretical Bose-Einstein correlation function in accordance with normalization that is applied by experimentalists: $C^{\text{exp}}(\mathbf{k}, \mathbf{q}) \rightarrow 1$ for $|\mathbf{q}| \rightarrow \infty$.

The assumption of the potential velocity field (27) and transformations (28) and (29) allow one to apply for calculations of the one- and two-particle spectra the same technique that was used in Ref. [6] for a nonexpanding system. We start by using expectation values of operators α and α^\dagger (these expressions are calculated in Ref. [6] and are provided below for the reader's convenience),

$$\langle \alpha^\dagger(\mathbf{j}_1) \alpha(\mathbf{j}_2) \rangle_N = \delta_{\mathbf{j}_1 \mathbf{j}_2} \sum_{s=1}^N e^{-s\beta\epsilon_{\mathbf{j}_2}} \frac{Z_{N-s}^0}{Z_N^0}, \quad (47)$$

$$\begin{aligned} & \langle \alpha^\dagger(\mathbf{j}_1) \alpha^\dagger(\mathbf{j}_2) \alpha(\mathbf{j}_3) \alpha(\mathbf{j}_4) \rangle_N \\ &= (\delta_{\mathbf{j}_1 \mathbf{j}_4} \delta_{\mathbf{j}_2 \mathbf{j}_3} + \delta_{\mathbf{j}_1 \mathbf{j}_3} \delta_{\mathbf{j}_2 \mathbf{j}_4}) \sum_{s=1}^{N-1} \sum_{s'=1}^{N-s} e^{-s\beta\epsilon_{\mathbf{j}_4}} e^{-s'\beta\epsilon_{\mathbf{j}_3}} \frac{Z_{N-s-s'}^0}{Z_N^0}, \end{aligned} \quad (48)$$

where for notational simplicity we write \mathbf{j} instead of (n, k, l) . The next step is to utilize Eqs. (47), (48), and (36) to obtain expectation values $\langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}(\mathbf{r}_2) \rangle_N$ and $\langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}^\dagger(\mathbf{r}_2) \hat{\Psi}(\mathbf{r}_3) \hat{\Psi}(\mathbf{r}_4) \rangle_N$. Then, for example,

$$\begin{aligned} & \langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}(\mathbf{r}_2) \rangle_N \\ &= \sum_{s=1}^N \frac{Z_{N-s}^0}{Z_N^0} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \phi_n^*(x_1) \phi_k^*(y_1) \phi_l^*(z_1) \phi_n(x_2) \\ & \quad \times \phi_k(y_2) \phi_l(z_2) e^{-\frac{3}{2}s\beta\omega} e^{-s\beta\omega(n+k+l)}, \end{aligned} \quad (49)$$

where the eigenfunctions are defined by Eq. (39). In order to keep things as simple as possible, here and below we assume that $\omega_x = \omega_y = \omega_z = \omega$, then $b_x = b_y = b_z = b$ and

$$b = (m\omega)^{-1/2}, \quad (50)$$

see Eqs. (39) and (41). Utilizing the integral representation of the Hermite function,

$$H_n\left(\frac{x}{b}\right) = \left(\frac{b}{i}\right)^n \frac{b}{2\sqrt{\pi}} e^{\frac{x^2}{b^2}} \int_{-\infty}^{+\infty} v^n e^{-\frac{1}{4}b^2 v^2 + i x v} dv, \quad (51)$$

one can simplify corresponding expressions. This was done in Ref. [6], and the results can be written as

$$\langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}(\mathbf{r}_2) \rangle_N = \sum_{s=1}^N \frac{Z_{N-s}^0}{Z_N^0} \hat{\Phi}(\mathbf{r}_1, \mathbf{r}_2, \beta\omega s), \quad (52)$$

and

$$\begin{aligned} & \langle \hat{\Psi}^\dagger(\mathbf{r}_1) \hat{\Psi}^\dagger(\mathbf{r}_2) \hat{\Psi}(\mathbf{r}_3) \hat{\Psi}(\mathbf{r}_4) \rangle_N \\ &= \sum_{s=1}^{N-1} \sum_{s'=1}^{N-s} \frac{Z_{N-s-s'}^0}{Z_N^0} (\hat{\Phi}(\mathbf{r}_1, \mathbf{r}_3, \beta\omega s) \hat{\Phi}(\mathbf{r}_2, \mathbf{r}_4, \beta\omega s') \\ & \quad + \hat{\Phi}(\mathbf{r}_1, \mathbf{r}_4, \beta\omega s) \hat{\Phi}(\mathbf{r}_2, \mathbf{r}_3, \beta\omega s')), \end{aligned} \quad (53)$$

where

$$\begin{aligned} \hat{\Phi}(\mathbf{r}_1, \mathbf{r}_2, \beta\omega s) &= \frac{1}{(2\pi)^{3/2}} \frac{1}{b^3} (\sinh(\beta\omega s))^{-3/2} \\ & \quad \times \exp\left(-\frac{\mathbf{r}_1^2 + \mathbf{r}_2^2}{2b^2 \tanh(\beta\omega s)}\right) \\ & \quad \times \exp\left(\frac{\mathbf{r}_1 \mathbf{r}_2}{b^2 \sinh(\beta\omega s)}\right). \end{aligned} \quad (54)$$

To evaluate particle momentum spectra for an expanding system, one needs first to specify the velocity profile. Here, for the purpose of illustration, we chose flow profile in the linear isotropic form,

$$\mathbf{u}(\mathbf{r}) = \kappa \mathbf{r}. \quad (55)$$

Then the solution of Eq. (27) can be written as

$$\theta(\mathbf{r}) = -\frac{m\kappa \mathbf{r}^2}{2}. \quad (56)$$

Equation (56) allows us to relate Ψ with $\hat{\Psi}$, see Eqs. (28) and (29), and thereby to define the expectation values $\langle \Psi^\dagger(\mathbf{r}_1) \Psi(\mathbf{r}_2) \rangle_N$ and $\langle \Psi^\dagger(\mathbf{r}_1) \Psi^\dagger(\mathbf{r}_2) \Psi(\mathbf{r}_3) \Psi(\mathbf{r}_4) \rangle_N$ for an expanding system. It is then a simple matter to perform Fourier transformations and calculate one- and two-particle momentum spectra, which are defined as corresponding expectation values for Fourier-transformed field operators (14) and (15). The results are

$$\langle \Psi^\dagger(\mathbf{p}_1)\Psi(\mathbf{p}_1) \rangle_N = \sum_{s=1}^N \frac{Z_{N-s}^0}{Z_N^0} \Phi(\mathbf{p}_1, \mathbf{p}_1, \beta\omega s, \kappa), \quad (57)$$

and

$$\begin{aligned} & \langle \Psi^\dagger(\mathbf{p}_1)\Psi^\dagger(\mathbf{p}_2)\Psi(\mathbf{p}_1)\Psi(\mathbf{p}_2) \rangle_N \\ &= \sum_{s=1}^{N-1} \sum_{s'=1}^{N-s} \frac{Z_{N-s-s'}^0}{Z_N^0} (\Phi(\mathbf{p}_1, \mathbf{p}_1, \beta\omega s, \kappa)\Phi(\mathbf{p}_2, \mathbf{p}_2, \beta\omega s', \kappa) \\ &+ \Phi(\mathbf{p}_1, \mathbf{p}_2, \beta\omega s, \kappa)\Phi(\mathbf{p}_2, \mathbf{p}_1, \beta\omega s', \kappa)), \end{aligned} \quad (58)$$

respectively. Here we introduce notation

$$\begin{aligned} & \Phi(\mathbf{p}_1, \mathbf{p}_2, \beta\omega s, \kappa) \\ &= \frac{b^3(1+m^2\kappa^2b^4)^{-3/2}}{(2\pi \sinh(\beta\omega s))^{3/2}} \exp\left(-\frac{b^2}{4(1+m^2\kappa^2b^4)}\right. \\ & \times \left[(\mathbf{p}_1 + \mathbf{p}_2)^2 \tanh\left(\frac{\beta\omega s}{2}\right) + \frac{(\mathbf{p}_2 - \mathbf{p}_1)^2}{\tanh\left(\frac{\beta\omega s}{2}\right)} \right. \\ & \left. \left. + 2i(\mathbf{p}_2^2 - \mathbf{p}_1^2)m\kappa b^2 \right] \right). \end{aligned} \quad (59)$$

One can see that, for a nonexpanding system, i.e., for $\kappa = 0$, Eqs. (57)–(59) are reduced to the corresponding expressions presented in Ref. [6].⁵ Furthermore, one can easily see that

$$|\Phi(\mathbf{p}_1, \mathbf{p}_2, \beta\omega s, \kappa)| = \xi^3 |\Phi(\xi\mathbf{p}_1, \xi\mathbf{p}_2, \beta\omega s, 0)|, \quad (60)$$

where

$$\xi = \frac{1}{\sqrt{1+m^2\kappa^2b^4}} \leq 1. \quad (61)$$

Finally, to completely specify the two-boson correlation function (46), one needs to estimate the normalization constant G_N . It can be realized by means of the limit $|\mathbf{q}| \rightarrow \infty$ at fixed \mathbf{k} in the corresponding expression. One can readily see that proper normalization is reached if

$$G_N = \frac{Z_N^0}{Z_{N-2}^0} \left(\frac{Z_{N-1}^0}{Z_N^0} \right)^2. \quad (62)$$

This value coincides with normalization constant calculated in Ref. [6] for a nonexpanding system. Equations (46), (57)–(59), and (62) serve in the next section as the starting point for the investigation of multiplicity and flow dependencies of two-particle Bose-Einstein momentum correlations at fixed particle number constraint.

⁵Note that here we slightly simplified and optimized the notations as compared to Ref. [6].

IV. TWO-BOSON MOMENTUM CORRELATIONS AT FIXED MULTIPLICITIES IN THE THERMAL MODEL OF AN EXPANDING SYSTEM

In the following, we focus on the multiplicity and flow dependencies of the correlation function (46). Below we assume that the model provides qualitatively reasonable estimations of these dependencies beyond the nonrelativistic region $p^2/m^2 \ll 1$. To discuss relations to $p + p$ collisions at the LHC, we utilize for numerical calculations the set of parameters corresponding roughly to the values at the system's breakup in $p + p$ collisions at the LHC energies. For specificity, we take the particle's mass as of a charged pion, $m = 139.57$ MeV, and the temperature $T = 150$ MeV. Following Ref. [6], we introduce parameter R such as

$$\omega = \frac{1}{R\sqrt{\beta m}} \quad (63)$$

and treat R as a free parameter instead of ω . For R we use 1.5 fm. Using Eq. (63), one gets

$$\beta\omega = \frac{1}{R} \sqrt{\frac{\beta}{m}} = \frac{\Lambda_T}{R}, \quad (64)$$

and

$$b = \frac{1}{\sqrt{m\omega}} = \sqrt{\Lambda_T R}, \quad (65)$$

where Λ_T is the thermal wavelength, which we defined as

$$\Lambda_T = \frac{1}{\sqrt{mT}}. \quad (66)$$

It is convenient to relate parameter κ in Eq. (55) with a physically meaningful parameter in relativistic particle and nucleus collisions, namely, with mean flow velocity of the system at fixed particle number constraint $\sqrt{\langle \mathbf{u}^2 \rangle_N}$, where

$$\langle \mathbf{u}^2 \rangle_N = \frac{\int dx dy dz \mathbf{u}^2 \langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \rangle_N}{\int dx dy dz \langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \rangle_N}. \quad (67)$$

Substituting (55) into the right-hand side of Eq. (67), we have

$$\langle \mathbf{u}^2 \rangle_N = \kappa^2 \langle \mathbf{r}^2 \rangle_N = 3\kappa^2 \langle x^2 \rangle_N, \quad (68)$$

where

$$\langle x^2 \rangle_N = \frac{\int dx dy dz x^2 \langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \rangle_N}{\int dx dy dz \langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \rangle_N}. \quad (69)$$

Because of relations (28) and (29), $\langle \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) \rangle_N = \langle \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r}) \rangle_N$, thereby the mean spatial size of the system

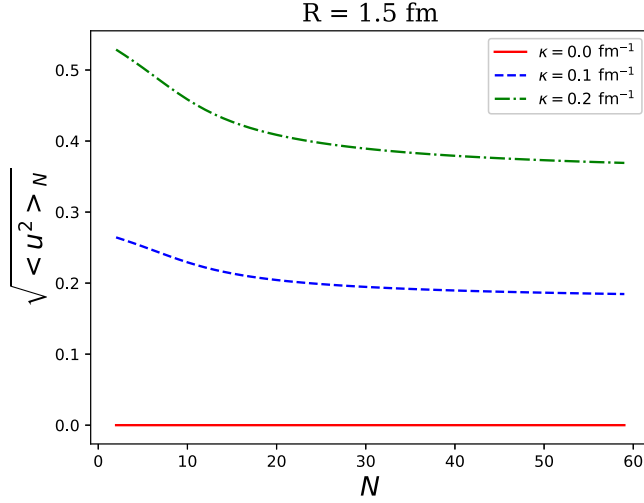


FIG. 1. The $\sqrt{\langle \mathbf{u}^2 \rangle_N}$ dependence on N at different κ .

at fixed multiplicity $\sqrt{\langle x^2 \rangle_N}$ does not depend on intensity of flow. A corresponding expression has been calculated in Ref. [6]. For the used set of parameter values, $\sqrt{\langle x^2 \rangle_N}$ is close to R . Figure 1 shows mean flow velocity at fixed multiplicity $\sqrt{\langle \mathbf{u}^2 \rangle_N}$ as a function of N for several different values of the strength of the expansion parameter κ . For κ we use 0.0, 0.1, and 0.2 fm^{-1} .

Then we investigate how the two-boson momentum correlation function (46) is affected by the flow. The results are plotted in Fig. 2 for various values of κ at $k = 0.25 \text{ GeV}/c$. One can see that the intercept of the correlation function $C_N(\mathbf{k}, \mathbf{0})$ decreases when the strength of the expansion parameter κ increases.

To have some insight into why it happens, it is useful to calculate the ground-state contribution to particle momentum spectra, $n_N^0(\mathbf{p}, \kappa)$. To derive this expression, one needs

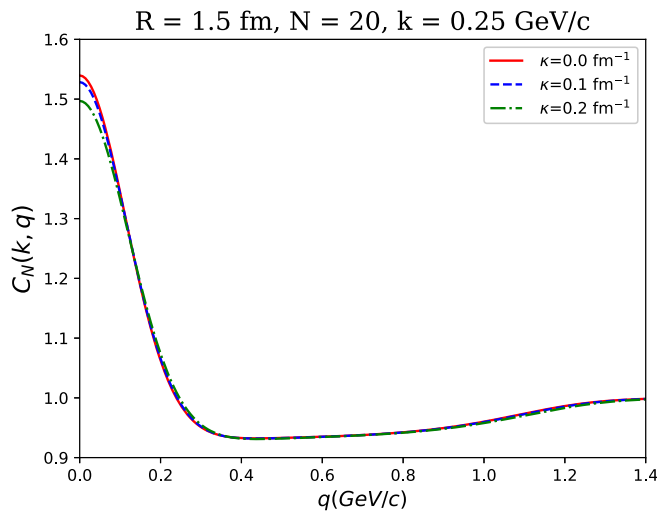


FIG. 2. Correlation functions with $k = 0.25 \text{ GeV}/c$, $N = 20$, $R = 1.5 \text{ fm}$ at different κ . See the text for details.

to take $k = l = n = 0$ in Eq. (49), and then follow the derivation of the one-particle momentum spectrum $n_N(\mathbf{p}, \kappa) = \langle \Psi^\dagger(\mathbf{p})\Psi(\mathbf{p}) \rangle_N$, see Eq. (57). The result is

$$n_N^0(\mathbf{p}, \kappa) = \sum_{s=1}^N \frac{Z_{N-s}^0}{Z_N^0} \frac{b^3 e^{-\frac{3}{2}\beta\omega s}}{\pi^{\frac{3}{2}}(1+m^2\kappa^2 b^4)^{3/2}} \exp\left(-\frac{b^2 \mathbf{p}^2}{1+m^2\kappa^2 b^4}\right). \quad (70)$$

While $n_N(\mathbf{p}, \kappa)$ and $n_N^0(\mathbf{p}, \kappa)$ both depend on ξ and thereby on κ ,

$$n_N(\mathbf{p}, \kappa) = \xi^3 n_N(\xi \mathbf{p}, 0), \quad (71)$$

$$n_N^0(\mathbf{p}, \kappa) = \xi^3 n_N^0(\xi \mathbf{p}, 0), \quad (72)$$

see Eqs. (57), (59)–(61), and (70), the occupation of the ground state $N_0 = \int d^3 p n_N^0(\mathbf{p}, \kappa)$ and the ground-state condensate fraction N_0/N do not depend on κ at fixed N . On the other hand, because $n_N^0(\mathbf{p}, 0)/n_N(\mathbf{p}, 0)$ is a decreasing function of particle momentum, we obtain that $n_N^0(\xi \mathbf{p}, 0)/n_N(\xi \mathbf{p}, 0)$ increases when ξ decreases. Accounting for Eq. (61), we then conclude that an increase of κ results in an increase of the $n_N^0(\mathbf{p}, \kappa)/n_N(\mathbf{p}, \kappa)$ ratio, because

$$\frac{n_N^0(\mathbf{p}, \kappa)}{n_N(\mathbf{p}, \kappa)} = \frac{n_N^0(\xi \mathbf{p}, 0)}{n_N(\xi \mathbf{p}, 0)}, \quad (73)$$

see Eqs. (71) and (72). In Fig. 3 we plot this ratio as a function of particle momentum for several different values of the κ parameter. The curves show that the ground-state fraction of the particle momentum spectra increases at moderately high momenta when κ increases, signaling the

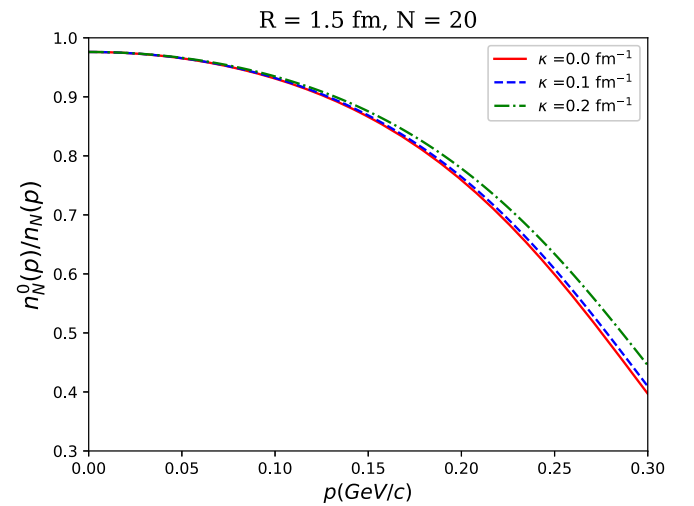


FIG. 3. The $n_N^0(\mathbf{p})/n_N(\mathbf{p})$ ratio as a function of particle momentum p for $N = 20$, $R = 1.5 \text{ fm}$, and for several different values of κ . See the text for details.

increasing importance of the ground-state contribution to particle momentum spectra. This implies that particle emission at such momenta becomes more coherent when intensity of flow increases, leading thereby to the decrease of the intercept of the two-boson momentum correlation function: It is well known that the intercept of the two-boson momentum correlation function for a chaotic emission is equal to 2, and that the intercept is equal to 1 for a coherent emission; see, e.g., Ref. [11].

One observes from Fig. 2 the essential non-Gaussianity of the correlation functions beyond the region of the correlation peak. Such a non-Gaussianity was discussed for a nonexpanding system in Ref. [6], where it was demonstrated that $C_N(\mathbf{k}, \mathbf{q})$ can be rather well fitted by the two-Gaussian expression. If the fitting procedure is restricted to the correlation peak region, then the correlation function is well fitted by the one-Gaussian expression

$$C_N^{1g}(\mathbf{k}, \mathbf{q}) = 1 + \lambda(\mathbf{k}, N) e^{-q^2 R_{\text{HBT}}^2(\mathbf{k}, N)}, \quad (74)$$

where $1 + \lambda(\mathbf{k}, N)$ is equal to the intercept of the correlation function $C_N(\mathbf{k}, \mathbf{0})$. In order to make contact with the previous findings of Ref. [6], one can relate correlation functions of an expanding system $C_N(\mathbf{k}, \mathbf{q}, \kappa)$ with the ones for a nonexpanding system $C_N(\mathbf{k}, \mathbf{q}, 0)$. This can be accomplished using Eqs. (46) and (57)–(60). The result is

$$C_N(\mathbf{k}, \mathbf{q}, \kappa) = C_N(\xi \mathbf{k}, \xi \mathbf{q}, 0). \quad (75)$$

This relation means, in particular, that

$$\lambda(\mathbf{k}, N, \kappa) = \lambda(\xi \mathbf{k}, N, 0), \quad (76)$$

$$R_{\text{HBT}}(\mathbf{k}, N, \kappa) = \xi R_{\text{HBT}}(\xi \mathbf{k}, N, 0). \quad (77)$$

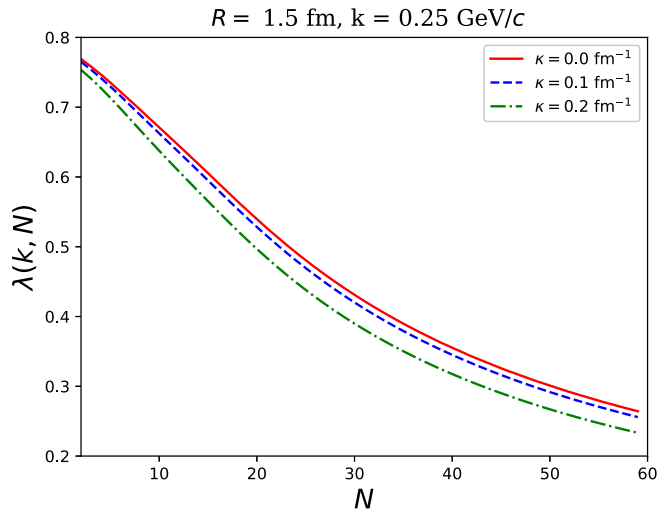


FIG. 4. The λ at $k = 0.25$ GeV/ c for several different values of κ . See the text for details.

It follows from Eq. (77) that increase of κ at fixed $\xi \mathbf{k}$ results in decrease of R_{HBT} .

Figure 4 displays the λ parameter as a function of N for various values of κ . All three curves reveal a consistent trend: increase of N results in decrease of the λ parameter, i.e., the intercept of the correlation function is reduced. Reasons for such a behavior were discussed in detail in Ref. [6]. In short, increase of N results in an increase of the value of the ground-state fraction N_0/N , leading for fairly high N to the noticeable Bose-Einstein condensation in the corresponding ground state of the fixed N canonical ensemble state. Such a condensation strengthens the coherent properties of the canonical ensemble state and results in the decrease of the intercept of the two-boson momentum correlation function when N increases.

Figure 5 shows R_{HBT} for $\kappa = 0.0$ and $\kappa = 0.2$ fm $^{-1}$ as a function on k for several different values of N . One can see that, unlike the mean spatial size, R_{HBT} depends on the intensity of flow.⁶ Also, one observes from this figure that, similar to zero flow results presented in Ref. [6], the interferometry radii are independent of N at moderately high pair momenta.

Finally, let us discuss possible relations of this model with high-multiplicity $p + p$ collision events at the LHC. For the purpose of illustration, we show in Fig. 6 some experimental data presented by the ATLAS [3] and CMS [4] Collaborations. It is worth noting that analysis procedures applied by the ATLAS and CMS Collaborations are quite different, therefore additional adjustment (which is not done in Fig. 6) is needed for direct comparison of the results, see Ref. [4] for details. First of all, one can deduce from the published data (see Fig. 6, left) that the two-boson momentum correlation radius parameters are small,⁷ compatible with the pion thermal wavelength, and do not change much with the collision energy. The latter seems to be natural if the actual size of the system is related to the mean multiplicity,⁸ because at high energies, where increase of the collision energy might be accompanied by the increase of the expansion rate, the mean multiplicity increases rather weakly with energy of collisions. Notice that small size of the system, together with the high rate of expansion (see, e.g., Ref. [13]), allow one to expect that there is no prolonged post-thermal stage of hadronic kinetic evolution, and therefore observed particle momentum spectra are not strongly influenced by the final-state hadronic rescatterings (apart from the Coulomb final-state

⁶Note that decrease of interferometry radii when intensity of flow increases can be interpreted as the decrease of “homogeneity lengths” [12] (sizes of the effective emission region).

⁷The results for exponential fits are shown. To compare the values of the radius parameters obtained from exponential and Gaussian fits, the R value of the Gaussian should be compared with $R/\sqrt{\pi}$ of the exponential form, see Ref. [3].

⁸We do not consider here effects conditioned by shape fluctuation of nucleon, see, e.g., Ref. [2] and references therein.

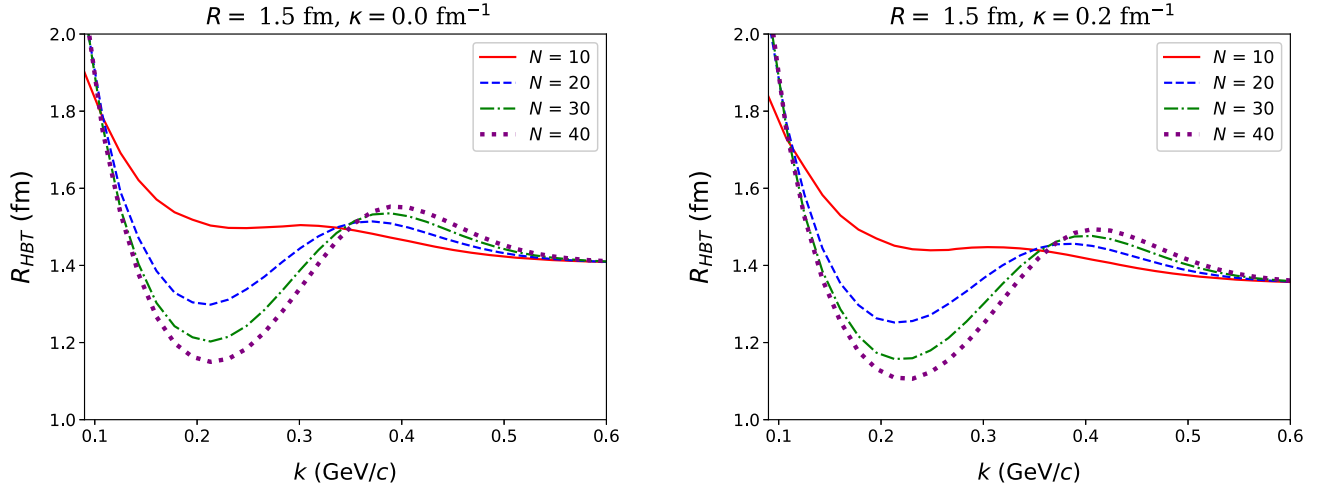


FIG. 5. HBT radii obtained from the one-Gaussian fit of the two-boson correlation function for several different values of N , as a function of the pair average momentum k .

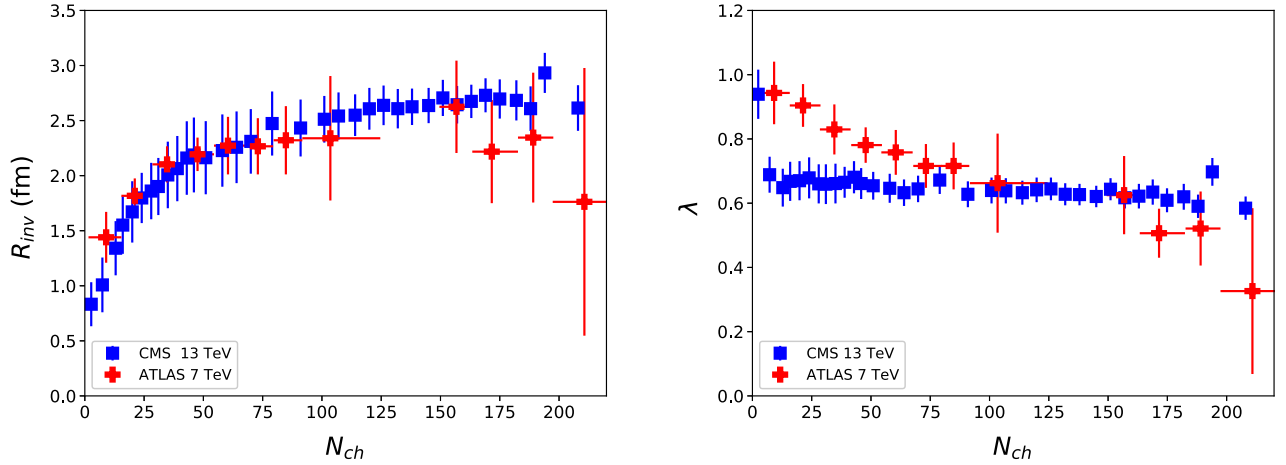


FIG. 6. The radius parameters obtained from exponential fits (left) and λ parameters (right), as a function of multiplicity. See Refs. [3,4] for details.

interactions, decays of resonances, etc.). Then, the saturation of the radius parameter with charged-particle multiplicity (see Fig. 6, left, and Refs. [3,4]) can indicate increase of the particle number density at momentum freeze-out for large values of charged-particle multiplicity. The latter, according to our analysis, results in the ground-state condensation. Such a condensation enhances the coherent properties of particle emission and, therefore, leads to decrease of the λ parameter. Interestingly enough, the experimental λ (see Fig. 6, right, and Refs. [3,4]) are rather small and, in fact, smaller than in relativistic heavy ion collisions, indicating the possibility of the formation of condensates in high-multiplicity $p + p$ collision events. This observation is not conclusive, however, because the λ parameter absorbs and reflects many effects, in particular, particle misidentification, contribution from decay of long-lived resonances, etc. Because of these complications,

theoretical description and model fitting of the λ parameters have so far received little attention, especially in comparison with the HBT radii. It seems, however, that in order to reveal ground-state condensate contribution to particle momentum spectra one needs to discriminate different contributions to the λ , and fit the λ parameters for various energies of collisions.

V. CONCLUSIONS

In the present paper we study two-boson momentum correlations at fixed particle number constraint in a simple analytically solvable model of a small thermal expanding system. For specificity, we use parameter values that correspond roughly to the values at the system's breakup in $p + p$ collisions at the LHC energies. We show that correlation strength parameter λ decreases with multiplicity

and that the HBT radius parameter tends to a constant at moderately large momenta when multiplicity increases. Both effects take place also at zero expansion velocity, see Ref. [6], and are associated with the increase of the ground-state fraction N_0/N at fairly large N when N increases. Furthermore, we find that the interferometry radius parameter at fixed multiplicity decreases when the flow increases and that the same is valid for correlation strength parameter λ . While the decrease of the interferometry radius parameter takes also place for averaged over multiplicities inclusive measurements of emission from thermalized expanding systems [12], the decrease of the λ parameter is specific for multiplicity-dependent measurements. We argue that the decrease of the λ parameter is conditioned by the increase of the ground-state contribution to the particle momentum spectra when the flow increases, i.e., by the increasing values of the $n_N^0(\mathbf{p})/n_N(\mathbf{p})$ ratio. We expect the main points of our analysis, such as N dependencies of particle momentum spectra and correlations, to hold if relativistic corrections are taken into account and suggest that certain features of the multiplicity-dependent measurements of the Bose-Einstein momentum correlations

in high-multiplicity $p + p$ collision events at the LHC can be conditioned by the presence of ground-state condensates.

We do not discuss here momentum dependencies of the correlation parameters at fixed multiplicity. For such an analysis considered simple nonrelativistic quantum-field model of the quasiequilibrium state cannot be applied, and relativistic extension of the model should be necessary. We hope that our paper will help stimulate research efforts in this and related directions.

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- [1] C. Shen, *Nucl. Phys.* **A1005**, 121788 (2021).
 - [2] B. Schenke, *Rep. Prog. Phys.* **84**, 082301 (2021).
 - [3] ATLAS Collaboration, *Eur. Phys. J. C* **75**, 466 (2015).
 - [4] A. M. Sirunyan *et al.* (CMS Collaboration), *Phys. Rev. C* **97**, 064912 (2018); *J. High Energy Phys.* 03 (2020) 14.
 - [5] V. M. Shapoval, P. Braun-Munzinger, Iu. A. Karpenko, and Yu. M. Sinyukov, *Phys. Lett. B* **725**, 139 (2013).
 - [6] M. D. Adzhymambetov, S. V. Akkelin, and Yu. M. Sinyukov, *Phys. Rev. D* **103**, 116012 (2021).
 - [7] D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics* (New York, Plenum Press, 1974); D. Zubarev, V. Morozov, and G. Röpke, *Statistical Mechanics of Nonequilibrium Processes. Volume 1: Basic Concepts. Kinetic Theory* (Akademie Verlag, Berlin, 1996); *Statistical Mechanics of Nonequilibrium Processes. Volume 2: Relaxation and Hydrodynamic Processes* (Akademie Verlag, Berlin, 1997); W. T. Grandy, Jr., *Phys. Rep.* **62**, 175 (1980).
 - [8] E. T. Jaynes, *Phys. Rev.* **106**, 620 (1957); **108**, 171 (1957).
 - [9] M. H. Namjoo, A. H. Guth, and D. I. Kaiser, *Phys. Rev. D* **98**, 016011 (2018).
 - [10] P. T. Landsberg, *Thermodynamics* (Interscience, New York, 1961); P. Borrmann and G. Franke, *J. Chem. Phys.* **98**, 2484 (1993).
 - [11] M. Gyulassy, S. K. Kauffmann, and L. W. Wilson, *Phys. Rev. C* **20**, 2267 (1979); M. I. Podgoretsky, *Fiz. Elem. Chastits At. Yadra* **20**, 628 (1989) [*Sov. J. Part. Nucl.* **20**, 266 (1989)]; D. H. Boal, C.-K. Gelbke, and B. K. Jennings, *Rev. Mod. Phys.* **62**, 553 (1990); U. A. Wiedemann and U. Heinz, *Phys. Rep.* **319**, 145 (1999); R. M. Weiner, *Phys. Rep.* **327**, 249 (2000); *Introduction to Bose-Einstein Correlations and Subatomic Interferometry* (Wiley, New York, 2000); M. Lisa, S. Pratt, R. Soltz, and U. Wiedemann, *Annu. Rev. Nucl. Part. Sci.* **55**, 357 (2005); R. Lednický, *Phys. Part. Nucl.* **40**, 307 (2009); Yu. M. Sinyukov and V. M. Shapoval, *Phys. Rev. D* **87**, 094024 (2013).
 - [12] Yu. M. Sinyukov, *Nucl. Phys.* **A566**, 589c (1994); in *Hot Hadronic Matter: Theory and Experiment* edited by J. Letessier, H. H. Gutbrod, and J. Rafelski (Plenum, New York, 1995), p. 309; S. V. Akkelin and Yu. M. Sinyukov, *Phys. Lett. B* **356**, 525 (1995); *Z. Phys. C* **72**, 501 (1996).
 - [13] T. Kalaydzhyan and E. Shuryak, *Phys. Rev. C* **91**, 054913 (2015).